Letters in Mathematical Physics **40:** 135–145, 1997.

Determinants, Grassmannians and Elliptic Boundary Problems for the Dirac Operator

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(Received: 26 November 1996)

Abstract. We study the relations between different determinants of the Dirac operator over a manifold with boundary considered as sections of a holomorphic line bundle over the Grassmannian of boundary conditions of Atiyah–Patodi–Singer type.

Mathematics Subject Classifications (1991). 58G11, 58G25, 57R90, 35J55, 35S35.

Key words: Dirac operator, Grassmannian of boundary conditions, ζ-determinant, Quillen determinant, canonical determinant, Calderon projection, Atiyah–Patodi–Singer condition.

0. Introduction

Recent studies in Quantum Field Theory (see, for instance, [1, 14, 16, 17]) have stressed the importance of the correct definition of the renormalized determinant of the Dirac operator over a closed manifold. With new developments in the mathematical understanding of QFTs as modified cohomology theories [1, 19], the need to extend the study of the determinant of the Dirac operator to manifolds with boundary has become clear. In [13], Quillen gave a construction of the determinant line bundle over a space of Fredholm operators and explained that, without making further choices, the determinant has to be viewed as a canonical section of this bundle. For a family of $\bar{\partial}$ operators over a Riemann surface Quillen identified a canonical trivialization of the determinant bundle by defining a metric and holomorphic connection through ζ-function renormalization and computing the curvarture, thus identifying the determinant up to a phase with a specific holomorphic function. These constructions have been used in many different contexts since then (see, for instance, [4, 16, 17]).

This Letter announces recent progress made in understanding these constructions for the ζ-determinant over the Grassmannian $\mathcal{G}r_\infty(A)$ of elliptic boundary conditions for a generalized Dirac operator A. This is especially important in view of recent results which show that Quillen's determinant satisfies a pasting law (see [14, 15]) which is naturally formulated in terms of a Fock space bilinear pairing associated to the Grassmannian and may explain the nature of the pasting axiom in Fermionic Field Theory.

In Section 1, we study the geometry of the determinant line bundle. We consider the infinite Grassmannian $\mathcal{G}r_\infty(A)$ of elliptic boundary conditions to be the parameter space for a holomorphic family of first-order elliptic differential operators. This follows the approach taken by Bismut and Freed [14] who extended the results of [13] to a general smooth family of Dirac operators over a closed manifold and showed that the curvature of the ζ-metric is the 2-form component of the local family index density. The method we use is to link up the Quillen–Bismut–Freed analysis with the holomorphic geometry of the Grassmannian as elucidated by Booss-Bavnbek and Wojciechowski (see [5, 6]) and Segal (see [12, 16]). More precisely, we find that the ζ-metric determines the same geometry as the canonical metric on the fundamental holomorphic line bundle over $\mathcal{G}r_\infty(A)$. To do this we identify the ζ metric with a metric constructed by a natural algebraic regularization of the Laplacian determinant and calculate its curvature. This is naturally understood as a statement of the *Local Family Index Theorem* for $\mathcal{G}r_\infty(A)$. The case of a family of varying Dirac operators with fixed Atiyah–Patodi–Singer boundary condition has been studied by Bismut Cheeger [3] and Melrose and Piazza [11]. By considering the dual situation of a fixed Dirac operator with varying boundary condition we are able to take advantage of the well-understood properties of $\mathcal{G}r_\infty(A)$ which effectively reduces the heat kernel analysis of the determinant to an algebraic problem. It is this reduction which allows the pasting to be formulated as a bilinear pairing on the Fermionic Fock space over $\mathcal{G}r_\infty(A)$. These constructions fit neatly into the framework of functorial QFT (see [1]) with the Fock space arising through geometric quantization of the above classical geometric (symplectic) theory. The details of this and the pasting will be presented in [15], though see also [14].

When A is defined over an odd-dimensional manifold, we find that there is a dense open subset U_{Graph} of the index zero component of the Grassmannian parameterising W for which the algebraic regularization of the determinant is defined for the elliptic boundary value problem A_W itself, and not just for its associated Dirac Laplacian. This is called the canonical determinant and denoted $\det_{\mathcal{C}} A_W$. In particular, U_{Graph} contains the real submanifold $\text{Gr}^*_\infty(A)$ parameterizing self-adjoint boundary conditions for A. In Section 2, we present the most important part of the proof of the following theorem which forms a crucial component in the proof of the pasting property.

THEOREM 0.1. *For W in $\text{Gr}^*_\infty(A)$ there is a constant c, independent of W, such that $\det_\zeta A_W = c\cdot\det_{\mathcal{C}} A_W$. Further, the phase of the canonical determinant is equal to the integral of the variation of the η-invariant. More precisely, let $\{W_r\}_{0\leqslant r\leqslant 1}$ denote a one-parameter family in $\text{Gr}^*_\infty(A)$ such that $W_0 = H(A)$ and $W_1 = W$ and let $\mathcal{A}_r$ denote the operator A_{W_r}, then the phase of the canonical determinant is equal to $\int_0^1 \mathrm{d}/\mathrm{d}r(\eta_{\mathcal{A}_r})\,\mathrm{d}r$.*

Such a result has previously only been obtained at the level of the determinant line (see [7]), Theorem 0.1 extends it explicitly to the level of the trivialization.

To explain the idea of the proof, we prove in Section 2 the second statement of the theorem for W in $\mathrm{Gr}^*_\infty(A)$. This is, for a self-adjoint boundary condition, the phase of the ζ-determinant in the odd-dimensional case is up to a constant equal to the phase of the determinant defined by the algebraic regularization. The complete proof of Theorem 0.1 is achieved through a generalization of these heat kernel techniques. Details of the proofs will be published in the forthcoming paper [15].

1. Geometry of the Determinant Line Bundle over the Grassmannian

Let M denote a compact odd-dimensional manifold with boundary Y. Let $A : C^\infty(S) \to C^\infty(S)$ denote a compatible Dirac operator acting on the space of sections of a bundle of Clifford modules S over M (see [6]). We discuss the case of a product metric structure in a neighborhood of the boundary. More precisely, we assume that the Riemannian metric on M and the Hermitian product on S are products in $N = [0,1] \times Y$, the collar neighborhood of Y in M. In this case, A has the form

$$A = \Gamma(\partial_u + B), \tag{1.1}$$

over N, where $\Gamma : S|Y \to S|Y$ is a unitary bundle automorphism (Clifford multiplication by the unit normal vector) and $B : C^\infty(Y; S|Y) \to C^\infty(Y; S|Y)$ is the tangential part of A on Y. Here B is the corresponding Dirac operator on Y and hence is a self-adjoint elliptic operator of first order. Furthermore, Γ and B do not depend on u and satisfy the following identities.

$$\Gamma^2 = -\mathrm{Id} \quad \text{and} \quad \Gamma B = -B\Gamma. \tag{1.2}$$

In particular, $S|Y$ decomposes into the direct sum $S^+ \oplus S^-$ of subbundles of eigenvectors of Γ corresponding to the eigenvalues $\pm i$. With respect to this decomposition, the operator B has the representation

$$B = \begin{bmatrix} 0 & B^- = (B^+)^* \\ B^+ & 0 \end{bmatrix}. \tag{1.3}$$

We assume that ker B = { 0 } in order to avoid unnecessary technical details in the presentation. The obvious modification to the general case will be presented elsewhere. Let $\Pi_>$ denote the spectral projection of B onto the subspace of $L^2(Y; S|Y)$ spanned by the eigenvectors corresponding to the positive eigenvalues of B. It is well known (see [2, 6]) that $\Pi_>$ is an elliptic boundary condition for the operator A, which means that the operator $A_{\Pi_>}$ defined by

$$\begin{aligned} &A_{\Pi_>} = A, \\ &\mathrm{dom}\, A_{\Pi_>} = \{s \in H^1(M; S|M) : \Pi_>(s|Y) = 0\}, \end{aligned} \tag{1.4}$$

is an unbounded operator, such that $A_{\Pi_>}: \operatorname{dom}(A_{\Pi_>}) \to L^2(M;S)$ is a Fredholm operator and the kernel of $A_{\Pi_>}$ and its cokernel consist of smooth sections of S. It is also well known that index $A_{\Pi_>} = \dim\ker B^+$ and, hence, in our case it is equal to 0. The operator $A_{\Pi_>}$ is now a self-adjoint operator and it is a particular example from the class of self-adjoint boundary problems which appear naturally within this context.

We define the Grassmannian of elliptic boundary value problems $\mathcal{G}r_\infty(A)$ as follows. The elements of $\mathcal{G}r_\infty(A)$ are pseudodifferential projections P acting on $C^\infty(Y;S|Y)$, such that they are orthogonal ($P = P^2 = P^*$), and such that the difference $P - \Pi_>$ is an operator with a smooth kernel. We can identify any projection P_W in the Grassmannian with its range $W \subset L^2(Y;S|Y)$. A preferred element of $\mathcal{G}r_\infty(A)$ is provided by the Calderon projection $P(A)$ of the operator A. This is the orthogonal projection onto the subspace $\mathcal{H}(A)$ of $C^\infty(Y;S|Y)$ defined as

$$\mathcal{H}(A) = \{v \in C^\infty(Y;S|Y) : \exists_{s\in C^\infty(M;S)} As = 0 \quad \text{and} \quad s|Y = v\}.$$

We refer to [6] (see also [5] and [14]) for details on $P(A)$ and $\mathcal{H}(A)$.

It was explained in [14] that we can construct the determinant line bundle over $\mathcal{G}r_\infty(A)$ in many different ways. The Quillen determinant line bundle $\mathcal{L}$ is the holomorphic pullback of the determinant line bundle from a space of Fredholm operators under the map $W \to A_{P_W}$, where P_W denotes the orthogonal projection onto the subspace $W \subset L^2(Y;S|Y)$.

We can also use the construction of the determinant bundle due to Segal [16]. Let $T : H_0 \to H_1$ denote a Fredholm operator with index equal to 0 acting on separable Hilbert spaces H_0 and H_1. Let Fred_T denote the space of all Fredholm operators which differ from T by a trace class operator. We define $\operatorname{Det} T = \mathrm{Fred}_T \times C$, where the relation is defined as follows. Let $S : H_0 \to H_1$ denote an invertible operator such that $S - T$ is an operator of trace class. Then any operator $Q \in \mathrm{Fred}_T$ is of the form $Q = S(\mathrm{Id} + q)$, where $q : H_0 \to H_0$ is a trace class operator. We identify

$$(S(\mathrm{Id} + q), z) \cong (S, z \cdot \det{}_F(\mathrm{Id} + q)),$$

where $\det_F R$ denotes the Fredholm determinant of the operator R. For a smooth family of such Fredholm operators, the lines fit together to define a line bundle canonically isomorphic to $\mathcal{L}$. Under this isomorphism, the canonical determinant section, defined over the index zero component by $T \to [(T,1)]$ maps to the canonical determinant section of $\mathcal{L}$.

In order to study the determinant bundle $\mathcal{L}$ associated to the family of elliptic boundary-value problems A_W parameterized by the Grassmannian, we use the operators

$$\rho_W : P_W P(A) : \mathcal{H}(A) \to W \tag{1.5}$$

The operator ρ_W is Fredholm with index equal to index A_W and with corresponding determinant line Det ρ_W. Globally we obtain a holomorphic line bundle Det isomorphic to $\mathcal{L}$ with determinant section canonically identified with that of $\mathcal{L}$ (see [14] for the details).

We now restrict our attention to the connected component $\mathcal{G}r^0_\infty(A)$ of $\mathcal{G}r_\infty(A)$ parameterizing subspaces W with index $A_W = 0$. Locally we may work over the open dense subset of $\mathcal{G}r^0_\infty(A)$ consisting of all those W which are the graphs of operators $T\colon\ L^2(Y;S^+) \to L^2(Y;S^-)$ such that $T - (B^+B^-)^{-1/2}B^+$ is an operator with a smooth kernel.

Let K denote the operator such that $\mathcal{H}(A) = \text{Graph}(K)$. Then we have a canonical trivialization of Det over U_{Graph} coming from the natural identification of W and K with F^+ defined by orthogonal projection. Specifically, the trivialization is the anti-holomorphic section

$$W = \text{Graph}(T) \to [B_W, 1], \tag{1.6}$$

where $B_W\colon\ \mathcal{H}(A) \to W$ is given relative to the splitting of $L^2(Y;S)$ into positive and negative spinor fields by

$$B_W = \begin{bmatrix} ((\text{Id} + T^*T)/2)^{-1} & 0 \\ 0 & T((\text{Id} + T^*T)/2)^{-1}K^{-1} \end{bmatrix}. \tag{1.7}$$

The isomorphism Det $\cong \mathcal{L}$ identifies det A_W with det ρ_W and, hence, a canonically renormalized determinant of A_W defined with respect to this local gauge by the formula $\det_{\mathcal{C}} A_W = \det_F(B^{-1}\rho_W)$ (the 'canonical determinant'). One computes

$$\det{}_{\mathcal{C}} A_W = \det{}_F(\tfrac{1}{2}(\text{Id} + KT^*)). \tag{1.8}$$

To connect this with the global geometry of the determinant bundle we define a Hermitian metric on $\mathcal{L}$ via the Laplacians

$$\Delta_{A_W} = A_W^* A_W, \qquad \Delta_{\rho W} = \rho_W^* \rho_W.$$

It is not difficult to see that there exists a natural isomorphism between the determinant lines Det Δ_{A_W} and Det $\Delta_{\rho W}$ preserving the canonical sections. The Laplacian $\Delta_{\rho W}\colon\ \mathcal{H}(A) \to \mathcal{H}(A)$ is an operator of the form $\text{Id}_{\mathcal{H}(A)} +$ *smoothing operator* and, hence, has a well-defined Fredholm determinant as a number in C. Therefore, we have a natural regularization of the determinant of Δ_{A_W} defined by

$$\det{}_{\mathcal{C}} \Delta_{A_W} = \det{}_F \Delta_{\rho W}. \tag{1.9}$$

PROPOSITION 1.1. *There is a natural inner product on $\mathcal{L}$ given over the index* 0 *component of $\mathcal{G}r_\infty(A)$ by*

$$\|\det A_W\|^2_{\mathcal{C}} = \det{}_{\mathcal{C}} \Delta_{A_W}, \tag{1.10}$$

where A_W is invertible and 0 *otherwise.*

We also have the usual Quillen norm $\|.\|_\zeta$ defined on $\mathcal{L}$ by $\|\det A_W\|_\zeta^2 = \det_\zeta \Delta_{A_W}$, where the right-hand side denotes the regularised zeta-function determinant. It is important to know the relation of the ζ norm to the $\mathcal{C}$ norm of Proposition 1.1. A holomorphic line bundle $\mathcal{L}$ with a Hermitian inner-product has a canonical connection compatible with the two structures whose curvature is the (1,1) form equal to $\bar{\partial}\partial \log \|s\|^2$ for any holomorphic section s. The elliptic Grassmannian $\mathcal{G}r_\infty(A)$ is endowed with a preferred form of type (1,1), namely the Kahler form

$$\omega = \frac{i}{2\pi} \mathrm{Tr}\, P\, \mathrm{d}P \wedge \mathrm{d}P. \tag{1.11}$$

Since any $P \in \mathcal{G}r_\infty(A)$ is of the form $\Pi_> +$ *smoothing operator* the trace on the right side is well-defined.

THEOREM 1.2. *The metrics $\|.\|_\zeta$ and $\|.\|_\mathcal{C}$ on the determinant line bundle have curvature equal to $-2\pi i$ times the Kähler form ω.*

The proof for the $\mathcal{C}$ metric follows from straightforward computations, the case of the Quillen metric uses heat kernel methods generalizing those presented in Section 2, details will be presented in [15]. On the other hand, a computation over U_{Graph} reveals the explicit formula

$$\det_F \Delta_{\rho W} = \frac{\det_F \frac{1}{2}(\mathrm{Id} + K^*T) \cdot \det_F \frac{1}{2}(\mathrm{Id} + T^*K)}{\det_F \frac{1}{2}(\mathrm{Id} + T^*T)}.$$

That is

THEOREM 1.3. *Over U_{Graph}, we have*

$$\det_\mathcal{C} \Delta_{A_W} = \frac{|\det_\mathcal{C} A_W|^2}{\det_F \frac{1}{2}(\mathrm{Id} + T^*T)}.$$

Thus, from Theorems 1.2 and 1.3 the canonical determinant of A_W on the set U_{Graph} is related to the global geometry of $\mathcal{L}$ by

$$\det_\mathcal{C} \Delta_{A_W} = \exp(-k) |\det_\mathcal{C} A_W|^2,$$

where k is the standard Kähler potential. Equivalently, by taking a section over U_{Graph} flat for the $\mathcal{C}$ metric connection, $\det_\mathcal{C} A_W$ is the function defined relative to this trivialization up to a scalar of absolute one. We note that the local anomaly formula of Theorem 1.2 is a measure of the failure of the canonical determinant to be multiplicative. Thus the fact that $\det_\mathcal{C} \Delta_{A_W}$ is not of the form

$|$*holomorphic function*$|^2$ is equivalent to the nontriviality of the determinant line bundle.

The local formula of Theorem 1.3 is also true for the Quillen metric up to a constant scale factor. The proof is modelled on the proof of theorem 0.1 (see [15] for the detailed exposition). To illustrate this, consider the simplest case of the operator $A = i(\mathrm{d}/\mathrm{d}x)$ on $M = [0, 1]$ with boundary conditions parametrized by CP^1. Then $W = \mathrm{Graph}(a)$ for $a \in C\backslash\{0\}$ corresponds to the homogeneous coordinate $[1, -\bar{a}^{-1}] \in CP^1$. An elementary calculation shows that the ζ-determinant of $\Delta_{i(\mathrm{d}/\mathrm{d}x)_{\mathrm{Graph}(a)}}$ is equal to

$$\det{}_\zeta \Delta_{i(\mathrm{d}/\mathrm{d}x)_{\mathrm{Graph}(a)}} = 2\frac{|1-a|^2}{1+|a|^2}$$

which coincides with the result of Theorem 2 up to factor 4.

2. The Phase of the Determinant on the Self-Adjoint Grassmannian

In this section we restrict ourselves to the situation studied in [14] (see also [21]). We study the canonical determinant and the ζ-determinant on the real submanifold of $\mathcal{G}r_\infty(A)$ parameterizing self-adjoint generalized Atiyah–Patodi–Singer boundary conditions for A. Specifically we define

$$\mathcal{G}r^*_\infty(A) = \{P \in \mathcal{G}r_\infty(A) : P \quad \text{is orthogonal and} \quad -\Gamma P \Gamma = \mathrm{Id} - P\} \tag{2.1}$$

The second condition implies that the range of the projection in $L^2(Y; S|Y)$ is a Lagrangian subspace with respect to the symplectic structure defined on $L^2(Y; S|Y)$ by the involution Γ. The projection $P_>$ is an element of $\mathcal{G}r^*_\infty(A)$ (in the case of invertible operator B). Another important example is provided by $P(A)$, the Calderon projection of the operator A (see [14]).

In [21] it was proved for $P \in \mathcal{G}r^*_\infty(A)$ that $\eta_{A_P}(s)$ and $\zeta_{A^2_P}(s)$ behave exactly like the η-function and the ζ-function of the Dirac operator on a closed manifold. In particular $\eta_{A_P}(0), \zeta_{A^2_P}(0)$ and $\mathrm{d}/\mathrm{d}s(\zeta_{A^2_P})|_{s=0}$ are well-defined and, hence, the ζ-determinant of A_P equal to

$$\det{}_\zeta A_P = \mathrm{e}^{(i\pi/2)\eta_{A_P}(0)}\, \mathrm{e}^{-\mathrm{d}/\mathrm{d}s(\zeta_{A^2_P})|_{s=0}} \tag{2.2}$$

is well-defined.

One has $\mathcal{G}r^*_\infty(A) \subset U_{\mathrm{Graph}}$ and so according to our work in Section 1 the determinant line bundle restricted to $\mathcal{G}r^*_\infty(A)$ is a canonically trivial line bundle. Now we make use of the result that elements of $\mathrm{Gr}^*_\infty(A)$ correspond precisely to those maps $S\colon\ L^2(Y; S^+) \to L^2(Y; S^-)$ which are L^2-unitary isomorphisms and differ from the operator $(B^+B^-)^{-1/2}B^+$ by a smoothing operator. The correspondence is given by

$$S \to P = \frac{1}{2}\begin{bmatrix} \mathrm{Id} & S^{-1} \\ S & \mathrm{Id} \end{bmatrix}. \tag{2.3}$$

It is obvious that $\operatorname{Ran}(P) = \operatorname{Graph}(S)$. From (1.8) we have

$$\det_{\mathcal{C}} A_P = \det_F \tfrac{1}{2}(\mathrm{Id} + KS^{-1}). \tag{2.4}$$

Let us assume for the moment that the operator $KS^{-1} : L^2(Y; S^-) \to L^2(Y; S^-)$ is of the form $\mathrm{e}^{i\alpha}$, where $\alpha : L^2(Y; S^-) \to L^2(Y; S^-)$ is a self-adjoint operator with a smooth kernel. The space of such operators α is a Lie algebra of $\mathrm{U}_\infty(S^-)$, the group of the unitary operators of the form Id plus *smoothing operator* acting on spinors of negative 'chirality' on Y, and so this is always the case when KS^{-1} is close to the Id in $\mathrm{GL}_\infty(S^-)$. This means

$$\begin{aligned} \det \tfrac{1}{2}(\mathrm{Id} + KS^{-1}) &= \det\left(\mathrm{e}^{(i\alpha/2)} \frac{\mathrm{e}^{(i\alpha/2)} + \mathrm{e}^{-(i\alpha/2)}}{2}\right) \\ &= \mathrm{e}^{(i/2)\mathrm{Tr}(\alpha)} \det \cos\frac{\alpha}{2}. \end{aligned} \tag{2.5}$$

This formula explains the structure of the canonical determinant, which is similar to the structure of the ζ-determinant (see (2.2)). The latter has phase determined by the η-invariant and modulus equal to the exponent of $\zeta'_{A^2_W}(0)$. The relation between the two determinants is given by Theorem 0.1, stated in the introduction. The proof of the Theorem consists of two parts. First we compute the variation of the η-invariant for a specific family of boundary conditions. Then we show that the result is independent of all choices and deformations made. We sketch the proof of the first part. Let

$$\begin{aligned} P &= \frac{1}{2}\begin{bmatrix} \mathrm{Id} & S^{-1} \\ S & \mathrm{Id} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} \mathrm{Id} & 0 \\ 0 & SK^{-1} \end{bmatrix}\begin{bmatrix} \mathrm{Id} & K^{-1} \\ K & \mathrm{Id} \end{bmatrix}\begin{bmatrix} \mathrm{Id} & 0 \\ 0 & KS^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{e}^{-i\alpha} \end{bmatrix} P(A) \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{e}^{i\alpha} \end{bmatrix}. \end{aligned}$$

We show that $(1/\pi)\,\mathrm{Tr}\,\alpha$ is the integral of the variation of the η-invariant for some family of the boundary conditions. We define the operator $\mathcal{A}_r$ as A_{P_r}, where the projection P_r is given by the formula

$$P_r = \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{e}^{-ir\alpha} \end{bmatrix} P(A) \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{e}^{ir\alpha} \end{bmatrix} \tag{2.6}$$

We will show that in this case

$$\int_0^1 \mathrm{d}/\mathrm{d}r(\eta_{\mathcal{A}_r})\,\mathrm{d}r = \frac{1}{\pi}\mathrm{Tr}\,\alpha. \tag{2.7}$$

In fact, it is not difficult to see that we can replace Calderon projection by the spectral projection. We refer to [15] for the details. In the following, we consider the family of projections defined by the formula

$$\Pi_r = \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{e}^{-ir\alpha} \end{bmatrix} \Pi_> \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{e}^{ir\alpha} \end{bmatrix}, \tag{2.8}$$

and we define the operator $\tilde{\mathcal{A}}_r$ as A_{Π_r}. Employing the method used in papers [8] (see Appendix 1) and [10], we perform a *Unitary Twist* on the operator $\tilde{\mathcal{A}}_r$. The operator we obtain has the same spectrum as $\tilde{\mathcal{A}}_r$ and, moreover, the new family has a fixed domain. We define a specific unitary transformation $U_r\colon\ L^2(M;S) \to L^2(M;S)$ as follows. First introduce a smooth nonnegative function $f\colon\ [0,1] \to [0,1]$ equal to 1 for $0 \leqslant u \leqslant \frac{1}{8}$ and equal to 0 for $\frac{7}{8} \leqslant u \leqslant 1$. We define

$$U_r = \begin{cases} \mathrm{Id} & \text{on } M\backslash N, \\ \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{e}^{irf(u)\alpha} \end{bmatrix} & \text{on } \{u\} \times Y. \end{cases} \tag{2.9}$$

It is obvious that the operator

$$\tilde{\mathcal{A}}_r = A_{\Pi_r} = A_{\begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{e}^{-ir\alpha} \end{bmatrix} \Pi_> \begin{bmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{e}^{ir\alpha} \end{bmatrix}}$$

has the same spectrum as the operator $(U_r A U_r^{-1})_{\Pi_>}$ and so we can study the variation of the η-invariant of the family $\{(U_r A U_r^{-1})_{\Pi_>}\}$. We follow the strategy of the papers [8] and [10] (see also [9, 20, 21]) and show that the only contribution to the variation of the η-invariant comes from the cylinder. The operator $U_r A U_r^{-1}$ is given by the formula

$$U_r A U_r^{-1} = A + \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & B^-(\mathrm{e}^{-irf(u)\alpha} - \mathrm{Id}) \\ (\mathrm{e}^{irf(u)\alpha} - \mathrm{Id})B^+ & 0 \end{bmatrix} + \\ + \begin{bmatrix} 0 & 0 \\ 0 & -rf'(u)\alpha \end{bmatrix},$$

and the operator $\mathrm{d}(U_r A U_r^{-1})/\mathrm{d}r$ has the following form

$$\frac{\mathrm{d}(U_r A U_r^{-1})}{\mathrm{d}r}$$

$$= \begin{bmatrix} 0 & f(u)B^{-}\alpha\, \mathrm{e}^{-irf(u)\alpha} \\ f(u)\alpha\, \mathrm{e}^{irf(u)\alpha}B^{+} & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & f'(u)\alpha \end{bmatrix}. \quad (2.10)$$

We use this representation in order to study

$$\frac{\mathrm{d}}{\mathrm{d}r}\eta_{(U_r A U_r^{-1})_{\Pi_>}}|_{r=0}$$
$$= -\frac{2}{\sqrt{\pi}} \lim_{\epsilon\to 0} \sqrt{\epsilon}\mathrm{Tr}\frac{\mathrm{d}}{\mathrm{d}r}(U_r A U_r^{-1})|_{r=0}\, \mathrm{e}^{-\epsilon((U_r A U_r^{-1})^2_{\Pi_>})|_{r=0}}. \quad (2.11)$$

The contribution due to the first term on the right side of (2.10) is equal to 0. This follows from (1.2). Hence, we only have to study the trace of the operator.

$$-f'(u)\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \mathrm{e}^{-\epsilon((U_r A U_r^{-1})^2_{\Pi_>})}|_{r=0}.$$

The function $f'(u)$ is nonzero only for $\frac{1}{8} \leqslant u \leqslant \frac{7}{8}$ and we can use Duhamel's Principle and replace the original heat kernel by the heat kernel of the Dirac operator $\Gamma(\partial_u + B)$ on the cylinder $(-\infty, +\infty) \times Y$. Now we obtain the formula

$$\frac{\mathrm{d}}{\mathrm{d}r}\eta_{(U_r A U_r^{-1})_{\Pi_>}}|_{r=0}$$
$$= -\frac{2}{\sqrt{\pi}} \lim_{\epsilon\to 0} \sqrt{\epsilon}\cdot \mathrm{Tr}\frac{\mathrm{d}}{\mathrm{d}r}(U_r A U_r^{-1})|_{r=0}\, \mathrm{e}^{-\epsilon((U_r A U_r^{-1})^2_{\Pi_>})}|_{r=0}$$
$$= \frac{2}{\sqrt{\pi}} \lim_{\epsilon\to 0} \sqrt{\epsilon}\cdot \mathrm{Tr}\, f'(u)\begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \mathrm{e}^{-\epsilon(-\partial_u^2 + B^2)}$$
$$= \frac{2}{\sqrt{\pi}} \lim_{\epsilon\to 0} \sqrt{\epsilon}\frac{1}{\sqrt{4\pi\epsilon}}\mathrm{Tr}\, f'(u)\alpha\, \mathrm{e}^{-\epsilon B^{+}B^{-}}$$
$$= \frac{1}{\pi}\int_0^1 \mathrm{d}u \lim_{\epsilon\to 0} \mathrm{Tr}\, f'(u)\alpha\, \mathrm{e}^{-\epsilon B^{+}B^{-}} = \frac{1}{\pi}\mathrm{Tr}\,\alpha. \quad (2.12)$$

In fact, using the unitary twist, we are able to show that $(\mathrm{d}/\mathrm{d}r)\eta_{(U_r A U_r^{-1})_{\Pi_>}(0)} = 1/\pi\, \mathrm{Tr}\,\alpha$ for any $0 \leqslant r \leqslant 1$. We use a similar argument to show that we can replace $\Pi_>$ by the Calderon projection and also to show that the integral depends only on the end-points of the family. Details will appear in [15].

Acknowledgements

We are grateful to Graeme Segal for originally suggesting to us the idea of a pasting formula for the determinant in terms of a bilinear pairing on Fock space.

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