

# THE RESIDUE DETERMINANT

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## Abstract

The purpose of this paper is to present the construction of a canonical determinant functional on elliptic pseudodifferential operators ( $\psi$ dos) associated to the Guillemin-Wodzicki residue trace.

The resulting residue determinant functional is multiplicative, a local invariant, and not defined by a regularization procedure. The residue determinant is consequently a quite different object to the zeta function determinant, which is non-local and non-multiplicative. Indeed, the residue determinant does not arise as the derivative of a trace on the complex power operators, and does not depend on a choice of spectral cut. The identification of a certain residue determinant with the index of an elliptic  $\psi$ do shows the residue determinant to be topologically significant.<sup>1</sup>

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<sup>1</sup>This work arose following conversations with Steve Rosenberg concerning higher Chern-Weil invariants, my thanks to him for his support and interest. I am also indebted to Kate Okikiolu for a helpful suggestion, the essential role of [Ok1, Ok2] in the current work is evident. I am grateful to Gerd Grubb for helpful comments and for pointing out a number of technical improvements. My thanks to Sylvie Paycha for interesting discussions and, in particular, for pointing out the fact in Remark(1.11).

## 1. DEFINITION AND PROPERTIES OF THE RESIDUE DETERMINANT

Let  $A$  be a  $\psi$ do of order  $\alpha \in \mathbb{R}$  acting on the space of smooth sections  $C^\infty(E)$  of a rank  $N$  vector bundle  $E$  over a compact boundaryless manifold  $M$  of dimension  $n$ . This means that in each local trivialization of  $E$  with  $U \times \mathbb{R}^N$ , with  $U$  an open subset of  $M$  identified with an open set in  $\mathbb{R}^n$ , and smooth functions  $\phi, \psi$  with  $\text{supp}(\phi), \text{supp}(\psi) \subset U$ , then for  $x \in U$  and  $f \in C_c^\infty(U, \mathbb{R}^N)$

$$(1.1) \quad (\phi A \psi)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_U e^{i(x-y) \cdot \xi} \mathbf{a}(x, \xi) f(y) dy d\xi ,$$

where  $\mathbf{a} \in S^\alpha(U)$ . We may write  $A = \text{OP}(\mathbf{a})$  on  $U$ .

Here,  $S^\alpha(U)$  is the symbol space of functions  $\mathbf{a}(x, \xi) \in C^\infty(U \times \mathbb{R}^n, (\mathbb{R}^N)^* \otimes \mathbb{R}^N)$  with values in  $N \times N$  matrices such that for all multi-indices  $\mu, \nu \in \mathbb{N}^n$ ,  $\partial_x^\mu \partial_\xi^\nu \mathbf{a}(x, \xi)$  is  $O((1 + |\xi|)^{\alpha - |\nu|})$ , uniformly in  $\xi$ , and, on compact subsets of  $U$ , uniformly in  $x$ . Write  $S(U)$  for  $\cup_{\alpha \in \mathbb{R}} S^\alpha(U)$  and  $S^{-\infty}(U)$  for  $\cap_{\alpha \in \mathbb{R}} S^\alpha(U)$ . Symbols  $\mathbf{a}, \mathbf{b} \in S(U)$  are said to be equivalent if  $\mathbf{a} - \mathbf{b} \in S^{-\infty}(U)$ , written  $\mathbf{a} \sim \mathbf{b}$ .

A symbol  $\mathbf{a} \in S^\alpha(U)$  is classical (1-step polyhomogeneous) of degree  $\alpha$  if there is a sequence  $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \dots$  with  $\mathbf{a}_j \in C^\infty(U \times \mathbb{R}^n \setminus \{0\}, (\mathbb{R}^N)^* \otimes \mathbb{R}^N)$  homogeneous in  $\xi$  of degree  $\alpha - j$  for  $|\xi| \geq 1$  such that  $\mathbf{a}(x, \xi) \sim \sum_{j=0}^{\infty} \mathbf{a}_j(x, \xi)$ ; thus,  $\mathbf{a}_j(x, t\xi) = t^{\alpha-j} \mathbf{a}_j(x, \xi)$  for  $t \geq 1, |\xi| \geq 1$ , and

$$\mathbf{a}(x, \xi) - \sum_{j=0}^{J-1} \mathbf{a}_j(x, \xi) \in S^{\alpha-J}(U) .$$

We may then write  $\mathbf{a} \sim (\mathbf{a}_0, \mathbf{a}_1, \dots)$ . A symbol  $\mathbf{b} \in S(U)$  is called *logarithmic* of type  $c \in \mathbb{R}$  if it has the form

$$\mathbf{b}(x, \xi) \sim c \log[\xi] I + \mathbf{q}(x, \xi) ,$$

where  $\mathbf{q} \sim (\mathbf{q}_0, \mathbf{q}_1, \dots) \in S^0(U)$  is a degree 0 classical symbol, and  $[\ ] : \mathbb{R}^n \rightarrow \mathbb{R}_+$  is a strictly positive function with  $[\xi] = |\xi|$  for  $|\xi| \geq 1$ .

A  $\psi$ do  $A$  on  $C^\infty(E)$  is classical of degree  $\alpha$  (*resp.* logarithmic of type  $c$ ) if the local symbol of  $A$  in each local trivialization of  $E$  is classical of degree  $\alpha$  (*resp.* logarithmic of type  $c$ ). A logarithmic  $\psi$ do has order  $\varepsilon$  for any  $\varepsilon > 0$ . We denote the space of classical  $\psi$ dos of order  $\alpha$  (*resp.* less than  $\alpha$ ) by  $\Psi^\alpha(E)$  (*resp.*  $\Psi^{<\alpha}(E)$ ), and the algebra of all integer order classical  $\psi$ dos by  $\Psi^{\mathbb{Z}}(E)$ .

The various homogeneous terms  $\mathbf{a}_j(x, \xi)$  (*resp.*  $\mathbf{q}_j(x, \xi)$ ) in the local symbol of a classical (*resp.* logarithmic)  $\psi$ do do not, in general, have a global invariant meaning as bundle endomorphisms over  $T^*M$ . However, it was observed by Guillemin [Gu] and Wodzicki [Wo2] for classical  $\psi$ dos, and extended to logarithmic operators by Okikiolu [Ok2], that if  $\sigma(A)_{-n}(x, \xi)$  is the term of homogeneity  $-n$  (so  $\sigma(A)_{-n}(x, \xi) = a_{\alpha+n}(x, \xi)$  if  $A$  is classical of degree  $\alpha$ , while if  $A$  is logarithmic

$\sigma(A)_{-n}(x, \xi) = q_n(x, \xi)$ , then

$$\frac{1}{(2\pi)^n} \left( \int_{|\xi|=1} \sigma(A)_{-n}(x, \xi) dS(\xi) \right) dx$$

with  $dS(\xi)$  the sphere measure on  $S^{n-1}$ , defines a global density on  $M$ . The number

$$(1.2) \quad \text{res}(A) = \frac{1}{(2\pi)^n} \int_M \int_{|\xi|=1} \text{tr}(\sigma(A)_{-n}(x, \xi)) dS(\xi) dx$$

is the Guillemin-Wodzicki *residue trace* of the  $\psi$ do  $A$ . Evidently, if  $A$  is a classical  $\psi$ do of order  $\alpha$ , then

$$(1.3) \quad \alpha \notin \mathbb{Z} \quad \text{or} \quad \alpha < -n \quad \Rightarrow \quad \text{res}(A) = 0 ,$$

and  $\text{res}$  drops down to a map on the quotient algebra

$$(1.4) \quad \text{res} : \Psi^{\mathbb{Z}}(E)/\Psi^{-\infty}(E) \longrightarrow \mathbb{C} .$$

The following linearity properties of the residue trace are immediate from its definition.

**Lemma 1.1.** *Let  $A, B$  be  $\psi$ dos. If  $A$  and  $B$  are both logarithmic, or, if  $A$  is classical of order  $\alpha \in \mathbb{Z}$  and  $B$  is logarithmic, or, if  $A$  is classical of order  $\alpha \in \mathbb{R}$  and  $B$  is classical of order  $\beta \in \mathbb{R}$  such that<sup>2</sup>  $\alpha - \beta \in \mathbb{Z}$ , then*

$$(1.5) \quad \text{res}(A + B) = \text{res}(A) + \text{res}(B) .$$

The characterizing tracial property of  $\text{res}$  is due to Guillemin, Wodzicki and extended to include logarithmic operators by Okikiolu:

**Proposition 1.2.** [Wo2],[Gu],[Ok2]. *Let  $A, B$  be classical or logarithmic  $\psi$ dos. Then*

$$(1.6) \quad \text{res}([A, B]) = 0 .$$

It follows that (1.4) is a trace functional. It is, moreover, projectively unique.

This has consequences for determinants.

Let  $A$  be a classical  $\psi$ do  $A$  of order  $\alpha$  admitting a principal angle  $\theta$ , meaning that the principal symbol  $\mathfrak{a}_0(A)(x, \xi)$  considered as a bundle endomorphism over  $T^*X \setminus 0$  has no eigenvalue on the spectral cut  $R_\theta = \{re^{i\theta} \mid r \geq 0\}$ ; in particular,  $A$  is

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<sup>2</sup>If  $A$  and  $B$  are classical and  $\alpha - \beta \notin \mathbb{Z}$  then  $A + B$  is not a classical  $\psi$ do (the symbol is then not 1-step, its expansion does not drop in integer orders). Consequently, though  $\Psi^{\mathbb{Z}}(E)$  is an algebra, the space  $\Psi(E) = \cup_{\alpha \in \mathbb{R}} \Psi^\alpha(E)$  is not, but forms a semi-group with respect to the usual composition product. This is relevant for linearity properties of traces on classical  $\psi$ dos, and the reason why  $\sigma(\log A)(x, \xi) = -(d/ds)|_{s=0} \sigma(A^{-s})(x, \xi)$ , as a limit of differences of classical symbols, is not quite classical.

elliptic. Then, as recalled below, the functional calculus constructs the log symbol  $(\log_\theta \mathbf{a})_{-n}(x, \xi)$  of homogeneity  $-n$  and

$$\frac{1}{(2\pi)^n} \left( \int_{|\xi|=1} (\log_\theta \mathbf{a})_{-n}(x, \xi) dS(\xi) \right) dx$$

defines a global density on  $M$  [Ok2]. If  $\alpha > 0$ , then  $\log_\theta A$  exists as a logarithmic  $\psi$ do of type  $\alpha$  and  $(\log_\theta \mathbf{a})_{-n}(x, \xi) = \sigma(\log_\theta A)_{-n}(x, \xi)$ . We can hence define canonically the following determinant functional on classical  $\psi$ dos.

**Definition 1.3.** *The residue determinant  $\det_{\text{res}} A$  of a classical  $\psi$ do  $A$  with principal angle  $\theta$  is the complex number*

$$(1.7) \quad \log \det_{\text{res}} A := \text{res}(\log A) ,$$

that is,

$$(1.8) \quad \log \det_{\text{res}} A := \frac{1}{(2\pi)^n} \int_M \int_{|\xi|=1} \text{tr} ((\log_\theta \mathbf{a})_{-n}(x, \xi)) dS(\xi) dx .$$

**Remark 1.4.** *From Lemma 1.1, one has the linearity*

$$(1.9) \quad \text{res}(\log A + \log B) = \text{res}(\log A) + \text{res}(\log B)$$

for all classical  $\psi$ dos  $A$  and  $B$ , of any real orders  $\alpha, \beta \in \mathbb{R}$ .

The properties of the residue determinant are as follows.

**Proposition 1.5.** *The residue determinant  $\det_{\text{res}} A$  is a local invariant, depending only on the first  $n+1$  homogeneous terms  $\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n$  in the local symbol of  $A$ , and is independent of the choice of principal angle  $\theta$  used to define  $\log_\theta A$ .*

The subscript  $\theta$  is therefore omitted in the notation (1.7).

**Corollary 1.6.** *Let  $A \in \Psi^\alpha(E)$ ,  $S \in \Psi^\sigma(E)$  with  $\alpha > \sigma + n$ . Then*

$$(1.10) \quad \det_{\text{res}}(A + S) = \det_{\text{res}}(A) .$$

The residue determinant is multiplicative:

**Theorem 1.7.** *Let  $A, B$  be classical  $\psi$ dos of order  $\alpha, \beta \in \mathbb{R}$ , and suppose that  $A, B, AB$  admit principal angles. Then*

$$(1.11) \quad \det_{\text{res}}(AB) = \det_{\text{res}}(A) \cdot \det_{\text{res}}(B) .$$

The residue determinant does not vanish on non-invertible operators, however.

In fact, since an elliptic operator is invertible modulo smoothing operators, the above properties imply that it never vanishes.

If  $A \in \Psi(E)$  has order  $\alpha > 0$  and is invertible, then  $\zeta(A, 0)|^{\text{mer}}$ , the meromorphically continued spectral zeta-function of  $A$  evaluated at  $s = 0$  (see Sect.2), is known to have the properties in Proposition 1.5; the relation with  $\det_{\text{res}} A$  is the following.

**Theorem 1.8.** *Let  $A$  be a classical  $\psi$ do of order  $\alpha > 0$  with principal angle. If  $A$  is invertible*

$$(1.12) \quad \det_{\text{res}}(A) = e^{-\alpha \zeta(A, 0)|^{\text{mer}}} .$$

*When  $A$  is not invertible*

$$(1.13) \quad \det_{\text{res}}(A) = e^{-\alpha (\zeta(A, 0)|^{\text{mer}} + h_0(A))} ,$$

where  $h_0(A) = \text{Tr}(\Pi_0(A))$  and  $\Pi_0(A)$  is a projection onto the finite-dimensional generalized 0-eigenspace  $E_0(A) = \{\tau \in C^\infty(E) \mid A^N \tau = 0 \text{ some } N \in \mathbb{N}\}$  (the projection  $\Pi_0(A)$  is defined in (3.33)). If  $\text{Ker}(A^2) = \text{Ker}(A)$ , in particular for  $A$  self-adjoint, then  $h_0(A) = \dim \text{Ker}(A)$ .

The additional  $h_0(A)$  term on the right-side of (1.13) thus corrects for the discontinuities of  $\zeta(A, 0)|^{\text{mer}}$  at non-invertible  $A$ . Notice, also, that  $\zeta(A, 0)|^{\text{mer}}$  is locally determined only if  $h_0(A) = 0$ , otherwise it is  $\zeta(A, 0)|^{\text{mer}} + h_0(A)$  which is local; this follows from Proposition 1.5 and (1.12), and is seen directly in the proof.

**Remark 1.9.** [1] *The number  $\zeta(A, 0)|^{\text{mer}} := \text{Tr}(I.A^{-s})|_{s=0}^{\text{mer}}$  defines a quasi- (or weighted-) trace of the identity operator  $I$  on  $C^\infty(E)$  and hence (1.12) associates  $\det_{\text{res}}$  with a notion of regularized dimension, rather than regularized volume.*

[2] *The continuity of  $\det_{\text{res}}$  on families of admissible operators in  $\Psi(E)$  contrasts with the  $\zeta$ -determinant, which is continuous only on families of invertible operators.*

**Remark 1.10.** *Subsequently two proofs of (1.13) of independent interest have been given. In [Pa] (see also [PaSc]) the identity is proved using a microlocal result of [KoVi]. In [Gr2] a proof is obtained via the resolvent trace  $\text{Tr}((A - \lambda I)^{-k})$ .*

**Remark 1.11.** *Since  $\log_\theta A$  is ‘almost’ in the subalgebra  $\Psi^{\leq 0}(E) \cap \Psi^{\mathbb{Z}}(E)$  on which the residue trace is not the unique trace [PaRo],  $\det_{\text{res}}$  is not quite the unique multiplicative functional on elliptic  $\psi$ dos. Indeed  $\det_0$  defined by  $\log \det_0(A) = (\text{vol}_\sigma(S^*M))^{-1} \int_{S^*M} \log \det(a_0(x, \xi)) d\sigma$ , where  $a_0(x, \xi)$  is the leading symbol of  $A$  and  $d\sigma$  is a volume form on the cosphere bundle  $S^*M$ , is multiplicative.*

**Example.** Let  $\Sigma$  be a closed Riemann surface and  $E$  a complex vector bundle of degree  $\deg(E) = \int_\Sigma c_1(E)$ . Let  $\bar{\partial}_\Sigma : C^\infty(E) \rightarrow C^\infty(E \otimes T^{0,1}\Sigma)$  be an invertible  $\bar{\partial}$ -operator; thus locally,  $\bar{\partial}_\Sigma = (\partial_{\bar{z}} + a(z))d\bar{z}$ . Then, from (1.12) and [Gi] Thm(4.1.6)(see also [Bo] §1.5), we have

$$(1.14) \quad \det_{\text{res}}(\bar{\partial}_\Sigma^* \bar{\partial}_\Sigma) = \exp \left( -\deg(E) - \frac{\chi(\Sigma) \text{rk}(E)}{3} \right) ,$$

with  $\chi(\Sigma)$  the Euler number, and  $\text{rk}(E)$  the rank of  $E$ .

This is independent of  $\overline{\partial}_\Sigma^* \overline{\partial}_\Sigma$ , but in general the residue determinant of a second-order differential operator over a surface will depend on the complete symbol. In the case of an invertible operator  $\Delta_g$  of Laplace-type one has for  $t \in \mathbb{R}$

$$(1.15) \quad \det_{\text{res}}(\Delta_g + tI) = \exp \left( \frac{A_g(\Sigma) \text{rk}(E)}{2\pi} t - \frac{1}{2\pi} \int_\Sigma \text{tr}(\varepsilon_x(\Delta_g)) dx - \frac{\chi(\Sigma) \text{rk}(E)}{3} \right),$$

where  $A_g(\Sigma)$  is the surface area of  $\Sigma$  with respect to a Riemannian metric  $g$ , and  $\varepsilon(\Delta_g)$  is the unique element of  $C^\infty(\text{End}(E))$  and  $\nabla$  the unique connection such that  $\Delta_g = -\sum_{i,j} g^{ij}(x) \nabla_i \nabla_j + \varepsilon_x(\Delta_g)$ . On the other hand, if  $(M, g)$  is a 4-manifold and  $\Delta_g$  the Laplace-Beltrami operator then

$$(1.16) \quad \det_{\text{res}}(\Delta_g + tI) = \exp \left( -\frac{\text{vol}_g(M)}{16\pi^2} t^2 + \frac{1}{24\pi^2} \int_M \kappa_M dx t \right) \det_{\text{res}}(\Delta_g),$$

with  $\text{vol}_g(M)$  the Riemannian volume,  $\kappa_M$  the scalar curvature. More generally, if  $(M, g)$  is a Riemannian manifold of dimension  $2m$ ,  $E$  a vector bundle, and  $\Delta_g$  an operator on  $C^\infty(E)$  of Laplace-type, meaning  $\Delta_g$  is a second-order differential operator with scalar principle symbol

$$(1.17) \quad \sigma(\Delta_g)_2(x, \xi) = |\xi|_{g(x)}^2 I := -\sum_{i,j=1}^{2m} g^{ij}(x) \xi_i \xi_j I,$$

where  $\xi = (\xi_1, \dots, \xi_{2m}) \in \mathbb{R}^{2m}$ , then for  $\varepsilon > 0$  the heat operator  $e^{-\varepsilon \Delta_g}$  is a smoothing operator with heat trace expansion as  $\varepsilon \rightarrow 0+$

$$(1.18) \quad \text{Tr}(e^{-\varepsilon \Delta_g}) = \frac{c_{-m}(\Delta_g)}{\varepsilon^m} + \dots + \frac{c_{-1}(\Delta_g)}{\varepsilon} + c_0(\Delta_g) + O(\varepsilon)$$

with locally determined coefficients  $c_j(\Delta_g)$ ; specifically, (1.17) easily implies

$$(1.19) \quad c_{-m}(\Delta_g) = \frac{\text{vol}_g(M) \text{rk}(E)}{(4\pi)^m},$$

while, by standard transition formulae (see for example [GrSe], [Gr]), (1.13) becomes

$$(1.20) \quad \det_{\text{res}}(\Delta_g) = e^{-2c_0(\Delta_g)}.$$

(If  $M$  is odd-dimensional  $\det_{\text{res}}(\Delta_g) = 1$ .) On the other hand, for  $t \in \mathbb{R}$  one has

$$\text{Tr}(e^{-\varepsilon(\Delta_g + tI)}) = e^{-\varepsilon t} \text{Tr}(e^{-\varepsilon \Delta_g}),$$

and so from (1.18)

$$(1.21) \quad c_0(\Delta_g + tI) = \frac{(-1)^m}{m!} c_{-m}(\Delta_g) t^m + \frac{(-1)^{m-1}}{(m-1)!} c_{-m+1}(\Delta_g) t^{m-1} + \dots + c_0(\Delta_g).$$

Thus (1.19) and (1.20) yield

$$\det_{\text{res}}(\Delta_g + tI) = \exp \left( \frac{2(-1)^{m+1}}{(4\pi)^m m!} \text{vol}_g(M) \text{rk}(E) t^m + 2 \sum_{j=1}^{m-1} \frac{(-1)^{m-j+1}}{(m-j)!} c_{-m+j}(\Delta_g) t^{m-j} \right) \det_{\text{res}}(\Delta_g) ,$$

the specific formulas (1.15), (1.16) now following from [Gi] Thm(4.1.6) and [McSi].

**Remark 1.12.** *Let  $\omega \in C^\infty(M)$  and let  $g_\omega = e^{2\omega}g$ . A consequence of (1.20) is that the Generalized Polyakov Formula of [BrOr] for the relative zeta-determinant of  $\Delta_{g_\omega}$  may be stated as*

$$\log \frac{\det_\zeta(\Delta_{g_\omega})}{\det_\zeta(\Delta_{g_0})} = \int_0^1 \log \det_{\text{res}}(\Delta_{g_{e\omega}}) d\epsilon .$$

Turning matters around, (1.13) can be used to deduce properties of  $\zeta(A, 0)|^{\text{mer}}$ .

Since the proof of Theorem 1.8 demonstrates that the equality in (1.13) holds logarithmically

$$(1.22) \quad \text{res}(\log A) = -\alpha (\zeta(A, 0)|^{\text{mer}} + h_0(A)) ,$$

(in fact, it holds pointwise on  $M$  as an equality between densities as shown in the proof of Theorem 1.8,) then, combined with (1.11) which also holds logarithmically, (but not as a pointwise identity of densities) we have the following.

**Corollary 1.13.** *Let  $A, B$  be classical  $\psi$ dos, admitting principal angles, and which have positive orders  $\alpha, \beta \in \mathbb{R}_+$ . Then the function  $Z(\sigma(A)) := -\alpha (\zeta(A, 0)|^{\text{mer}} + h_0(A))$  is additive:*

$$(1.23) \quad Z(\sigma(AB)) = Z(\sigma(A)) + Z(\sigma(B)) .$$

**Remark 1.14.** [1] *The additivity (1.23) is referred to in [Ka] and in the introduction to [KoVi] (whose notation is respected here) as a property known to Wodzicki, though no proof appeared. On the other hand, Wodzicki defined in [Wo3] a determinant for order zero  $\psi$ dos path connected to the identity which, for such operators, coincides with  $\det_{\text{res}}$ .*

[2] *In [PaSc] it is shown that (1.22) extends as an exact splitting formula into local and global components of the generalized zeta function  $\zeta_\theta(A, Q, s)|^{\text{mer}} = \text{Tr}(AQ^{-s})|^{\text{mer}}$ . On the other hand, Grubb [Gr2] has recently shown using resolvent methods that (1.22) extends to certain classes of boundary value problems.*

Conversely, (1.13), (1.23) combine to prove (1.11) when  $\alpha, \beta \in \mathbb{R}_+$ .

From the Atiyah-Bott-Seeley  $\zeta$ -function index formula (see for example [Sh]), a further immediate consequence of (1.12) is the following ‘super’ residue determinant

formula for the index  $\text{Index}(D) = \dim \text{Ker}(D) - \dim \text{Ker}(D^*)$  of a general elliptic operator  $D : C^\infty(E^+) \longrightarrow C^\infty(E^-)$  of order  $d > 0$ .

**Corollary 1.15.**

$$(1.24) \quad \frac{\det_{\text{res}}(D^*D + I)}{\det_{\text{res}}(DD^* + I)} = e^{-2d \text{Index}(D)} .$$

Equivalently,

$$(1.25) \quad \text{Index}(D) = \frac{1}{2d} (\text{res log}(DD^* + I) - \text{res log}(D^*D + I)) .$$

**Remark 1.16.** *In contrast, there is not a formula for the index using the residue trace of a classical (non-logarithmic)  $\psi$ do. The identities (1.13) and (1.25) lead to an alternative elementary proof of the local Atiyah-Singer index theorem [ScZa].*

For order zero operators we have:

**Theorem 1.17.** *If  $A$  has order  $\alpha = 0$  and has the form  $A = I + Q$  with  $Q$  a classical  $\psi$ do of negative integer order  $k < 0$ , then*

$$(1.26) \quad \det_{\text{res}}(I + Q) = \prod_{j=1}^{\left[ \frac{n}{|k|} \right]} e^{\frac{(-1)^{j+1}}{j} \text{res}(Q^j)} .$$

**Remark 1.18.** [1] *For order zero operators the zeta-function at zero in (1.13) is thus replaced by a relative zeta function at zero (see Theorem 1.20, and comments around (2.17)).*

[2] *Generalizing the Fredholm determinant ( $p=1$ ) there is a well-known notion of  $p$ -determinant  $\det_p(I+Q)$  for  $Q$  in the  $p^{\text{th}}$  Schatten ideal  $L_p$ , which is not multiplicative for  $p > 1$ , but for  $Q \in L_1$  the following formula holds [Si] in analogy to (1.26)*

$$\frac{\det_p(I + Q)}{\det_1(I + Q)} = \prod_{j=1}^{p-1} e^{\frac{(-1)^j}{j} \text{Tr}(Q^j)} .$$

Theorem 1.17 evidently implies:

**Corollary 1.19.**

$$(1.27) \quad \det_{\text{res}}(I + Q) = 1 \quad \text{if} \quad \text{ord}(Q) < -n , \quad \text{ord}(Q) \in \mathbb{Z} .$$

*Hence  $\det_{\text{res}}$  drops down to a multiplicative function on the ‘determinant Lie group’  $\tilde{G} = \Psi^*(E)/(I + \Psi^{-\infty}(E))$ , of Kontsevich-Vishik [KoVi], where  $\Psi^*(E)$  is the group of invertible elliptic  $\psi$ dos. ((1.27) also follows from Corollary 1.6.)*

**Example.** Let  $\Delta_g$  be an invertible generalized Laplacian on a closed Riemannian manifold  $(M, g)$  of dimension  $2m$ . Thus  $\Delta_g$  has principal symbol as in (1.17), and

with that notation, we therefore have  $\sigma(\Delta_g^{-m})_{-2m}(x, \xi) = |\xi|_{g(x)}^{-2m} I$ . Whence, with  $S^{2m-1}$  the Euclidean  $(2m-1)$ -sphere,

$$\begin{aligned}
\text{res}(\Delta_g^{-m}) &= \int_M \frac{1}{(2\pi)^{2m}} \left( \int_{|\xi|_{g(x)}=1} \text{tr}(I) dS_{g(x)}(\xi) \right) dx \\
&= \frac{1}{(2\pi)^{2m}} \int_M \sqrt{\det(g(x))} \left( \int_{S^{2m-1}} dS(\xi) \right) dx \text{rk}(E) \\
(1.28) \quad &= \frac{\text{vol}(S^{2m-1})}{(2\pi)^{2m}} \text{vol}_g(M) \text{rk}(E) .
\end{aligned}$$

Since  $\text{vol}(S^{2m-1}) = 2\pi^m / (m-1)!$  we therefore have from (1.26)

$$(1.29) \quad \det_{\text{res}}(I + \Delta^{-m}) = \exp \left( \frac{\text{vol}_g(M) \text{rk}(E)}{2^{2m-1} (m-1)! \pi^m} \right) .$$

**Example.** It is instructive to check how the multiplicativity property of  $\det_{\text{res}}$  works for this class of  $\psi$ dos. As a simple case, consider  $\psi$ dos  $\mathbf{Q}_1, \mathbf{Q}_2$  for which  $3 \text{ord}(\mathbf{Q}_i) < -n \leq 2 \text{ord}(\mathbf{Q}_i) < 0$  – for example, operators of order  $-2$  on a  $4$ -manifold. Then according to Theorem 1.17

$$(1.30) \quad \log \det_{\text{res}}(I + \mathbf{Q}_i) = \sum_{p=1}^2 \frac{(-1)^{p+1}}{p} \text{res}(\mathbf{Q}_i^p) = \text{res}(\mathbf{Q}_i) - \frac{1}{2} \text{res}(\mathbf{Q}_i^2) .$$

So

$$\begin{aligned}
(1.31) \quad &\log \det_{\text{res}}(I + \mathbf{Q}_1) + \log \det_{\text{res}}(I + \mathbf{Q}_2) \\
&= \text{res}(\mathbf{Q}_1) + \text{res}(\mathbf{Q}_2) - \frac{1}{2} \text{res}(\mathbf{Q}_1^2) - \frac{1}{2} \text{res}(\mathbf{Q}_2^2) .
\end{aligned}$$

On the other hand,  $(I + \mathbf{Q}_1)(I + \mathbf{Q}_2) = I + (\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_1\mathbf{Q}_2)$  and (1.30) applies with  $\mathbf{Q}_i$  replaced by  $\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_1\mathbf{Q}_2$  so that

$$\begin{aligned}
(1.32) \quad &\log \det_{\text{res}}((I + \mathbf{Q}_1)(I + \mathbf{Q}_2)) \\
&= \text{res}(\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_1\mathbf{Q}_2) - \frac{1}{2} \text{res}((\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_1\mathbf{Q}_2)^2) .
\end{aligned}$$

Since the  $\mathbf{Q}_i$  have integer order

$$\text{res}(\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_1\mathbf{Q}_2) = \text{res}(\mathbf{Q}_1) + \text{res}(\mathbf{Q}_2) + \text{res}(\mathbf{Q}_1\mathbf{Q}_2)$$

while by the tracial property (1.6) of  $\text{res}$

$$\begin{aligned}
\text{res}((\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_1\mathbf{Q}_2)^2) &= \text{res}(\mathbf{Q}_1^2 + \mathbf{Q}_2^2 + \mathbf{Q}_1\mathbf{Q}_2 + \mathbf{Q}_2\mathbf{Q}_1 + \text{terms of order } < -n) \\
&= \text{res}(\mathbf{Q}_1^2) + \text{res}(\mathbf{Q}_2^2) + 2 \text{res}(\mathbf{Q}_1\mathbf{Q}_2) .
\end{aligned}$$

Hence (1.31) and (1.32) are equal.

An application of properties in the previous theorems yields the following formula.

**Theorem 1.20.** *Let  $A \in \Psi^\alpha(E)$ ,  $B_0 \in \Psi^{\beta_0}(E)$ ,  $\dots$ ,  $B_d \in \Psi^{\beta_d}(E)$  be classical  $\psi$ dos. Assume  $\alpha = \text{ord}(A) > 0$  and  $\alpha - \beta_j \in \mathbb{N} \setminus \{0\}$  (strictly positive integer). For  $t \in \mathbb{R}$  the polynomial  $B[t] = B_0 + B_1 t + \dots + B_d t^d \in \Psi^\beta(E)$  is a classical  $\psi$ do of order  $\beta = \max\{\beta_j \mid j = 0, \dots, d\}$ . Suppose  $A$  admits a principal angle. Then  $\zeta(A + B[t]) + \text{h}_0(A + B[t])$  is a polynomial of degree  $d \lfloor n/(\alpha - \beta) \rfloor$  in  $t$  with local coefficients. Specifically, let  $Q \in \Psi^{-\alpha}(E)$  be a two-sided parametrix for  $A$ , then*

$$(1.33) \quad \begin{aligned} & \zeta(A + B[t], 0)|^{\text{mer}} + \text{h}_0(A + B[t]) = \zeta(A, 0)|^{\text{mer}} + \text{h}_0(A) \\ & - \frac{1}{\alpha} \sum_{k=1}^{\lfloor \frac{n}{\alpha-\beta} \rfloor} \sum_{I_k} \frac{(-1)^k}{k} \text{res}(QB_{i_1}QB_{i_2}\dots QB_{i_k}) t^{|I_k|} , \end{aligned}$$

where the inner sum is over  $k$ -tuples  $I_k = (i_1, \dots, i_k)$  of  $k$  (not necessarily distinct) elements  $i_j \in \{1, \dots, d\}$ , and  $|I_k| = i_1 + \dots + i_k$ . If  $A$  is invertible then  $Q$  in (1.33) can be replaced by  $A^{-1}$ . In particular, if  $A$  is invertible

$$(1.34) \quad \zeta(A + tI, 0)|^{\text{mer}} = \zeta(A, 0)|^{\text{mer}} - \frac{1}{\alpha} \sum_{k=1}^n \frac{(-1)^k}{k} \text{res}(A^{-k}) t^k .$$

Note that (1.33) implies that if  $B \in \Psi^\beta(E)$  and  $\alpha - \beta - n \in \mathbb{N} \setminus \{0\}$ , then

$$\zeta(A + B, 0)|^{\text{mer}} + \text{h}_0(A + B) = \zeta(A, 0)|^{\text{mer}} + \text{h}_0(A) .$$

Equation (1.34) is equivalent to equation (6.5) of [Wo1].

**Example.** If we apply (1.34) to a Laplace-type differential operator  $\Delta_g$  on a  $2m$ -dimensional manifold, then comparing with (1.21) we infer for  $k \geq 1$

$$(1.35) \quad \frac{1}{2} \text{res}(\Delta_g^{-k}) = \frac{c_{-k}(\Delta_g)}{(k-1)!} .$$

Specifically, for  $k = m$

$$\frac{1}{2} \text{res}(\Delta_g^{-m}) = \frac{\text{vol}_g(M) \text{rk}(E)}{(4\pi)^m (m-1)!}$$

which coincides with (1.28), while, for example, for the Laplace-Beltrami operator on a 4-manifold with scalar curvature  $\kappa_M$

$$\text{res}(\Delta_g^{-2}) = \frac{1}{24\pi^2} \int_M \kappa_M dx .$$

With a little extra work it is easy to see using these methods for  $A \in \Psi^{\alpha \neq 0}(E)$  not necessarily positive but admitting a principal angle, that

$$\frac{1}{\alpha} \text{res}(A^{-z_0}) = \text{Res}_{s=z_0}^{\mathbb{C}} \zeta(A, s)|_{s=z_0}^{\text{mer}} ,$$

where the right side is the usual complex residue of a meromorphic function. This formula is well-known [Wo2] and (1.35) is a particular case.

## 2. PRELIMINARIES

To explain the nature of the residue determinant we begin with the sub-algebra  $\Psi^0(E)$  of classical  $\psi$ dos of order zero. For  $A \in \Psi^0(E)$  with spectrum disjoint from the ray  $R_\theta = \{re^{i\theta} \mid r \geq 0\}$  the complex powers for any  $s \in \mathbb{C}$  are defined by

$$(2.1) \quad A_\theta^{-s} = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda_\theta^{-s} (A - \lambda I)^{-1} d\lambda \in \Psi^0(E),$$

where  $\Gamma_\theta$  is a bounded ‘keyhole’ contour enclosing the spectrum of  $A$  but not enclosing any part of  $R_\theta$  (or the origin), while the logarithm of  $A$  is defined by

$$(2.2) \quad \log_\theta A = \frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda (A - \lambda I)^{-1} d\lambda \in \Psi^0(E).$$

Here  $\lambda_\theta^{-s}$ ,  $\log_\theta \lambda := -(d/ds)\lambda_\theta^{-s}|_{s=0}$  are the branches defined by  $\lambda_\theta^{-s} = |\lambda|^{-s} e^{-is \arg(\lambda)}$  with  $\theta - 2\pi \leq \arg(\lambda) < \theta$ . These formulas are valid in any Banach algebra.

Sitting inside  $\Psi^0(E)$  is the ideal  $\Psi^{<-n}(E)$ . An operator  $Q \in \Psi^{<-n}(E)$  is trace class with an absolutely integrable matrix-valued kernel

$$(2.3) \quad K(Q, x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi} \sigma(Q)(x, \xi) d\xi$$

over  $M \times M$  smooth away from the diagonal (so for all  $\psi$ dos) and continuous along  $\{(x, x) \mid x \in M\}$ . Consequently,  $Q$  has  $L^2$  trace

$$(2.4) \quad \text{Tr}(Q) = \int_M \text{tr}(K(Q, x, x)) dx.$$

This is a *non-local* spectral invariant; from (2.3) the trace (2.4) depends on the complete symbol  $\sigma(Q)$ , not just on finitely many homogeneous terms.

Taken together, (2.2) and (2.4) define a determinant on the ring  $I + \Psi^{<-n}(E)$  of  $\psi$ dos which differ from the identity by an element of  $\Psi^{<-n}(E)$

$$(2.5) \quad \det_{\text{Tr}} : I + \Psi^{<-n}(E) \longrightarrow \mathbb{C}, \quad \log \det_{\text{Tr}}(A) := \text{Tr}(\log_\theta(A)).$$

Here, one uses

$$(2.6) \quad A \in I + \Psi^{<-n}(E) \quad \Rightarrow \quad \log_\theta A \in \Psi^{<-n}(E).$$

$\det_{\text{Tr}}$  extends the usual determinant in finite-dimensions to the group  $(I + \Psi^{<-n}(E))_{\text{inv}}$  of invertible operators in  $I + \Psi^{<-n}(E)$ :

**Lemma 2.1.** *On the group  $(I + \Psi^{<-n}(E))_{\text{inv}}$  the determinant  $\det_{\text{Tr}}$  is the Fredholm determinant; it is independent of the choice of  $\theta$  and is multiplicative.  $\det_{\text{Tr}}$  is a non-local spectral invariant, and has no (multiplicative) extension from  $(I + \Psi^{<-n}(E))_{\text{inv}}$  to the group  $\Psi^*(E)$  of invertible elliptic  $\psi$ dos.*

*Proof.* The first sentence follows, for example, from the comments around (2.17), below, and [Sc] Prop(2.21). Alternatively, one can prove these properties directly; independence of  $\theta$  by the method of proof of Proposition 1.5, multiplicativity by the Campbell-Hausdorff theorem (cf. proof of Theorem 1.7). Since the  $L^2$  trace  $\text{Tr}$  is non-local and has no extension to a trace functional on  $\Psi(E)$ , the second statement easily follows.  $\square$

Since  $\log_\theta A$  vanishes on  $\text{Ker}(A)$ , the functional  $\det_{\text{Tr}}$  is discontinuous at non-invertible elements of the ring  $I + \Psi^{<-n}(E)$ , and therefore differs from the Fredholm determinant which is continuous and vanishes on such elements. On the other hand, from (1.3) and (2.6), or from (1.10):

**Lemma 2.2.** *The residue determinant is trivial on  $I + \Psi^{<-n}(E)$*

$$(2.7) \quad A \in I + \Psi^{<-n}(E) \quad \Rightarrow \quad \det_{\text{res}} A = 1 .$$

The relation of the residue determinant (1.7) to the classical determinant (2.5) is thus structurally the same as that of the residue trace to the classical  $L^2$  operator trace.

Because adding a smoothing operator to  $A \in \Psi(E)$  does not affect  $\det_{\text{res}} A$  the invertibility of the operator is not detected in either (1.26) or (2.7).

Next, let  $A$  be a classical  $\psi$ do of order  $\alpha > 0$  with principal angle  $\theta$ . We assume further for simplicity that  $\theta$  is an Agmon angle, meaning that  $A - \lambda I$  is invertible in a neighborhood of  $R_\theta$ ; in particular,  $A$  is elliptic and invertible. Since the  $L^2$  operator norm of  $(A - \lambda I)^{-1}$  is  $O(|\lambda|^{-1})$  as  $|\lambda| \rightarrow \infty$  one can define for  $\text{Re}(s) > 0$

$$(2.8) \quad A_\theta^{-s} = \frac{i}{2\pi} \int_{\mathcal{C}_\theta} \lambda^{-s} (A - \lambda I)^{-1} d\lambda , \quad \text{Re}(s) > 0 ,$$

where now  $\mathcal{C}_\theta$  is a contour travelling in along  $R_\theta$  from infinity to a small circle around the origin, clockwise around the circle, and then back out along  $R_\theta$  to infinity.

In Seeley's 1967 paper [Se] on the complex powers  $A_\theta^{-s}$  of an elliptic operator, two quite different extensions of (2.8) to the whole complex plane were explained.

In the first of these, for  $s \in \mathbb{C}$  choosing  $k \in \mathbb{N}$  with  $\text{Re}(s) + k > 0$  and setting

$$(2.9) \quad A_\theta^{-s} := A^k A_\theta^{-s-k} \in \Psi^{-\alpha s}(E) ,$$

where  $A_\theta^{-s-k}$  on the right side is defined by (2.8), defines, independently of  $k$ , a group of elliptic classical  $\psi$ dos with

$$(2.10) \quad A_\theta^0 = I, \quad A_\theta^m = A^m, \quad m \in \mathbb{Z} .$$

An important consequence is the construction of the logarithm of  $A$ , a logarithmic  $\psi$ do of type  $\alpha$  [Ok1, KoVi], defined by

$$(2.11) \quad \log_\theta A := - \left. \frac{d}{ds} \right|_{s=0} A_\theta^{-s} ,$$

and satisfying  $\frac{d}{ds} A_\theta^{-s} = -\log_\theta A \cdot A_\theta^{-s}$ . Since  $\log_\theta A$  and  $A_\theta^{-s}$  for  $\operatorname{Re}(s) < 0$  are unbounded (note (2.10)), and hence far from trace class, this extension has been of little direct interest. It is, though, precisely the object of interest, and, with the residue trace at hand, all that is needed to define the residue determinant.

Traditionally, however, one does something quite different and attempts to extend the determinant (2.5) from  $I + \Psi^{<-n}(E)$  to  $\Psi(E)$  by extending the  $L^2$  trace from  $\Psi^{<-n}(E)$  to  $\Psi(E)$  using spectral zeta-functions. The latter leads necessarily to a quasi-trace on  $\Psi(E)$  (i.e. a functional which is non-tracial), and hence to a quasi-determinant (i.e. a functional which is non-multiplicative), the zeta-determinant. This is achieved through the meromorphic extension of the complex powers  $A_\theta^{-s}$  from  $\operatorname{Re}(s) > n/\alpha$ , the second of Seeley's extensions, and has been the object of enormous interest. It is, though, quite irrelevant to the construction of the residue determinant.

Nevertheless, spectral zeta functions have a part to play in what follows and we need to recall something of these constructions. We include a  $\psi$ do coefficient  $B$ , which for the moment we assume to be classical of order  $\beta$ , and we assume  $A$  to have order  $\alpha > 0$  and to be invertible. The kernel  $K(B A_\theta^{-s}, x, y)$  of  $B A_\theta^{-s}$ , which is continuous in  $(x, y)$  and holomorphic in  $s$  for  $\operatorname{Re}(s) > (n + \beta)/\alpha$ , has along the diagonal a meromorphic extension  $K(B A_\theta^{-s}, x, x)|^{\text{mer}}$  to all  $s \in \mathbb{C}$  with at most simple poles located on the real axis at the points indicated in (2.12). Consequently the zeta function  $\zeta_\theta(B, A, s) := \operatorname{Tr}(B A_\theta^{-s})$  is holomorphic for  $\operatorname{Re}(s) > (n + \beta)/\alpha$  and extends to a meromorphic function

$$\zeta_\theta(B, A, s)|^{\text{mer}} := \int_M \operatorname{tr}(K(B A_\theta^{-s}, x, x)|^{\text{mer}}) dx$$

on  $\mathbb{C}$ . It has pole structure ([Se], [GrSe], [Gr])

$$(2.12) \quad \Gamma(s) \zeta_\theta(B, A, s)|^{\text{mer}} \sim \sum_{j=0}^{\infty} \frac{c_j}{s + \frac{j-n-\beta}{\alpha}} + \sum_{k=0}^{\infty} \left( \frac{c'_k}{(s+k)^2} + \frac{c''_k}{s+k} \right) ,$$

where the terms  $c_j, c'_j$ , are local, depending on just finitely many homogeneous terms of the symbols of both  $A$  and  $B$ , while the  $c''_k$  are global, depending on the complete symbols. Around zero (2.12) implies a Laurent expansion

$$\zeta_\theta(B, A, s)|^{\text{mer}} = \frac{c'_0}{s} + (c_{n+\beta} + c''_0) + O(s) , \quad \text{as } s \longrightarrow 0 ,$$

the simple pole determining the zeta-function formula for the residue trace of the classical  $\psi$ do  $B$

$$(2.13) \quad \text{res}(B) = \alpha \text{Res}_{s=0}^{\mathbb{C}} (\zeta_{\theta}(B, A, s)|^{\text{mer}}) = \alpha c'_0 ,$$

while the constant term defines the ‘ $A$ -weighted zeta-trace’

$$(2.14) \quad \text{Tr}_{\zeta}^A(B) = c_{n+\beta} + c''_0 .$$

If  $\beta < -n$  one has  $\text{res}(B) = 0$  and so  $\zeta_{\theta}(B, A, s)|^{\text{mer}}$  is then holomorphic near  $s = 0$ , while  $\text{Tr}_{\zeta}^A(B) = \zeta_{\theta}(B, A, 0)|^{\text{mer}}$  and is equal to the  $L^2$  trace  $\text{Tr}(B) = c''_0$ . If  $\beta \in \mathbb{R} \setminus \mathbb{Z}$  then once more  $\text{res}(B) = 0$  and  $\text{Tr}_{\zeta}^A(B) = \zeta_{\theta}(B, A, 0)|^{\text{mer}}$  is independent of  $A$  [KoVi], and again equal to the global term  $c''_0$  [Gr], and vanishes on commutators for which the sum of operator orders is non-integral [KoVi]. These properties also hold on the subalgebra of odd-class  $\psi$ dos [KoVi].  $\text{Tr}_{\zeta}^A$  is a quasi-trace in so far as it is not tracial on the full algebra  $\Psi(E)$ , but does extend the  $L^2$  trace to a trace on the above subclasses.

If  $B$  is a *logarithmic*  $\psi$ do then  $\zeta_{\theta}(B, A, s)|^{\text{mer}}$  again extends meromorphically to  $\mathbb{C}$  but now with  $\beta = 0$  in (2.12) and with possible additional poles  $c_{j,1}(s + \frac{j-n}{\alpha})^{-2}$  [Gr]. When  $B = \log_{\theta} A$  there is no pole at  $s = 0$  and a (quasi-) determinant, the zeta determinant  $\det_{\zeta, \theta}(A)$ , may be defined by taking the zeta trace <sup>3</sup> of  $\log_{\theta} A$

$$(2.15) \quad \begin{aligned} \log \det_{\zeta, \theta}(A) &:= \text{Tr}_{\zeta}^A(\log A) \\ &= \zeta_{\theta}(\log A, A, 0)|^{\text{mer}} . \end{aligned}$$

If  $\alpha > 0$  and one sets  $\zeta(A, s)|^{\text{mer}} := \zeta(I, A, s)|^{\text{mer}} = \text{Tr}(A^{-s})|^{\text{mer}}$ , then equivalently

$$(2.16) \quad \log \det_{\zeta, \theta}(A) = -\frac{d}{ds} \zeta_{\theta}(A, s)|_{s=0}^{\text{mer}} , \quad \alpha > 0 .$$

If  $\alpha = 0$  then from (2.6) and the comment following (2.14), the determinant (2.15) is defined on  $I + \Psi^{<-n}(E)$  and equal there to the Fredholm determinant. On the other hand,  $\zeta_{\theta}(A, s)$ , and hence the right side of (2.16), is then not defined for any  $s$ . Nevertheless, the relative zeta function

$$(2.17) \quad \zeta_{\theta}^{\text{rel}}[A, B](s) := \text{Tr}(A_{\theta}^{-s} - B_{\theta}^{-s})$$

is defined on  $I + \Psi^{<-n}(E)$ , and one has there  $\text{Tr} \log_{\theta}(A) = -\frac{d}{ds} \zeta_{\theta}^{\text{rel}}[A, I](s)|_{s=0}$ . For elliptic  $A_0, A_1$  of non-zero order  $\zeta_{\theta}^{\text{rel}}[A_0, A_1](s) = \zeta_{\theta}(A_0, s) - \zeta_{\theta}(A_1, s)$ ; relative determinants are studied in [Mu], [Sc].

The zeta determinant  $\det_{\zeta, \theta}(A)$  is non-local, depends on the spectral cut  $R_{\theta}$  [Wo1] and is not multiplicative [Ok2, KoVi].

The residue determinant has a quite different nature.

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<sup>3</sup>For  $L$  a logarithmic  $\psi$ do, such as  $\log_{\theta} A$ , the dependence on the choice of regularizing operator  $\text{Tr}_{\zeta}^{A_1}(L) - \text{Tr}_{\zeta}^{A_2}(L) = \text{Tr}(L(A_1^{-s} - A_2^{-s}))|_{s=0}^{\text{mer}}$  is computed in [KoVi], [Ok2], [PaSc] as a residue trace, leading to the multiplicative anomaly formula for the zeta-determinant.

First, the *construction* of the residue determinant takes place completely independently of spectral zeta-functions, and it is in this distinction that its non-triviality lies.

Specifically, using the spectral  $\zeta$ -function formula (2.13) with  $B = \log_\theta A$  to define a putative residue determinant, rather than the symbolic definition (1.7), leads to a trivial determinant (equal to 1); the triviality is equivalent to  $\zeta(A, s)|^{\text{mer}}$  being holomorphic at zero, and thus (2.15) being defined.

Further, since the residue zeta function  $\zeta_{\text{res}}(A, s) := \text{res}(A^{-s})$  is highly discontinuous – for, from (1.3),  $\text{res}(A^{-s})$  can be non-zero only for  $s \cdot \text{ord}(A) \in \mathbb{Z} \cap (-\infty, n]$  – there is no residue analogue of (2.16).

The residue trace (1.7) on  $\log_\theta A$  thus does not arise a complex residue, but it does generalize the integral formula which for classical  $\psi$ dos coincides with the complex residue (2.13). However, if  $A, B$  have residue determinants and are of the same order then the difference  $\text{res}(\log A) - \text{res}(\log B)$  is given as a complex residue.

The residue determinant  $\det_{\text{res}}(A)$  is local, independent of the spectral cut  $R_\theta$  and multiplicative.

### 3. PROOFS

Let  $\mathbf{a} \sim (\mathbf{a}_0, \mathbf{a}_1, \dots) \in S^\alpha(U)$ ,  $\mathbf{b} \sim (\mathbf{b}_0, \mathbf{b}_1, \dots) \in S^\beta(U)$  be local classical (1-step polyhomogeneous) symbols of respective degrees  $\alpha, \beta \in \mathbb{R}$ . Then a product structure is defined on  $S(U)$

$$\mathbf{a} \circ \mathbf{b} \sim ((\mathbf{a} \circ \mathbf{b})_0, (\mathbf{a} \circ \mathbf{b})_1, \dots) \in S^{\alpha+\beta}(U) ,$$

with

$$(3.1) \quad (\mathbf{a} \circ \mathbf{b})_j = \sum_{|\mu|+k+l=j} \frac{1}{\mu!} \partial_\xi^\mu(\mathbf{a}_k) D_x^\mu(\mathbf{b}_l) ,$$

and multiplicative identity element

$$\mathbf{l} = (I, 0, 0, \dots) .$$

At the  $\psi$ do level this represents the operator product modulo smoothing operators; thus if in local coordinates  $\sigma(A), \sigma(B)$  are respectively equivalent symbols to  $\mathbf{a}, \mathbf{b}$ , then  $\sigma(AB)$  is equivalent to  $\mathbf{a} \circ \mathbf{b}$ . This is all that is needed to compute local quantities such as the residue determinant.

To do so for a classical elliptic  $\psi$ do  $A$  of order  $\alpha$ , standard methods [Gi, Se, Sh] construct a parametrix for  $A - \lambda I$  by inverting locally at the symbolic level. We consider a finite open cover of  $M$  by coordinate patches  $U_i, i \in I = \{1, \dots, m\}$ , over each of which  $E$  is trivialized, with a subordinate partition of unity  $\phi_i \in C^\infty(U_i)$

such that for  $i, j \in I$  there is a  $l_{ij} \in I$  with  $\text{supp}(\phi_i) \cup \text{supp}(\phi_j) \subset U := U_{l_{ij}}$ . Then

$$(3.2) \quad A = \sum_{i,j} \phi_i A \phi_j$$

with each summand a  $\psi$ do acting in a single coordinate patch, and it will be enough to work with a symbol  $\mathbf{a} = \sigma(\phi_i A \phi_j) \sim (\mathbf{a}_0, \mathbf{a}_1, \dots) \in S^\alpha(U)$  of each such local operator. A local resolvent symbol

$$\mathbf{r}(\lambda) \sim (\mathbf{r}(\lambda)_0, \mathbf{r}(\lambda)_1, \dots) \in S^{-\alpha}(U_\lambda)$$

is defined over  $U_\lambda = \{x \in U \mid \lambda \notin \text{spec}(\mathbf{a}_0(x, \xi)), \xi \in \mathbb{R}^n\}$  by the inductive formulae

$$(3.3) \quad \mathbf{r}(\lambda)_0(x, \xi) = (\mathbf{a}_0(x, \xi) - \lambda \mathbf{I})^{-1},$$

$$(3.4) \quad \mathbf{r}(\lambda)_j(x, \xi) = -\mathbf{r}(\lambda)_0(x, \xi) \sum_{\substack{|\mu|+k+l=j \\ l < j}} \frac{1}{\mu!} \partial_\xi^\mu \mathbf{a}_k(x, \xi) D_x^\mu \mathbf{r}(\lambda)_l(x, \xi).$$

For  $|\xi| \geq 1, t \geq 1$ , each  $\mathbf{r}(\lambda)_j(x, \xi)$  has the quasi-homogeneity property

$$\mathbf{r}(t^\alpha \lambda)_j(x, t\xi) = t^{-\alpha-j} \mathbf{r}(\lambda)_j(x, \xi),$$

and by construction

$$(3.5) \quad \mathbf{r}(\lambda) \circ (\mathbf{a} - \lambda \mathbf{I}) \sim (\mathbf{a} - \lambda \mathbf{I}) \circ \mathbf{r}(\lambda) \sim \mathbf{I}.$$

Consequently, if  $A$  has principal angle  $\theta$ , then  $\log_\theta A$  and  $A_\theta^{-s}$  are represented by symbols

$$(3.6) \quad \log_\theta \mathbf{a} \sim (\mathbf{q}_{\theta,0}, \mathbf{q}_{\theta,1}, \dots), \quad \mathbf{a}_\theta^{-s} \sim (\mathbf{a}_{\theta,0}^{-s}, \mathbf{a}_{\theta,1}^{-s}, \dots),$$

where, with  $\Gamma_\theta = \Gamma_\theta(x, \xi)$  a closed contour as in (2.1) chosen to enclose the spectrum of  $\mathbf{a}_0(x, \xi)$  avoiding the spectral cut and the origin,

$$(3.7) \quad \mathbf{q}_{\theta,j}(x, \xi) = \frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda \mathbf{r}(\lambda)_j(x, \xi) d\lambda,$$

$$(3.8) \quad \mathbf{a}_{\theta,j}^{-s}(x, \xi) = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda_\theta^{-s} \mathbf{r}(\lambda)_j(x, \xi) d\lambda.$$

We may also write

$$(\log \mathbf{a})_{\theta,j}(x, \xi) := \mathbf{q}_{\theta,j}(x, \xi).$$

For  $|\xi| \geq 1$ , it follows for  $j \geq 0$  that  $\mathbf{a}_{\theta,j}^{-s} \in S^{-\alpha s - j}(U)$  is homogeneous of degree  $-\alpha s - j$ , and for  $j \geq 1$  that  $\mathbf{q}_{\theta,j} \in S^{-j}(U)$  is homogeneous of degree  $-j$ . Since

$$\mathbf{q}_{\theta,j}(x, \xi) = -\partial_s|_{s=0} \mathbf{a}_{\theta,j}^{-s}(x, \xi)$$

and

$$\mathbf{a}_{\theta,j}^{-s}(x, \xi)|_{s=0} = \delta_{j,0} \mathbf{I},$$

then

$$(3.9) \quad \mathbf{a}_\theta^0 = \mathbf{l} = (I, 0, 0, \dots)$$

and

$$(3.10) \quad \mathbf{q}_{\theta,0}(x, \xi) = \alpha \log[\xi] I + \mathbf{p}_{\theta,0}(x, \xi)$$

with  $\mathbf{p}_{\theta,0} \in S^0(U)$  a classical symbol of degree 0. This means that the  $\psi$ dos  $\log_\theta A$  and  $A_\theta^{-s}$  can be approximated modulo smoothing operators as

$$(3.11) \quad \log_\theta A \sim \sum_{j=0}^{\infty} (\log_\theta A)_{[j]}, \quad A_\theta^{-s} \sim \sum_{j=0}^{\infty} A_{\theta,j}^{(-s)}$$

with  $(\log_\theta A)_{[j]} = \text{OP}(\mathbf{q}_{\theta,j})$ , an operator in  $\Psi^{-j}(U, E)$  for  $j \geq 1$ , and  $A_{\theta,j}^{(-s)} = \text{OP}(\mathbf{a}_{\theta,j}^{-s}) \in \Psi^{-s\alpha-j}(U, E)$  for  $j \geq 0$ . In particular, the local residue density associated to  $\mathbf{q}_{\theta,n}(x, \xi)$  is defined independently of the choice of local coordinates and

$$(3.12) \quad \log \det_{\text{res}} A := \frac{1}{(2\pi)^n} \int_M \int_{|\xi|=1} \text{tr}(\mathbf{q}_{\theta,n}(x, \xi)) dS(\xi) dx .$$

### Proof of Proposition 1.5

*Proof.* From (3.4) we have that  $r(\lambda)_n$  is computed only using the first  $n+1$  homogeneous terms  $\mathbf{a}_0, \dots, \mathbf{a}_n$ . Consequently, by (3.7) and (3.12) the same is true for the residue determinant.

Let  $\theta, \phi \in \mathbb{R}$  be two choices of principal angle with  $(\theta - \phi)/2\pi \in \mathbb{R} \setminus \mathbb{Z}$ . Then

$$(3.13) \quad \mathbf{q}_{\phi,j}(x, \xi) - \mathbf{q}_{\theta,j}(x, \xi) = 2\pi i m \mathbf{l}_j + \int_{\Gamma_{\phi,\theta}} r(\lambda)_j(x, \xi) d\lambda ,$$

where  $m = \pm[(\theta - \phi)/2\pi] \in \mathbb{Z}$  and the bounded contour  $\Gamma_{\phi,\theta} = \Gamma_{\phi,\theta}(x, \xi)$  can be taken of the form

$$\{\rho e^{i\theta} \mid R \geq \rho \geq r\} \cup \{r e^{it} \mid \phi \geq t \geq \theta\} \cup \{\rho e^{i(\theta-2\pi)} \mid r \leq \rho \leq R\} \cup \{R e^{it} \mid \phi \leq t \leq \theta\}$$

enclosing an annular region between the cuts  $R_\theta$ , and  $R_\phi$  and circles of radius  $r < R$ . This follows by a similar analysis to [Wo1]§3 for the symbols of the complex powers.

On the other hand, the contour integral on the right-side of (3.13) is  $-2\pi i$  times the homogeneous component of degree  $-j$  of the local symbol of a  $\psi$ do projection  $P_{\theta,\phi}(A)$  whose range contains the direct sum of those generalized eigenspaces of  $A$  with eigenvalues contained in  $\Gamma_{\phi,\theta}$ , and is zero if  $(\theta - \phi)/2\pi \in \mathbb{Z}$  (see [Bu], [Po]). Consequently, taking  $j = n$ , (3.12) and (3.13) imply

$$\text{res}(\log_\theta A) - \text{res}(\log_\phi A) = -2\pi i \text{res}(P_{\theta,\phi}(A)) .$$

Since the residue trace of any  $\psi$ do projection is zero [Wo2], we infer that  $\det_{\text{res}}$  is independent of the choice of principal angle.  $\square$

**Remark 3.1.** *The vanishing of  $\text{res}$  on  $\psi\text{do}$  projections is shown in [Wo2] to be equivalent to  $\zeta_\theta(A, 0)|^{\text{mer}}$  being independent of  $\theta$ .*

### Proof of Theorem 1.7

*Proof.* From (3.12) and (3.1),  $\text{res}(\log A)$  is seen to depend on only the first  $n + 1$  homogeneous terms in the local symbol expansion of  $A$ , and finitely many of their derivatives, while  $(\mathbf{a} \circ \mathbf{b})_n$  is determined using only  $\mathbf{a}_0, \dots, \mathbf{a}_n, \mathbf{b}_0, \dots, \mathbf{b}_n$ . The demonstration of multiplicativity can therefore be reduced to a certain finite-dimensional symbol algebra, introduced by Okikiolu [Ok1]§3, where the following standard Banach algebra version of the Campbell-Hausdorff Theorem [Ok1, Ja] can be applied.

**Theorem** *Let  $\mathcal{B}$  be a Banach algebra with norm  $\| \cdot \|$  and identity  $I$ . For invertible elements  $a, b \in \mathcal{B}$  and a choice of Agmon angles one can define using (2.2) elements  $\log(a)$ ,  $\log(b)$  and  $\log(ab)$  in  $\mathcal{B}$ . Then for real sufficiently small  $s, t > 0$*

$$(3.14) \quad \log(a^s b^t) = s \log(a) + t \log(b) + \sum_{k=1}^{\infty} C^{(k)}(s \log(a), t \log(b)) ,$$

where  $C^{(k)}(s \log(a), t \log(b))$  is the element of  $\mathcal{B}$

$$(3.15) \quad \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j+1} \sum \frac{(\text{Ad}(s \log(a))^{n_1} (\text{Ad}(t \log(b))^{m_1} \dots (\text{Ad}(s \log(a))^{n_j} (\text{Ad}(t \log(b))^{m_j} \log(b))))}{(1 + \sum_{i=1}^j m_i) n_1! \dots n_j! m_1! \dots m_j!}$$

and the inner sum is over  $j$ -tuples of pairs  $(n_i, m_i)$  such that  $n_i + m_i > 0$  and  $\sum_{i=1}^j n_i + m_i = k$ . For  $c \in \mathcal{B}$  the operator  $\text{Ad}(c)$  acts by  $\text{Ad}(c)(c') = [c, c']$ .

We denote by  $\mathcal{S}_{[n]}(U)$  the algebra of finite-symbol sequences of length  $n$ , introduced in [Ok1]. An element of  $\mathcal{S}_{[n]}(U)$  is an  $(n + 1)$ -tuple  $\mathbf{p} = (\mathbf{p}_0, \dots, \mathbf{p}_n)$  of polynomials

$$(3.16) \quad \mathbf{p}_j : U \times \mathbb{R}^n \longrightarrow \text{End}(\mathbb{R}^N) , \quad \mathbf{p}_j(x, \xi) = \sum_{|\mu|+|\nu| \leq n-1} p_{j,\mu,\nu} x^\mu \xi^\nu ,$$

with  $p_{j,\mu,\nu} \in \text{End}(\mathbb{R}^N)$ .  $\mathcal{S}_{[n]}(U)$  is a finite-dimensional vector space which relative to a fixed point  $(x_0, \xi_0) \in U \times \mathbb{R}^n$  can be endowed with an associative product structure, defined for  $\mathbf{p}, \tilde{\mathbf{p}} \in \mathcal{S}_{[n]}(U)$  by

$$(3.17) \quad (\mathbf{p} \circ \tilde{\mathbf{p}})_j = \pi_{n-j} \left( \sum_{|\mu|+k+l=j} \frac{1}{\mu!} \partial_\xi^\mu (\mathbf{p}_k) D_x^\mu (\tilde{\mathbf{p}}_l) \right) ,$$

where for a smooth function  $f$  defined in a neighborhood of  $(x_0, \xi_0) \in U \times \mathbb{R}^n$

$$(3.18) \quad \pi_m(f) = \sum_{|\mu|+|\nu| \leq m} \frac{1}{\mu! \nu!} \partial_\xi^\mu \partial^\nu (f)(x_0, \xi_0) (x - x_0)^\mu (\xi - \xi_0)^\nu$$

is the Taylor expansion of  $f$  around  $(x_0, \xi_0)$  to order  $m$ . Endowed with this product, relative to  $(x_0, \xi_0)$ ,  $\mathbf{S}_{[n]}(U)$  becomes an algebra which we denote by

$$\mathbf{S}_{[n]}(U)(x_0, \xi_0) .$$

The map from the symbol space  $S(U)$  to symbols of length  $n$

$$(3.19) \quad \pi : S(U) \longrightarrow \mathbf{S}_{[n]}(U)(x_0, \xi_0) , \quad \pi(\mathbf{a}) := (\pi_n(\mathbf{a}_0), \pi_{n-1}(\mathbf{a}_1), \dots, \pi_0(\mathbf{a}_n)) ,$$

where  $\mathbf{a} = (\mathbf{a}_0, \mathbf{a}_1, \dots)$ , is an algebra homomorphism, so that

$$(3.20) \quad (\pi(\mathbf{a} \circ \mathbf{b}))_j = (\pi(\mathbf{a}) \circ \pi(\mathbf{b}))_j ,$$

while, from (3.18), evaluation at the point  $(x_0, \xi_0)$  gives

$$(3.21) \quad (\pi(\mathbf{a}))_j(x_0, \xi_0) = \mathbf{a}_j(x_0, \xi_0) , \quad j \leq n .$$

The logarithm of an element  $\mathbf{p} = (\mathbf{p}_0, \dots, \mathbf{p}_n) \in \mathbf{S}_{[n]}(U)(x_0, \xi_0)$  admitting a principal angle can be defined by the procedure used in  $S(U)$ : Consider, by inclusion,  $\mathbf{p}$  as an element  $\tilde{\mathbf{p}}$  of  $S(U)$ . If  $\lambda \notin \text{spec}(\mathbf{p}_0(x, \xi))$  then  $\tilde{\mathbf{p}}$  has a resolvent  $\mathbf{r}(\lambda) \in S(U_\lambda)$  given by (3.4), while

$$(3.22) \quad \mathbf{r}_\pi(\lambda) := \pi(\mathbf{r}(\lambda)) \in \mathbf{S}_{[n]}(U)(x_0, \xi_0)$$

inverts  $\mathbf{p} - \lambda \mathbf{l}_n$  in  $\mathbf{S}_{[n]}(U)(x_0, \xi_0)$ ; that is, since  $\pi(\tilde{\mathbf{p}}) = \mathbf{p}$ , applying  $\pi$  to (3.5) and using (3.20) we have

$$(3.23) \quad \mathbf{r}_\pi(\lambda) \circ (\mathbf{p} - \lambda \mathbf{l}_n) = (\mathbf{p} - \lambda \mathbf{l}_n) \circ \mathbf{r}_\pi(\lambda) = \mathbf{l}_n ,$$

where  $\mathbf{l}_n = (I, 0, \dots, 0)$  is the identity symbol in  $\mathbf{S}_{[n]}(U)(x_0, \xi_0)$ . Set

$$(3.24) \quad (\log_\theta \mathbf{p})_j(x, \xi) = \frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda \mathbf{r}_\pi(\lambda)_j(x, \xi) d\lambda .$$

Since the entries of  $\mathbf{r}_\pi(\lambda)$  are finite Taylor expansions of  $\mathbf{r}(\lambda)_j(x, \xi)$  around  $(x_0, \xi_0)$  the only logarithmic term is a  $\log |\xi_0|$ , there is no  $\log |\xi|$  term. It follows that (3.24) is an element of  $\mathbf{S}_{[n]}(U)(x_0, \xi_0)$ . Moreover, it is clear ([Ok1] Lemma 3.6) that for  $\mathbf{a} \in S(U)$

$$(3.25) \quad (\pi(\log_\theta \mathbf{a}))_j = (\log_\theta(\pi(\mathbf{a})))_j .$$

Likewise,  $\mathbf{p}_{\theta,j}^{-s}(x, \xi) = \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda_\theta^{-s} \mathbf{r}_\pi(\lambda)_j(x, \xi) d\lambda \in \mathbf{S}_{[n]}(U)(x_0, \xi_0)$  if  $\mathbf{p} \in \mathbf{S}_{[n]}(U)(x_0, \xi_0)$ , and we find for  $\mathbf{a} \in S(U)$

$$(3.26) \quad (\pi(\mathbf{a}_\theta^{-s}))_j = ((\pi(\mathbf{a}))_\theta^{-s})_j .$$

Now for  $s, t \in [0, 1]$  and  $\mathbf{a}, \mathbf{b} \in S(U)$ , (3.20), (3.25) and (3.26) give

$$(3.27) \quad (\pi(\log_\theta(\mathbf{a}^s \circ \mathbf{b}^t)))_n = (\log_\theta(\pi(\mathbf{a})^s \circ \pi(\mathbf{b})^t))_n ,$$

omitting the  $\theta$  subscript. Since  $\mathbf{S}_{[n]}(U)(x_0, \xi_0)$  is a finite-dimensional algebra, for  $s, t \geq 0$  sufficiently small we have from (3.14) for the induced norm

$$\begin{aligned}
(3.28) \quad (\pi(\log_\theta(\mathbf{a}^s \circ \mathbf{b}^t)))_n &= s(\log \pi(\mathbf{a}))_n + t(\log \pi(\mathbf{b}))_n \\
&\quad + \sum_{k=1}^{\infty} (C^{(k)}(s \log \pi(\mathbf{a}), t \log \pi(\mathbf{b})))_n \\
&= s(\pi(\log \mathbf{a}))_n + t(\pi(\log \mathbf{b}))_n \\
&\quad + \sum_{k=1}^{\infty} (\pi(C^{(k)}(s \log \mathbf{a}, t \log \mathbf{b})))_n,
\end{aligned}$$

and so, evaluating at the point  $(x_0, \xi_0)$ , (3.21) implies

$$\begin{aligned}
(3.29) \quad \log_\theta(\mathbf{a}^s \circ \mathbf{b}^t)_n(x_0, \xi_0) &= s \log \mathbf{a}_n(x_0, \xi_0) + t \log \mathbf{b}_n(x_0, \xi_0) \\
&\quad + \sum_{k=1}^{\infty} (C^{(k)}(s \log \mathbf{a}, t \log \mathbf{b}))_n(x_0, \xi_0).
\end{aligned}$$

Since all terms in (3.29) lie in the symbol class  $S(U)$ , with uniformly continuous derivatives of all orders on compact subsets of  $U \times \mathbb{R}^n$ , the convergence in (3.29) as  $N \rightarrow \infty$  of

$$\log_\theta(\mathbf{a}^s \circ \mathbf{b}^t)_n - s \log \mathbf{a}_n - t \log \mathbf{b}_n - \sum_{k=1}^N (C^{(k)}(s \log \mathbf{a}, t \log \mathbf{b}))_n$$

and all its derivatives at the point  $(x_0, \xi_0)$  is also uniform in  $(x_0, \xi_0) \in U_c \times S^{n-1}$  for compact subsets  $U_c \subset U$ . Hence, taking a partition of unity we can interchange the sum with integration over  $S^*M$  to get

$$\begin{aligned}
(3.30) \quad &\int_M \int_{|\xi|=1} \text{tr}(\log(\mathbf{a}^s \circ \mathbf{b}^t)_n(x, \xi)) dS(\xi) dx = \\
&s \int_M \int_{|\xi|=1} \text{tr}(\log \mathbf{a}_n(x, \xi)) dS(\xi) dx + t \int_M \int_{|\xi|=1} \text{tr}(\log \mathbf{b}_n(x, \xi)) dS(\xi) dx \\
&\quad + \sum_{k=1}^{\infty} \int_M \int_{|\xi|=1} \text{tr}(C^{(k)}(s \log \mathbf{a}, t \log \mathbf{b})_n(x, \xi)) dS(\xi) dx.
\end{aligned}$$

But  $C^{(k)}(s \log A, t \log B)$  is classical  $\psi$ do of order 0 with symbol

$$\sigma(C^{(k)}(s \log A, t \log B)) \sim C^{(k)}(s \log \mathbf{a}, t \log \mathbf{b}),$$

and so, in particular,

$$\sigma(C^{(k)}(s \log A, t \log B))_n = C^{(k)}(s \log \mathbf{a}, t \log \mathbf{b})_n.$$

It is, furthermore, by definition a commutator of logarithmic  $\psi$ dos, and hence by Proposition 1.2

$$\frac{1}{(2\pi)^n} \int_M \int_{|\xi|=1} \text{tr} (C^{(k)}(s \log \mathbf{a}, t \log \mathbf{b})_n(x, \xi)) dS(\xi) dx = \text{res} (C^{(k)}(s \log A, t \log B)) = 0 .$$

Thus (3.30) says that for sufficiently small  $s, t \in [0, 1]$

$$(3.31) \quad \text{res}(\log(A^s B^t)) = s \text{res}(\log A) + t \text{res}(\log B) .$$

But (3.31) is analytic in  $s, t$  and so it holds for all  $s, t \in [0, 1]$ . Evaluating at  $s = t = 1$  completes the proof.  $\square$

### Proof of Theorem 1.8

*Proof.* Let  $\mathbf{a}(x, \xi) = \sigma(A)(x, \xi) \in S^\alpha(U)$  be the symbol of  $A$  localized over  $U$ , as above. The complex powers  $A_\theta^{-s} \in \Psi^{-\alpha s}(E)$  are classical  $\psi$ dos defined in the half-plane  $\text{Re}(s) > 0$  by (2.8), and elsewhere by (2.9), with local symbol

$$(3.32) \quad \sigma(A_\theta^{-s})(x, \xi) \sim \sum_{j \geq 0} \mathbf{a}_{\theta, j}^{-s}(x, \xi) .$$

If  $A$  is not invertible, then for  $s \neq 0$  (2.10) remains unchanged, while

$$(3.33) \quad A_\theta^0 = I - \Pi_0(A) ,$$

with  $\Pi_0(A)$  a (in general, non self-adjoint) projection onto the generalized eigenspace  $E_0(A)$  in the statement of Theorem 1.8.

The symbol  $\sigma(A_\theta^{-s})(x, \xi)$  is integrable in  $\xi$  for  $\text{Re}(s) > n/\alpha$  and, for such  $s$ ,  $K(A_\theta^{-s}, x, x) dx$  defines a  $C^\infty$  globally defined  $n$ -form on  $M$  with values in  $\text{End}(E)$ . For  $\text{Re}(s) > n/\alpha$  and any  $J \in \mathbb{N}$  we have with  $\hat{d}\xi := (2\pi)^{-n} d\xi$

$$(3.34) \quad \begin{aligned} K(A_\theta^{-s}, x, x) &= \int_{\mathbb{R}^n} \sigma(A_\theta^{-s})(x, \xi) \hat{d}\xi \\ &= \int_{\mathbb{R}^n} \left( \sigma(A_\theta^{-s})(x, \xi) - \sum_{j=0}^{J-1} \mathbf{a}_{\theta, j}^{-s}(x, \xi) \right) \hat{d}\xi + \sum_{j=0}^{J-1} \int_{\mathbb{R}^n} \mathbf{a}_{\theta, j}^{-s}(x, \xi) \hat{d}\xi . \end{aligned}$$

With  $A_\theta^{-s}$  defined for all  $s \in \mathbb{C}$  by (2.9), the difference

$$\sigma(A_\theta^{-s})(x, \xi) - \sum_{j=0}^{J-1} \mathbf{a}_{\theta, j}^{-s}(x, \xi) \in S^{-\alpha \text{Re}(s) - J}(U)$$

is integrable in  $\xi$  for

$$(3.35) \quad \text{Re}(s) > \frac{n - J}{\alpha} ,$$

and so the first integral on the right-side of (3.34) extends holomorphically to the half-plane (3.35). Hence choosing  $J = n + 1$  (or any  $J > n$ ) we can set  $s = 0$  in that integral to get, using (3.9) and (3.33),

$$\begin{aligned}
\int_{\mathbb{R}^n} \sigma(A_\theta^{-s})(x, \xi) - \sum_{j=0}^n \mathbf{a}_{\theta,j}^{-s}(x, \xi) \hat{d}\xi \Big|_{s=0}^{\text{mer}} &= \int_{\mathbb{R}^n} \left( \sigma(A_\theta^0)(x, \xi) - \sum_{j=0}^n \mathbf{a}_{\theta,j}^0(x, \xi) \right) \hat{d}\xi \\
&= \int_{\mathbb{R}^n} \left( \sigma(I - \Pi_0(A))(x, \xi) - \sum_{j=0}^n \mathbf{l}_{\theta,j}(x, \xi) \right) \hat{d}\xi \\
(3.36) \qquad \qquad \qquad &= \int_{\mathbb{R}^n} -\sigma(\Pi_0(A))(x, \xi) \hat{d}\xi .
\end{aligned}$$

The remaining objects of interest, then, are the local-kernels along the diagonal

$$\mathbf{K}_j^{-s}(x) = \int_{\mathbb{R}^n} \mathbf{a}_j^{-s}(x, \xi) \hat{d}\xi .$$

Splitting the integral into two parts we have

$$(3.37) \quad \mathbf{K}_j^{-s}(x) \Big|_{s=0}^{\text{mer}} = \int_{|\xi| \leq 1} \mathbf{a}_j^{-s}(x, \xi) \hat{d}\xi \Big|_{s=0}^{\text{mer}} + \int_{|\xi| \geq 1} \mathbf{a}_j^{-s}(x, \xi) \hat{d}\xi \Big|_{s=0}^{\text{mer}} .$$

We deal first with the second term on the right side of (3.37), for which the symbol is homogeneous in  $|\xi|$ , hence leading to only local poles (any  $s$ ). Changing to polar coordinates and using the homogeneity of  $\mathbf{a}_j^{-s}(x, \xi)$ , we have for  $\text{Re}(s) > (n - j)/\alpha$

$$(3.38) \quad \int_{|\xi| \geq 1} \mathbf{a}_j^{-s}(x, \xi) \hat{d}\xi = \int_1^\infty r^{-\alpha s - j + n - 1} dr \int_{|\xi|=1} \mathbf{a}_j^{-s}(x, \xi) \hat{d}S(\xi)$$

$$(3.39) \quad = \frac{1}{(\alpha s + j - n)} \int_{|\xi|=1} \mathbf{a}_j^{-s}(x, \xi) \hat{d}S(\xi) .$$

The meromorphic extension of the left side of (3.38) is defined by (3.39). When  $j \neq n$  then (3.39) is holomorphic around  $s = 0$  and so from (3.9)

$$(3.40) \quad \int_{|\xi| \geq 1} \mathbf{a}_j^{-s}(x, \xi) \hat{d}\xi \Big|_{s=0}^{\text{mer}} = 0, \quad j \neq 0, n ,$$

$$(3.41) \quad \int_{|\xi| \geq 1} \mathbf{a}_0^{-s}(x, \xi) \hat{d}\xi \Big|_{s=0}^{\text{mer}} = -\frac{1}{(2\pi)^n} \cdot \frac{\text{vol}(S^{n-1})}{n} .$$

For  $j = n$  we use [Ok2] Lemma(2.1) which states that there is an equality

$$\mathbf{a}_j^{-s}(x, \xi) = \sum_{k=0}^{\infty} \frac{(-s)^k}{k!} ((\log \mathbf{a})^k)_j(x, \xi) ,$$

where  $(\log \mathbf{a})^k := \log \mathbf{a} \circ \log \mathbf{a} \circ \dots \circ \log \mathbf{a}$  ( $k$  times) and the right-side is convergent as a function of  $(s, x, \xi)$  in the standard Frechet topology on  $C^\infty(\mathbb{C} \times U, (\mathbb{R}^N)^* \otimes \mathbb{R}^N)$ .

So we obtain

$$\begin{aligned} \int_{|\xi| \geq 1} \mathbf{a}_n^{-s}(x, \xi) \hat{d}\xi \Big|_{s=0}^{\text{mer}} &= \frac{1}{\alpha s} \int_{|\xi|=1} (l_n(x, \xi) - s (\log \mathbf{a})_n(x, \xi) + o(s)) \hat{d}\xi \\ &= -\frac{1}{\alpha} \int_{|\xi|=1} (\log \mathbf{a})_n(x, \xi) \hat{d}\xi + o(s^0). \end{aligned}$$

Hence

$$(3.42) \quad \int_{|\xi| \geq 1} \mathbf{a}_n^{-s}(x, \xi) \hat{d}\xi \Big|_{s=0}^{\text{mer}} = -\frac{1}{\alpha} \int_{|\xi|=1} (\log \mathbf{a})_n(x, \xi) \hat{d}\xi .$$

For general  $s \in \mathbb{C}$  the first (non-homogeneous) term on the right side of (3.37) is a more complicated expression leading to global poles. However, at  $s = 0$  this is local and, from (3.9), given by

$$\int_{|\xi| \leq 1} \mathbf{a}_j^{-s}(x, \xi) \hat{d}\xi \Big|_{s=0}^{\text{mer}} = \int_{|\xi| \leq 1} l_j(x, \xi) \hat{d}\xi .$$

Hence

$$(3.43) \quad \int_{|\xi| \leq 1} \mathbf{a}_j^{-s}(x, \xi) \hat{d}\xi \Big|_{s=0}^{\text{mer}} = 0 \quad j \neq 0 .$$

$$(3.44) \quad \int_{|\xi| \leq 1} \mathbf{a}_0^{-s}(x, \xi) \hat{d}\xi \Big|_{s=0}^{\text{mer}} = \frac{1}{(2\pi)^n} \cdot \text{vol}(B^n) ,$$

where  $B^n$  is the  $n$ -ball.

Thus from (3.34), (3.36), (3.37), (3.40), (3.41), (3.42), (3.43), (3.44) we have

$$\begin{aligned} K(A_\theta^{-s}, x, x) \Big|_{s=0}^{\text{mer}} &= - \int_{\mathbb{R}^n} \sigma(\Pi_0(A))(x, \xi) \hat{d}\xi dx \\ &\quad - \frac{1}{\alpha} \int_{|\xi|=1} (\log \mathbf{a})_n(x, \xi) \hat{d}S(\xi) - \frac{1}{(2\pi)^n} \cdot \frac{\text{vol}(S^{n-1})}{n} + \frac{1}{(2\pi)^n} \cdot \text{vol}(B^n) \\ &= - \int_{\mathbb{R}^n} \sigma(\Pi_0(A))(x, \xi) \hat{d}\xi dx - \frac{1}{\alpha} \int_{|\xi|=1} (\log \mathbf{a})_n(x, \xi) \hat{d}S(\xi) . \end{aligned}$$

Hence

$$\begin{aligned} &\int_M \int_{|\xi|=1} \text{tr}((\log \mathbf{a})_n(x, \xi)) \hat{d}S(\xi) dx = \\ &-\alpha \left( \int_M \text{tr}(K(A_\theta^{-s}, x, x) \Big|_{s=0}^{\text{mer}}) dx + \int_M \int_{\mathbb{R}^n} \text{tr}(\sigma(\Pi_0(A))(x, \xi)) \hat{d}\xi dx \right) , \end{aligned}$$

that is,

$$\text{res}(\log A) = -\alpha (\zeta(A, 0) \Big|_{s=0}^{\text{mer}} + \text{Tr}(\Pi_0(A))) .$$

□

### Proof of Theorem 1.17

*Proof.* We have,

$$\log(I + \mathbf{Q}) = \frac{i}{2\pi} \int_{\Gamma_\theta} \log \lambda (I + \mathbf{Q} - \lambda I)^{-1} d\lambda ,$$

where the finite contour  $\Gamma_\theta$  encloses, in particular 1. Iterating

$$(I + \mathbf{Q} - \lambda I)^{-1} = (1 - \lambda)^{-1} I - (1 - \lambda)^{-1} \mathbf{Q} (I + \mathbf{Q} - \lambda I)^{-1}$$

yields

$$(I + \mathbf{Q} - \lambda I)^{-1} = \sum_{j=0}^m (-1)^j (1 - \lambda)^{-j-1} \mathbf{Q}^j + (-1)^{m+1} (1 - \lambda)^m \mathbf{Q}^m (I + \mathbf{Q} - \lambda I)^{-1} ,$$

and so

$$(3.45) \quad \log(I + \mathbf{Q}) = \sum_{j=0}^m (-1)^j \mathbf{Q}^j \frac{i}{2\pi} \int_{\Gamma_\theta} \log \lambda (1 - \lambda)^{-j-1} d\lambda + R(\mathbf{Q}, m) ,$$

where

$$R(\mathbf{Q}, m) = (-1)^{m+1} \mathbf{Q}^m \frac{i}{2\pi} \int_{\Gamma_\theta} \log \lambda (1 - \lambda)^m (I + \mathbf{Q} - \lambda I)^{-1} d\lambda .$$

$R(\mathbf{Q}, m)$  is a classical  $\psi$ do of order  $-km$  and so for any positive integer  $m$  with  $mk < -n$  we have  $\text{res}(R(\mathbf{Q}, m)) = 0$ . All operators in (3.45) are integer order and so we can use the linearity of  $\text{res}$  in Lemma 1.1 to find

$$\text{res}(\log(I + \mathbf{Q})) = \sum_{j=1}^{\lfloor \frac{n}{|k|} \rfloor} (-1)^j \text{res}(\mathbf{Q}^j) \frac{i}{2\pi} \int_{\Gamma_\theta} \log \lambda (1 - \lambda)^{-j-1} d\lambda ,$$

the summation beginning now from  $j = 1$ , since  $\text{res}(I) = 0$  and the contour integral is zero for  $j = 0$ . Since  $\Gamma_\theta$  encloses 1, then for  $j \geq 1$

$$\frac{i}{2\pi} \int_{\Gamma_\theta} \log \lambda (1 - \lambda)^{-j-1} d\lambda = \frac{1}{j} \frac{i}{2\pi} \int_{\Gamma_\theta} \lambda^{-1} (1 - \lambda)^{-j} d\lambda = -\frac{1}{j}$$

and we reach the conclusion.  $\square$

### Proof of Theorem 1.20

*Proof.* Let  $A \in \Psi^\alpha(E)$  be an elliptic  $\psi$ do, admitting a principal angle. Let  $Q$  be a parametrix for  $A$ , so that

$$(3.46) \quad AQ - I = s_\infty \in \Psi^{-\infty}(E) , \quad QA - I = \tilde{s}_\infty \in \Psi^{-\infty}(E)$$

are smoothing operators. For any smoothing operator  $\kappa_\infty \in \Psi^{-\infty}(E)$  one has by Corollary 1.6

$$(3.47) \quad \det_{\text{res}}(A + \kappa_\infty) = \det_{\text{res}}(A) .$$

Let  $B \in \Psi^\alpha(E)$  with  $\alpha - \beta \in \mathbb{N}$ . Then by (1.11), which from the proof of Theorem 1.11 is seen to hold logarithmically,

$$\begin{aligned} \log \det_{\text{res}}(A + B) &= \log \det_{\text{res}}(AQ - s_\infty)(A + B) \\ &= \log \det_{\text{res}}(AQA + AQB + t_\infty) \\ &= \log \det_{\text{res}}(AQA + AQB) \\ &= \log \det_{\text{res}} A + \log \det_{\text{res}}(QA + QB) \\ &= \log \det_{\text{res}} A + \log \det_{\text{res}}(I + QB) , \end{aligned}$$

where  $t_\infty \in \Psi^{-\infty}(E)$  and for the final equality we use (3.46) and (3.47). Rewriting in terms of (1.13) and (1.26) this reads

$$-\alpha (\zeta(A+B, 0)|^{\text{mer}} + h_0(A+B)) = -\alpha (\zeta(A, 0)|^{\text{mer}} + h_0(A)) + \sum_{j=0}^M \frac{(-1)^j}{j} \text{res} ((QB)^j) .$$

The sum terminates when  $\text{ord}(QB).j < -n$ , so we may take

$$M = \left\lceil \frac{n}{\alpha - \beta} \right\rceil .$$

Replacing  $B$  by  $B[t] = B_0 + B_1 t + \dots B_d t^d$  now proves (1.33).  $\square$

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