

# THE $\zeta$ -DETERMINANT AND $\mathcal{C}$ -DETERMINANT ON THE GRASSMANNIAN IN DIMENSION ONE

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ABSTRACT. We discuss the relation of the  $\zeta$ -determinant of the Dirac operator on the interval to the *canonical*-determinant, which appears naturally in this situation. This note is a pilot for papers in preparation on the  $\zeta$  determinant defined on the infinite-dimensional Grassmannian of elliptic boundary problems for a Dirac operator in dimensions greater than one (see [9] for the statement of the result and preliminary discussion).

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## INTRODUCTION

A long standing question in mathematics and mathematical physics is:

*How natural is the  $\zeta$ -renormalization procedure leading to the definition of the determinant of the Dirac operator?*

We offer a detailed discussion of this question in the general case in the paper [10] (for work in progress, see also [9]). In this note we give a presentation of the 1-dimensional *toy model* for the general theory. The answer is positive, in the sense that we show using heat kernel methods that the  $\zeta$ -determinant is, up to a multiplicative constant, equal to a canonically defined algebraic determinant. Moreover, we are able to demonstrate all our analytical tools at work in this simple situation and also to explain several conceptual problems which arise in the theory. Therefore this note serves as an announcement and a pilot for a general analysis of the  $\zeta$ -determinant of an elliptic boundary value problem to be presented in [10] and to be elaborated in [2].

We avoid discussion of the *general* theory of the  $\zeta$ -determinant in dimension 1. Our analytical results, though they were not published before, can be obtained by using different methods. We refer to [5] for related results and an extensive bibliography of the subject.

We study the  $\zeta$ -determinant of the operator  $-i\frac{d}{dx}$ , or more generally  $-i\frac{d}{dx} + B(x)$ , where  $B(x)$  is a self-adjoint  $n \times n$  matrix, acting on  $\mathbf{C}^n$ -valued functions on the interval

$[0, 2\pi]$ . We have to pose a boundary condition in order to obtain a self-adjoint operator with a discrete spectrum. Such boundary conditions are parameterised by unitary operators  $T : \mathbf{C}^n \rightarrow \mathbf{C}^n$ , defining  $\mathcal{D}_T$  as the closed self-adjoint extension of the operator  $\mathcal{D} := -i \frac{d}{dx}$  with the domain

$$(1) \quad \{s \in C^\infty([0, 2\pi]; \mathbf{C}^n) \mid s(2\pi) = Ts(0)\}.$$

To make a connection with the Grassmannian description of the space of boundary conditions used in [9], let us observe that the space of boundary data can be identified with  $\mathbf{C}^n \oplus \mathbf{C}^n = \{(s(0), s(2\pi)) \mid s \in C^\infty([0, 2\pi]; \mathbf{C}^n)\}$  and there is an orthogonal projection onto the set of boundary data determined by condition  $T$

$$P_T = \frac{1}{2} \begin{pmatrix} \text{Id} & T^{-1} \\ T & \text{Id} \end{pmatrix}.$$

We define the canonical determinant  $\det_{\mathcal{C}} \mathcal{D}_T$  of the operator  $\mathcal{D}_T$ , by the formula

$$(2) \quad \det_{\mathcal{C}} \mathcal{D}_T := \det \frac{\text{Id}_{\mathbf{C}^n} - T^{-1}}{2}.$$

Let us recall that the  $\zeta$ -determinant  $\det_{\zeta} \mathcal{D}_T$  of the operator  $\mathcal{D}_T$ , introduced by Ray and Singer in [7] (see also [11], [12]), is given by the formula

$$(3) \quad \det_{\zeta} \mathcal{D}_T = e^{\frac{i\pi}{2}(\eta_{\mathcal{D}_T}(0) - \zeta_{\mathcal{D}_T^2}(0))} \cdot e^{-1/2 \cdot (d/ds(\zeta_{\mathcal{D}_T^2})|_{s=0})}.$$

Here is the main result of this note:

**Theorem 0.1.** *There exists a constant  $C$  such that for any unitary  $T$  the following equality holds*

$$\det_{\mathcal{C}} \mathcal{D}_T = C \cdot \det_{\zeta} \mathcal{D}_T.$$

Actually, for the choices of (2) and (3), we shall find (in Proposition 2.5 below) that the constant  $C$  takes the value  $2^n$ .

The Theorem shows that the  $\zeta$ -determinant can be obtained by a “healthy” algebraic procedure.

**Remark 0.2.** (a) *The canonical determinant, as defined above, appears naturally in the higher dimensional case (see [9] and [10]). The determinant line bundle over the infinite-dimensional Grassmannian of elliptic boundary conditions for a Dirac operator is a non-trivial complex line bundle with canonical determinant section, as defined by Quillen. This bundle restricted to the sub-Grassmannian of self-adjoint conditions becomes a trivial line bundle, and the canonical section becomes a function once we fix a trivialization. In [9] it was shown that there is a natural choice of trivialization; the*

canonical determinant of an elliptic boundary value problem for the Dirac operator is precisely the value of the canonical section in this trivialization.

Now, in the 1-dimensional case, the Grassmannian is finite-dimensional and there are no trivializations determined by the Calderón projection or the tangential operator. Nevertheless we observe that

- The operator  $\mathcal{D}_{-\text{Id}}$  has spectrum  $\{\frac{2k+1}{2}\}_{k \in \mathbf{Z}}$ , hence it corresponds to the operator  $-i\frac{d}{dx} + \frac{1}{2}$  on  $S^1$ , which is the Dirac operator defined by the non-trivial Spin-structure on  $S^1$ . This makes it somewhat natural to assume that  $\det_C \mathcal{D}_{-\text{Id}}$  is equal to 1.
- The operator  $\mathcal{D}_{\text{Id}}$  has spectrum equal to the set of integers (more precisely equal to the direct sum of  $n$  copies of  $\mathbf{Z}$ ), hence it is non-invertible and its determinant should be equal to 0. More generally,  $\mathcal{D}_T$  is non-invertible, whenever 1 is an eigenvalue of the matrix  $T$ . This narrows a reasonable choice to

$$\det \frac{\text{Id}_{\mathbf{C}^n} - T^{\pm 1}}{2} .$$

- The choice of the  $-1$  in the exponent of  $T$  in the formula is also motivated by the higher dimensional case. It makes (2) consistent with the definition of the canonical determinant given in formula (2.4) of [9] if we assume the transformation  $K$  in (2.4) of [9], which determines the Calderon projection, corresponds to  $-\text{Id}$ , and the transformation  $S$  defining the boundary condition corresponds to  $T$ .

(b) The essential novelty of the results presented here is the method. The variational formula of [5] is with respect to a variation of an operator with a fixed boundary condition and is obtained via a contour integral. Here we deal with the harder problem of proving the variational equality with respect to the boundary condition and using the heat kernel representation of the spectral  $\zeta$ -function. Thus we prove the projective equality of the determinants as functions on the unitary group  $U(n)$  considered as the parameter space of self-adjoint boundary conditions.

We study the variation of the  $\zeta$ -determinant to prove Theorem 0.1. We actually show that the variation of the phase of  $\det_\zeta$  is equal to the variation of the phase of  $\det_C$  and that the variation of the modulus of  $\det_\zeta$  is equal to the variation of the modulus of  $\det_C$ .

**Remark 0.3.** The proof suggests that one can fix the value of  $\det_\zeta$  of the operator  $\mathcal{D}_{-\text{Id}}$  as 1 and use the integral from the variation in order to define  $\det_\zeta$  on the whole of  $U(n)$ . This modified  $\zeta$ -determinant is equal to  $\det_C$ .

In Section 1 we present formulas for the variation of  $\det_C$ . In Section 2 we discuss the variation of  $\det_\zeta$  and, in order to determine the constant  $C$ , compute  $\det_\zeta(\mathcal{D}_{-\text{Id}})$ .

## 1. THE VARIATION OF THE CANONICAL DETERMINANT

In this Section we discuss the variation of  $\det_{\mathcal{C}} \mathcal{D}_T$  at a fixed boundary condition  $T$ . We replace  $T$  by  $T_r := e^{ir\alpha} T$ , where  $\alpha = \alpha^*$  is a self-adjoint  $n \times n$  matrix and compute

$$d/dr \{ \det_{\mathcal{C}} \mathcal{D}_{T_r} \} |_{r=0} .$$

The result is stated in the following Proposition.

**Proposition 1.1.** *Let  $\mathcal{R}_T(\alpha)$  denote  $d/dr \{ \ln \det_{\mathcal{C}} \mathcal{D}_{T_r} \} |_{r=0}$ , then the phase of the variation  $d/dr \{ \det_{\mathcal{C}} \mathcal{D}_{T_r} \} |_{r=0}$  is given by the formula*

$$(4) \quad \text{Im } \mathcal{R}_T(\alpha) = -\frac{\text{tr } \alpha}{2}$$

and the modulus is equal to

$$(5) \quad \text{Re } \mathcal{R}_T(\alpha) = -\frac{i}{2} \cdot \text{tr } \alpha (\text{Id} + T)(\text{Id} - T)^{-1} .$$

*Proof.* We use the formula

$$(6) \quad \frac{d}{dr} \{ \ln \det S_r \} = \text{Tr} \left( \frac{d}{dr} \{ S_r \} S_r^{-1} \right) ,$$

which in the case  $S_r := \frac{\text{Id} - T_r^{-1}}{2}$  gives

$$\mathcal{R}_T(\alpha) = d/dr \{ \ln \det_{\mathcal{C}} \mathcal{D}_{T_r} \} |_{r=0} = -i \cdot \text{tr } \alpha (\text{Id} - T)^{-1} .$$

It follows that the phase of the variation of the determinant is equal to

$$\begin{aligned} \text{Im } \mathcal{R}_T(\alpha) &= \frac{1}{2i} \cdot \text{tr} \{ -i\alpha (\text{Id} - T)^{-1} - (-i\alpha (\text{Id} - T)^{-1})^* \} \\ &= -\frac{1}{2} \cdot \text{tr} \{ \alpha (\text{Id} - T)^{-1} + (\text{Id} - T^{-1})^{-1} \alpha \} \\ &= -\frac{1}{2} \cdot \text{tr} \{ \alpha (\text{Id} - T)^{-1} - \alpha (\text{Id} - T)^{-1} T \} = -\frac{\text{tr } \alpha}{2} . \end{aligned}$$

We compute the modulus of  $\mathcal{R}_T(\alpha)$  in the same way

$$\begin{aligned} \text{Re } \mathcal{R}_T(\alpha) &= \frac{1}{2} \cdot \text{tr} \{ -i\alpha (\text{Id} - T)^{-1} + (-i\alpha (\text{Id} - T)^{-1})^* \} \\ &= -\frac{i}{2} \cdot \text{tr} \{ \alpha (\text{Id} - T)^{-1} - \alpha (\text{Id} - T^{-1})^{-1} \} = -\frac{i}{2} \cdot \text{tr} \alpha (\text{Id} - T)^{-1} (\text{Id} + T) . \end{aligned}$$

□

2. THE VARIATION OF THE ζ-DETERMINANT

Now we study the variation of the ζ-determinant under the change of the boundary condition described at the beginning of Section 1. We use a *unitary twist* in order to keep the boundary condition fixed and vary the operator inside the interval away from the boundary. This method was used for the first time in this context in [3] and since then has been crucial in obtaining several interesting results in spectral geometry (see [6], [9], [12]). We introduce a smooth cut-off function  $f : [0, 2\pi] \rightarrow [0, 1]$  equal to 1 for  $0 \leq x \leq \pi/2$  and equal to 0 for  $3\pi/2 \leq x \leq 2\pi$  and define a unitary transformation of the (trivial) bundle  $S^1 \times \mathbf{C}^n$  as follows

$$(7) \quad U_r(x) := T^{-1} e^{irf(x)\alpha} T .$$

The operator  $\mathcal{D}_{T_r}$  is unitarily equivalent to the operator

$$\mathcal{D}_r := (U_r(-i \frac{d}{dx}) U_r^{-1})_T ,$$

hence we can compute the variation of the ζ-determinant of  $\mathcal{D}_{T_r}$  by computing the variation of the family of operators

$$\mathcal{D}_r = -i \frac{d}{dx} - r f'(x) T^{-1} \alpha T ,$$

with fixed domain  $\{s \mid s(2\pi) = T s(0)\}$ . Equivalently, we can study the operator

$$-i \frac{d}{dx} - r f'(x) T^{-1} \alpha T : C^\infty(S^1; V_T) \rightarrow C^\infty(S^1; V_T),$$

where  $V_T$  denotes the complex bundle over  $S^1$  defined by  $T$ . Now, we can (as in [9]) directly compute the value of the invariants of the operator  $\mathcal{D}_r$  which contribute to the phase. Alternatively, we can use the result from the case of closed manifolds, see [4]. In any case we have the following well-known result.

**Lemma 2.1.** *For any  $T \in U(n)$  and for any self-adjoint  $n \times n$  matrix  $\alpha$  the following equalities hold*

$$(8) \quad \zeta_{\mathcal{D}_T^2}(0) = 0 \quad \text{and} \quad \frac{d}{dr} \{\eta_{\mathcal{D}_r}(0)\}|_{r=0} = -\frac{\text{tr } \alpha}{\pi} .$$

**Corollary 2.2.** *The variation of the phase of  $\det_\zeta$  is equal to the variation of the phase of  $\det_{\mathcal{C}}$ .*

We need to do more extensive work in order to compute the variation of the modulus of  $\det_\zeta$ . We have to study the variation of  $\zeta'_{\mathcal{D}_T^2}(0)$ , which is given by the *regularized* integral

$$\int_0^\infty \frac{1}{t} \cdot \text{Tr } e^{-t\mathcal{D}_T^2} dt = \lim_{s \rightarrow 0} \left\{ \int_0^\infty t^{s-1} \cdot \text{Tr } e^{-t\mathcal{D}_T^2} dt - \frac{\zeta_{\mathcal{D}_T^2}(0)}{s} \right\} .$$

Actually,  $\zeta_{\mathcal{D}_T^2}(0)$  vanishes by Lemma 2.1 and the integral on the left side of the identity is well-defined. Now we differentiate with respect to the parameter and obtain

$$\begin{aligned} \frac{d}{dr} \{\zeta'_{\mathcal{D}_r^2}(0)\}|_{r=0} &= \int_0^\infty \frac{1}{t} \cdot \text{Tr}(-2t\dot{\mathcal{D}}_0\mathcal{D}_0e^{-t\mathcal{D}_0^2})dt \\ &= -2 \int_0^\infty \text{Tr} \dot{\mathcal{D}}_0\mathcal{D}_0^{-1}\mathcal{D}_0^2e^{-t\mathcal{D}_0^2}dt \\ &= 2 \int_0^\infty \text{Tr} \dot{\mathcal{D}}_0\mathcal{D}_0^{-1} \frac{d}{dt}(e^{-t\mathcal{D}_0^2})dt \\ &= 2 \cdot \lim_{\varepsilon \rightarrow 0} (\text{Tr} \dot{\mathcal{D}}_0\mathcal{D}_0^{-1}e^{-t\mathcal{D}_0^2})|_{t=\varepsilon}^{t=1/\varepsilon} = -2 \cdot \lim_{\varepsilon \rightarrow 0} \text{Tr} \dot{\mathcal{D}}_0\mathcal{D}_0^{-1}e^{-\varepsilon\mathcal{D}_0^2} . \end{aligned}$$

This gives us the following formula for the variation of the modulus

$$(9) \quad \frac{d}{dr} \left\{ -\frac{1}{2} \zeta'_{\mathcal{D}_r^2}(0) \right\} |_{r=0} = \lim_{\varepsilon \rightarrow 0} \text{Tr} \dot{\mathcal{D}}_0\mathcal{D}_0^{-1}e^{-\varepsilon\mathcal{D}_0^2} .$$

We only need an explicit formula for the kernel of the operator  $\mathcal{D}_0^{-1} = \mathcal{D}_T^{-1}$ , to evaluate this formula.

**Lemma 2.3.** *Let  $k_T(x, y)$  denote the kernel of the operator  $\mathcal{D}_T^{-1}$ . Then*

$$k_T(x, y) = \begin{cases} -i(\text{Id} - T)^{-1} & \text{for } x < y \\ i(\text{Id} - T^{-1})^{-1} & \text{for } x > y . \end{cases}$$

Now we can evaluate the variation of the modulus of  $\det_\zeta$ . We have:

$$\begin{aligned} \frac{d}{dr} \left\{ -\frac{1}{2} \zeta'_{\mathcal{D}_r^2}(0) \right\} |_{r=0} &= \lim_{\varepsilon \rightarrow 0} \text{Tr} \dot{\mathcal{D}}_0\mathcal{D}_0^{-1}e^{-\varepsilon\mathcal{D}_0^2} \\ &= - \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} dx \text{tr} \left( f'(x)T^{-1}\alpha T \int_0^{2\pi} k_T(x, y)\mathcal{E}(\varepsilon; y, x)dy \right) , \end{aligned}$$

where  $\mathcal{E}(\varepsilon; y, x)$  denotes the kernel of the operator  $e^{-\varepsilon\mathcal{D}_0^2}$ . It is immediate from the properties of the heat kernel that

$$(10) \quad \lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} dx \text{tr} \left( f'(x)T^{-1}\alpha T \int_{\{y; |x-y| > \sigma\}} k_T(x, y)\mathcal{E}(\varepsilon; y, x)dy \right) = 0$$

for any  $\sigma > 0$ . Moreover,  $f'(x)$  is equal to 0 for  $x$  outside of the interval  $[\pi/2, 3\pi/2]$ , hence we use *Duhamel's Principle* and replace the original heat kernel by the heat kernel

of the operator  $-d^2/dx^2$  on the real line. We have

$$\begin{aligned}
 & \frac{d}{dr} \left\{ -\frac{1}{2} \zeta_{\mathcal{D}_r^2}(0) \right\} \Big|_{r=0} \\
 &= -\lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} dx \operatorname{tr} \left( f'(x) T^{-1} \alpha T \int_{\{y; |x-y| < \sigma\}} k_T(x, y) \mathcal{E}(\varepsilon; y, x) dy \right) \\
 &= -\lim_{\varepsilon \rightarrow 0} \int_0^{2\pi} dx \operatorname{tr} \left( f'(x) T^{-1} \alpha T \left\{ -i(\operatorname{Id} - T)^{-1} \int_0^\sigma \frac{1}{\sqrt{4\pi\varepsilon}} e^{-r^2/4\varepsilon} dr \right. \right. \\
 &\quad \left. \left. + i(\operatorname{Id} - T^{-1})^{-1} \int_0^\sigma \frac{1}{\sqrt{4\pi\varepsilon}} e^{-r^2/4\varepsilon} dr \right\} \right) \\
 &= -\frac{i}{\sqrt{\pi}} \cdot \int_0^{2\pi} \operatorname{tr} \left( f'(x) T^{-1} \alpha T \{ (\operatorname{Id} - T^{-1})^{-1} - (\operatorname{Id} - T)^{-1} \} \right) dx \\
 &\quad \cdot \lim_{\varepsilon \rightarrow 0} \int_0^\sigma e^{-r^2/4\varepsilon} \frac{dr}{2\sqrt{\varepsilon}} \\
 &= \frac{i}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{2} \cdot \operatorname{tr} \left( T^{-1} \alpha T \{ (\operatorname{Id} - T^{-1})^{-1} - (\operatorname{Id} - T)^{-1} \} \right) \\
 &= -\frac{i}{2} \cdot \operatorname{tr} T^{-1} \alpha T (\operatorname{Id} - T)^{-1} (\operatorname{Id} + T).
 \end{aligned}$$

This ends the proof of Theorem 0.1.

A particular corollary of our computations, which may be of independent interest, (at least in dimensions higher than 1) is that the only critical point of the modulus of the determinant is the ‘normalizing’ and ‘diagonalizing’ boundary condition given in dimension 1 by  $T = -\operatorname{Id}$ :

**Corollary 2.4.** *The variation of the modulus of the determinants  $\det_\zeta$  and  $\det_C$  at  $T = -\operatorname{Id}$  is equal to 0.*

We determine the proportionality constant between  $\det_\zeta$  and  $\det_C$  by calculating the precise value of  $\det_\zeta(\mathcal{D}_T)$  at  $T = -\operatorname{Id}$ . Recall that by definition  $\det_C(\mathcal{D}_{-\operatorname{Id}}) = 1$ .

**Proposition 2.5.** *For the operator  $\mathcal{D} = -i \frac{d}{dx}$  acting on  $\mathbf{C}^n$ -valued functions on the interval  $[0, 2\pi]$  we have*

$$\det_\zeta(\mathcal{D}_{-\operatorname{Id}}) = 2^n.$$

*Proof.* We use well-known formulas for the Hurwitz  $\zeta$  function. As noted in Remark 0.2,  $\mathcal{D}_{-\operatorname{Id}}$  has the same spectrum as  $-i \frac{d}{dx} + \frac{1}{2} \operatorname{Id}$  acting on  $C^\infty(S^1; \mathbf{C}^n)$ . More generally, consider the operator  $D_A = -i \frac{d}{dx} + A$  where  $A$  is a  $n \times n$  diagonal matrix with diagonal entries  $0 < a_i \leq 1$ . Since obviously

$$(11) \quad \det_\zeta D_A = \prod_{i=1}^n \det_\zeta D_{a_i}$$

where  $D_{a_i} = -i\frac{d}{dx} + a_i$  is acting on  $C^\infty(S^1; \mathbf{C})$ , it will be enough for us to consider the rank  $n = 1$  case. So we are going to compute the  $\zeta$  determinant

$$(12) \quad \det_\zeta(D_a) = e^{\frac{i\pi}{2}\eta_{D_a}(0)} \cdot (\det_\zeta D_a^2)^{1/2}$$

of  $D_a = -i\frac{d}{dx} + a$  with  $0 < a \leq 1$ . To do so we use the Hurwitz zeta-function

$$(13) \quad \zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

So  $\zeta(s, 1) = \zeta(s)$ . This has the analytically continued values

$$(14) \quad \zeta(0, a) = \frac{1}{2} - a, \quad \zeta'(0, a) = \log \Gamma(a) - \frac{1}{2} \log(2\pi).$$

We then have (choosing  $(-1)^{-s} = e^{i\pi s}$ ):

$$(15) \quad \sum_{n \in \mathbf{Z}} \frac{1}{(n+a)^s} = \zeta(s, a) + e^{i\pi s} \zeta(s, 1-a).$$

We also have

$$(16) \quad \text{spec}(D_a) = \{n+a \mid n \in \mathbf{Z}\} \quad \text{and} \quad \text{spec}(D_a^2) = \{(n+a)^2 \mid n \in \mathbf{Z}\}.$$

So that:

$$(17) \quad \zeta_{D_a}(s) = \zeta(s, a) + e^{i\pi s} \zeta(s, 1-a) \quad \text{and} \quad \zeta_{D_a^2}(s) = \zeta(2s, a) + \zeta(2s, 1-a).$$

Hence using (14)

$$(18) \quad \zeta_{D_a}(0) = 0 \quad \text{and} \quad \zeta_{D_a^2}(0) = 0.$$

Using  $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$  and (14), an elementary computation from (17) gives

$$(19) \quad \zeta'_{D_a}(0) = -\log(2 \sin(\pi a)) - \frac{i\pi}{2}(1-2a)$$

and

$$(20) \quad \zeta'_{D_a^2}(0) = -\log(4 \sin^2(\pi a)).$$

Thus by (12)

$$(21) \quad \det_\zeta D_a = e^{\frac{i\pi}{2}(1-2a)} \cdot 2 \sin(\pi a) = e^{\frac{i\pi}{2}(1-2a)} (\det_\zeta D_a^2)^{1/2}.$$

One can easily check the exponent is  $\eta(0)$ . In particular, then for  $a = 1/2$

$$(22) \quad (\det_\zeta D_{a=1/2}^2)^{1/2} = 2 \sin(\pi/2) = 2$$

and from (11) we arrive at  $2^n$  for the modulus of the zeta determinant at  $T = -\text{Id}$ .  $\square$

**Remark 2.6.** *Proposition 2.5 also follows from [8] where contour integral methods were used to prove the equality*

$$(23) \quad \det_\zeta(\mathcal{D}_T) = \det(\text{Id} + R^{-1}) .$$

[Notice the different sign of  $R$  in (23), since in this note we are considering the boundary condition (i.e.  $f(2\pi) = Tf(0)$ ) orthogonal to the one considered in [8] (i.e.  $f(2\pi) = -Rf(0)$  or, equivalently,  $P_R(f(0), f(2\pi)) = 0$ ).

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