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THE ζ-DETERMINANT AND QUILLEN DETERMINANT FOR A DIRAC OPERATOR ON A MANIFOLD WITH BOUNDARY

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Abstract

Let $\mathcal{D}$ denote a Dirac operator on a compact odd-dimensional manifold M with boundary Y. The elliptic boundary value problem $\mathcal{D}_P$ is the operator $\mathcal{D}$ with domain determined by a boundary condition P from the smooth self-adjoint Grassmannian $Gr^*_\infty(\mathcal{D})$. It has a well-defined ζ-determinant (see [Wo5]). The determinant line bundle over $Gr^*_\infty(\mathcal{D})$ has a natural trivialization in which the canonical Quillen determinant section becomes a function, denoted by $\det_{\mathcal{C}} \mathcal{D}_P$, equal to the Fredholm determinant of a naturally associated operator on the space of boundary sections. In this paper we show that the ζ-regularized determinant $\det_\zeta \mathcal{D}_P$ is equal to $\det_{\mathcal{C}} \mathcal{D}_P$ modulo a natural multiplicative constant.

Introduction

Since the early stages of *Quantum Mechanics* there has been a fundamental need for a rigorous and workable definition of the determinant of an invertible linear operator acting on an infinite-dimensional space. The *Fredholm determinant* provides a natural extension from the finite-rank case to a linear operator $T : \mathcal{H} \to \mathcal{H}$ acting on a separable Hilbert space such that $T - Id$ is an operator of trace class. It is defined in a small neighbourhood of the identity Id by the formula

$$\det{}_{Fr}\, e^{\alpha} = e^{\operatorname{Tr} \alpha}\,, \tag{0.1}$$

where α denotes an operator of trace class, and for general $T = Id + \alpha$ with α trace-class, by the absolutely convergent series

$$\det{}_{Fr}(Id + \alpha) = 1 + \sum_{k=1}^{\infty} \operatorname{Tr}(\wedge^k \alpha)\,. \tag{0.2}$$

In particular, the Fredholm determinant retains the characteristic multiplicative property of the finite-rank algebraic determinant

$$\det{}_{Fr}\, T_0 T_1 = \det{}_{Fr}\, T_0 \det{}_{Fr}\, T_1\,. \tag{0.3}$$

The trace-class condition is, however, very restrictive and the class of operators with Fredholm determinants certainly does not include any class of elliptic differential operators. Nevertheless, important applications in *Physics*, particularly in *Quantum Field Theory* (see the fundamental papers [MS1,2], led to the idea that the Fredholm determinant allows one to study the *ratio* of the determinant of an elliptic operator to the determinant of a 'comparable' basepoint operator; for instance, the ratio of the determinant of a Hamiltonian with potential and the determinant of the free Hamiltonian. In this way a *regularized determinant* relative to a choice of basepoint operator may be defined.

On the other hand, in many important problems, such as in quantizing gauge theories, it is necessary to discuss directly a regularized determinant of an elliptic operator. The *Heuristic Approach* to the determinant in this context was first proposed by mathematicians for the case of a positive definite second-order elliptic differential operator

$$L : C^\infty(M;S) \to C^\infty(M;S)$$

acting on the sections of a smooth vector bundle S over a closed manifold M. The operator L has a discrete spectral resolution and so formally has determinant equal to the infinite product of its eigenvalues. The starting point in defining a regularized product is the following formula for an invertible finite-rank linear operator T:

$$ln \det T = -\tfrac{d}{ds}\{\operatorname{Tr} T^{-s}\}|_{s=0} \,. \tag{0.4}$$

For large $Re(s)$ the ζ-function of the operator L is just the trace occurring on the right side of (0.4)

$$\zeta_L(s) = \operatorname{Tr} L^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr} e^{-tL} dt \,. \tag{0.5}$$

It is a holomorphic function of s for $Re(s) > \dim M/2$ and has a meromorphic extension to the whole complex plane with only simple poles (see [Se1]). In particular $s = 0$ is not a pole. Hence $\zeta_L'(0) = \frac{d}{ds}\{\zeta_L(s)\}|_{s=0}$ is well-defined and we may define the ζ-*determinant* by

$$\det\nolimits_\zeta L = e^{-\zeta_L'(0)} \,. \tag{0.6}$$

This definition was introduced in 1971 in a famous paper of Ray and Singer [RS] in order to define *Analytic Torsion*, the analytical counterpart of the topological invariant *Franz–Reidemeister Torsion*. The equality of the two torsions was subsequently proved independently by Jeff Cheeger and Werner Müller (see [C1], [Mü]). Since then, there have been numerous applications of the ζ-determinant in physics and mathematics, beginning with the 1975 Hawking paper [H] on quantum gravity.

For positive-definite operators of Laplace type over a closed manifold the ζ-determinant provides a generally satisfactory regularization method. Though the fundamental multiplicative property of the Fredholm determinant (0.3) no longer holds; if L_1 and L_2 denote two positive elliptic operators of positive order on a Hilbert space H then in general

$$\det_\zeta L_1 L_2 \neq \det_\zeta L_1 \cdot \det_\zeta L_2 \,.$$

We refer to [KV] for a detailed study of the so-called Multiplicative Anomaly. In many physical applications, however, such as the quantization of Fermions, one encounters the more problematic task of defining the determinant of a first-order *Dirac operator*. These are not positive operators, and it is at this point that anomalies may arise due to the phase of the determinant [AtS]. For a Dirac operator $\mathcal{D} : C^\infty(M;S) \to C^\infty(M;S)$ acting on the sections of a bundle of Clifford modules over a closed (odd-dimensional) manifold M one proceeds in the following way (see [BoW4] for an introduction and notation). The operator $\mathcal{D}$ is an elliptic self-adjoint first-order operator and hence has infinitely many positive and negative eigenvalues. Let $\{\lambda_k\}_{k\in\mathbf{N}}$ denote the set of positive eigenvalues and $\{-\mu_k\}_{k\in\mathbf{N}}$ denote the set of negative eigenvalues. Once again, $\zeta_{\mathcal{D}}(s) = \mathrm{Tr}(\mathcal{D}^{-s})$ is well-defined and holomorphic for $Re(s) > \dim M$ and we have

$$\begin{aligned}\zeta_{\mathcal{D}}(s) &= \sum_k \lambda_k^{-s} + \sum_k (-1)^{-s}\mu_k^{-s}\\ &= \sum_k \left(\frac{\lambda_k^{-s}-\mu_k^{-s}}{2} + \frac{\lambda_k^{-s}+\mu_k^{-s}}{2}\right) + (-1)^{-s}\sum_k \left(\frac{\lambda_k^{-s}+\mu_k^{-s}}{2} - \frac{\lambda_k^{-s}-\mu_k^{-s}}{2}\right),\end{aligned}$$

which can be written as

$$\zeta_{\mathcal{D}}(s) = (-1)^{-s}\frac{\zeta_{\mathcal{D}^2}(s/2) - \eta_{\mathcal{D}}(s)}{2} + \frac{\eta_{\mathcal{D}}(s) + \zeta_{\mathcal{D}^2}(s/2)}{2}\,, \tag{0.7}$$

where $\eta_{\mathcal{D}}(s) = \sum_k \lambda_k^{-s} - \sum_k \mu_k^{-s}$ is the η-function of the operator $\mathcal{D}$ introduced by Atiyah, Patodi and Singer (see [AtPS]). Once again it is holomorphic for $Re(s)$ large and has a meromorphic extension to the whole complex plane with only simple poles. There is no pole at $s = 0$ and therefore we can study the derivative of $\zeta_{\mathcal{D}}(s)$ at $s = 0$. We have

$$\zeta'_{\mathcal{D}}(0) = \frac{\zeta'_{\mathcal{D}^2}(0)}{2} + \frac{d}{ds}\{(-1)^{-s}\}\Big|_{s=0} \cdot \frac{\zeta_{\mathcal{D}^2}(0) - \eta_{\mathcal{D}}(0)}{2}\,.$$

The ambiguity in defining $(-1)^{-s}$ (i.e. a choice of spectral cut) now leads to an ambiguity in the phase of the ζ-determinant. We have

$$(-1)^{-s} = e^{\pm i\pi s}\,,$$

and we pick the " $-$ " sign. This leads to the following formula for the

ζ-determinant of the Dirac operator $\mathcal{D}$

$$\det_\zeta \mathcal{D} = e^{\frac{i\pi}{2}(\zeta_{\mathcal{D}^2}(0)-\eta_{\mathcal{D}}(0))} \cdot e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2}(0)} . \tag{0.8}$$

We come back to the discussion of the choice of sign in $(-1)^s$ in the final section of the paper.

The purpose of this paper is to explain a direct and precise identity between the ζ-determinant of a self-adjoint elliptic boundary value problem for the Dirac operator over an odd-dimensional manifold with boundary and a regularization of the determinant as the Fredholm determinant of a canonically associated operator over the boundary. We consider an infinite-dimensional Grassmannian of elliptic boundary conditions commensurable with the Atiyah-Patodi-Singer condition. The latter regularization is naturally understood, in the sense explained below, as the *ratio* of the determinant of the elliptic boundary value problem to the determinant of a base-point elliptic boundary value problem. It is a regularization canonically constructed from the topology of the associated determinant line bundle and hence called the *canonical determinant*. The canonical determinant is a robust algebraic operator-theoretic object, while the ζ-determinant is a highly delicate analytic object, and so it is surprising that they coincide. (Though, the equality of the torsions mentioned above at least suggests that the ζ-determinant may be somehow related to Fredholm determinants.)

To formalise the construction of taking the ratios of determinants used to define the canonical determinant we use the machinery of the determinant line bundle. This was introduced in a fundamental paper of Quillen [Q] for a family of Cauchy-Riemann operators acting on a Hermitian bundle over a Riemann surface, as the pull-back of the corresponding 'universal' determinant bundle over the space of Fredholm operators on a separable Hilbert space. Without making further choices, the determinant arises not as a function on the parameter space of operators but as a canonical section $A \mapsto \det A$ of the associated determinant line bundle DET. More precisely, $\det A$ lives in the complex line $\operatorname{Det} A := \wedge^{\max}(\operatorname{Ker} A)^* \otimes \wedge^{\max} \operatorname{Coker} A$ and is non-zero if and only if A is invertible. Using ζ-function regularization, Quillen constructed a natural Hermitian metric on the determinant bundle for a family of Cauchy Riemann operators and computed its curvature. This was extended by Bismut and Freed to the context of general families of Dirac operators on closed manifolds (see [BF]) and the curvature identified with the 2-form component of the families index density. It did not, though, provide a straightforward correspondence between the ζ-determinant and the Quillen determinant section. More precisely, given that $\det_\zeta(A)$ is defined, the problem is to identify the non-zero section σ of DET such that $\det_\zeta(A) = \det A/\sigma(A)$. Clearly the global existence of such a section σ is equivalent to the triviality of the determinant line bundle. From this view point, the principal result of this paper is the exact identification of the 'basepoint' section σ for the class of elliptic boundary value problems considered here. In order to link this up with Fredholm determinants we use a construction of the determinant line bundle due to Segal [Seg].

We do not discuss in this paper the corresponding problem for a closed manifold. Perhaps surprisingly, it is easier to discuss the relation between the ζ-determinant and the Quillen determinant section on a manifold with boundary, because of reduction to the *boundary integral* (see [BoW4]). In early 1995 the second author as the follow-up to his work on the η-invariant on a manifold with boundary (see [Wo3,4]) showed the existence of the ζ-determinant on the Grassmannian of generalized Atiyah-Patodi-Singer boundary conditions. A little earlier the first author using the Segal construction of the determinant line bundle introduced the canonical $\mathcal{C}$-determinant on this Grassmannian and showed that it is equal to the ζ-determinant in the one dimensional case (see [S1]; see also [BoSW] for a discussion of the one dimensional case in the spirit of this exposition). The present paper contains the result of joint work, the proof of the equality of the ζ-determinant and $\mathcal{C}$-determinant up to a natural multiplicative constant in any odd dimension. Early progress was reported in the note [SW1] and the results of this paper were announced in [SW2]. We refer to [BoMSW] for a discussion of related topics in the even-dimensional case (see also the review [WoSMB]). The construction of a metric and compatible connection on the determinant line bundle using the canonical regularization for a family of Dirac operators over a closed manifold endowed with a partition is explained in [S2].

We now give a more detailed presentation of the situation discussed in this paper. Let $\mathcal{D} : C^\infty(M; S) \to C^\infty(M; S)$ denote a compatible Dirac operator acting on the space of sections of a bundle of Clifford modules S over a compact connected manifold M with boundary Y. It is not actually necessary to assume that $\mathcal{D}$ is a *compatible* Dirac operator; further technical comments are made in the final section of the paper. In the present paper we always assume that M is an odd-dimensional manifold; the even-dimensional case will be discussed separately. And we discuss only the *Product Case*. Namely we assume that the Riemannian metric on M and the Hermitian structure on S are products in a certain collar neighborhood of the boundary. Let us fix a parameterization $N = [0,1] \times Y$ of the collar. Then in N the operator $\mathcal{D}$ has the form

$$\mathcal{D}_{|N} = G(\partial_u + B)\,, \tag{0.9}$$

where $G : S|Y \to S|Y$ is a unitary bundle isomorphism (Clifford multiplication by the unit normal vector) and $B : C^\infty(Y; S|Y) \to C^\infty(Y; S|Y)$ is the corresponding Dirac operator on Y, which is an elliptic self-adjoint operator of first order. Furthermore, G and B do not depend on the normal coordinate u and they satisfy the identities

$$G^2 = -Id \quad \text{and} \quad GB = -BG\,. \tag{0.10}$$

Since Y has dimension $2m$ the bundle $S|Y$ decomposes into its positive and negative chirality components $S|Y = S^+ \bigoplus S^-$ and we have a corresponding splitting of the operator B into $B^\pm : C^\infty(Y; S^\pm) \to C^\infty(Y; S^\mp)$, where $(B^+)^* = B^-$. Equation (0.9) can be rewritten in the form

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \left(\partial_u + \begin{pmatrix} 0 & B^- \\ B^+ & 0 \end{pmatrix} \right).$$

In order to obtain a Fredholm operator with good elliptic regularity properties we have to impose a boundary condition on the operator $\mathcal{D}$. Let $\Pi_\geq$ denote the spectral projection of B onto the subspace of $L^2(Y; S|Y)$ spanned by the eigenvectors corresponding to the nonnegative eigenvalues of B. It is well known that $\Pi_\geq$ is an elliptic boundary condition for the operator $\mathcal{D}$ (see [AtPS], [BoW4]). The meaning of the ellipticity is as follows. We introduce the unbounded operator $\mathcal{D}_{\Pi_\geq}$ equal to the operator $\mathcal{D}$ with domain

$$\operatorname{dom} \mathcal{D}_{\Pi_\geq} = \left\{ s \in H^1(M; S) \mid \Pi_\geq(s|Y) = 0 \right\},$$

where H^1 denotes the first Sobolev space. Then the operator

$$\mathcal{D}_{\Pi_\geq} = \mathcal{D} : \operatorname{dom}(\mathcal{D}_{\Pi_\geq}) \to L^2(M; S)$$

is a Fredholm operator with kernel and cokernel consisting only of smooth sections.

The orthogonal projection $\Pi_\geq$ is a pseudodifferential operator of order 0 (see [BoW4]). In fact we can take any pseudodifferential operator R of order 0 with principal symbol equal to the principal symbol of $\Pi_\geq$ and obtain an operator $\mathcal{D}_R$ which satisfies the aforementioned properties. In order to explain this phenomenon, we give a short exposition of the necessary facts from the theory of elliptic boundary problems. In contrast to the case of an elliptic operator on a closed manifold, the operator $\mathcal{D}$ has an infinite-dimensional space of solutions. More precisely, the space

$$\left\{ s \in C^\infty(M; S) \;\middle|\; \mathcal{D}s = 0 \ \text{ in } M \setminus Y \right\}$$

is infinite-dimensional. We introduce the Calderon projection, which is the projection onto the L^2-closure of the *Cauchy Data* space $\mathcal{H}(\mathcal{D})$ of the operator $\mathcal{D}$

$$\mathcal{H}(\mathcal{D}) = \big\{f{\in}C^\infty(Y;S|Y) \;\big|\; \exists s{\in}C^\infty(M;S),\ \mathcal{D}(s){=}0 \text{ in } M\setminus Y \text{ and } s|Y = f\big\}.$$

The projection $P(\mathcal{D})$ is a pseudodifferential operator with principal symbol equal to the symbol of $\Pi_\geq$ (see [BoW4]). Moreover, in our situation $P(\mathcal{D})$ is an orthogonal projection in $L^2(Y;S|Y)$. This is not true in more general situations, for instance in the case of non-product metric structures near the boundary. The operator $\mathcal{D}$ has the *Unique Continuation Property*, and hence we have a one to one correspondence between solutions of the operator $\mathcal{D}$ and the traces of the solutions on the boundary Y, at least in the case of a connected manifold M. This explains roughly, why only the projection $\mathcal{P}_R$ onto the kernel of the boundary conditions R matters. If the difference $\mathcal{P}_R - P(\mathcal{D})$ is an operator of order -1, then it follows that we choose the domain of the operator $\mathcal{D}_R$ in such a way that we throw away almost all solutions of the operator $\mathcal{D}$ on $M\setminus Y$, with the possible exception of a finite-dimensional subspace. The above condition on $\mathcal{P}_R$ allows us also to construct a parametrix for the operator $\mathcal{D}_R$, hence we obtain regularity of the solutions of the operator $\mathcal{D}_R$. We refer to the monograph [BoW4] for more details.

In the following we do not discuss the determinant on the 'total space' of elliptic boundary conditions for the operator $\mathcal{D}$, we choose a smaller and more convenient space, mostly in order to avoid unpleasant technical questions. We restrict ourselves to the study of the Grassmannian $Gr(\mathcal{D})$ of all pseudodifferential projections which differ from $\Pi_\geq$ by an operator of order -1. The space $Gr(\mathcal{D})$ has infinitely many connected components and two boundary conditions P_1 and P_2 belong to the same connected component if and only if

$$index\,\mathcal{D}_{P_1} = index\,\mathcal{D}_{P_2}\,.$$

We are interested, however, in self-adjoint realizations of the operator $\mathcal{D}$. The anti-involution $G : S|Y \to S|Y$ equips $L^2(Y;S|Y)$ with a symplectic structure, and using Green's formula

$$(\mathcal{D}s_1, s_2) - (s_1, \mathcal{D}s_2) = -\int_Y \big\langle G(s_1|Y), s_2|Y\big\rangle dy\,, \tag{0.11}$$

it is shown in [BoW4] that the boundary condition R provides a self-adjoint realization $\mathcal{D}_R$ of the operator $\mathcal{D}$ if and only if $\ker R$ is a Lagrangian subspace of $L^2(Y;S|Y)$ (see [BoW3], [BoW4], [DoW2]). We may therefore restrict our attention to those elements of $Gr(\mathcal{D})$ which are projections onto Lagrangian subspaces of $L^2(Y;S|Y)$. More precisely, we introduce $Gr^*(\mathcal{D})$, the Grassmannian of orthogonal, pseudodifferential projections P such that $P - \Pi_{\geq}$ is an operator of order -1 and

$$-GPG = Id - P\,. \tag{0.12}$$

The space $Gr^*(\mathcal{D})$ is contained in the connected component of $Gr(\mathcal{D})$ parameterizing projections P with $index\,\mathcal{D}_P = 0$.

For analytical reasons associated with the existence of the ζ-determinant, in this paper we discuss only the *Smooth, Self-adjoint Grassmannian*, a dense subset of the space $Gr^*(\mathcal{D})$, defined by

$$Gr^*_\infty(\mathcal{D}) = \left\{P \in Gr^*(\mathcal{D}) \;\middle|\; P - \Pi_{\geq} \text{ has a smooth kernel}\right\}. \tag{0.13}$$

The spectral projection $\Pi_{\geq}$ is an element of $Gr^*_\infty(\mathcal{D})$ if and only if $\ker B = \{0\}$. On the other hand, it is well known that the (orthogonal) Calderon projection $P(\mathcal{D})$ is an element of $Gr^*(\mathcal{D})$ (see for instance [BoW3]). Moreover, it was proved in [S1, Proposition 2.2] (see also [DK, Appendix]) that $P(\mathcal{D}) - \Pi_{\geq}$ is a smoothing operator and hence that $P(\mathcal{D})$ is an element of $Gr^*_\infty(\mathcal{D})$. The finite-dimensional perturbations of $\Pi_{\geq}$ (see also [DoW2], [LW] and [Wo4]) provide further examples of boundary conditions from $Gr^*_\infty(\mathcal{D})$. The latter were introduced by Jeff Cheeger, who called them *Ideal Boundary Conditions* (see [C2,3]).

For any $P \in Gr^*(\mathcal{D})$ the operator $\mathcal{D}_P$ has a discrete spectrum nicely distributed on the real line (see [BoW3], [DoW2]). It was shown by the second author that for any $P \in Gr^*_\infty(\mathcal{D})$, $\eta_{\mathcal{D}_P}(s)$ and $\zeta_{\mathcal{D}_P^2}(s)$ are well-defined functions, holomorphic for $Re(s)$ large and with meromorphic extensions to the whole complex plane with only simple poles. In particular both functions are holomorphic in a neighborhood of $s = 0$. Therefore $\det_\zeta \mathcal{D}_P$ defined by formula (0.8) is a well-defined smooth function on $Gr^*_\infty(\mathcal{D})$ (see [Wo5]).

The canonical determinant is defined in the following way. The family of elliptic boundary value problems $\{\mathcal{D}_P \mid P \in Gr_\infty(\mathcal{D})\}$ parameterized by

$$Gr_\infty(\mathcal{D}) = \left\{P \in Gr(\mathcal{D}) \;\middle|\; P - \Pi_{\geq} \text{ has a smooth kernel}\right\}$$

has an associated determinant line bundle $DET(\mathcal{D}) \to Gr(\mathcal{D})$ with determinant section

$$P \mapsto \det \mathcal{D}_P \in \mathrm{Det}(\mathcal{D}_P) \cong \wedge^{\max}\mathrm{Ker}(\mathcal{D}_P)^* \otimes \wedge^{\max}\mathrm{Coker}(\mathcal{D}_P)$$

(see section 1). On the other hand, relative to the basepoint Calderon projection $P(\mathcal{D}) \in Gr_\infty(\mathcal{D})$, we have the smooth family of Fredholm operators over the boundary

$$\left\{\mathcal{S}(P) = PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to Ran\,P \;\middle|\; P \in Gr_\infty(\mathcal{D})\right\}$$

with associated (Segal) determinant line bundle $DET_{P(\mathcal{D})}$ again equipped with its determinant section $P \mapsto \det \mathcal{S}(P)$. From [S1,2] we know that there is a canonical line bundle isomorphism preserving these sections

$$DET(\mathcal{D}) \cong DET_{P(\mathcal{D})}\,, \quad \det \mathcal{D}_P \longleftrightarrow \det \mathcal{S}(P)\,. \tag{0.14}$$

and therefore, since $DET_{P(\mathcal{D})}$ is a non-trivial complex line bundle whose first Chern class generates $H^2(Gr_\infty(\mathcal{D});\mathbb{Z}) = \mathbb{Z}$, no global non-zero section of $DET(\mathcal{D})$ exists. However, from the computation of homotopy groups in [BoW3], [DoW2] we have $H^2(Gr^*_\infty(\mathcal{D});\mathbb{Z}) = 0$, and so the determinant line bundle does restrict to a trivializable complex line bundle over the smooth Grassmannian of self-adjoint boundary conditions.

The problem now is to identify which trivialization defines the ζ-determinant $\det_\zeta \mathcal{D}_P$. To make the presentation smoother, we assume henceforth that $\ker B = \{0\}$; this is in fact not a serious restriction and it will be explained in section 7 that we can easily relax this condition. The correct choice of trivialization is indicated by the fact that any elliptic boundary condition $P \in Gr^*_\infty(\mathcal{D})$ is described precisely by the property that its range is the graph of an elliptic unitary isomorphism $T : F^+ \to F^-$ such that $T - (B^+B^-)^{-1/2}B^+$ has a smooth kernel [S1], where $F^\pm$ are the spaces of chiral spinor fields over the boundary. In section 1 of this paper we explain how this defines a preferred non-zero 'basepoint' section $P \mapsto \sigma(P) \in \operatorname{Det} \mathcal{D}_P$. The *canonical determinant* is then defined to be the quotient taken in $\operatorname{Det} \mathcal{D}_P$

$$\det_{\mathcal{C}} \mathcal{D}_P = \tfrac{\det \mathcal{D}_P}{\sigma(P)}\,, \tag{0.15}$$

and this turns out to be the Fredholm determinant of an operator living on the boundary Y constructed from $\mathcal{S}(P)$. The main result of the paper is the following Theorem:

Theorem 0.1. *The following equality holds over $Gr^*_\infty(\mathcal{D})$*

$$\det_\zeta \mathcal{D}_P = \det_\zeta \mathcal{D}_{P(\mathcal{D})} \cdot \det_{\mathcal{C}} \mathcal{D}_P\,. \tag{0.16}$$

Remark 0.2. (1) Theorem 0.1 shows that, at least on $Gr^*_\infty(\mathcal{D})$, the ζ-determinant is an object which is a natural extension of the well-defined algebraic concept of the determinant.

(2) The identification of $\det_\zeta \mathcal{D}_{P(\mathcal{D})}$ with a regularized Fredholm determinant of the operator $\mathcal{S}(P)$ living on the boundary extends the corresponding result for the *index*, which is well known (see [BoW4, Theorem 20.8]).

(3) Theorem 0.1 suggests a new approach to the pasting formula for the ζ-determinant with respect to a partitioning of a closed manifold. The

pasting formula for $det_{\mathcal{C}}$ was introduced in [S1]. It is hoped that a new insight into the pasting mechanism of the ζ-determinant will be obtained by combining results of [S1] and formula (0.16). For a recent application of the results of this paper to an adiabatic pasting formula see [PW].

To prove Theorem 0.1 we follow a basic idea of Robin Forman [F] and study the variation of the relative determinants. More precisely, given two projections $P_1, P_2 \in Gr_\infty^*(\mathcal{D})$, we define two one-parameter families of boundary conditions $P_{i,r}$ and compute the relative variation

$$\tfrac{d}{dr}\{ln \det \mathcal{D}_{P_{1,r}} - ln \det \mathcal{D}_{P_{2,r}}\}|_{r=0}$$

for both the canonical determinant and the ζ-determinant and show that they coincide. Here we face the technical problem of dealing with a family of unbounded operators with varying domain. To circumvent this and make sense of the variation with respect to the boundary condition we follow Douglas and Wojciechowski [DoW2] and apply their *"Unitary Trick"* (see section 3). Finally, using the fact that the space of projections P in $Gr_\infty^*(\mathcal{D})$ such that $\mathcal{D}_P$ is invertible is actually path connected (see section 5 and [N]), we integrate the variational equality in order to obtain formula (0.16) of Theorem 0.1.

The paper is organized as follows. In section 1 we explain the construction of the canonical determinant. We follow here the exposition of [S1].

Assume that for given $P \in Gr_\infty^*(\mathcal{D})$ the operator $\mathcal{D}_P$ is invertible. In section 2 we present our construction of an inverse $\mathcal{D}_P^{-1}$. To do that we have to discuss certain aspects of the theory of elliptic boundary problems. We also introduce $\mathcal{K}$ the Poisson map of the operator $\mathcal{D}$, and $\mathcal{K}(P)$ the Poisson map of the operator $\mathcal{D}_P$. The first is used in the construction of the Calderon projection. The operator $\mathcal{K}(P)$ appears in several crucial places in our computation of the variation of the canonical determinant.

In section 3 we discuss the variation of the ζ-determinant and in section 4 we study the variation of the canonical determinant. It has already been mentioned that the work [Wo5] is crucial for the study here of the ζ-determinant, while in the calculation of the variation of the canonical determinant we were influenced by the work of Robin Forman [F].

With the variational equality at hand, section 5 contains the final steps of the proof of Theorem 0.1.

In section 6 we discuss an immediate application of our result to the modulus of the ζ-determinant, regarded as a function on the Grassmannian $Gr_\infty^*(\mathcal{D})$. We show that the Calderon projection is the only critical point of this function on the space $\tilde{G}r_\infty^*(\mathcal{D})$ of projections $P \in Gr_\infty^*(\mathcal{D})$ such that the operator $\mathcal{D}_P$ is invertible.

In section 7 we make some final comments on several technical issues we have to deal with in this paper. We discuss the choice of the sign of the phase of ζ-determinant. We also explain the necessary changes required for the case of a non-invertible tangential operator B. A more detailed explanation of the topological structure of $\tilde{G}r^*_\infty(\mathcal{D})$ is given.

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1 Canonical Determinant on the Grassmannian $Gr^*_\infty(\mathcal{D})$

In this section we review briefly the construction of the determinant line bundle and give an explicit construction of the canonical determinant.

The determinant line bundle over the space of Fredholm operators was first introduced in a seminal paper of Quillen [Q]. An equivalent construction which is better suited to our purposes here was subsequently given by Segal (see [Seg]), and so we follow his approach. Let $Fred(\mathcal{H})$ denote the space of Fredholm operators on a separable Hilbert space $\mathcal{H}$. We work first in the connected component $Fred_0(\mathcal{H})$ of this space parameterizing operators of *index* zero. For $A \in Fred_0(\mathcal{H})$ define

$$Fred_A = \big\{ S \in Fred(\mathcal{H}) \;\big|\; S - A \;\text{ is trace-class} \big\} .$$

Fix a trace-class operator $\mathcal{A}$ such that $S = A + \mathcal{A}$ is an invertible operator. Then the determinant line of the operator A is defined as

$$\operatorname{Det} A = Fred_A \times \mathbf{C}/_{\simeq} , \tag{1.1}$$

where the equivalence relation is defined by

$$(R, z) = \big(S(S^{-1}R), z\big) \simeq \big(S, z \cdot \det_{Fr}(S^{-1}R)\big) .$$

The Fredholm determinant of the operator $S^{-1}R$ is well-defined, as it is of the form $Id_{\mathcal{H}}$ plus a trace class operator. Denoting the equivalence class of a pair (R, z) by $[R, z]$, complex multiplication is defined on $\operatorname{Det} A$ by

$$\lambda \cdot [R, z] = [R, \lambda z] . \tag{1.2}$$

The *determinant element* is defined by

$$\det A := [A, 1] \tag{1.3}$$

and is non-zero if and only if A is invertible.

The complex lines fit together over $Fred_0(\mathcal{H})$ to define a complex line bundle $\mathcal{L}$, the determinant line bundle. To see this, observe first that over the open set $U_{\mathcal{A}}$ in $Fred_0(\mathcal{H})$ defined by

$$U_{\mathcal{A}} = \left\{ F \in Fred_0(\mathcal{H}) \;\middle|\; F + \mathcal{A} \text{ is invertible} \right\},$$

the assignment $F \to \det F$ defines a trivializing (non-vanishing) section of $\mathcal{L}_{|U_{\mathcal{A}}}$. The transition map between the canonical determinant elements over $U_{\mathcal{A}} \cap U_{\mathcal{B}}$ is the smooth (holomorphic) function

$$g_{\mathcal{AB}}(F) = \det{}_{Fr}\left((F+\mathcal{A})(F+\mathcal{B})^{-1}\right).$$

This defines $\mathcal{L}$ globally as a complex line bundle over $Fred_0(\mathcal{H})$, endowed with its determinant section $A \to \det A$. If $ind\, A = d$ we define $\operatorname{Det} A$ to be the determinant line of $A \oplus 0$ as an operator $\mathcal{H} \to \mathcal{H} \oplus \mathbf{C}^d$ if $d > 0$, or $\mathcal{H} \oplus \mathbf{C}^{-d} \to \mathcal{H}$ if $d < 0$ and the construction extends in the obvious way to the other components of $Fred(\mathcal{H})$. Note that the determinant section is zero outside of $Fred_0(\mathcal{H})$.

We use this construction in order to define the determinant line bundle over $Gr_\infty(\mathcal{D})$. For each projection $P \in Gr_\infty(\mathcal{D})$ we have the (Segal) determinant line $\operatorname{Det}(\mathcal{S}(P))$ of the operator

$$\mathcal{S}(P) := PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to Ran\, P$$

and the determinant line $\operatorname{Det} \mathcal{D}_P$ of the boundary value problem $\mathcal{D}_P : \operatorname{dom}(\mathcal{D}_P) \to L^2(M;S)$. These lines fit together in the manner explained above to define determinant line bundles $DET_{P(\mathcal{D})}$ and $DET(\mathcal{D})$ over the Grassmannian (some care has to be taken as the operator acts between two different Hilbert spaces, but with the obvious notational modifications we once again obtain well-defined determinant line bundles). The canonical isomorphism (0.14) identifies the two line bundles and preserves the determinant elements. The bundle $DET_{P(\mathcal{D})}$ is a non-trivial line bundle over $Gr_\infty(\mathcal{D})$, but when restricted to the Grassmannian $Gr^*_\infty(\mathcal{D})$ it is canonically trivial.

We use the specific trivialization introduced in [S1]. Recall that we work here with orthogonal projections onto the Lagrangian subspaces of $L^2(Y;S|Y)$, which are a compact perturbation of the Cauchy data space $\mathcal{H}(\mathcal{D})$. We have assumed that $\ker B = \{0\}$, and hence $\Pi_>(B)$ is an element of $Gr^*_\infty(\mathcal{D})$. The range of $\Pi_>(B)$ is actually the graph of the unitary operator $V_> : F^+ \to F^-$ given by the formula:

$$V_> = (B^+B^-)^{-1/2}B^+ . \tag{1.4}$$

This identification extends to the whole Grassmannian $Gr^*_\infty(\mathcal{D})$: elements are in 1 to 1 correspondence with unitary maps $V : F^+ \to F^-$, such that the difference $V - V_{>}$ is an operator with a smooth kernel. The corresponding orthogonal projection P is given by the formula

$$P = \frac{1}{2}\begin{pmatrix} Id_{F^+} & V^{-1} \\ V & Id_{F^-} \end{pmatrix}.$$

By choosing a basepoint, the correspondence defined above allows us to establish an isomorphism between $Gr^*_\infty(\mathcal{D})$ and the group $U^\infty(F^-)$ of unitaries acting on $F^- = L^2(Y;S^-)$ which differ from Id_{F^-} by an operator with a smooth kernel. It is convenient for us to work with the Calderon projection as basepoint, hence let $K : C^\infty(Y;S^+) \to C^\infty(Y;S^-)$ denote the unitary such that $\mathcal{H}(\mathcal{D})$ is equal to $graph(K)$. For any projection $P \in Gr^*_\infty(\mathcal{D})$ there exists $T = T(P) : F^+ \to F^-$ such that $Ran\, P = graph(T)$, and so we have a natural isomorphism $Gr^*_\infty(\mathcal{D}) \cong U^\infty(F^-)$ defined by the map $P \to TK^{-1}$. This is expressed in terms of the homogeneous structure of the Grassmannian by

$$P = \begin{pmatrix} Id_{F^+} & 0 \\ 0 & TK^{-1} \end{pmatrix} P(\mathcal{D}) \begin{pmatrix} Id_{F^+} & 0 \\ 0 & KT^{-1} \end{pmatrix}. \tag{1.5}$$

Now we can define a non-vanishing section l of the determinant line bundle $DET_{P(\mathcal{D})}$ over $Gr^*_\infty(\mathcal{D})$. The value of l at the projection P is the class in $\mathrm{Det}(\mathcal{S}(P))$ of the couple

$$\left(U(P) := \begin{pmatrix} Id_{F^+} & 0 \\ 0 & TK^{-1} \end{pmatrix} \; ; \; 1 \right),$$

where the operator $U(P)$ acts from $\mathcal{H}(\mathcal{D})$ to $Ran(P)$. That is, $l(P) = \det U(P)$. The fact that $l(P)$ is an element of $\mathrm{Det}(\mathcal{S}(P))$ follows from the following elementary result.

Lemma 1.1. *The difference between $U(P)$ and the operator $\mathcal{S}(P) = PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to Ran\, P$ is an operator with a smooth kernel, hence $\det U(P) = [U(P), 1]$ is an element of $\mathrm{Det}\, \mathcal{S}(P)$.*

Proof. The operator $U(P)$ acts from $graph(\mathcal{K}) = \mathcal{H}(\mathcal{D})$ to $graph(T) = Ran(P)$ and acts by

$$\begin{pmatrix} x \\ Kx \end{pmatrix} \to \begin{pmatrix} Id_{F^+} & 0 \\ 0 & TK^{-1} \end{pmatrix} \begin{pmatrix} x \\ Kx \end{pmatrix} = \begin{pmatrix} x \\ Tx \end{pmatrix}.$$

The operator $PP(\mathcal{D})$ is given by the following formula

$$PP(\mathcal{D}) = \frac{1}{4}\begin{pmatrix} Id_{F^+} + T^{-1}K & T^{-1} + K^{-1} \\ T + K & Id_{F^-} + TK^{-1} \end{pmatrix},$$

leading to the following expression for the operator $\mathcal{S}(P) = PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to Ran(P)$

$$\mathcal{S}(P)\begin{pmatrix} x \\ Kx \end{pmatrix} = \begin{pmatrix} \frac{Id_{F^+} + T^{-1}K}{2} x \\ \frac{Id_{F^-} + TK^{-1}}{2} Kx \end{pmatrix} = \begin{pmatrix} \frac{Id_{F^+} + T^{-1}K}{2} & 0 \\ 0 & \frac{Id_{F^-} + TK^{-1}}{2} \end{pmatrix} \begin{pmatrix} x \\ Kx \end{pmatrix}. \tag{1.6}$$

Because $T^{-1}K$ (resp. TK^{-1}) differs from Id_{F^+} (resp. Id_{F^-}) by a smoothing operator, it is now obvious that the difference $U(P) - \mathcal{S}(P)$ is an operator with a smooth kernel. □

The discussion above allows us to now define the *Canonical Determinant* over $Gr^*_\infty(\mathcal{D})$. Let $A : \mathcal{H}(\mathcal{D}) \to Ran\, P$ denote an invertible Fredholm operator such that $A - \mathcal{S}(P)$ is an operator of trace class. We have:

$$\begin{aligned} \det A &:= [(A,1)] \\ &= \big[(U(P)(U(P)^{-1}A),1)\big] \\ &= \big[(U(P); \det_{Fr}(U(P)^{-1}A)\big] \\ &= \det_{Fr}\big(U(P)^{-1}A\big)\big[(U(P);1\big] \\ &:= \det_{Fr}\big(U(P)^{-1}A\big)\cdot \det U(P)\,. \end{aligned}$$

where we use equations (1.2) and (1.3). The above identity means we can define the determinant of the operator A as the ratio in $\operatorname{Det} A$ of the non-vanishing canonical elements $\det A$ and $\det U(P)$, or equivalently as the *Fredholm determinant* of the operator $U(P)^{-1}A$. This leads to the following definition of the canonical determinant of the operator $\mathcal{D}_P$.

Definition 1.2. We define the *Canonical Determinant* of the elliptic boundary value problem $\mathcal{D}_P$ by:

$$\det_{\mathcal{C}} \mathcal{D}_P := \det_{\mathcal{C}} \mathcal{S}(P) := \det_{Fr}\big(U(P)^{-1}\mathcal{S}(P)\big)\,. \tag{1.7}$$

The naturality of this definition lies in the identification of the abstract determinants of the Fredholm operators $\mathcal{D}_P$ and $\mathcal{S}(P)$ by the isomorphism (0.14); the section σ in (0.15) is just the image of $\det U(P)$ under (0.14). In fact, from the proof of Lemma 1.1 we see that the determinant on the right side of the equality (1.7) is the Fredholm determinant of the operator

$$\begin{pmatrix} \frac{Id_{F^+}+T^{-1}K}{2} & 0 \\ 0 & \frac{Id_{F^-}+TK^{-1}}{2} \end{pmatrix}$$

acting on the graph of the operator K. Hence we obtain:

Lemma 1.3.

$$\det_{\mathcal{C}} \mathcal{D}_P = \det_{Fr}\left(\frac{Id + KT^{-1}}{2}\right), \tag{1.8}$$

where the Fredholm determinant on the right side is taken on F^-.

We may therefore reformulate Theorem 0.1 as:

Theorem 1.4. *The following equality holds over $Gr^*_\infty(\mathcal{D})$*

$$\det_\zeta \mathcal{D}_P = \det_\zeta \mathcal{D}_{P(\mathcal{D})}\cdot \det_{Fr}\left(\frac{Id + KT^{-1}}{2}\right), \tag{1.9}$$

Equivalently, (*since* $\det_{\mathcal{C}} \mathcal{D}_{P(\mathcal{D})} = 1$)

$$\frac{\det_\zeta \mathcal{D}_P}{\det_\zeta \mathcal{D}_{P(\mathcal{D})}} = \frac{\det_{\mathcal{C}} \mathcal{D}_P}{\det_{\mathcal{C}} \mathcal{D}_{P(\mathcal{D})}} . \tag{1.10}$$

REMARK 1.5. In section 4 we also use determinants of operators of the form

$$\mathcal{S}(P_1)\mathcal{S}(P_2)^{-1} : Ran\, P_2 \to Ran\, P_1 ,$$

under the assumption that the operator $\mathcal{S}(P_2)$ is invertible. From the discussion presented above, it follows that for any Fredholm operator $A : Ran\, P_2 \to Ran\, P_1$ such that the difference between A and the operator $P_1P_2 : Ran\, P_2 \to Ran\, P_1$ is of trace class we can define the canonical determinant of A using the formula

$$\det_{\mathcal{C}} A = \det_{Fr} \begin{pmatrix} Id_{F^+} & 0 \\ 0 & T_2T_1^{-1} \end{pmatrix} A , \tag{1.11}$$

where $Ran\, P_i$ is equal to $graph\, T_i$.

2 Boundary Problems Defined by $Gr^*_\infty(\mathcal{D})$: Inverse Operator and Poisson Maps

For any $P \in Gr(\mathcal{D})$ the operator $\mathcal{D}_P$ is a Fredholm operator, hence it has closed range. As a consequence, we can define an inverse to the induced operator $\operatorname{dom} \mathcal{D}_P / \ker \mathcal{D}_P \to L^2(M; S)/\operatorname{coker} \mathcal{D}_P$. If we assume that P is an element of $Gr^*(\mathcal{D})$ then the operator $\mathcal{D}_P$ is self-adjoint and $\ker \mathcal{D}_P = \operatorname{coker} \mathcal{D}_P$. It follows that if we assume $\ker \mathcal{D}_P = \{0\}$, then there exists an inverse $\mathcal{D}_P^{-1}$ to the operator $\mathcal{D}_P$.

In this section we give an explicit formula for the operator $\mathcal{D}_P^{-1}$. This formula plays a key role in the proof of the main result of the paper. The operator $\mathcal{D}_P^{-1}$ is a sum of two operators. The first is the *interior inverse* of $\mathcal{D}^{-1}$. The second is a correction term which lives on the boundary.

We start with the *"interior"* part of the inverse. Let $\tilde{M} = M_- \cup_Y M$ denote the closed double of the manifold M (M_- is a copy of M with reversed orientation). The bundle of Clifford modules S extends to a bundle $\tilde{S}$ of Clifford modules over $\tilde{M}$ and the operator $\mathcal{D}$ determines a compatible Dirac operator $\tilde{\mathcal{D}}$ over $\tilde{M}$ (equal to $\mathcal{D}$ on M and $-\mathcal{D}$ on M_-). We refer to [Wo1], [DoW1] for the details of these constructions and applications to the analytic realization of $K-homology$. The operator

$$\tilde{\mathcal{D}} : C^\infty(\tilde{M}; \tilde{S}) \to C^\infty(\tilde{M}; \tilde{S})$$

is an invertible self-adjoint operator, hence its inverse $\tilde{\mathcal{D}}^{-1}$ is a well-defined elliptic operator of order -1 over the manifold $\tilde{M}$. We also have natural extension and restriction maps acting on sections of S and $\tilde{S}$. The extension (*by zero*) operator $e_+ : L^2(M;S) \to L^2(\tilde{M};\tilde{S})$ is given by the formula:

$$e_+(f) := \begin{cases} f & \text{on } \tilde{M} \setminus M_- \\ 0 & \text{on } M_- \,. \end{cases} \tag{2.1}$$

The restriction operator $r_+ : H^s(\tilde{M};\tilde{S}) \to H^s(M;S)$, where H^s denotes the s^{th} *Sobolev space*, is given by $\tilde{f} \to f = \tilde{f}|M$. To simplify the notation in the following we always write

$$\mathcal{D}^{-1} = r_+ \tilde{\mathcal{D}}^{-1} e_+ \,. \tag{2.2}$$

The operator $\mathcal{D}^{-1}$ is the interior part of the inverse. It is used in several crucial constructions in the theory of boundary problems. It maps $L^2(M;S)$ into $H^1(M;S)$, however the range is not necessarily inside the domain of $\mathcal{D}_P$. For this reason we have to introduce an additional term to obtain an operator with the correct range. To do this, we need to study the situation in a neighborhood of the boundary Y. The restriction of smooth sections to the boundary extends to a continuous map

$$\gamma_0 : H^s(M;S) \to H^{s-\frac{1}{2}}(Y;S|Y)\,,$$

which is well-defined for $s > 1/2$ (see [BoW4]). The corresponding adjoint operator γ_0^* (in the distributional sense) provides us with a well-defined map

$$\gamma_0^* : H^s(Y;S|Y) \to H^{s-\frac{1}{2}}(M;S)\,,$$

for $s < 0$. Now for any real s the mapping

$$\mathcal{K} = r_+ \tilde{\mathcal{D}}^{-1} \gamma_0^* G : C^\infty(Y;S|Y) \to C^\infty(M;S)$$

extends to a continuous map $\mathcal{K} : H^{s-1/2}(Y;S|Y) \to H^s(M;S)$, with range equal to the space

$$\ker(\mathcal{D},s) = \big\{ f \in H^s(M;S) \;\big|\; \mathcal{D}f = 0 \text{ in } M \setminus Y \big\}\,.$$

In fact, the map

$$\mathcal{K} : \mathcal{H}(\mathcal{D},s) = Ran\big\{ P(\mathcal{D}) : H^{s-1/2}(Y;S|Y) \to H^{s-1/2}(Y;S|Y) \big\} \to \ker(\mathcal{D},s) \tag{2.3}$$

is an isomorphism (see [BoW4]). We have the following equality:

$$\mathcal{D}^{-1}\mathcal{D} = Id - \mathcal{K}\gamma_0\,, \tag{2.4}$$

which holds on the space of smooth sections (see [BoW4, Lemma 12.7]).

The operator $\mathcal{K}$ is called the *Poisson operator of* $\mathcal{D}$. It defines the Calderon projection:

$$P(\mathcal{D}) = \gamma_0 \mathcal{K} \tag{2.5}$$

(see [BoW4, Theorem 12.4]).

Remark 2.1. Formula (2.5) gives, a priori, only a projector, not an orthogonal projection, onto $\mathcal{H}(\mathcal{D})$. In the situation discussed in this paper, however, the resulting projector is orthogonal. We refer to [BoW4] for the details of the construction, which is originally due to Calderon and Seeley.

To construct the correction term to the operator $\mathcal{D}^{-1}$ we use the inverse of the operator $\mathcal{S}(P)$. The invertibility of $\mathcal{S}(P)$ is equivalent to $\mathcal{D}_P$ being invertible:

Lemma 2.2. *The operator $\mathcal{D}_P$ is an invertible operator if and only if the operator $\mathcal{S}(P) = PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to Ran\, P$ is invertible.*

Proof. The Grassmannian $Gr^*_\infty(\mathcal{D})$ is a subspace of the *"big" Grassmannian* $Gr(\mathcal{D})$ (see [Wo2], [BoW3], [DoW2, Appendix B]). The space $Gr(\mathcal{D})$ has countably many connected components distinguished by the index of the operator $\mathcal{S}(P)$, i.e. P_1 and P_2 belong to the same connected component of $Gr(\mathcal{D})$ if and only if $index\, \mathcal{S}(P_1) = index\ \mathcal{S}(P_2)$. The space $Gr^*_\infty(\mathcal{D})$ is contained in the *index zero* component of $Gr(\mathcal{D})$. Now we have

$$\ker \mathcal{S}(P) = \{ f \mid P(\mathcal{D})f = f \ \text{ and } \ P(f) = 0 \}$$

and

$$\operatorname{coker} \mathcal{S}(P) = \{ g \mid Pg = g \ \text{ and } \ P(\mathcal{D})g = 0 \} .$$

If P is an element of $Gr^*(\mathcal{D})$, then $index\, \mathcal{S}(P) = index\, \mathcal{D}_P = 0$. We see that the operator $\mathcal{S}(P)$ is invertible if and only if it has trivial kernel. Similarly a self-adjoint Fredholm operator $\mathcal{D}_P$ is invertible only if it has trivial kernel. On the other hand, the operator $\mathcal{K}$ defines an isomorphism

$$\mathcal{K} : \ker \mathcal{S}(P) \to \ker \mathcal{D}_P .$$

This ends the proof of lemma. □

Remark 2.3. Note that the lemma proves a somewhat stronger statement: via the Poisson operator $\mathcal{K}$, constructing solutions for the operator $\mathcal{S}(P)$ is equivalent to constructing solutions to the elliptic boundary value problem $\mathcal{D}_P$ (and the same for the adjoints). In particular this implies that the index of the two operators coincide. This is the underlying reason why it is easier to compute determinants on manifolds with boundary than on closed manifolds.

From now on we assume that $\mathcal{D}_P$ is invertible. The operator $\mathcal{S}(P)^{-1}$ is not a pseudodifferential operator, as it acts from $Ran\, P$ into $\mathcal{H}(\mathcal{D})$. However, we can show that it is a restriction of an elliptic pseudodifferential operator of order 0. More precisely, the operator $PP(\mathcal{D})+(Id-P)(Id-P(\mathcal{D}))$ is an elliptic pseudodifferential operator, which can be written as

$$\mathcal{S}(P) \oplus (Id - P)\big(Id - P(\mathcal{D})\big) : \mathcal{H}(\mathcal{D}) \oplus \mathcal{H}(\mathcal{D})^\perp \to W \oplus W^\perp ,$$

where $W = Ran\, P$. It can be seen that

$$\ker \mathcal{S}(P) = \operatorname{coker}(Id - P)\big(Id - P(\mathcal{D})\big)\,,$$

and

$$\operatorname{coker} \mathcal{S}(P) = \ker(Id - P)\big(Id - P(\mathcal{D})\big)\,.$$

Therefore if we assume that $\ker \mathcal{S}(P) = \{0\}$, then the operator $PP(\mathcal{D}) + (Id - P)(Id - P)(\mathcal{D}))$ is invertible. Its inverse is an elliptic operator (see for instance [BoW4]) and it follows that

$$\mathcal{S}(P)^{-1} = P(\mathcal{D})\big[PP(\mathcal{D}) + (Id - P)(Id - P(\mathcal{D}))\big]^{-1} P\,. \tag{2.6}$$

We can now present the formula for the inverse of the operator $\mathcal{D}_P$.

Theorem 2.4. *Assume that the operator $\mathcal{D}_P : \operatorname{dom} \mathcal{D}_P \to L^2(M;S)$ is invertible, then its inverse is given by the formula:*

$$\mathcal{D}_P^{-1} = \mathcal{D}^{-1} - \mathcal{K}\mathcal{S}(P)^{-1} P\gamma_0 \mathcal{D}^{-1}\,. \tag{2.7}$$

Proof. From (2.3) we have that $\mathcal{D}\mathcal{K}$ is equal to 0 in $M \setminus Y$, and hence that $\mathcal{D}\mathcal{D}_P^{-1}$ is equal to Id on $L^2(M;S)$. Now let $f \in L^2(M;S)$, then:

$$\begin{aligned} P\gamma_0(\mathcal{D}_P^{-1} f) &= P(\gamma_0(\mathcal{D}^{-1} f) - P\gamma_0 \mathcal{K}\mathcal{S}(P)^{-1} P\gamma_0 \mathcal{D}^{-1}(f) \\ &= P\gamma_0(\mathcal{D}^{-1} f) - PP(\mathcal{D})\mathcal{S}(P)^{-1} P\gamma_0 \mathcal{D}^{-1}(f) \\ &= P(\gamma_0(\mathcal{D}^{-1} f)) - P(\gamma_0(\mathcal{D}^{-1} f)) = 0\,, \end{aligned}$$

and hence $\mathcal{D}_P^{-1} f \in \operatorname{dom} \mathcal{D}_P$. We have shown that $\mathcal{D}_P \mathcal{D}_P^{-1} : L^2(M;S) \to L^2(M;S)$ is equal to Id_{L^2} and that $\mathcal{D}_P^{-1} : L^2(M;S) \to \operatorname{dom} \mathcal{D}_P$, hence the operator $\mathcal{D}_P^{-1}$ is indeed a right inverse of $\mathcal{D}_P$, and obviously since $index\, \mathcal{D}_P = 0$ it is also a left inverse. □

Corollary 2.5. *Let $P_1, P_2 \in Gr_\infty^*(\mathcal{D})$ such that the operators $\mathcal{D}_{P_1}$ and $\mathcal{D}_{P_2}$ are invertible. Then the difference $\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1}$ is an operator with smooth kernel.*

Proof. It follows from Theorem 2.4 that

$$\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1} = \mathcal{K}\big(\mathcal{S}(P_2)^{-1} P_2 - \mathcal{S}(P_1)^{-1} P_1\big)\gamma_0 \mathcal{D}^{-1}\,. \tag{2.8}$$

Now the fact that $P_1 - P_2$ is an operator with a smooth kernel and equation (2.6) implies that the operator $\mathcal{S}(P_2)^{-1} P_2 - \mathcal{S}(P_1)^{-1} P_1$ also has a smooth kernel. □

For the rest of this section we take a closer look at the operator $\mathcal{D}_P^{-1}\mathcal{D}$, as it allows us to introduce another important operator $\mathcal{K}(P)$, the Poisson operator of the operator $\mathcal{D}_P$. From formula (2.4) we have that

$$\mathcal{D}_P^{-1}\mathcal{D} = Id - \mathcal{K}\gamma_0 - \mathcal{K}\mathcal{S}(P)^{-1} P\gamma_0 (Id - \mathcal{K}\gamma_0)$$

$$= Id - \mathcal{K}\gamma_0 + \mathcal{K}\mathcal{S}(P)^{-1} PP(\mathcal{D})\gamma_0 - \mathcal{K}\mathcal{S}(P)^{-1} P\gamma_0 = Id - \mathcal{K}\mathcal{S}(P)^{-1} P\gamma_0\,.$$

Hence if $f \in \operatorname{dom} \mathcal{D}_P$, then $P\gamma_0(f) = 0$ and

$$\mathcal{D}_P^{-1}\mathcal{D}_P f = \big(Id - \mathcal{K}\mathcal{S}(P)^{-1}P\gamma_0\big)f = f\,.$$

We define the *Poisson operator of* $\mathcal{D}_P$ by

$$\mathcal{K}(P) = \mathcal{K}\mathcal{S}(P)^{-1}P\,, \tag{2.9}$$

The operator $\mathcal{K}(P)$ appeared in the second term of the operator $\mathcal{D}_P^{-1}\mathcal{D}_P$.

Let g denote an element in the range of the projection P. More precisely, assume that $g \in H^s(Y;S|Y)$ and $Pg = g$. Then $\mathcal{K}(P)g$ is a solution of $\mathcal{D}$, which belongs to the Sobolev space $H^{s+1/2}(M;S)$, and $\gamma_0\mathcal{K}(P)g$ is an element of the space of Cauchy data of $\mathcal{D}$, in general not equal to g. However, the part of $\gamma_0\mathcal{K}(P)g$ along P is in fact equal to the original element g:

$$P\gamma_0\mathcal{K}(P)g = P\gamma_0\mathcal{K}\mathcal{S}(P)^{-1}Pg = PP(\mathcal{D})\mathcal{S}(P)^{-1}Pg = Pg\,.$$

In section 4 we also need the following results.

LEMMA 2.6. *Let $P_1, P_2 \in Gr_\infty^*(\mathcal{D})$ such that the operators $\mathcal{D}_{P_1}$ and $\mathcal{D}_{P_2}$ are invertible. Let $f_1, f_2 \in Ran\,P_2$ and assume that*

$$P_1\gamma_0\mathcal{K}(P_2)f_1 = P_1\gamma_0\mathcal{K}(P_2)f_2\,.$$

Then, $f_1 = f_2$ and $\mathcal{K}(P_2)f_1 = \mathcal{K}(P_2)f_2$.

Proof. We have

$$P_1\gamma_0(\mathcal{K}(P_2)f_i) = \mathcal{S}(P_1)\mathcal{S}(P_2)^{-1}f_i\,,$$

hence the first equality follows from the invertibility of the operators $\mathcal{S}(P_1)$ and $\mathcal{S}(P_2)$. The second is a consequence of the *Unique Continuation Property* for Dirac operators. We have

$$\begin{aligned}\gamma_0(\mathcal{K}(P_2)f_1) &= P(\mathcal{D})\mathcal{S}(P_2)^{-1}f_1 = \mathcal{S}(P_1)^{-1}\mathcal{S}(P_1)\mathcal{S}(P_2)^{-1}f_1\\ &= \mathcal{S}(P_1)^{-1}\mathcal{S}(P_1)\mathcal{S}(P_2)^{-1}f_2 = \gamma_0(\mathcal{K}(P_2)f_2)\end{aligned}$$

and hence two solutions of $\mathcal{D}$ with the same *Cauchy* data, hence they are equal. □

PROPOSITION 2.7.

$$\mathcal{K}(P_1)P_1\gamma_0\mathcal{D}_{P_2}^{-1} = \mathcal{D}_{P_2}^{-1} - \mathcal{D}_{P_1}^{-1}\,. \tag{2.10}$$

Proof. We fix $f \in L^2(M;S)$. Let

$$h = \mathcal{K}(P_1)P_1\gamma_0(\mathcal{D}_{P_2}^{-1}f)\,.$$

Observe that the section h is the unique solution of $\mathcal{D}$ with boundary data along P_1 equal to $P_1\gamma_0(\mathcal{D}_{P_2}^{-1}f)$. Indeed

$$\begin{aligned}P_1(\gamma_0 h) &= P_1\big(g_0\mathcal{K}(P_1)P_1\gamma_0(\mathcal{D}_{P_2}^{-1}f)\big)\\ &= P_1P(\mathcal{D})\mathcal{S}(P_1)^{-1}P_1\gamma_0(\mathcal{D}_{P_2}^{-1}f) = P_1\gamma_0(\mathcal{D}_{P_2}^{-1}f)\,,\end{aligned}$$

and uniqueness is a consequence of Lemma 2.6. Now, the section $g = (\mathcal{D}_{P_2}^{-1} - \mathcal{D}_{P_1}^{-1})f$ is also a solution of $\mathcal{D}$ and the restriction of g to the boundary has P_1-component equal to

$$\begin{aligned} P_1(\gamma_0 g) &= P_1\big(\gamma_0\mathcal{K}(\mathcal{S}(P_1)^{-1}P_1 - \mathcal{S}(P_2)^{-1}P_2)\gamma_0\mathcal{D}^{-1}f\big) \\ &= P_1 P(\mathcal{D})\big(\mathcal{S}(P_1)^{-1}P_1 - \mathcal{S}(P_2)^{-1}P_2\big)\gamma_0\mathcal{D}^{-1}f \\ &= P_1\gamma_0\mathcal{D}^{-1}f - P_1\gamma_0\big(\mathcal{K}\mathcal{S}(P_2)^{-1}P_2\big)\gamma_0\mathcal{D}^{-1}f \\ &= P_1\gamma_0(\mathcal{D}_{P_2}^{-1})f \end{aligned}$$

and therefore h and g are the same section. □

REMARK 2.8. (1) The construction of the inverse presented in this section gives, in fact, a parametrix for any elliptic boundary problem for the Dirac operator. First of all, if P is an element of $Gr(\mathcal{D})$ we can still use formula (2.7) in order to construct the aforementioned parametrix. The operator $\mathcal{S}(P)^{-1}$ has to be replaced by an operator $\mathcal{R}(P)$ of the form

$$P(\mathcal{D})RP : Ran\, P \to \mathcal{H}(\mathcal{D})\,,$$

where R denotes any parametrix of the elliptic operator $PP(\mathcal{D}) + (Id - P)(Id - P(\mathcal{D}))$. The formula

$$\mathcal{C}_P = \mathcal{D}^{-1} - \mathcal{K}\mathcal{R}(P)P\gamma_0\mathcal{D}^{-1}$$

now gives an operator such that $\mathcal{D}_P\mathcal{C}_P - Id$ and $\mathcal{C}_P\mathcal{D}_P - Id$ have smooth kernels.

(2) More generally, this formula gives a parametrix for any elliptic boundary problem $\mathcal{D}_T$ as defined in [BoW4] (where the authors were following Seeley's exposition [Se2]). The reason is that $\mathcal{N}_T$, the kernel of the boundary condition T, and $\mathcal{H}(\mathcal{D})$ form a Fredholm pair of subspaces of $L^2(Y; S|Y)$, which allows a parametrix R to be constructed. This fact was well known to Booss and Wojciechowski and is implicit in their work [BoW1] and [BoW2] (see also Proposition 1.4. in [BoMSW]). Last but not least, we are not really restricted in this construction of the parametrix only to Dirac operators. This construction holds for any first order elliptic differential operator on a compact manifold with boundary. The details will be presented elsewhere.

(3) The explicit construction of the parametrix presented in this paper may be found elsewhere in the literature in related contexts. It was used for instance by Peter Gilkey and Lance Smith in their work on the η-invariant for a class of (local) elliptic boundary problems (see [GS]). See also the work of Forman [F].

3 Variation of the ζ-determinant on $Gr^*_\infty(\mathcal{D})$

In this section we study the variation of the ζ-determinant of the operator $\mathcal{D}_P$, where $P \in Gr^*_\infty(\mathcal{D})$, under a change of boundary condition. From section 1 we know that the Grassmannian $Gr^*_\infty(\mathcal{D})$ can be identified with the group $U^\infty(F^-)$. If we fix a base projection, for instance $P(\mathcal{D})$, then any other projection is of the form:

$$P = \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g \end{pmatrix} P(\mathcal{D}) \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g^{-1} \end{pmatrix},$$

where $g : F^- \to F^-$ is a unitary operator such that $g - Id$ has a smooth kernel.

We introduce a smooth one-parameter family $\{g_r\}_{-\varepsilon<r<\varepsilon}$ of operators from $U^\infty(F^-)$ with $g_0 = Id_{F^-}$. Let $\{P_r\}$ denote the corresponding family of projections:

$$P_r = \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g_r \end{pmatrix} P \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g_r^{-1} \end{pmatrix}.$$

We want to compute the variation

$$\tfrac{d}{dr}\{ln \det\nolimits_\zeta \mathcal{D}_{P_r}\}|_{r=0}\,.$$

For the purposes of this paper it is enough to solve an easier problem. Let us fix two elements of the Grassmannian P_1 and P_2 such that $\mathcal{D}_{P_1}$ and $\mathcal{D}_{P_2}$ are invertible operators. The family $\{g_r\}$ determines two 1-parameter families of projections

$$P_{i,r} = \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g_r \end{pmatrix} P_i \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g_r^{-1} \end{pmatrix} \tag{3.1}$$

with respect to which we may study the relative variation

$$\tfrac{d}{dr}\{ln \det\nolimits_\zeta \mathcal{D}_{P_{1,r}} - ln \det\nolimits_\zeta \mathcal{D}_{P_{2,r}}\}|_{r=0}\,. \tag{3.2}$$

The first obstacle here is that the domains of the unbounded operators $\mathcal{D}_{P_{i,r}}$ vary with the parameter r. It was explained in [DoW2] and [LW] how to solve this problem by applying a *"Unitary Twist"*. The point is that we may extend the family of unitary isomorphisms $\{g_r\}$ on the boundary sections to a family of unitary transformations $\{U_r\}$ on $L^2(M;S)$. To do that, fix a smooth non-decreasing function $\kappa(u)$ such that

$$\kappa(u) = 1 \;\text{ for }\; u < 1/4 \quad\text{and}\quad \kappa(u) = 0 \;\text{ for }\; u > 3/4\,,$$

and for each r introduce the 2-parameter family

$$g_{r,u} = g_{r\kappa(u)} \;\text{ for }\; 0 \le u \le 1\,. \tag{3.3}$$

Now we define a transformation U_r as follows:

$$U_r := \begin{cases} \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g_{r,u} \end{pmatrix} & \text{on } \{u\} \times Y \subset N = [0,1] \times Y \\ Id & \text{on } M \setminus N\,. \end{cases} \tag{3.4}$$

We then have the following elementary result.

LEMMA 3.1. *The operators $\mathcal{D}_{P_{i,r}}$ and $(U_r^{-1}\mathcal{D}U_r)_{P_i}$ are unitarily equivalent operators.*

Clearly, the operators U_r depend upon the choice of the extension function κ, however by Lemma 3.1 the ζ-determinant does not, which is all that we need. The canonical determinant is also independent of the choice of the family $\{U_r\}$. This follows from the fact that $P_{i,r}P(\mathcal{D})$ and the operator $P_iP(\mathcal{D}_r)$ are unitarily equivalent:

$$P_{i,r}P(\mathcal{D}) = g_rP_ig_r^{-1}P(\mathcal{D}) = g_r\bigl(P_ig_r^{-1}P(\mathcal{D})g_r\bigr)g_r^{-1} = g_rP_iP(\mathcal{D}_r)g_r^{-1}. \tag{3.5}$$

In the following we use the notation

$$\mathcal{D}_r = U_r^{-1}\mathcal{D}U_r \quad \text{and} \quad \dot{\mathcal{D}}_0 = \tfrac{d}{dr}\mathcal{D}_r|_{r=0}\,. \tag{3.6}$$

The main result of this section is the following theorem:

Theorem 3.2. *For any pair of projections $P_1, P_2 \in Gr_\infty^*(\mathcal{D})$ such that $\mathcal{D}_{P_1}$ and $\mathcal{D}_{P_2}$ are invertible operators and for any smooth 1-parameter family $\{g_r\}$ of operators from $U^\infty(F^-)$ with $g_o = Id$, the following equality holds:*

$$\tfrac{d}{dr}\{ln\det{}_\zeta\,\mathcal{D}_{P_{1,r}} - ln\det{}_\zeta\,\mathcal{D}_{P_{2,r}}\}|_{r=0} = \operatorname{Tr}\dot{\mathcal{D}}_0(\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1})\,. \tag{3.7}$$

Notice that although the variation $\dot{\mathcal{D}}_0$ is localized in N, the variation of the ζ-determinant is not. It depends on global data because of the term $\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1}$ and it is this that makes the ζ-determinant a more difficult spectral invariant than the η-invariant, which corresponds to the phase of the determinant. Indeed, formula (3.7) contains only a variation of the difference of logarithms of the modulus of the ζ-determinant. The reason is that the variation of the phase of the determinant of $\mathcal{D}_{P_{1,r}}$ is equal to the variation of the phase of the determinant of $\mathcal{D}_{P_{2,r}}$:

Theorem 3.3. *The variation of the phase of the ζ-determinant of the operator $\mathcal{D}_{P_r}$ depends only on the family of the unitaries $\{g_r\}$ on F^- such that*

$$P_r = \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g_r \end{pmatrix} P_0 \begin{pmatrix} Id_{F^+} & 0 \\ 0 & g_r^{-1} \end{pmatrix},$$

not on the choice of the base-point projection P_0. More specifically, $\zeta_{\mathcal{D}_P^2}(0)$ is a constant function of the projection P, and the variation of the η-invariant depends only on the family $\{g_r\}$.

Proof. The theorem follows from two technical results proved in the work of the second author [Wo5]. The phase of the determinant is equal to

$$\exp\bigl\{\tfrac{i\pi}{2}\bigl(\zeta_{\mathcal{D}_{P_r}^2}(0) - \eta_{\mathcal{D}_{P_r}}(0)\bigr)\bigr\}.$$

It was shown in [Wo5] (Proposition 0.5) that $\zeta_{\mathcal{D}_P^2}(0)$ is constant on $Gr_\infty^*(\mathcal{D})$, hence the variation of the logarithm of the phase is equal to the variation of the η-invariant *times* $-i\pi/2$. The formula for the variation of the η-invariant was derived in the proof of Theorem 4.3. in [Wo5]. We have

$$\frac{d}{dr}\eta_{\mathcal{D}_{P_{i,r}}}(0)|_{r=0} = \frac{i}{\pi}\int_0^1 du \,\mathrm{Tr}\left(\frac{d}{dr}\left(g_{r,u}^{-1}\frac{\partial g_{r,u}}{\partial u}\right)\Big|_{r=0}\right). \tag{3.8}$$

In particular the right side of (3.8) does not depend on P_i. □

REMARK 3.4. A special case of the formula (3.8) was discussed in the paper [SW1].

Next we study the logarithm of the modulus of the determinant

$$ln|\det_\zeta \mathcal{D}_P| = -\tfrac{1}{2}\zeta'_{\mathcal{D}_P^2}(0).$$

It is well known (see section 3 of [Wo5]) that

$$\zeta'_{\mathcal{D}_P^2}(0) = \lim_{s\to 0}\left\{\int_0^\infty t^{s-1}\,\mathrm{Tr}\,e^{-t\mathcal{D}_P^2}dt - \frac{\zeta_{\mathcal{D}_P^2}(0)}{s}\right\} - \gamma\cdot\zeta_{\mathcal{D}_P^2}(0), \tag{3.9}$$

where γ denotes the Euler constant. The fact that $\zeta_{\mathcal{D}_P^2}(0)$ does not depend on P allows us to study just the variation of the integral in formula (3.9) and with the help of *Duhamel's Principle* we obtain

$$\begin{aligned}\frac{d}{dr}\big(\zeta'_{\mathcal{D}^2_{P_{i,r}}}(0)\big)|_{r=0} &= \int_0^\infty \frac{1}{t}\cdot\mathrm{Tr}(-2t\dot{\mathcal{D}}_0\mathcal{D}_{P_i}e^{-t\mathcal{D}^2_{P_i}})dt\\ &= -2\int_0^\infty \mathrm{Tr}\,\dot{\mathcal{D}}_0\mathcal{D}_{P_i}^{-1}\mathcal{D}^2_{P_i}e^{-t\mathcal{D}^2_{P_i}}dt = 2\int_0^\infty\frac{d}{dt}(\mathrm{Tr}\,\dot{\mathcal{D}}_0\mathcal{D}_{P_i}^{-1}e^{-t\mathcal{D}^2_{P_i}})dt\\ &= 2\cdot\lim_{\varepsilon\to 0}(\mathrm{Tr}\,\dot{\mathcal{D}}_0\mathcal{D}_{P_i}^{-1}e^{-t\mathcal{D}^2_{P_i}})|_\varepsilon^{1/\varepsilon} = -2\cdot\lim_{\varepsilon\to 0}\mathrm{Tr}\,\dot{\mathcal{D}}_0\mathcal{D}_{P_i}^{-1}e^{-\varepsilon\mathcal{D}^2_{P_i}}.\end{aligned}$$

We therefore have

LEMMA 3.5.

$$\tfrac{d}{dr}\big(-\tfrac{1}{2}\zeta'_{\mathcal{D}^2_{P_{i,r}}}(0)\big)\big|_{r=0} = \lim_{\varepsilon\to 0}\mathrm{Tr}\,\dot{\mathcal{D}}_0\mathcal{D}_{P_i}^{-1}e^{-\varepsilon\mathcal{D}^2_{P_i}}. \tag{3.10}$$

In general the limit on the right hand of the equation (3.10) is just the constant term in the asymptotic expansion of the trace. However, since we discuss the difference (3.2), in this situation we actually obtain the true operator trace:

$$\begin{aligned}&\tfrac{d}{dr}\{ln\det_\zeta\mathcal{D}_{P_{1,r}} - ln\det_\zeta\mathcal{D}_{P_{2,r}}\}|_{r=0}\\ &\qquad = \lim_{\varepsilon\to 0}\mathrm{Tr}\,\dot{\mathcal{D}}_0\mathcal{D}_{P_1}^{-1}e^{-\varepsilon\mathcal{D}^2_{P_1}} - \lim_{\varepsilon\to 0}\mathrm{Tr}\,\dot{\mathcal{D}}_0\mathcal{D}_{P_2}^{-1}e^{-\varepsilon\mathcal{D}^2_{P_2}}\\ &\qquad = \lim_{\varepsilon\to 0}\mathrm{Tr}\,\dot{\mathcal{D}}_0(\mathcal{D}_{P_1}^{-1}-\mathcal{D}_{P_2}^{-1})e^{-\varepsilon\mathcal{D}^2_{P_1}}\\ &\qquad = \mathrm{Tr}\,\dot{\mathcal{D}}_0(\mathcal{D}_{P_1}^{-1}-\mathcal{D}_{P_2}^{-1}),\end{aligned}$$

where for the final step we use Corollary 2.5. This completes the proof of Theorem 3.2.

4 Variation of the Canonical Determinant

In this section we prove that the relative variation of the canonical determinant coincides with the relative variation of the ζ-determinant (3.7). We begin with the following result:

PROPOSITION 4.1. *The following formula holds for any $P_1, P_2 \in Gr^*_\infty(\mathcal{D})$ such that $\mathcal{D}_{P_1}$ and $\mathcal{D}_{P_2}$ are invertible operators.*

$$\det_{\mathcal{C}} \mathcal{D}_{P_{1,r}} (\det_{\mathcal{C}} \mathcal{D}_{P_{2,r}})^{-1} = \det_{Fr} \begin{pmatrix} Id & 0 \\ 0 & T_2 T_1^{-1} \end{pmatrix} \mathcal{S}_r(P_1) \mathcal{S}_r(P_2)^{-1}, \quad (4.1)$$

where $\mathcal{S}_r(P_i)$ denotes the operator $P_i P(\mathcal{D}_r) : \mathcal{H}(\mathcal{D}_r) \to Ran\, P_i$.

Proof. Let

$$U_{T_1,T_2} = \begin{pmatrix} Id & 0 \\ 0 & T_2 T_1^{-1} \end{pmatrix} : Ran\, P_1 \to Ran\, P_2$$

and observe that

$$U_{T_1,T_2} U_{T_3,T_1} = U_{T_3,T_2} \,, \qquad U^{-1}_{T_1,T_2} = U_{T_2,T_1} \quad (4.2)$$

and that if $A : Ran\, P_1 \to Ran\, P_1$ is of the form Id plus trace-class then

$$\det_{Fr} A = \det_{Fr} U^{-1}_{T_1,T_2} A U_{T_1,T_2} \,, \quad (4.3)$$

where the determinant on the left-side is taken on $Ran\, P_1$ and the determinant on the right-side is taken on $Ran\, P_2$. Then since $U(P) = U_{K,T}$, we have using the invariance (3.5) of the canonical determinant under a unitary twist and the multiplicativity (0.3) of the Fredholm determinant

$$\begin{aligned} \det_{\mathcal{C}} \mathcal{D}_{P_{1,r}} (\det_{\mathcal{C}} \mathcal{D}_{P_{2,r}})^{-1} &= \det_{Fr} \left(U^{-1}_{K_r,T_1} \mathcal{S}_r(P_1) \right) \det_{Fr} \left((U^{-1}_{K_r,T_2} \mathcal{S}_r(P_2))^{-1} \right) \\ &= \det_{Fr} \left(U^{-1}_{K_r,T_1} \mathcal{S}_r(P_1) \mathcal{S}(P_2)^{-1} U_{K_r,T_2} \right) \\ &= \det_{Fr} i \left(U^{-1}_{K_r,T_2} U_{T_1,T_2} \mathcal{S}_r(P_1) \mathcal{S}(P_2)^{-1} U_{K_r,T_2} \right) \\ &= \det_{Fr} \left(U_{T_1,T_2} \mathcal{S}_r(P_1) \mathcal{S}_r(P_2)^{-1} \right) , \end{aligned}$$

where the last two lines use (4.2) and (4.3), respectively. □

Hence, setting

$$\mathcal{S}_r = U_{T_1,T_2} \mathcal{S}_r(P_1) \mathcal{S}_r(P_2)^{-1} P_2 : Ran\, P_2 \to Ran\, P_2 \,,$$

we have proved

COROLLARY 4.2.

$$\tfrac{d}{dr} \{ ln \det_{\mathcal{C}} \mathcal{D}_{P_{1,r}} - ln\ \det_{\mathcal{C}} \mathcal{D}_{P_{2,r}} \}|_{r=0} = \mathrm{Tr} \left(\left(\tfrac{d}{dr} \mathcal{S}_r \right) \mathcal{S}_r^{-1} \right) \Big|_{r=0} = \mathrm{Tr}\, \dot{\mathcal{S}}_0 \mathcal{S}_0^{-1} \,. \quad (4.4)$$

LEMMA 4.3.

$$\mathrm{Tr}\, \dot{\mathcal{S}}_0 \mathcal{S}_0^{-1} = \mathrm{Tr}\, P_1 \gamma_0 \left(\tfrac{d}{dr} \mathcal{K}_r(P_2) \right) \Big|_{r=0} P_2 \gamma_0 \mathcal{K}(P_1) \,. \quad (4.5)$$

Proof. We compute

$$\begin{aligned}\tfrac{d}{dr} ln \det \mathcal{S}_r\big|_{r=0} &= \mathrm{Tr}\left(\left(\tfrac{d}{dr}\mathcal{S}_r\right)\mathcal{S}_r^{-1}\right)\big|_{r=0} = \mathrm{Tr}\,\dot{\mathcal{S}}_0\mathcal{S}_0^{-1}\\ &= \mathrm{Tr}\begin{pmatrix} Id & 0\\ 0 & T_2T_1^{-1}\end{pmatrix}\left\{\tfrac{d}{dr}\big(P_1P(\mathcal{D}_r)(P_2P(\mathcal{D}_r))^{-1}P_2\big)\big|_{r=0}\right\}\\ &\qquad\cdot\big(P_2P(\mathcal{D})(P_1P(\mathcal{D}))^{-1}\big)\begin{pmatrix} Id & 0\\ 0 & T_1T_2^{-1}\end{pmatrix}\\ &= \mathrm{Tr}\,\tfrac{d}{dr}\big(P_1\gamma_0\mathcal{K}\mathcal{S}_r(P_2)^{-1}P_2\big)\big|_{r=0}P_2\gamma_0\mathcal{K}\mathcal{S}(P_1)^{-1}P_1\\ &= \mathrm{Tr}\,\tfrac{d}{dr}\big(P_1\gamma_0\mathcal{K}_r(P_2)\big)\big|_{r=0}P_2\gamma_0\mathcal{K}(P_1)\\ &= \mathrm{Tr}\,P_1\gamma_0\left(\tfrac{d}{dr}\mathcal{K}_r(P_2)\right)\big|_{r=0}P_2\gamma_0\mathcal{K}(P_1)\,.\end{aligned}$$

The lemma is proved. □

The next lemma takes care of the variation of the operator $\mathcal{K}_r(P_2)$.

Lemma 4.4. *The following formula holds at $r=0$*

$$\dot{\mathcal{K}}_0(P_2) := \tfrac{d}{dr}\mathcal{K}_r(P_2)|_{r=0} = -\mathcal{D}_{P_2}\dot{\mathcal{D}}_0\mathcal{K}(P_2)\,. \tag{4.6}$$

Proof. Let us fix $f \in Ran\,P_2$ and let $s_r = \mathcal{K}_r(P_2)f$. We have

$$\mathcal{D}_r s_r = 0 \quad\text{and}\quad P_2\gamma_0 s_r = f\,,$$

hence differentiation with respect to r gives

$$\left(\tfrac{d}{dr}\mathcal{D}_r\right)s_r = -\mathcal{D}_r\left(\tfrac{d}{dr}s_r\right) \quad\text{and}\quad \tfrac{d}{dr}(P_2(\gamma_0 s_r)) = P_2\big(\gamma_0\tfrac{d}{dr}s_r\big) = 0\,,$$

hence $\frac{d}{dr}s_r \in \mathrm{dom}\,\mathcal{D}_{P_2}$. We obtain

$$\tfrac{d}{dr}\mathcal{K}_r(P_2)f = \tfrac{d}{dr}s_r = -\mathcal{D}^{-1}_{r,P_2}\left(\tfrac{d}{dr}\mathcal{D}_r\right)s_r = -\mathcal{D}^{-1}_{r,P_2}\left(\tfrac{d}{dr}\mathcal{D}_r\right)\mathcal{K}_r(P_2)f\,.$$

This gives at $r=0$

$$\dot{\mathcal{K}}_0(P_2) = -\mathcal{D}^{-1}_{P_2}\dot{\mathcal{D}}_0\mathcal{K}(P_2)\,.$$ □

The trace of $\dot{\mathcal{S}}_0\mathcal{S}_0^{-1}$ is therefore given by the following formula

$$\mathrm{Tr}\,\dot{\mathcal{S}}_0\mathcal{S}_0^{-1} = \mathrm{Tr}\,P_1\gamma_0(-\mathcal{D}^{-1}_{P_2})\dot{\mathcal{D}}_0\mathcal{K}(P_2)P_2\gamma_0\mathcal{K}(P_1)\,. \tag{4.7}$$

The next important step is to change the order of the operators under the trace:

$$\begin{aligned}&\mathrm{Tr}\,P_1\gamma_0(-\mathcal{D}^{-1}_{P_2})\dot{\mathcal{D}}_0\mathcal{K}(P_2)P_2\gamma_0\mathcal{K}(P_1)\\ &\qquad= \mathrm{Tr}\,\big(P_1\gamma_0(-\mathcal{D}^{-1}_{P_2})\dot{\mathcal{D}}_0\mathcal{K}(P_2)P_2\big)\big(P_2\gamma_0\mathcal{K}(P_1)\big)\\ &\qquad= \mathrm{Tr}\,\big(P_2\gamma_0\mathcal{K}(P_1)\big)\big(P_1\gamma_0(-\mathcal{D}^{-1}_{P_2})\dot{\mathcal{D}}_0\mathcal{K}(P_2)P_2\big)\,.\end{aligned}$$

The exchange is justified by the fact that

$$P_2\gamma_0\mathcal{K}(P_1) = P_2P(\mathcal{D})\mathcal{S}(P_1)^{-1}P_1$$

is a pseudodifferential operator of order 0 (with the symbol equal to the symbol of $P(\mathcal{D})$), and hence that it is a bounded operator on $L^2(Y;S|Y)$. Thus we have

$$\mathrm{Tr}\,\dot{\mathcal{S}}_0\mathcal{S}_0^{-1} = \mathrm{Tr}\,\big(P_2\gamma_0\mathcal{K}(P_1)\big)\big(P_1\gamma_0(-\mathcal{D}^{-1}_{P_2})\dot{\mathcal{D}}_0\mathcal{K}(P_2)P_2\big)\,. \tag{4.8}$$

Now from equation (4.8) and Proposition 2.7 we have that

$$\operatorname{Tr} \dot{\mathcal{S}}_0 \mathcal{S}_0^{-1} = \operatorname{Tr} P_2 \gamma_0 (\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1}) \dot{\mathcal{D}}_0 \mathcal{K}(P_2) P_2 \,. \tag{4.9}$$

The operator on the right side of (4.9) has a smooth kernel (see Corollary 2.5) and so we can again switch the order of operators:

$$\begin{aligned} &\operatorname{Tr}\big(P_2\gamma_0(\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1})\dot{\mathcal{D}}_0\big)\big(\mathcal{K}(P_2)P_2\big) = \operatorname{Tr}\big(\mathcal{K}(P_2)P_2\big)\big(P_2\gamma_0(\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1})\dot{\mathcal{D}}_0\big) \\ &\quad = \operatorname{Tr} \mathcal{K}\big(\mathcal{S}(P_2)^{-1}P_2\gamma_0\mathcal{K}\big)\big(\mathcal{S}(P_2)^{-1}P_2 - \mathcal{S}(P_1)^{-1}P_1\big)\gamma_0\mathcal{D}^{-1}\dot{\mathcal{D}}_0 \\ &\quad = \operatorname{Tr} \mathcal{K}\big(\mathcal{S}(P_2)^{-1}P_2 - \mathcal{S}(P_1)^{-1}P_1\big)\gamma_0\mathcal{D}^{-1}\dot{\mathcal{D}}_0 = \operatorname{Tr}(\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1})\dot{\mathcal{D}}_0 \end{aligned}$$

where we have used (2.5) and (2.8).

Thus we have

$$\operatorname{Tr} \dot{\mathcal{S}}_0 \mathcal{S}_0^{-1} = \operatorname{Tr}(\mathcal{D}_{P_1}^{-1} - \mathcal{D}_{P_2}^{-1})\dot{\mathcal{D}}_0 \,.$$

This completes the proof of the following theorem.

Theorem 4.5. *With the assumptions of Theorem 3.2 one has*

$$\begin{aligned} &\tfrac{d}{dr}\{ln \det{}_\zeta \mathcal{D}_{P_{1,r}} - ln \det{}_\zeta \mathcal{D}_{P_{2,r}}\}|_{r=0} \\ &\qquad = \tfrac{d}{dr}\{ln \det{}_{\mathcal{C}} \mathcal{D}_{P_{1,r}} - ln \det{}_{\mathcal{C}} \mathcal{D}_{P_{2,r}}\}|_{r=0} \,. \end{aligned} \tag{4.10}$$

5 Equality of the Determinants

In this section we prove the main result of the paper. First, we need the following elementary result, which allows us to integrate the equality (4.10). (We refer to [N] for a more detailed discussion of the topology of $\tilde{Gr}_\infty^*(\mathcal{D})$, see also the remarks in sections 6 and 7).

Proposition 5.1. *The space $\tilde{Gr}_\infty^*(\mathcal{D})$, which consists of projections $P \in Gr_\infty^*(\mathcal{D})$ such that the operator $\mathcal{D}_P$ is invertible, is path connected.*

Proof. We show that for any $P \in \tilde{Gr}_\infty^*(\mathcal{D})$ there exists a path $\{P_r\}_{0\le r\le 1} \subset \tilde{Gr}_\infty^*(\mathcal{D})$ such that

$$P_0 = P \quad \text{and} \quad P_1 = P(\mathcal{D}) \,.$$

Let H denote the range of the projection P. Lemma 2.2 tells us that if $\mathcal{D}_P$ is invertible then

$$H^\perp \oplus \mathcal{H}(\mathcal{D}) = L^2(Y;S|Y) \quad \text{and} \quad H^\perp \cap \mathcal{H}(\mathcal{D}) = \{0\} \,. \tag{5.1}$$

Equivalently we can write the first equality in (5.1) as

$$H \oplus \mathcal{H}(\mathcal{D})^\perp = L^2(Y;S|Y) \,.$$

The equality above implies the existence of a linear operator $T : \mathcal{H}(\mathcal{D}) \to \mathcal{H}(\mathcal{D})^\perp$, such that $H = graph(T)$. In fact T is defined in the following way. We introduce the operator

$$(P(\mathcal{D})P)^{-1} = (\mathcal{S}(P)^*)^{-1} : \mathcal{H}(\mathcal{D}) \to H \,.$$

It follows now easily that H is a graph of the operator

$$T = (\mathcal{S}(P)^*)^{-1} - P(\mathcal{D})\,. \tag{5.2}$$

The fact that H is Lagrangian gives the equality

$$T^*G + GT = 0 \tag{5.3}$$

where the bundle anti-involution G (see (0.9), (0.10)) determines the symplectic structure on $L^2(Y; S|Y)$. If we now write the projection P with respect to the decomposition $L^2(Y; S|Y) = \mathcal{H}(\mathcal{D}) \oplus \mathcal{H}(\mathcal{D})^{\perp}$, we obtain

$$P = \begin{pmatrix} (Id + T^*T)^{-1} & (Id + T^*T)^{-1}T^* \\ T(Id + T^*T)^{-1} & T(Id + T^*T)^{-1}T^* \end{pmatrix}. \tag{5.4}$$

Since $P \in Gr^*_\infty(\mathcal{D})$ then $P - P(\mathcal{D})$ is a smoothing operator and so the operator T has a smooth kernel. For each value of the parameter r we define the operator $T_r = rT$ and the corresponding projection

$$P_r = \begin{pmatrix} (Id + T_r^*T_r)^{-1} & (Id + T_r^*T_r)^{-1}T_r^* \\ T_r(Id + T_r^*T_r)^{-1} & T_r(Id + T_r^*T_r)^{-1}T_r^* \end{pmatrix}.$$

It is obvious that

$$\ker P(\mathcal{D})P_r \cong \operatorname{coker} \mathcal{S}(P_r) \cong Graph(T_r) \cap H(\mathcal{D})^{\perp} = \{0\}\,.$$

We know that $index\, \mathcal{S}(P_r)$ is equal to 0 and hence that $\mathcal{S}(P_r)$ also has a trivial kernel. Therefore the operator $\mathcal{D}_{P_r}$ is invertible for each $0 \le r \le 1$. The operators T_r satisfy condition (5.3), which shows that $H_r = Ran\, P_r$ is a Lagrangian subspace satisfying condition (5.1). It follows that the operators $\mathcal{D}_{P_r}$ are self-adjoint. Moreover $P_0 = P(\mathcal{D})$, which ends the proof. □

The next result is a consequence of Theorem 4.5 and Proposition 5.1.

Proposition 5.2. *Assume that we have $P_1, P_2 \in Gr^*_\infty(\mathcal{D})$ and $g \in U^\infty(F^-)$ such that all four operators $\mathcal{D}_{P_1}$, $\mathcal{D}_{UP_1U^{-1}}$, $\mathcal{D}_{P_2}$, $\mathcal{D}_{UP_2U^{-1}}$ are invertible, then*

$$\frac{\det_\zeta \mathcal{D}_{UP_1U^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{UP_1U^{-1}}}{\det_\zeta \mathcal{D}_{P_1} / \det_{\mathcal{C}} \mathcal{D}_{P_1}} = \frac{\det_\zeta \mathcal{D}_{UP_2U^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{UP_2U^{-1}}}{\det_\zeta \mathcal{D}_{P_2} / \det_{\mathcal{C}} \mathcal{D}_{P_2}}\,. \tag{5.5}$$

In particular, the ratio of the determinants does not depend on the choice of the base projection.

Proof. From Proposition 5.1, given any two projections from $Gr^*_\infty(\mathcal{D})$ such that $\mathcal{D}_{P_1}$ and $\mathcal{D}_{P_2}$ are invertible operators, we can find a path $\{P_r\}$ in the subspace $\tilde{G}r^*_\infty(\mathcal{D})$ which connects P_1 and P_2. Hence we can use Theorem 4.5 and integrate equation (4.10), which gives the identity

$$\frac{\det_\zeta \mathcal{D}_{P_{1,1}} / \det_{\mathcal{C}} \mathcal{D}_{P_{1,1}}}{\det_\zeta \mathcal{D}_{P_{1,0}} / \det_{\mathcal{C}} \mathcal{D}_{P_{1,0}}} = \frac{\det_\zeta \mathcal{D}_{P_{2,1}} / \det_{\mathcal{C}} \mathcal{D}_{P_{2,1}}}{\det_\zeta \mathcal{D}_{P_{2,0}} / \det_{\mathcal{C}} \mathcal{D}_{P_{2,0}}}\,, \tag{5.6}$$

where by construction

$$P_{i,1} = gP_{i,0}g^{-1} = gP_ig^{-1}\,, \qquad P_{i,0} = P_i\,.$$ □

We introduce an invariant $\mathcal{A}(g)$ using (5.5):

$$\mathcal{A}(g) = \frac{\det_\zeta \mathcal{D}_{UPU^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{UPU^{-1}}}{\det_\zeta \mathcal{D}_P / \det_{\mathcal{C}} \mathcal{D}_P} \,. \tag{5.7}$$

The next result follows from Proposition 5.2 and gives the first formula directly relating $\det_\zeta$ to $\det_{\mathcal{C}}$.

Theorem 5.3. *There is the following relation between* $\det_\zeta$ *and* $\det_{\mathcal{C}}$ *on* $Gr^*_\infty(\mathcal{D})$*:*

$$\det_\zeta \mathcal{D}_P = \det_\zeta \mathcal{D}_{P(\mathcal{D})} \cdot \det_{\mathcal{C}} \mathcal{D}_P \cdot \mathcal{A}(g) \,, \tag{5.8}$$

where, as before, $P = \begin{pmatrix} Id & 0 \\ 0 & g \end{pmatrix} P(\mathcal{D}) \begin{pmatrix} Id & 0 \\ 0 & g^{-1} \end{pmatrix}$.

Proof. The result is immediate from the identity (5.5) with $P_1 = P(\mathcal{D})$ and $P_2 = P = \begin{pmatrix} Id & 0 \\ 0 & g \end{pmatrix} P(\mathcal{D}) \begin{pmatrix} Id & 0 \\ 0 & g^{-1} \end{pmatrix}$. □

The main result of this section is the following theorem.

Theorem 5.4. *The function* $\mathcal{A}(g)$ *is the trivial character on the group* $U^\infty(F^-)$*, i.e. for any* $g \in U^\infty(F^-)$

$$\mathcal{A}(g) = 1 \,.$$

Proof. Let g and h be elements of $Gr^*_\infty(\mathcal{D})$ such that $\mathcal{D}_{U_g P(\mathcal{D}) U_g^{-1}}$, $\mathcal{D}_{U_h P(\mathcal{D}) U_h^{-1}}$ and $\mathcal{D}_{U_h U_g P(\mathcal{D}) U_g^{-1} U_h^{-1}}$ are invertible. We have

$$\begin{aligned} \mathcal{A}(hg) &= \frac{\det_\zeta \mathcal{D}_{U_{hg} P U_{hg}^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{U_{hg} P U_{hg}^{-1}}}{\det_\zeta \mathcal{D}_P / \det_{\mathcal{C}} \mathcal{D}_P} \\ &= \frac{\det_\zeta \mathcal{D}_{U_{hg} P U_{hg}^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{U_{hg} P U_{hg}^{-1}}}{\det_\zeta \mathcal{D}_{U_g P U_g^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{U_g P U_g^{-1}}} \cdot \frac{\det_\zeta \mathcal{D}_{U_g P U_g^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{U_g P U_g^{-1}}}{\det_\zeta \mathcal{D}_P / \det_{\mathcal{C}} \mathcal{D}_P} \\ &= \mathcal{A}(h)\mathcal{A}(g) \,, \end{aligned}$$

hence $\mathcal{A}(g)$ is a multiplicative character. It is well known that there are only two non-trivial characters on the group $U^\infty(F^-)$

$$\mathcal{A}^+(g) = \det_{Fr} g \quad \text{and} \quad \mathcal{A}^-(g) = (\det_{Fr}\ g)^{-1} \,. \tag{5.9}$$

We study the variation of $\det_\zeta$ at the Calderon projection $P(\mathcal{D})$ to show that $\mathcal{A}(g)$ is actually the trivial character. Let $\alpha : F^- \to F^-$ denote a self-adjoint operator with a smooth kernel. We define the 1-parameter smooth family of operators $\{g_r = e^{ir\alpha}\}$ in $U^\infty(F^-)$ and the corresponding family of operators on M

$$U_r = \begin{cases} Id & \text{on} \quad M \setminus N \\ \begin{pmatrix} Id & 0 \\ 0 & e^{ir\kappa(u)\alpha} \end{pmatrix} & \textit{on} \quad N \,. \end{cases}$$

with κ as in equation (3.3). The variation of the phase of the ζ-determinant is equal to the variation of the η-invariant times the factor $-(i\pi/2)$. It follows now from formula (3.8) that

$$\frac{d}{dr}\eta_{\mathcal{D}_{P_{i,r}}}(0)|_{r=0} = \frac{i}{\pi}\int_0^1 du\ \mathrm{Tr}\left(\frac{d}{dr}\left(g_{r,u}^{-1}\frac{\partial g_{r,u}}{\partial u}\right)\Big|_{r=0}\right)$$
$$= \frac{i}{\pi}\int_0^1 du\ \mathrm{Tr}\,\frac{d}{dr}(ir\kappa'(u)\alpha) = -\frac{\mathrm{Tr}\,\alpha}{\pi}\int_0^1 \kappa'(u)du = \frac{\mathrm{Tr}\,\alpha}{\pi}\,, \quad (5.10)$$

and so we see that variation of the ζ-determinant in this case is equal to $-i\cdot\mathrm{Tr}\,\alpha/2$. On the other hand, the canonical determinant of $\mathcal{D}_{g_r P(\mathcal{D})g_r^{-1}}$ is equal to

$$\det{}_{\mathcal{C}}\,\mathcal{D}_{g_r P(\mathcal{D})g_r^{-1}} = \det{}_{Fr}\,\frac{Id+KT_r^{-1}}{2} = \det{}_{Fr}\,\frac{Id+e^{-ir\alpha}}{2}$$
$$= \det{}_{Fr}\left(e^{-r\frac{i\alpha}{2}}\,\frac{e^{r\frac{i\alpha}{2}}+e^{-r\frac{i\alpha}{2}}}{2}\right)$$
$$= \det{}_{Fr}\left(e^{-r\frac{i\alpha}{2}}\cos r\tfrac{\alpha}{2}\right) = e^{-\frac{ir}{2}\,\mathrm{Tr}\,\alpha}\det{}_{Fr}\left(\cos r\tfrac{\alpha}{2}\right)\,. \quad (5.11)$$

Therefore the variation of the phase of the canonical determinant is equal to the variation of the phase of the ζ-determinant. From equation (0.1), the variation of the only two non-trivial characters (5.9) of the group $U(F^-)$ are in our case equal to

$$\tfrac{d}{dr}(\mathcal{A}^{\pm}(g_r))|_{r=0} = \pm i\cdot\mathrm{Tr}\,\alpha\,, \quad (5.12)$$

and hence $\mathcal{A}(g)$ is the trivial character of the group $U^{\infty}(F^-)$. □

This completes the proof of the main theorem.

6 Calderon Projection is the Only Critical Point of the Modulus of $\det_\zeta$ on the Space $\tilde{Gr}^*_\infty(\mathcal{D})$

Let us denote the modulus by $|\det_\zeta \mathcal{D}_P| = e^{-\frac{1}{2}\zeta'_{\mathcal{D}_P^2}(0)}$. We consider $|\det_\zeta \mathcal{D}_P|$ as a smooth function on the subspace $\tilde{Gr}^*_\infty(\mathcal{D}) \subset Gr^*_\infty(\mathcal{D})$. The striking property of this function is the following:

Theorem 6.1. *The only critical point of the function $|\det_\zeta|$ on $\tilde{Gr}^*_\infty(\mathcal{D})$ is the Calderon projection $P(\mathcal{D})$. In other words, at the point $P(\mathcal{D})$ the variation of $|\det_\zeta|$ at any direction in $Gr^*_\infty(\mathcal{D})$ is equal to 0. For any other $P \in \tilde{Gr}^*_\infty(\mathcal{D})$ there exists $\beta : F^- \to F^-$ such that the variation of $|\det_\zeta|$ at the point P in the direction of β is not equal to* 0.

Proof. It follows from Theorem 0.1 that it is enough to prove this result for $\det_{\mathcal{C}}$. Let us study the variation of $\det_{\mathcal{C}}$ in the direction of $\beta : F^- \to F^-$, where β is a self-adjoint operator with a smooth kernel. Let us fix $P \in \tilde{G}r^*_\infty(\mathcal{D})$. The projection P is given by formula (1.5) for some T, and now we consider the family $\{P_r\}$ of projections:

$$P_r = \begin{pmatrix} Id_{F^+} & 0 \\ 0 & e^{ir\beta} \end{pmatrix} P \begin{pmatrix} Id_{F^+} & 0 \\ 0 & e^{-ir\beta} \end{pmatrix} = \begin{pmatrix} Id_{F^+} & 0 \\ 0 & (e^{ir\beta}T)K^{-1} \end{pmatrix} P(\mathcal{D}) \begin{pmatrix} Id_{F^+} & 0 \\ 0 & K(e^{ir\beta}T)^{-1} \end{pmatrix}. \quad (6.1)$$

Let $\mathcal{S}_r$ denote the operator $\frac{Id+K(Te^{ir\beta})^{-1}}{2}$. Then

$$\frac{d}{dr}(ln \det{}_{\mathcal{C}} \mathcal{D}_{P_r})|_{r=0} = \frac{d}{dr}(ln \det{}_{Fr} \mathcal{S}_r)|_{r=0} = -i \cdot \operatorname{Tr} KT^{-1}\beta(Id + KT^{-1})^{-1}. \quad (6.2)$$

Now the phase of the right side of (6.2) equals $-\operatorname{Tr} \beta/2$ and the modulus is

$$\frac{1}{2} \operatorname{Tr} \big(-iKT^{-1}\beta(Id + KT^{-1})^{-1} + (-iKT^{-1}\beta(Id + KT^{-1})^{-1})^*\big) = -\frac{i}{2} \cdot \operatorname{Tr} \beta(Id + TK^{-1})^{-1}(Id - TK^{-1}). \quad (6.3)$$

It follows that for any β (6.3) is equal to 0 when $T = K$, hence the variation of the modulus of $\det_\zeta$ at $P(\mathcal{D})$ is equal to 0.

We have to work a little harder in order to show that the variation is non-trivial at $P \neq P(\mathcal{D})$. The operator $TK^{-1} : F^- \to F^-$ is a Fredholm operator and 1 is the only element of the essential spectrum of this operator. In the case that the operator TK^{-1} not equal to Id_{F^-} there exists μ in the spectrum of TK^{-1}, such that $\mu \neq 1$. Moreover TK^{-1} is unitary, which implies that $|\mu| = 1$. The invertibility of $\frac{Id+TK^{-1}}{2}$ implies also that $\mu \neq -1$.

The operator TK^{-1} is Fredholm and of the form Id_{F^-} *plus smoothing operator*, hence the *Fredholm alternative* shows the existence of a section $s \in F^-$, such that

$$TK^{-1}s = \mu s\,.$$

We may assume that $\|s\|_{L^2} = 1$ and we fix an orthonormal basis $\{\phi_k\}_{k\in\mathbf{Z}}$ of F^- such that $\phi_0 = s$. Now we define the operator β as follows

$$\beta(\phi_0) = \phi_0 \quad \text{and} \quad \beta(\phi_k) = 0 \quad \text{for} \quad k \neq 0\,.$$

Equation (6.3) shows that the variation of the modulus of $\det_{\mathcal{C}}$ at P in the direction of β is equal to

$$\begin{aligned} -\frac{i}{2}\cdot \operatorname{Tr} \beta(Id + TK^{-1})^{-1}(Id - TK^{-1}) &= -\frac{i}{2}\cdot \sum_{k\in\mathbf{Z}} \big(\beta(Id + TK^{-1})^{-1}(Id - TK^{-1})\phi_k; \phi_k\big) \\ &= -\frac{i}{2}\cdot\frac{1-\mu}{1+\mu} \\ &= -\frac{Im(\mu)}{2(1 + Re(\mu))} \neq 0\,, \end{aligned}$$

and the proof of the theorem is now complete. □

7 Comments and Concluding Remarks

In this section we comment on some technical issues around the proof of Theorem 0.1.

7.1 The choice of spectral cut. This determines the sign of the phase of the ζ-determinant and refers to the choice of the branch of $(-1)^{-s}$. Bearing in mind Theorem 5.4, it is natural to choose the *minus* sign in the representation $(-1)^{-s} = e^{\pm i\pi s}$, and therefore obtain that the phase of the ζ-determinant (see (0.8)) is equal to

$$\tfrac{i\pi}{2}\big(\zeta_{\mathcal{D}_P^2}(0) - \eta_{\mathcal{D}_P}(0)\big)\,.$$

This choice is opposite to the one made in the original reference (see [Si, p. 331]). One can argue that the original choice was dictated by applications in Quantum Field Theory. However, the discussion of the phase of the determinant in the fundamental work of Witten (see [W, section 2]), extended later by others, suggested that the choice of the sign, or more generally the parameter which determines the phase, should depend on the particular model being discussed (see for instance [AS1,2]).

Anyway, if we make the opposite choice, so that $(-1)^{-s} = e^{+i\pi s}$, then the ζ-determinant is equal to

$$\det{}_\zeta \mathcal{D} = e^{\frac{i\pi}{2}(\eta_{\mathcal{D}_P}(0) - \zeta_{\mathcal{D}_P^2}(0))} \cdot e^{-\frac{1}{2}\zeta'_{\mathcal{D}^2}(0)}\,. \tag{7.1}$$

Reviewing the proof of Theorem 5.4, equations (5.10) and (5.11) show that in this situation the variation of the phase of the ζ-determinant is equal to $\frac{i\cdot \operatorname{Tr}\alpha}{2}$, which is now minus the variation of the $\mathcal{C}$-determinant. It follows now from (5.12), that we have the following result.

Theorem 7.1. *Assume that we define* $\det_\zeta \mathcal{D}_P$ *using formula* (7.1), *then the following equality holds on* $Gr^*_\infty(\mathcal{D})$*:*

$$\det{}_\zeta \mathcal{D}_P = \det{}_{Fr} g \cdot \det{}_{\mathcal{C}} \mathcal{D}_P \cdot \det{}_\zeta \mathcal{D}_{P(\mathcal{D})}\,, \tag{7.2}$$

where, as before, g denotes the unique element of $U^\infty(F^-)$ such that

$$P = \begin{pmatrix} Id & 0 \\ 0 & g \end{pmatrix} P(\mathcal{D}) \begin{pmatrix} Id & 0 \\ 0 & g^{-1} \end{pmatrix}.$$

7.2 Contractibility of $\tilde{Gr}^*_\infty(\mathcal{D})$. One of the important technical results used in the paper is Proposition 5.1, which allows us to integrate equation (4.10) in order to obtain Theorem 0.1. The idea of the proof belongs to Liviu Nicolaescu (see [N, Proposition 3.12]). In fact the following theorem is a special case of a result proved by Nicolaescu.

Theorem 7.2 (see [N, Proposition C7]). *The space $\tilde{G}r^*_\infty(\mathcal{D})$, which consists of projections $P \in Gr^*_\infty(\mathcal{D})$ such that the operator $\mathcal{D}_P$ is invertible, is weakly contractible, i.e. any continuous map*

$$f : S^n \to \tilde{G}r^*_\infty(\mathcal{D})$$

is homotopic to a constant map.

7.3 The case of non-invertible B. Next we discuss the modification which has to be made in the case of a non-invertible tangential operator B (see the decomposition formula (0.9)). There are two important points which have to be addressed here.

First, we have to know if the difference $P(\mathcal{D}) - \Pi_>$ is a smoothing operator. It was assumed in the original proof (see [S1]) that the tangential operator B is invertible. The general case, however requires only a slight modification and we refer to the appendix in [DK] for the details.

Second, we need a replacement for the operator $V_> = (B^+B^-)^{-1/2}B^+$ (see (1.4)) , which provides us with a unitary transformation used in the construction of the trivialization of the determinant line bundle over $Gr^*_\infty(\mathcal{D})$. We employ the *Cobordism theorem* for Dirac operators (see for instance [BoW4, Theorem 21.5]). Namely, if Y is a boundary of a compact manifold M and the operator $B = \begin{pmatrix} 0 & B^-=(B^+)^* \\ B^+ & 0 \end{pmatrix}$ is the boundary component of a Dirac operator $\mathcal{D}$ on M, then *index* $B^+ = 0$, which implies

$$\dim\ker B^+ = \dim\ker B^- . \tag{7.3}$$

Now we fix an orthonormal basis $\{\varphi_k^\pm\}_{k=1}^{k=\dim\ker B^+}$ of $\ker B^\pm$ and define a unitary transformation $\sigma : \ker B \to \ker B$ by the formula

$$\sigma(\varphi_k^\pm) = \pm\varphi_k^\mp .$$

We define a modified operator $B_\sigma = B + \sigma$ and

$$V_\sigma = (B_\sigma^+ B_\sigma^-)^{-1/2} B_\sigma^+ . \tag{7.4}$$

The orthogonal projection

$$\Pi_\sigma = \frac{1}{2}\begin{pmatrix} Id_{F^+} & V_\sigma^{-1} \\ V_\sigma & Id_{F^-} \end{pmatrix} .$$

is an element of $Gr^*_\infty(\mathcal{D})$ (it satisfies (0.12) and it is a modification of $\Pi_>$ by a smoothing operator). One then proves that $Gr^*_\infty(\mathcal{D})$ consists of the graphs of unitary operators $V : F^+ \to F^-$, such that $V - V_\sigma$ is an operator with a smooth kernel, in the same way as in [S1].

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