

FUNCTORIAL QFT, GAUGE ANOMALIES AND THE DIRAC DETERMINANT BUNDLE

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ABSTRACT. Using properties of the determinant line bundle for a family of elliptic boundary value problems, we explain how the Fock space functor defines an axiomatic quantum field theory which formally models the Fermionic path integral. The ‘sewing axiom’ of the theory arises as an algebraic pasting law for the determinant of the Dirac operator. We show how representations of the boundary gauge group fit into this description and that this leads to a Fock functor description of certain gauge anomalies.

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1. INTRODUCTION

Advances in the construction of topological invariants for low-dimensional manifolds using methods from gauge theory have led to a great deal of interest in the construction of quantum field theories as modified cohomology theories [1, 22, 23, 25, 26]; that is, as generalized functors from manifolds to vector spaces. The purpose of this paper is to explain a construction of a functorial quantum field theory (FQFT) using the Fock functor, generalizing a construction suggested by Segal [22] in (1+1)-dimensions. This may be of particular interest in view of recent developments in the theory of branes in superstring theory. In doing so, we realize the higher-dimensional gauge group representations of [14] in terms of a $d+1$ -dimensional FQFT, while the gluing law of the FQFT arises as an algebraic pasting law for the determinant of a Dirac operator with respect to a partition of the underlying manifold.

The aim of FQFT is to abstract the algebraic structure that the path integral would create if it existed as a rigorous mathematical object. With respect to a partition of the underlying manifold the functoriality formally encodes intriguing formal gluing laws for spectral or topological invariants realized as expectation values. The prototypical situation we consider is for a family of chiral Dirac operators over an even-dimensional manifold with closed odd-dimensional boundary. The parameter space \mathcal{A} in this case is an affine space of gauge potentials cross Riemannian metrics, acting on which one has a group \mathcal{G} of gauge transformations. To a spin manifold X with boundary Y endowed with an admissible decomposition $H_Y = W \oplus W^\perp$ of the space of boundary spinor fields, the Fock functor associates a Fock space \mathcal{F}_W of *holomorphic* sections of a relative determinant line bundle over the restricted Grassmannian defined by the polarization W . Globally the functor associates a bundle \mathcal{F} of Fock spaces to the parameter space \mathcal{A} and the sewing properties of the determinant line bundle for a family of elliptic boundary value problems explained in [20] translate into the required functorial properties of the FQFT. The constructions of [14] arise in this situation in terms of two “orthogonal” \mathcal{G} -anomalies. First, there is the “bulk” even-dimensional chiral anomaly measuring the obstruction to a \mathcal{G} -equivariant determinant regularization. Second, associated to the gauge group on the boundary one has the odd-dimensional Mickelsson-Faddeev commutator anomaly. In the FQFT context the \mathcal{G} -action is lifted from \mathcal{A} to a projective *bundle map* on the Fock bundle, rather than an automorphism of the whole (fixed) space of sections of the determinant line bundle [14]. FQFT formally encodes the relation between the path integral and Hamiltonian approaches to second quantization, the Fock functor FQFT

we consider provides a coherent framework in which to describe simultaneously the path integral (determinant line) description of gauge anomalies and their Hamiltonian (Fock space) realization. We hope this may serve to clarify some of the underlying mathematical structures of the QFT.

In the remainder of the Introduction we recall from [1, 23, 26] the axiomatic characterization of a QFT and the heuristic path integral formulae this aims to encode. In Section 2 we explain some facts about determinant bundles and Fock spaces associated to families of elliptic boundary value problems. In Section 3 we define the Fock space functor in general and outline its fundamental properties. In Section 4 we apply this to the interactive Yang-Mills gauge theory associated to \mathcal{A} and discuss the boundary gauge group action and the chiral anomaly and commutator anomalies. In Section 5 we outline the path integral formulae for an elliptic boundary value problem which the Fock functor aims to model, and present a concrete $0 + 1$ -dimensional example which relates our constructions to the finite-dimensional Fermionic (Berezin) integral.

1.1. Axiomatic QFT. A $(d+1)$ -dimensional FQFT means a functor from the category C_d of d -dimensional closed manifolds and cobordisms to the category of vector spaces and linear maps, which satisfies certain natural axioms suggested by path integral formulae. A morphism in C_d between d -dimensional manifolds Y_0, Y_1 is a $(d + 1)$ -dimensional manifold X with boundary $\partial X = Y_0 \sqcup Y_1$. The orientation on the ‘incoming’ boundary Y_0 is assumed to be induced by the orientation of X and the inward directed normal vector field on the boundary, whereas for the ‘outgoing’ boundary Y_1 the orientation is fixed by the outward directed normal vector field. Let C_{vect} denote the category whose objects are topological vector spaces and whose morphisms are homomorphisms.

A $d + 1$ -dimensional FQFT means a functor $Z : C_d \rightarrow C_{vect}$ assigning to each d -dimensional manifold Y a vector space $Z(Y)$ and to each cobordism X a vector $Z_X \in Z(\partial X)$. By fiat $Z(\emptyset) = \mathbb{C}$, so that if X is closed then Z_X is a complex number. Z is required to satisfy the following axioms.

For $(d + 1)$ -dimensional manifolds X, X_0, X_1 and d -dimensional manifolds Y, Y_0, Y_1 :

A1. Multiplicativity:

$$Z_{X_0 \sqcup X_1} = Z_{X_0} \otimes Z_{X_1}, \quad Z(Y_0 \sqcup Y_1) = Z(Y_0) \otimes Z(Y_1).$$

A2. Duality: If \bar{Y} denotes Y with reversed orientation then

$$Z(\bar{Y}) = Z(Y)^*.$$

A3. Associativity: If $M = X_0 \cup_Y X_1$ with $\partial X_0 = \overline{Y_0} \sqcup Y$ and $\partial X_1 = \overline{Y} \sqcup Y_1$, then

$$Z_M = Z_{X_1} \circ Z_{X_0}.$$

A4. Hermitian: $Z_{\overline{X}} = \overline{Z_X}$.

The associativity property refers to the fact that axioms A1 and A2 mean that a cobordism $X \in C_d$ induces a linear transformation $Z_X \in C_{vect}$ through the identifications

$$Z_X \in Z(\overline{Y_0} \sqcup Y_1) = Z(\overline{Y_0}) \otimes Z(Y_1) = Z(Y_0)^* \otimes Z(Y_1) = \text{Hom}(Z(Y_0), Z(Y_1)).$$

Thus morphisms in C_d are taken to morphisms in C_{vect} . In particular, we have then a canonical pairing

$$Z(\overline{Y_0}) \otimes Z(Y) \otimes Z(\overline{Y}) \otimes Z(Y_1) \longrightarrow Z(\overline{Y_0}) \otimes Z(Y_1).$$

When $Y_0 = Y_1 = \emptyset$, so M is closed, this becomes a pairing

$$(1.1) \quad (,) : Z(Y) \otimes Z(\overline{Y}) \longrightarrow \mathbb{C},$$

and A3 implies the *sewing property*

$$(1.2) \quad Z_M = (Z_{X_0}, Z_{X_1}).$$

This is perhaps the most striking feature of a FQFT, it states that by partitioning the manifold M into ‘simpler’ codimension 0 submanifolds, the number Z_M can be computed by evaluating over the submanifolds and then sewing together the results via the bilinear pairing. The bilinear pairing further implies that if $\partial X = Y \sqcup \overline{Y}$ and $f : Y \rightarrow Y$ is an orientation reversing diffeomorphism, then

$$(1.3) \quad \text{Tr}(Z_X(f)) = Z_{X_f}.$$

Here X_f is the closed manifold obtained by identifying the boundary components via f , and $Z_X(f) \in \text{End}(Z(Y))$ is induced by functoriality (and in (1.3) is implicitly assumed to be trace class).

The Hermitian axiom A4 applies to unitary FQFTs. This means there is a Hermitian structure $\langle , \rangle : Z(Y) \otimes \overline{Z(Y)} \longrightarrow \mathbb{C}$, and hence a canonical isomorphism $Z(Y)^* \cong \overline{Z(Y)}$. A4 is the corresponding expected behaviour of Z_X .

These axioms are idealized, and in practice some modifications are needed. This is illustrated in the FQFT we consider in Section 3.

1.2. Heuristic Path Integral Formulae. The above framework aims to algebraicize the relation between the Feynman path integral formulation of QFT and its Hilbert space formulation. The following heuristic interpretation is useful to bear in mind. If X has connected boundary, the vector Z_X represents the partition function, which is given by a formal path integral

$$(1.4) \quad Z_X : \mathcal{E}(Y) \longrightarrow \mathbb{C}, \quad Z_X(f) = \int_{\mathcal{E}_f(X)} e^{-S(\psi)} \mathcal{D}\psi,$$

where $\mathcal{D}\psi$ is a formal measure. Here $S : \mathcal{E}_f(X) \rightarrow \mathbb{C}$ is an action functional on a space of fields on X , which for definiteness we shall take to be the space of C^0 functions on X , with boundary value $f \in \mathcal{E}(Y)$. The vector space $Z(Y)$ is a space of functions on $\mathcal{E}(Y)$ and forms the Hilbert space of the theory, and Z_X is the vacuum state.

To a cobordism $X \in C_d$ with $\partial X = \overline{Y}_0 \sqcup Y_1$ one has $f = (f_0, f_1)$, and then

$$Z_X(f_0, f_1) = \int_{\mathcal{E}_{(f_0, f_1)}(X)} e^{-S(\psi)} \mathcal{D}\psi,$$

is the kernel of the linear operator $Z_X \in \text{Hom}(Z(Y_0), Z(Y_1))$ defined by

$$Z_X(\xi_0)(f_1) = \int_{\mathcal{E}(Y_0)} Z_X(f_0, f_1) \xi_0(f_0) \mathcal{D}f_0.$$

If $Y_0 = Y_1 = Y$ we hence obtain the bilinear form corresponding to (1.1):

$$(\xi_0, \xi_1) = \int_{\mathcal{E}(Y)} \xi_1(f) Z_X(\xi_0)(f) \mathcal{D}f.$$

In the case of a closed manifold $M = X_0 \cup_Y X_1$ we can express the space of C^0 functions on M as a fibre product $\mathcal{E}(M) = \mathcal{E}(X_0) \times_{\mathcal{E}(Y)} \mathcal{E}(X_1)$ and so formally one expects an equality

$$(1.5) \quad \begin{aligned} \int_{\mathcal{E}(M)} e^{-S(\psi)} \mathcal{D}\psi &= \int_{\mathcal{E}(Y)} \mathcal{D}f \int_{\mathcal{E}_f(X_0)} e^{-S(\psi_0)} \mathcal{D}\psi_0 \int_{\mathcal{E}_f(X_1)} e^{-S(\psi_1)} \mathcal{D}\psi_1 \\ &= \int_{\mathcal{E}(Y)} Z_{X_0}(f) Z_{X_1}(f) \mathcal{D}f. \end{aligned}$$

which is the path integral version of the algebraic sewing formula (1.2). The Hamiltonian of the theory is defined by the Euclidean time evolution operator $e^{-tH} = Z_{Y \times [0, t]} \in \text{End}(Z(Y))$, and to compute the trace one has the integral formulae

$$\text{Tr}(e^{-tH}) = \int_{\mathcal{E}(Y)} \text{Tr} Z_X(f, f) \mathcal{D}f,$$

and corresponding to (1.3)

$$(1.6) \quad \mathrm{Tr}(e^{-tH(f)}) = \int_{\mathcal{E}(X_f)} e^{-S(\psi)} \mathcal{D}\psi.$$

The sewing formula (1.5) says that the partition function on M is the vacuum-vacuum expectation value calculated from the partition functions on the two halves. Equivalently: the invariant Z_M is obtained from $Z_{X_0}(f)$ and $Z_{X_1}(f)$ by integrating (‘averaging’) away the choice of boundary data f . In the case of determinants of Dirac operators this formalism provides some insight to sewing formulae relative to a partition of the underlying manifold (see Section 5). First, we need to review some facts about determinants and Fock bundles for families of Dirac operators.

2. DETERMINANT LINE BUNDLES AND FOCK SPACES

The determinant of a family of first-order elliptic operators arises canonically not as a function, but as a section of a complex line bundle called the determinant line bundle. The anomalies we shall discuss may be realized as obstructions to constructing appropriate trivializations of that bundle. Equivalently, we can view the determinant line of an operator as a ray in the associated Fock space (via the Plücker embedding), and globally the determinant bundle as rank 1 subbundle of an infinite-dimensional Fock bundle to which the gauge group lifts as a projective bundle map.

First, recall the construction of the determinant line bundle for a family of Dirac-type operators over a closed compact manifold X . Such a family can be specified by a smooth fibration of manifolds $\pi : M \rightarrow B$ with fibre diffeomorphic to X , endowed with a Riemannian metric $g_{M/B}$ along the fibres and a vertical bundle of Clifford modules $S(M/B)$ which we may identify with the vertical spinor bundle tensored with an external vertical gauge bundle ξ . We assume that ξ is endowed with a Hermitian structure with compatible connection. The manifold B is not required to be compact. We refer to this data as a *geometric fibration*.

Associated to a geometric fibration one has a smooth elliptic family of Dirac operators $\mathbb{D} = \{D_b \mid b \in B\} : \mathcal{H} \rightarrow \mathcal{H}$, where $\mathcal{H} = \pi_*(S(M/B))$ is the infinite-dimensional Hermitian vector bundle on B whose fibre at b is the Frechet space of smooth sections $\mathcal{H}_b = C^\infty(M_b, \mathcal{S}_b)$, and \mathcal{S}_b a bundle of Clifford modules. If X is even-dimensional there is a \mathbb{Z}_2 bundle grading $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ into positive and negative chirality fields and we then have a family of chiral Dirac operators $\mathbb{D} : \mathcal{H}^+ \rightarrow \mathcal{H}^-$. The Quillen determinant line bundle $\mathrm{DET}(\mathbb{D})$ is a complex line bundle over B with fibre at b canonically isomorphic to the complex line $\mathrm{Det}(\mathrm{Ker} D_b)^* \otimes \mathrm{Det} \mathrm{Coker}(D_b)$ [5, 18], where for a finite-dimensional vector space V , $\mathrm{Det} V$ is the complex line $\wedge^{\max} V$.

The bundle structure is defined relative to the covering of B by open subsets U_λ , with $\lambda \in \mathbb{R}_+$, parameterising those operators D_b for which λ is not in the spectrum of the Laplacian $D_b^*D_b$. Over each U_λ are smooth finite-rank vector bundles H_λ^+, H_λ^- equal to the sum of eigenspaces of $D_b^*D_b$ (resp. of $D_bD_b^*$) for eigenvalues less than λ , and one defines

$$(2.1) \quad \text{DET}(\mathbb{D})|_{U_\lambda} = \text{Det}(H_\lambda^+)^* \otimes \text{Det} H_\lambda^-.$$

The locally defined line bundles patch together naturally over the overlaps $U_\lambda \cap U_{\lambda'}$ [4]. This ‘spectral’ construction of the determinant line bundle is designed to allow one to define the Quillen ζ -function metric and a compatible connection whose curvature R^ζ is identified with the 2-form component of the Bismut family’s index density:

$$(2.2) \quad R^\zeta = (2\pi i)^{-n/2} \left[\int_{M/B} \widehat{A}(M/B) \text{ch}(\xi) \right]_2,$$

where $\widehat{A}(M/B)$ is the vertical A-hat form and $\text{ch}(\xi)$ the Chern character, see [18, 5, 4].

There is, however, a natural alternative construction of the determinant line bundle, due to Segal [24, 17], and applied to Dirac families in [20], which allows us to consider more general smooth families of Fredholm operators, which need not be elliptic operators. Let $\alpha : H^0 \rightarrow H^1$ be a Fredholm operator of index zero. Then a point of the complex line over α is an equivalence class $[A, \lambda]$ of pairs (A, λ) , where the invertible operator $A : H^0 \rightarrow H^1$ is such that $A^{-1}\alpha \in \text{End}(H^0)$ is trace-class, $\lambda \in \mathbb{C}$, and the equivalence relation is defined by $(Aq, \lambda) \sim (A, \det_F(q)\lambda)$ for $q \in \text{End}(H^0)$ an operator of the form identity plus trace-class, and \det_F denotes the Fredholm determinant. If $\text{ind } \alpha = d$ we define $\text{Det}(\alpha) := \text{Det}(\alpha \oplus 0)$ with $\alpha \oplus 0$ acting $H^0 \rightarrow H^1 \oplus \mathbb{C}^d$ if $d > 0$, or $H^0 \oplus \mathbb{C}^{-d} \rightarrow H^1$ if $d < 0$. Note that, by definition, a Fredholm operator of index zero has an approximation by an invertible operator A such that $A - \alpha$ has finite rank. We work with the larger ideal of trace-class operators in order to be able to use the (complete) topology determined by the trace norm.

With the above notation, the abstract determinant of α is defined to be the canonical element $\det \alpha := [A, \det_F(A^{-1}\alpha)] \in \text{Det}(\alpha)$. For an admissible smooth family of Fredholm operators $\mathcal{A} = \{\alpha_b \mid b \in B\} : \mathcal{H}^0 \rightarrow \mathcal{H}^1$ acting between (weak) vector bundles \mathcal{H}^i [20, 24], the union $\text{DET}(\mathcal{A})$ of the determinant lines is naturally a complex line bundle. The bundle structure is defined relative to a denumerable covering of open sets U_τ , where $\tau : \mathcal{H}^0 \rightarrow \mathcal{H}^1$ is finite-rank and U_τ parameterizes those b for which $A_b = \alpha_b + \tau_b$ is invertible, via the local trivialization $b \mapsto \det(\alpha_b + \tau_b)$ over U_τ . On the intersection $U_{\tau_1} \cap U_{\tau_2}$ the transition function is $b \rightarrow \det_F((\alpha_b + \tau_b^1)^{-1}(\alpha_b + \tau_b^2))$.

For a family of elliptic operators, such as \mathbb{D} , there is a canonical isomorphism between the two constructions of the determinant bundle described above which preserves the determinant section $b \mapsto \det D_b$, and we may therefore use them interchangeably [20].

This is important when we consider the determinant line bundle for a family of elliptic boundary value problems (EBVPs). To define such a family we proceed initially as for the case of a closed manifold with a geometric fibration $\pi : M \rightarrow B$ of connected manifolds, but with fibre diffeomorphic to a compact connected manifold X with boundary $\partial X = Y$. Note that the boundary manifolds ∂M and ∂X may possibly be disconnected. Globally we obtain as before a family of Dirac operators $\mathbb{D} = \{D_b \mid b \in B\} : \mathcal{H} \rightarrow \mathcal{H}$. We assume that the geometry in a neighbourhood $\mathcal{U} \equiv \partial M \times [0, 1]$ of the boundary is a pull-back of the geometry induced on the boundary geometric fibration of closed boundary manifolds $\partial\pi : \partial M \rightarrow B$. This means that all metrics and connections on $T(M/B)$ and $S(M/B)$ restricted to U are geometric products composed of the trivial geometry in the normal u -coordinate direction, and the boundary geometry in tangential directions, so $g_{M/B} = du^2 + g_{\partial M/B}$ and so forth. In $\mathcal{U}_b := U \equiv \partial X_b \times [0, 1]$ the Dirac operator D_b then has the form

$$(2.3) \quad D_{b|U} = G_b \left(\frac{\partial}{\partial u} + D_{Y,b} \right),$$

where $D_{Y,b}$ is a boundary Dirac operator and G_b is a unitary bundle automorphism. The family of boundary Dirac operators $\mathbb{D}_Y = \{D_{Y,b} \mid b \in B\} : \mathcal{H}_Y \rightarrow \mathcal{H}_Y$, where \mathcal{H}_Y is the bundle with fibre $C^\infty(Y_b, S_{Y_b})$ at $b \in B$, is identified with family defined by the fibration $\partial\pi : \partial M \rightarrow B$.

In contrast to the closed manifold case, the operators D_b are not Fredholm. The crucial analytical property underlying the following determinant line and Fock bundle identifications is the existence of a canonical identification between the infinite-dimensional space $\text{Ker}(D_b)$ of solutions to the Dirac operator and the boundary traces $K(D_b) = \gamma \text{Ker}(D_b)$, where $\gamma : C^\infty(X_b, S_b) \rightarrow C^\infty(Y_b, S_{Y_b})$ is the operator restricting sections to the boundary. More precisely, the *Poisson operator* $\mathcal{K}_b : C^\infty(Y_b, S_{Y_b}) \rightarrow C^\infty(X_b, S_b)$ restricts to define the above isomorphism. It extends to a continuous operator $\mathcal{K}_b : H^{s-1/2}(Y_b; S|Y_b) \rightarrow H^s(X_b; S)$ on the Sobolev completions with range

$$\text{Ker}(D_b, s) = \{f \in H^s(X_b; S) \mid D_b f = 0 \text{ in } X_b \setminus Y_b\},$$

and the restriction

$$(2.4) \quad \mathcal{K}_b^s : K(D_b, s) \rightarrow \text{Ker}(D_b, s)$$

is an isomorphism (see [6, 11]). The Poisson operator of D_b defines the *Calderon projection*:

$$(2.5) \quad P(D_b) = \gamma \mathcal{K}_b^1.$$

$P(D_b)$ is a pseudodifferential projection on $L^2(Y_b, S_{Y_b})$ which we can take to be orthogonal with range equal to $\text{Ker}(D, s)$. The construction depends smoothly on the parameter $b \in B$ and so globally we obtain a smooth map $P(\mathbb{D}) : B \rightarrow \text{End}(\mathcal{H}_Y)$ defining, equivalently, a smooth Frechet subbundle $K(\mathbb{D})$ of \mathcal{H}_Y with fibre $K(D_b) = \text{ran}(P(D_b))$. Because of the tubular boundary geometry, $P(D_b)$ in fact differs from the APS spectral projection Π_b by only a smoothing operator [19, 11]. Recall that there is a polarization $H_{Y_b} = H_b^+ \oplus H_b^-$ into the non-negative and negative energy modes of the elliptic self-adjoint boundary Dirac operator $D_{Y,b}$ and Π_b is defined to be the orthogonal projection onto H_b^+ . Hence $P(D_b)$ is certainly an element of the pseudodifferential Grassmannian $Gr_b = Gr(H_{Y_b})$ parameterizing projections on $H_{Y,b}$ which differ from Π_b by a pseudodifferential operator of order $< -\dim M/2$, where by projection we mean self-adjoint idempotent. Gr_b is a dense submanifold of the Hilbert-Schmidt Grassmannian. Associated to $P \in Gr_b$ we have the elliptic boundary value problem (EBVP) for D_b

$$(2.6) \quad D_{P,b} = D_b : \text{dom}(D_{P,b}) \rightarrow L^2(X_b; S^1)$$

with domain $\text{dom } D_{P,b} = \{s \in H^1(X_b; S_b^0) \mid P(s|_{Y_b}) = 0\}$. The operator $D_{P,b}$ is Fredholm with kernel and cokernel consisting of smooth sections, see [6] for a general account of EBVPs in index theory. The smooth family of EBVPs $D_{Gr_b} := \{D_{P,b} \mid P \in Gr_b\}$ defines a smooth admissible family of Fredholm operators and hence an associated determinant line bundle $\text{DET}(D_{Gr_b}) \rightarrow Gr_b$. On the other hand, for each choice of a basepoint $P_0 \in Gr_b$ we have the smooth family of Fredholm operators

$$\{P_{W_0, W} := P \circ P_0 : W_0 \rightarrow W \mid P \in Gr_b\},$$

where $\text{ran}(P_0) = W_0, \text{ran}(P) = W$, and hence a relative (Segal) determinant line bundle $\text{DET}_{W_0} \rightarrow Gr_b$ based at W_0 . The bundles so defined for different choices of basepoint are all isomorphic, but not quite canonically. More precisely, from [22, 20], given $P_0, P_1 \in Gr_b$ there is a canonical line bundle isomorphism

$$(2.7) \quad \text{DET}_{W_0} \cong \text{DET}_{W_1} \otimes \text{DET}(W_0, W_1),$$

where $\text{DET}(W_0, W_1)$ means the trivial line bundle with fibre the relative determinant line $\text{DET}(W_0, W_1) := \text{Det}(P_{W_0, W_1})$. The identification (2.4) defines a canonical line bundle isomorphism

$$(2.8) \quad \text{DET}(D_{Gr_b}) \cong \text{DET}_{K_b},$$

preserving the determinant sections

$$\det(D_{Gr_b}) \longleftrightarrow \det(S_b(P)),$$

where $S_b(P) := PP(D_b) : K(D_b) \rightarrow \text{ran}(P)$. (see [20]).

The translation of these facts into global statements follows by observing that the restricted Grassmannians Gr_b , defined for each $b \in B$, fit together to define a fibration $\mathcal{G}r_Y \rightarrow B$. A *spectral section* (or, *Grassmann section* [20]) $\mathbb{P} = \{P_b : b \in B\}$ for the family \mathbb{D} is defined to be a smooth section of $\mathcal{G}r_Y$, and we denote the space of such sections by $Gr(M/B)$. By cobordism, such sections always exist. In particular, the family of Dirac operators \mathbb{D} defines canonically the Calderon section $P(\mathbb{D}) \in Gr(M/B)$. In this sense one may think of the parameter space B as a generalized Grassmannian (parameterizing the subspaces $K(D_b)$), and the usual Grassmannian as a ‘universal moduli space’. Notice, however, that the map $b \rightarrow \Pi_b$ is generically not a smooth spectral section because of the flow of eigenvalues of the boundary family. Indeed, it is this elementary fact that is the source of gauge anomalies, see [15] and Section 4. A spectral section has a number of consequences for determinants:

First: We obtain a smooth family of EBVPs

$$(\mathbb{D}, \mathbb{P}) = \{D_{P,b} := (D_b)_{P_b} \mid b \in B\}$$

which has an associated determinant line bundle $\text{DET}(\mathbb{D}, \mathbb{P}) \rightarrow B$ with determinant section $b \mapsto D_{P,b}$.

Second: A spectral section \mathbb{P} defines a smooth infinite-dimensional vector bundle \mathcal{W} with fibre $W_b = \text{ran}(P_b)$, and associated to \mathbb{P} we have the smooth family of Fredholm operators

$$\mathbb{S}(\mathbb{P}) : K(\mathbb{D}) \rightarrow \mathcal{W},$$

parameterizing the operators $S(P_b) := P_{K(D_b), W_b} : K(D_b) \rightarrow W_b$. This also has a determinant line bundle $\text{DET}(\mathbb{S}(\mathbb{P}))$, and corresponding to (2.8), there is a canonical line bundle isomorphism

$$(2.9) \quad \text{DET}(\mathbb{D}, \mathbb{P}) \cong \text{DET}(\mathbb{S}(\mathbb{P})), \quad \det(D_{P,b}) \longleftrightarrow \det(S(P_b)),$$

preserving the determinant sections. Given a pair of sections $\mathbb{P}_1, \mathbb{P}_2 \in Gr(M/B)$ there is the smooth family of admissible Fredholm operators $(\mathbb{P}^1, \mathbb{P}^2) : \mathcal{W}^1 \rightarrow \mathcal{W}^2$, and corresponding to (2.7) and (2.9) one finds a canonical isomorphism

$$(2.10) \quad \text{DET}(\mathbb{D}, \mathbb{P}^1) \cong \text{DET}(\mathbb{D}, \mathbb{P}^2) \otimes \text{DET}(\mathbb{P}^2, \mathbb{P}^1),$$

which does not preserve the determinant sections [20].

Third: We obtain a bundle of Fock spaces $\mathcal{F}_{\mathbb{P}}$ over B . To see this, return for a moment to the case of a single operator and its Grassmannian Gr_b .

By choosing a basepoint $P_0 \in Gr_b$, we obtain the determinant line bundle $\text{DET}_{W_0} \rightarrow Gr_b$. This is a holomorphic line bundle, but has no global holomorphic sections. The dual bundle $\text{DET}_{W_0}^*$, on the other hand, has an infinite-dimensional space of holomorphic sections, and this, by definition, is the Fock space based at W_0 :

$$(2.11) \quad \mathcal{F}_{W_0,b} := \Gamma_{hol}(Gr_b; \text{DET}_{W_0}^*).$$

This is the natural quantization defined by geometric (Kähler) quantization. Actually, a Fock space comes together with a vacuum vector and a representation of the canonical anticommutation relations; we shall return to this at the end of the section. Taking the union $\mathcal{F}_b := \cup_{W \in Gr_b} \mathcal{F}_{W,b}$ we obtain the Fock bundle over Gr_b . This bundle is topologically completely determined by ‘the’ determinant bundle DET_{W_0} , in fact this is the most direct way to define the bundle structure on \mathcal{F}_b . To be precise, if we change the basepoint we find, dropping the b subscript, a canonical isomorphism

$$\begin{aligned} \mathcal{F}_{W_1} &= \Gamma_{hol}(Gr; \text{DET}_{W_1}^*) \\ &\cong \Gamma_{hol}(Gr_b; \text{DET}_{W_0}^* \otimes \text{DET}(W_1, W_0)^*) \\ &\cong \Gamma_{hol}(Gr_b; \text{DET}_{W_0}^*) \otimes \text{DET}(W_1, W_0)^* \\ &\cong \mathcal{F}_{W_0} \otimes \text{DET}(W_0, W_1), \end{aligned}$$

where we use (2.7). Hence relative to a basepoint $W_0 \in Gr_b$ we have a canonical isomorphism

$$(2.12) \quad \mathcal{F}_b \cong \mathcal{F}_{W_0} \otimes \text{DET}_{W_0},$$

where the first factor on the right-side is the trivial bundle with fibre \mathcal{F}_{W_0} . Hence the topological type of the Fock bundle \mathcal{F}_b is determined by that of the determinant line bundle DET_{W_0} for any basepoint W_0 . One moves between the isomorphisms for different basepoints via (2.7).

As an abstract vector bundle, a Fock bundle is always trivial (but not necessarily canonically); this is because of the fact that (according to Kuiper’s theorem) the unitary group in an infinite-dimensional Hilbert space is contractible. However, as already mentioned, the Fock spaces are equipped with additional structure, the vacuum vectors related to a choice of a family of (Dirac) Hamiltonians, which will modify this statement. In the case of (2.12) we have a preferred line bundle (the ‘vacuum bundle’) inside of the Hilbert bundle and the structure group is reduced giving a nontrivial Fock bundle; this will be discussed in more detail in section 4.

Now as we let b vary we obtain a vertical Fock bundle $\mathcal{F}(M/B)$ over the total space $\mathcal{G}r_Y$ of the Grassmann fibration, which restricted to the fibre Gr_b of $\mathcal{G}r_Y$ coincides with \mathcal{F}_b . The bundle structure is obvious from the local triviality of the fibration $\mathcal{G}r_Y \rightarrow B$. A spectral section \mathbb{P} is a smooth

cross section of that fibration, and hence by pull-back we get a Fock bundle over B associated to \mathbb{P} :

$$(2.13) \quad \mathcal{F}_{\mathbb{P}} := \mathbb{P}^*(\mathcal{F}(M/B)) \longrightarrow B,$$

with fibre $\mathcal{F}_{P_b} = \Gamma_{hol}(Gr_b; \text{DET}_{W_b}^*)$, $W_b = \text{ran}(P_b)$ at $b \in B$. In the following we may at times also write $\mathcal{F}_{\mathbb{P}} = \mathcal{F}_{\mathcal{W}}$, where $\mathcal{W} \rightarrow B$ is the bundle associated to \mathbb{P} . Moreover, from the equivalences above we get that the various Fock bundles are related in the following way.

Proposition 2.1. *For spectral sections $\mathbb{P}^1, \mathbb{P}^2 \in \mathcal{G}r(M/B)$, there is a canonical isomorphism of Fock bundles*

$$(2.14) \quad \mathcal{F}_{\mathbb{P}^1} \cong \mathcal{F}_{\mathbb{P}^2} \otimes \text{DET}(\mathbb{P}^2, \mathbb{P}^1).$$

Notice, in a similar way to $\mathcal{F}_{\mathbb{P}}$, we can also identify the determinant line bundle $\text{DET}(\mathbb{P}^1, \mathbb{P}^2)$ as a pull-back bundle. For, associated to the section \mathbb{P}^1 we have a vertical determinant line bundle $\text{DET}_{\mathbb{P}^1} \rightarrow \mathcal{G}r_Y$, which restricts to $\text{DET}_{W_b^1}$ over Gr_b , where $W_b^1 = \text{ran}(P_b^1)$. Then $\text{DET}(\mathbb{P}^1, \mathbb{P}^2) = (\mathbb{P}^2)^*(\text{DET}_{\mathbb{P}^1})$. In particular, associated to the family \mathbb{D} of Dirac operators parameterized by B , we have the canonical spectral section $P(\mathbb{D})$, and by (2.9) we have a canonical isomorphism $\text{DET}(\mathbb{D}, \mathbb{P}) \cong \mathbb{P}^*(\text{DET}_{P(\mathbb{D})})$. At the Fock space level we have a Fock bundle $\mathcal{F}_{\mathbb{D}}$ canonically associated to the family \mathbb{D} , independently of an extrinsic choice of spectral section, whose fibre at $b \in B$ is $\mathcal{F}_{D_b} := \Gamma_{hol}(Gr_b; \text{DET}(D_{Gr_b})^*)$. From (2.8) and (2.14) we obtain the Fock space version of (2.9) and (2.10):

Proposition 2.2. *There is a canonical isomorphism of Fock bundles*

$$(2.15) \quad \mathcal{F}_{\mathbb{D}} \cong \mathcal{F}_{P(\mathbb{D})}.$$

For $\mathbb{P} \in \mathcal{G}r(M/B)$:

$$(2.16) \quad \mathcal{F}_{\mathbb{D}} \cong \mathcal{F}_{\mathbb{P}} \otimes \text{DET}(\mathbb{D}, \mathbb{P}).$$

Thus the topology of Fock bundle and the determinant line bundle are intimately related. This is the topological reason relating the Schwinger terms in the Hamiltonian anomaly to the index density.

The Fock space \mathcal{F}_{W_0} based at $W_0 \in Gr_b$ can be thought of more concretely in terms of equivariant functions on the Stiefel frame bundle over Gr_b . To describe this we fix an orthonormal basis of eigenvectors of $D_{Y,b}$ in $H_{Y,b}$ such that $e_i \in H_b^-$ for $i \leq 0$ and $e_i \in H_b^+$ for $i > 0$. A point in the fibre of the Stiefel bundle St_b based at H_b^+ over W in the index zero component of Gr_b is a linear isomorphism $\xi : H_b^+ \rightarrow W$ such that $\Pi_b \circ \xi : H_b^+ \rightarrow H_b^+$ has a Fredholm determinant. ξ is also referred to as an ‘admissible basis’ for W (relative to H_b^+), in so far as it transforms $e_i, i > 0$ to a basis for W . ξ can be thought of as a matrix $\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$ with columns labeling the

elements of the basis and rows the coordinates in the standard basis e_i . If ξ, ξ' are two admissible bases for W then $\xi' = \xi.g$ where g is an element of the restricted general linear group Gl^1 consisting of invertible linear maps $g : H_b^+ \rightarrow H_b^+$ such that $g - I$ is trace-class. Gl^1 acts freely on $St_b \times \mathbb{C}$ by $(\xi, \lambda).g = (\xi.g, \lambda \det_F(g)^{-1})$ and we obtain the alternative construction of

$$(2.17) \quad \text{DET}_{H_b^+} = St_b \times_{Gl^1} \mathbb{C}.$$

Similarly, for $W \in Gr_b$ we have $\text{DET}_W = St_W \times_{Gl^1} \mathbb{C}$, where St_W is the corresponding frame bundle based at W . In particular, notice that an isomorphism $St_{W_0} \rightarrow St_{W_1}$ is specified by an invertible operator $A : W_0 \rightarrow W_1$ such that $P_1 P_0 - A$ is trace-class, from which we once again have the identification (2.7).

In this description, an element of the Fock space \mathcal{F}_{W_b} is a holomorphic function $\psi : St_{W_b} \rightarrow \mathbb{C}$ transforming equivariantly under the Gl^1 action as $\psi(\xi.g) = \psi(\xi) \det_F(g)$. The distinguished element

$$(2.18) \quad \nu_{W_b}(\xi) = \det_F(P_{W_b} \xi)$$

is the *vacuum vector*. Equivalently, an element of \mathcal{F}_{W_b} is a holomorphic function $f : \text{DET}_{W_b} \rightarrow \mathbb{C}$ which is linear on each fibre, and from this view point the vacuum vector is the function

$$(2.19) \quad \nu_{W_b}([\alpha, \lambda]) = \lambda \det_F(P_{W_b} \alpha),$$

for any representative (α, λ) of the equivalence class $[\alpha, \lambda] \in \text{Det}(W_b, W)$.

A generalization of this leads to the Plücker embedding. First, for $W \in Gr_b$, fix an orthonormal basis $\{e_i\}_{i \in \mathbb{Z}}$ of H_b such that $e_i \in W^\perp$ for $i \leq 0$ and $e_i \in W$ for $i > 0$; Let \mathcal{S} be the set of all increasing sequences of integers $S = (i_1, i_2, \dots)$ with $S - \mathbb{N}$ and $\mathbb{N} - S$ finite. For each sequence S we have an admissible basis $\xi(S) = \{e_{i_1}, e_{i_2}, \dots\} \in St_W$, and the Fredholm index of the operator $P_S P_W : W \rightarrow H_S$, where H_S is the closed subspace spanned by $\xi(S)$ and P_S the corresponding orthogonal projection, defines a bijection $\pi_0(Gr_b) \rightarrow \mathbb{Z}$. The Plücker coordinates of the basis $\omega \in St_W$ are the collection of complex numbers $\psi_S(\omega) = \det_F(P_S \omega) = \det_F(\omega_S)$, where ω_S is the matrix formed from the rows of $\omega = \begin{pmatrix} \omega_+ \\ \omega_- \end{pmatrix}$ labeled by S . In particular $\psi_{\mathbb{N}}(\omega)$ is the coordinate defined by the vacuum vector. If ω is a basis for $W' \in Gr_b$, then the Plücker coordinates of a second admissible basis ω_1 differ from those of ω by the Fredholm determinant of the matrix relating the two bases. The Plücker coordinates therefore define a projective embedding $Gr_b \rightarrow \mathcal{F}_W$. This is prescribed equivalently by the map

$$(2.20) \quad \phi : St_{W_b} \times St_{W_b} \longrightarrow \mathbb{C}, \quad \phi(\tau, \omega) = \det_F(\tau^* \omega) = \det_F(\tau_+^* \omega_+ + \tau_-^* \omega_-),$$

with respect to which the Plücker coordinates are $\psi_S(\omega) = \phi(\xi(S), \omega)$. ϕ is the same thing as the map on the determinant bundle

$$(2.21) \quad g_\phi : \text{DET}_{W_b} \times \text{DET}_{W_b} \longrightarrow \mathbb{C}, \quad g_\phi([\alpha, \lambda], [\beta, \mu]) = \bar{\lambda}\mu \det_F(\alpha^* P_W \beta),$$

where $\alpha : W_b \rightarrow W$, (resp. $\beta : W_b \rightarrow W'$), is antiholomorphic (resp. holomorphic) and antilinear (resp. linear) in the first (resp. second) variable. We then have the Plücker embedding map

$$(2.22) \quad \text{DET}_{W_b} - \{0\} \longrightarrow \mathcal{F}_{W_b}$$

which maps $[\omega, \lambda] \mapsto \bar{\lambda}\phi(\omega, \cdot)$, or, using the Segal definition of the determinant line

$$(2.23) \quad [\alpha, \lambda] \longmapsto \psi_{[\alpha, \lambda]}$$

defined for $[\alpha, \lambda] = [P_W \alpha, \lambda] \in \text{DET}(W_b, W)$ and $\xi : W_b \rightarrow W'$ by

$$(2.24) \quad \psi_{[\alpha, \lambda]}(\xi) = \bar{\lambda} \det_F(\alpha^* \circ \xi) = \bar{\lambda} \det_F(\alpha^* \circ P_W \circ \xi).$$

Notice that

$$(2.25) \quad \det(id_{W_b}) \longmapsto \nu_{W_b},$$

where $id_{W_b} := P_{W_b, W_b}$. The map (2.22) thus defines a projective embedding $Gr_b \rightarrow \mathcal{F}_W$

The map (2.21) restricted to a linear map $\text{DET}_{W_b} \times \text{DET}_{W_b} \rightarrow \mathbb{C}$, defines a canonical metric on DET_{W_b} by $\|[\alpha, \lambda]\|^2 = |\lambda|^2 \det_F(\alpha^* \alpha)$, and globally, via (2.9), we get the *canonical metric* of [20] on $\text{Det}(\mathbb{D}, \mathbb{P})$

$$(2.26) \quad \|\det D_{P_b}\|^2 := g_\phi(S(P_b), S(P_b)) = \det_F(S(P_b)^* S(P_b)).$$

On the other hand, we can use the map ϕ (or g_ϕ) to put a unitary structure on \mathcal{F}_W with respect to which (2.22) is an isometry. To do that we use the fact that any section in \mathcal{F}_W can be written as a linear combination of the $\psi_{[\alpha, \lambda]}$, and set

$$(2.27) \quad \langle \psi_{[\alpha, \lambda]}, \psi_{[\beta, \mu]} \rangle_W = g_\phi([\alpha, \lambda], [\beta, \mu]).$$

In particular, the finite linear combinations of the sections $\psi_S, S \in \mathcal{S}$ are dense in \mathcal{F}_W with respect to the topology of uniform convergence on compact subsets, and one has

$$\langle \psi_S, \psi_{S'} \rangle = \phi(\xi(S), \xi(S')) = \delta_{SS'}.$$

Notice further the identities

$$(2.28) \quad \langle \nu_W, \nu_W \rangle_W = 1 \quad \text{and} \quad \langle \psi_{[\alpha, \lambda]}, \psi_{[\alpha, \lambda]} \rangle_W = \|[\alpha, \lambda]\|^2,$$

the latter being the statement that (2.22) is an isometry. For further details see [17] and [14].

There is a different way of thinking about Fock spaces which is perhaps more familiar to physicists, as an infinite-dimensional exterior algebra (fermionic Fock space). Recall [14] that a polarization W of the Hilbert space H_b fixes a representation of the canonical anticommutation relations (CAR) in a Fock space $\mathcal{F}(H_b, W)$, whose only non-zero anticommutators are

$$(2.29) \quad a^*(v)a(u) + a(u)a^*(v) = \langle u, v \rangle .$$

The defining property of this irreducible representation is that there is a vacuum vector $|W \rangle$ with the property

$$(2.30) \quad a(u)|W \rangle = 0 = a^*(v)|W \rangle \text{ for all } u \in W, v \in W^\perp .$$

One has

$$(2.31) \quad \mathcal{F}(H_b, W) = \wedge(W) \otimes \wedge((W^\perp)^*) = \sum_{d=q-p \in \mathbb{Z}} \wedge^p(W) \otimes \wedge^q((W^\perp)^*) .$$

The vacuum $|W \rangle$ is represented as the unit element in the exterior algebra. For $u \in W$, $a(u)$ corresponds to interior multiplication by u , the creation operator $a^*(u)$ is given by exterior multiplication. For $u \in W^\perp$ the operator $a(u)$ (resp., $a^*(u)$) is given by exterior (resp., interior) multiplication by Ju ; here $J : H \rightarrow H^*$ is the canonical antilinear isomorphism from a complex Hilbert space to its dual. The vacuum $|W \rangle$ has then the characteristic property $a(u)|W \rangle = 0, u \in W$, and $a^*(v)|W \rangle = 0, v \in W^*$.

If we choose a different $W' \in Gr_b$ then there is a complex vacuum line in $\mathcal{F}(H_b, W)$ corresponding to the new polarization W' . The different vacuum lines parameterized by the planes W' form another realization for the determinant bundle DET_W over Gr_b as a subbundle of the trivial Fock bundle with fibre \mathcal{F}_W . The Plücker embedding $\text{DET}_W \rightarrow \mathcal{F}(H_b, W)$ is defined by mapping $(\omega, \lambda) \in St_W \times \mathbb{C}$ to $\lambda \sum_{S \in \mathcal{S}} \det \omega_S \psi_S$, where ω_S is as before. A Hermitian metric on $\mathcal{F}(H_b, W)$ is again defined by $\langle \psi_S, \psi_{S'} \rangle = \delta_{SS'}$. On the other hand, the finite-dimensional matrix identity for $\alpha : \mathbb{C}^m \rightarrow \mathbb{C}^n, \beta : \mathbb{C}^n \rightarrow \mathbb{C}^m$ with $n \leq m$:

$$(2.32) \quad \det(\alpha\beta) = \sum_{(i)} \det(\alpha(i)) \det(\beta(i)),$$

the sum being over all sequences $(i) = \{1 \leq i_1 < i_2 \dots i_n \leq m\}$, with $\alpha(i)$ (resp. $\beta(i)$) the matrix obtained from A (resp. B) by selecting the columns of A (resp. rows of B) labeled by S , implies the pairing of Fock space vectors

$$(2.33) \quad \langle \psi_{[\alpha, \lambda]}, \psi_{[\beta, \mu]} \rangle = \sum_{S \in \mathcal{S}} \psi_{[\alpha, \lambda]}(\xi(S))^* \psi_{[\beta, \mu]}(\xi(S)) .$$

The metrics so defined on the CAR construction $\mathcal{F}(H_b, W)$ and the geometric construction \mathcal{F}_W of the Fock space, then correspond under the algebraic isomorphism defined by associating to each section $\psi_S \in \mathcal{F}_W$ the vector

$$a(e_{i_1}) \dots a(e_{i_p}) a^*(e_{j_1}) \dots a^*(e_{j_q}) |W \rangle \in \mathcal{F}(H_b, W),$$

where $i_1 < i_2 < \dots i_p \leq 0$ is the set of negative indices in the sequence S and $0 < j_1 < j_2 < \dots j_q$ is the set of *missing positive* indices, giving a dense inclusion $\mathcal{F}(H_b, W) \longrightarrow \mathcal{F}_W$.

Returning to the case of a family \mathbb{D} of Dirac-type operators parameterized by a manifold B , if we are given a spectral section $\mathbb{P} \in \mathcal{G}r(M/B)$ then we have the global version of the above properties. Associated to \mathbb{P} we have a Fock bundle $\mathcal{F}_{\mathcal{W}} \rightarrow B$ and this is endowed with a unitary structure $\langle \cdot, \cdot \rangle_{\mathbb{P}}$, given on the fibre \mathcal{F}_{W_b} by (2.27). The bundle $\mathcal{F}_{\mathcal{W}}$ has a distinguished section, the *vacuum section* $\nu_{\mathbb{P}} = \nu_{\mathcal{W}}$, assigning to $b \in B$ the vacuum vector ν_{W_b} , with unit norm in the fibres. Associated to the canonical Calderon section $P(\mathbb{D})$ defining the Fock bundle $\mathcal{F}_{\mathcal{K}}$ we then have a determinant line bundle $\text{DET}(\mathbb{D}, \mathbb{P}) \cong \text{DET}(\mathbb{S}(\mathbb{P}))$ and a generalized Plücker embedding

$$(2.34) \quad \text{DET}(\mathbb{D}, \mathbb{P}) \cong \text{DET}(\mathbb{S}(\mathbb{P})) \longrightarrow \mathcal{F}_{\mathcal{K}},$$

corresponding to the view point on the parameter space B as a generalized Grassmannian. More generally, for any pair of spectral sections $\mathbb{P}_1, \mathbb{P}_2$, there is a generalized Plücker embedding

$$(2.35) \quad \text{DET}(\mathbb{P}_1, \mathbb{P}_2) \longrightarrow \mathcal{F}_{\mathbb{P}_1},$$

defined fibrewise by the embeddings $\text{Det}(W_{1,b}, W_{2,b}) \hookrightarrow \text{DET}_{W_{1,b}} \rightarrow \mathcal{F}_{W_{1,b}}$, which according to (2.22) is the map $[\alpha, \lambda] \mapsto \psi_{[\alpha, \lambda]} \in \mathcal{F}_{\mathbb{P}_1}$. So, a section of the determinant bundle $\text{DET}(\mathbb{P}_1, \mathbb{P}_2)$ defines a section of the Fock bundle $\mathcal{F}_{\mathbb{P}_1}$. In particular, the vacuum section is the image of the determinant section in the ‘trivial’ case

$$(2.36) \quad \text{DET}(\mathbb{P}_1, \mathbb{P}_1) \longrightarrow \mathcal{F}_{\mathbb{P}_1},$$

Associated to the family of Dirac operators \mathbb{D} , we have a canonical vacuum section $\nu_{\mathcal{K}} \in \mathcal{F}_{\mathcal{K}}$ with $\langle \nu_{K_b}, \nu_{K_b} \rangle_{K_b} = 1$, and if we choose an external spectral section \mathbb{P} , then via (2.34) we have a canonical section $\psi_{\mathcal{K}, \mathbb{P}}$ of $\mathcal{F}_{\mathcal{K}}$ corresponding to the determinant section $b \mapsto \det(\mathbb{D}_{P_b}) \leftrightarrow \det(S(P_b))$ of $\text{DET}(\mathbb{D}, \mathbb{P})$, with

$$(2.37) \quad \langle \psi_{\mathcal{K}, \mathbb{P}}, \psi_{\mathcal{K}, \mathbb{P}} \rangle_{K_b} = \|\det(S(P_b))\|^2.$$

That is, the generalized Plücker embedding (2.34) is an isometry with respect to the canonical metric on $\text{DET}(\mathbb{D}, \mathbb{P})$. This follows by construction from (2.28).

As we already mentioned, as an abstract vector bundle the Fock bundle is trivial. However, the non-triviality of the construction lies in the (locally defined) physical vacuum subbundle defined by the family of Hamiltonians. As an example, assume that we have a family of Dirac Hamiltonians parameterized by the set \mathcal{A} of smooth vector potentials. Given a real number λ we can define $W_0(A)$ as the subspace of the boundary Hilbert space corresponding to the spectral restriction $D_{Y,A} > \lambda$ for the boundary Hamiltonian; $A \mapsto W_0(A)$ is a smooth Grassmann section over the set $U_\lambda \subset \mathcal{A}$ of Hamiltonians with $\lambda \notin \text{Spec}(D_{Y,A})$. Let $A \mapsto W_1(A)$ be a globally defined Grassmann section. For each $A \in U_\lambda$ we have a well-defined vacuum line $|A\rangle \in \mathcal{F}_{W_1(A)}$. This line is just the image of the determinant line $\text{DET}(W_1(A), W_0(A))$ with respect to the map (2.22). If $\dim Y = 1$ the Grassmannian Gr_A does not depend on the parameter A and we may take $W_1(A)$ as a constant section. Anyway, the bundle of vacua over U_λ can be identified as the relative determinant bundle $\text{DET}(W_1, W_0)$ and the twisting of this bundle depends solely on the twisting of the local section $A \mapsto W_0(A)$.

3. CONSTRUCTION OF THE FQFT

In this section we utilize the facts presented in the previous section to piece together a FQFT, generalized from the two dimensional case proposed by Segal [22]. As in Section 1.1, the constructions are mathematical and do not refer to any particular physical system. In the next section we explain how the chiral anomaly and commutator anomaly arise in this context.

3.1. Strategy. We define a projective functor from a subcategory \mathcal{C}_d of the category of spin manifolds to the category \mathcal{C}_{vect} of \mathbb{Z} -graded vector spaces and linear maps, which factors through the category \mathcal{C}_{Gr} of *linear relations*:

$$\begin{array}{ccc} \mathcal{C}_d & \longrightarrow & \mathcal{C}_{Gr} \\ & \searrow & \downarrow \\ & & \mathcal{C}_{vect} \end{array}$$

The combination of these functors is the Fock functor defining the FQFT.

3.2. Projective representations of categories. By a category \mathcal{C} we mean a set $\text{Ob}(\mathcal{C})$ of elements called the objects of \mathcal{C} , and for any two elements $a, b \in \text{Ob}(\mathcal{C})$ a set $\text{Mor}_{\mathcal{C}}(a, b)$ of morphisms $a \mapsto b$, such that for $a, b, c \in \text{Ob}(\mathcal{C})$ there is a multiplication defined

$$\text{Mor}_{\mathcal{C}}(a, b) \times \text{Mor}_{\mathcal{C}}(b, c) \longrightarrow \text{Mor}_{\mathcal{C}}(a, c), \quad (f_{a,b}, f_{b,c}) \longmapsto f_{b,c} \circ f_{a,b}.$$

The product is required to be associative, so that if $f_{c,d} \in \text{Mor}_{\mathcal{C}}(c, d)$, then $f_{c,d}(f_{b,c}f_{a,b}) = (f_{c,d}f_{b,c})f_{a,b}$. One usually also asserts the existence of an

identity morphism $id_b \in \text{Mor}_{\mathcal{C}}(b, b)$ which satisfies $id_b \circ f_{a,b} = f_{a,b}$ and $f_{b,c} \circ id_b = f_{b,c}$.

A (covariant) *functor* Ψ from a category \mathcal{C} to a category \mathcal{C}' means a map $\Psi : \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{C}')$ and for each pair $a, b \in \text{Ob}(\mathcal{C})$ a map

$$\Psi_{a,b} : \text{Mor}_{\mathcal{C}}(a, b) \rightarrow \text{Mor}_{\mathcal{C}'}(\Psi(a), \Psi(b))$$

such that

$$(3.1) \quad \Psi_{a,c}(f_{b,c}f_{a,b}) = \Psi_{b,c}(f_{b,c})\Psi_{a,b}(f_{a,b}).$$

If \mathcal{C}' is the category of vector spaces and linear maps, then Ψ is a *representation* of the category \mathcal{C} . Hence, from the view point of FQFT, quantization may be roughly characterized as a representation of the category \mathcal{C}_d of manifolds defined by classical geometric data.

A classical result of Wigner tells us that in quantum systems we must content ourselves with projective representations of symmetry groups. Similarly, with the Fock functor we have to consider projective category representations. This means that there is essentially a scalar ambiguity in defining the map $\Psi_{a,b}$, so that (3.1) is replaced by

$$(3.2) \quad \Psi_{a,c}(f_{b,c}f_{a,b}) = c(f_{b,c}, f_{a,b})\Psi_{b,c}(f_{b,c})\Psi_{a,b}(f_{a,b}),$$

where the ‘cocycle’ $c(f_{b,c}, f_{a,b})$ takes values in $\mathbb{C} - \{0\}$. To explain the meaning here of ‘essentially’, recall that a projective representation of a group G is a true representation of an extension group \hat{G} of G by \mathbb{C}^\times . The group \hat{G} forms a \mathbb{C}^\times bundle over G whose Lie algebra cocycle is the first Chern class of the associated line bundle. Equivalently, \hat{G} is defined by assigning to each $g \in G$ a complex line L_g such that $L_{g_1g_2} = L_{g_1} \otimes L_{g_2}$. (A well-known instance of this occurs for loop groups, see Example 4.1 below, and more generally we shall in Section 4 give the gauge group representations for a Yang-Mills action functional a similar description.) Likewise, a projective representation of a category is a true representation of an extension category $\hat{\mathcal{C}}$ constructed by assigning to each $f_{a,b} \in \text{Mor}_{\mathcal{C}}(a, b)$ a complex line $L_{f_{a,b}}$, and given $f_{b,c} \in \text{Mor}_{\mathcal{C}}(b, c)$, an identification

$$(3.3) \quad L_{f_{a,b}} \otimes L_{f_{b,c}} \longrightarrow L_{f_{a,b}f_{b,c}},$$

which is associative in the natural sense. One then has

$$(3.4) \quad \text{Ob}(\hat{\mathcal{C}}) = \text{Ob}(\mathcal{C}), \quad \text{Mor}_{\hat{\mathcal{C}}}(a, b) = \{(f, \lambda) \mid f \in \text{Mor}_{\mathcal{C}}(a, b), \lambda \in L_f\}.$$

3.3. The category \mathcal{C}_d . An element of $\text{Ob}(\mathcal{C}_d)$ is a pair (\mathcal{Y}, W) , where $\mathcal{Y} = (Y, g_Y, S_Y \otimes \xi_Y)$, with Y is a closed, smooth and oriented d -dimensional spin manifold, g_Y a Riemannian metric on Y , S_Y a spinor bundle over Y , ξ_Y a Hermitian bundle over Y with compatible gauge connection, and W is an admissible polarization of the ‘one-particle’ Hilbert space $H_Y = W \oplus W^\perp$ to a pair of closed infinite-dimensional subspaces. Here $H_Y = L^2(Y, S_Y \otimes$

ξ_Y) and admissible means that $P_W \in Gr_Y$, where P_W is the orthogonal projection onto W and Gr_Y is the Hilbert-Schmidt Grassmannian defined with respect to the energy polarization $H_Y = H^+ \oplus H^-$ into positive, resp. negative, energies of the Dirac operator D_Y .

Let $(\mathcal{Y}_i, W_i) \in \text{Ob}(\mathcal{C}_d)$, $i = 1, 2$, where $\mathcal{Y}_i = (Y_i, g_{Y_i}, S_{Y_i} \otimes \xi_{Y_i})$. An element of $\text{Mor}_{\mathcal{C}_d}((\mathcal{Y}_1, W_1), (\mathcal{Y}_2, W_2))$ is a triple $\mathcal{X} = (X, g_X, S_X \otimes \xi_X)$, where X smooth and oriented $(d+1)$ -dimensional spin manifold with boundary $\partial X = Y_1 \sqcup Y_2$, g_X a Riemannian metric on X with $(g_X)|_{Y_i} = g_{Y_i}$, S_X a spinor bundle and ξ_X a Hermitian bundle over X with compatible gauge connection, such that $(S_X \otimes \xi_X)|_{Y_i} \cong S_{Y_i} \otimes \xi_{Y_i}$ and the connections metrics correspond under the isomorphism. We refer to \mathcal{X} as a *geometric cobordism* from \mathcal{Y}_1 to \mathcal{Y}_2 . We assume that:

- In a collar neighbourhood of the boundary $U = U_1 \sqcup U_2$, where $U_i = ([0, 1] \times Y_i)$ all metrics, connections are of product-type. Recall this means that near the boundary the metric becomes the product of the standard metric on the real axis and the boundary metric. Similarly, the gauge connection approaches smoothly the connection on the boundary such that at the boundary all the normal derivatives vanish. Thus $\xi_{X|U_i}$ is a pull-back of the boundary bundle (ξ_{Y_i}) , and similarly all metrics, connections, etc are pull-backs of their boundary counterparts, so $g_{X|U_i} = du^2 + g_{Y_i}$ etc.
- The orientation on the ‘ingoing’ boundary Y_1 is assumed to be induced by the orientation of X and the inward directed normal vector field on the boundary, whereas for the ‘outgoing’ boundary Y_2 the orientation is fixed by the outward directed normal vector field.

For notational brevity we may write $S_i := S_{Y_i} \otimes \xi_{Y_i}$, $g_i := g_{Y_i}$ etc, and $S_{1,2} := S_X \otimes \xi_X$, $g_{1,2} := g_X$ etc, in the following.

We augment \mathcal{C}_d by including the empty set $\emptyset \in \text{Ob}(\mathcal{C}_d)$, and for each $(\mathcal{Y}, W) \in \text{Ob}(\mathcal{C}_d)$ we also allow $\emptyset := id$ as an element of $\text{Mor}_{\mathcal{C}_d}((\mathcal{Y}, W), (\mathcal{Y}, W))$. In particular, a geometric cobordism $\mathcal{X} \in \text{Mor}_{\mathcal{C}_d}(\emptyset, (\mathcal{Y}, W))$ means $d + 1$ -dimensional manifold X with boundary Y (plus bundles, connections etc). Thus a morphism in \mathcal{C}_d may have disconnected, connected, or empty (X is closed) boundary, according as Y is disconnected, connected or empty.

For $(\mathcal{Y}_i, W_i) \in \text{Ob}(\mathcal{C}_d)$, $i = 1, 2, 3$, there is an associative product map

$$(3.5) \quad \begin{aligned} \text{Mor}_{\mathcal{C}}((\mathcal{Y}_1, W_1), (\mathcal{Y}_2, W_2)) \times \text{Mor}_{\mathcal{C}}((\mathcal{Y}_2, W_2), (\mathcal{Y}_3, W_3)) \\ \longrightarrow \text{Mor}_{\mathcal{C}}((\mathcal{Y}_1, W_1), (\mathcal{Y}_3, W_3)) \end{aligned}$$

taking a pair $(\mathcal{X}_{1,2}, \mathcal{X}_{2,3})$ to the geometric cobordism

$$(3.6) \quad \mathcal{X}_{1,2} \cup_{Y_2} \mathcal{X}_{2,3} = (X_{1,2} \cup_{Y_2} X_{2,3}, g_{1,2} \cup g_{2,3}, S_{1,2} \cup_{\sigma} S_{2,3}).$$

This ‘sewing together’ of bundles is defined in the usual way, the crucial point being that all geometric data in the collar is pulled back from the

boundary and is hence compatible under sewing of manifolds. Briefly, the collar neighbourhood $U_r = [0, 1) \times Y_2$ of the boundary of Y_2 in $X_{2,3}$ is a copy of the collar neighbourhood $U_l = (-1, 0] \times Y_2$ of the boundary of Y_2 in $X_{1,2}$ but with orientation reversed. Hence we may glue together the manifolds $X_{1,2}$ and $X_{2,3}$ along Y_2 to get the ‘doubled’ manifold $X_{1,2} \cup_{Y_2} X_{2,3}$ with a tubular neighbourhood of the partition Y_2 which we may parameterize as $U = (-1, 1) \times Y_2$. Associated to the geometric data we have Dirac operators $D_{1,2}$ and $D_{2,3}$ acting respectively on sections of the Clifford bundles $S_{1,2}$ and $S_{2,3}$. Over U_l the operator $D_{1,2}$ takes the product form $\sigma(\partial/\partial u + D_{Y_2})$, because of the change of orientation, and hence chirality, $(D_{2,3})|_{U_r} = \sigma^{-1}(\partial/\partial v + \sigma D_{Y_2})\sigma^{-1}$. Over Y_2 we construct $S_{1,2} \cup_\sigma S_{2,3}$ by gluing $S_{1,2}$ to $S_{2,3}$ via the unitary isomorphism σ , identifying $s \in (S_{1,2})|_Y$ with $\sigma s \in (S_{2,3})|_Y$. (Thus σ takes positive spinors to negative spinors.) A section of $S_{1,2} \cup_\sigma S_{2,3}$ is a pair (ψ, ϕ) with ψ (resp. ϕ) is a smooth section of $S_{1,2}$ (resp. of $S_{2,3}$) such that the normal derivatives of all orders match-up: $\frac{\partial^k}{\partial u^k} \psi(0, y) = (-1)^k \sigma(y) \frac{\partial^k}{\partial v^k} \phi(0, y)$. We then have the Dirac-type operator $(D_{1,2} \cup D_{2,3})(\psi, \phi) = (D_{1,2}\psi, D_{2,3}\phi)$ acting on $C^\infty(X_{1,2} \cup_{Y_2} X_{2,3}, S_{1,2} \cup_\sigma S_{2,3})$, associated to the induced geometric data on $X_{1,2} \cup_{Y_2} X_{2,3}$.

3.4. The category \mathcal{C}_{Gr} . An element of $\text{Ob}(\mathcal{C}_{Gr})$ is a pair (H, W) with H a Hilbert space, and W a polarization of H into a pair of closed orthogonal infinite-dimensional subspaces $H = W \oplus W^\perp$. A morphism $(E, \epsilon) \in \text{Mor}_{\mathcal{C}_{Gr}}((\overline{H}_1, W_1^\perp), (H_2, W_2))$ is a closed subspace $E \subset \overline{H}_1 \oplus H_2$ such that $P_E - P_{W_1^\perp \oplus W_2}$ is a Hilbert-Schmidt operator, where $P_E, P_{W_1^\perp \oplus W_2}$ are the orthogonal projections with range $E, W_1^\perp \oplus W_2$ respectively, along with an element ϵ of the relative determinant line $\text{Det}(W_1^\perp \oplus W_2, E)$. (It is convenient here to use the ‘reverse’ polarization W_1^\perp of H_1 in order to account for boundary orientations later on, see below). Thus there is an identification

$$(3.7) \quad \text{Mor}_{\mathcal{C}_{Gr}}((H_1, W_1^\perp), (H_2, W_2)) = \text{DET}_{W_1^\perp \oplus W_2},$$

where the right-side is the determinant line *bundle* based at $W_1^\perp \oplus W_2$ over the trace-class Grassmannian $Gr(\overline{H}_1 \oplus H_2)$. Here \overline{H}_1 serves to remind us that we are considering the reverse polarization W_1^\perp ; we may write

$$Gr(\overline{H}_1 \oplus H_2) = Gr(\overline{H}_1 \oplus H_2, W_1^\perp \oplus W_2), \quad \text{DET}_{W_1^\perp \oplus W_2} = \text{DET}_{W_1^\perp \oplus W_2}(\overline{H}_1 \oplus H_2)$$

if we wish to emphasize the polarization. We also allow $\emptyset \in \text{Ob}(\mathcal{C}_{Gr})$ as an object, and define

$$(3.8) \quad \begin{aligned} \text{Mor}_{\mathcal{C}_{Gr}}(\emptyset, (H, W)) &= \text{DET}_W(H), \\ \text{Mor}_{\mathcal{C}_{Gr}}((\overline{H}, W^\perp), \emptyset) &= \text{DET}_{W^\perp}(\overline{H}), \\ \text{Mor}_{\mathcal{C}_{Gr}}(\emptyset, \emptyset) &= \mathbb{C}. \end{aligned}$$

To define the product of morphisms in \mathcal{C}_{Gr} , first recall when $H_0 \neq \emptyset \neq H_2$, from the ‘category of linear relations’, the ‘join’ product rule

$$(3.9) \quad Gr(\overline{H}_0 \oplus H_1, W_0^\perp \oplus W_1) \times Gr(\overline{H}_1 \oplus H_2, \widetilde{W}_1^\perp \oplus W_2) \longrightarrow Gr(\overline{H}_0 \oplus H_2, W_0^\perp \oplus W_2),$$

$$(E_{01}, E_{12}) \longmapsto E_{01} * E_{12},$$

where $\widetilde{W}_1 \in Gr(H_1, W_1)$, defined by

$$E_{01} * E_{12} = \{(u, v) \in H_0 \oplus H_2 \mid \exists w \in H_1 \text{ such that } (u, w) \in E_{01}, (w, v) \in E_{12}\}.$$

The join is a generalized composition law of graphs of linear operators, but here the morphisms E are not in general everywhere defined, but $\text{dom}(E) = \text{range}(P_{H_1} P_E : E \rightarrow H_1)$, and may also be ‘multi-valued’. The composition may therefore be discontinuous. From [22] we recall that for continuity one requires that: (i) the map $E_{01} \oplus E_{12} \rightarrow H_1$, $((u, w), (w', v)) \mapsto w - w'$ is surjective, and (ii) $E_{01} \oplus E_{12} \rightarrow H_0 \oplus H_1 \oplus H_2$, $((u, w), (w', v)) \mapsto (u, w - w', v)$ is injective. The crucial fact is the following:

Proposition 3.1. *With the above notation, when $(\overline{H}_0, W_0^\perp) = \emptyset = (H_2, W_2)$ there is a canonical pairing, linear and holomorphic on the fibres in the first and second variables,*

$$(3.10) \quad \kappa : \text{DET}_{W_1} \times \text{DET}_{\widetilde{W}_1^\perp} \longrightarrow \text{Det}(W_1, \widetilde{W}_1).$$

If $W_1 = \widetilde{W}_1$, then

$$(3.11) \quad \kappa : \text{DET}_{W_1} \times \text{DET}_{W_1^\perp} \longrightarrow \mathbb{C}.$$

More generally, if (i) and (ii) hold, then one has such a pairing

$$(3.12) \quad \kappa : \text{DET}_{W_0^\perp \oplus W_1}(\overline{H}_0 \oplus H_1) \times \text{DET}_{\widetilde{W}_1^\perp \oplus W_2}(\overline{H}_1 \oplus H_2) \longrightarrow$$

$$\text{DET}_{W_0^\perp \oplus W_2}(\overline{H}_0 \oplus H_2) \otimes \text{Det}(W_1, \widetilde{W}_1),$$

which respects the join multiplication: on each fibre

$$(3.13) \quad \kappa : \text{Det}(W_0^\perp \oplus W_1, E_{01}) \times \text{Det}(\widetilde{W}_1^\perp \oplus W_2, E_{12}) \longrightarrow$$

$$\text{Det}(W_0^\perp \oplus W_2, E_{01} * E_{12}) \otimes \text{Det}(W_1, \widetilde{W}_1).$$

(Here the second factor on the right-side of (3.12) denotes the trivial bundle with fibre $\text{Det}(W_1, \widetilde{W}_1)$.) If $W_1 = \widetilde{W}_1$, then

$$(3.14) \quad \kappa : \text{DET}_{W_0^\perp \oplus W_1}(\overline{H}_0 \oplus H_1) \times \text{DET}_{W_1^\perp \oplus W_2}(\overline{H}_1 \oplus H_2)$$

$$\longrightarrow \text{DET}_{W_0^\perp \oplus W_2}(\overline{H}_0 \oplus H_2).$$

Proof. As before, we denote by $P_{W,W'}$ the orthogonal projection onto W restricted to the subspace W' . Given $E \in Gr(H_1, W_1)$, $E' \in Gr(\widetilde{H}_1, \widetilde{W}_1^\perp) = Gr(\widetilde{H}_1, W_1^\perp)$, we can represent elements $\epsilon \in \text{Det}(W_1, E)$ and $\delta \in \text{Det}(\widetilde{W}_1^\perp, E')$ as the determinant elements of linear operators $a_\epsilon : W_1 \rightarrow E$ and $b_\delta : \widetilde{W}_1^\perp \rightarrow E'$ with $P_{W_1}a_\epsilon - id_{W_1}$ and $P_{\widetilde{W}_1^\perp}b_\delta - id_{\widetilde{W}_1^\perp}$ trace-class. We define

$$(3.15) \quad \kappa : \text{Det}(W_1, E) \times \text{Det}(\widetilde{W}_1^\perp, E') \longrightarrow \text{Det}(W_1, \widetilde{W}_1),$$

by

$$(3.16) \quad \kappa(\epsilon, \delta) = \det(P_1 a_\epsilon + \widetilde{P}_1^\perp b_\delta) \in \text{Det}(W_1 \oplus \widetilde{W}_1^\perp, H_1) \cong \text{Det}(W_1, \widetilde{W}_1),$$

where P_1, \widetilde{P}_1 are the projections on W_1, \widetilde{W}_1 , and $\text{Det}(W_1 \oplus \widetilde{W}_1^\perp, H_1)$ is the determinant line of $P_1 + \widetilde{P}_1^\perp : W_1 \oplus \widetilde{W}_1^\perp \rightarrow H_1$. This operator differs from $P_1 a_\epsilon + \widetilde{P}_1^\perp b_\delta$ by an operator of trace-class (so (3.15) is well-defined) because $P_{W_1}a_\epsilon - id_{W_1}$ and $P_{\widetilde{W}_1^\perp}b_\delta - id_{\widetilde{W}_1^\perp}$ are trace-class.

The canonical isomorphism on the right-side of (3.16) is expressed via the diagram of commutative maps with exact rows and Fredholm columns

$$\begin{array}{ccccccc} 0 & \longrightarrow & W_1 & \longrightarrow & W_1 \oplus \widetilde{W}_1^\perp & \longrightarrow & \widetilde{W}_1^\perp \longrightarrow 0 \\ & & & & \downarrow \widetilde{P}_1 P_1 & & \downarrow id \\ 0 & \longrightarrow & \widetilde{W}_1 & \longrightarrow & H_1 & \longrightarrow & \widetilde{W}_1^\perp \longrightarrow 0 \end{array}$$

where the horizontal maps are the obvious ones. Such a diagram defines an isomorphism between the determinant line of the centre map with the tensor product of the lines defined by the outer columns, mapping the determinant elements to each other [22, 20]. Hence since $\text{Det}(id) = \mathbb{C}$ canonically, the isomorphism follows, and with $E = W_1$ and $E' = \widetilde{W}_1^\perp$

$$(3.17) \quad \kappa(\det(id_{W_1}), \det(id_{\widetilde{W}_1^\perp})) = \det(P_{\widetilde{W}_1, W_1}),$$

where $id_W = P_{W,W}$, which will be relevant later in this Section.

For the general case (3.12), suppose initially that $W_1 = \widetilde{W}_1$ and choose $\epsilon \in \text{Det}(W_0^\perp \oplus W_1, E_{01})$ and $\delta \in \text{DET}(W_1^\perp \oplus W_2, E_{12})$ identified with the determinant elements of linear operators $a_\epsilon : W_0^\perp \oplus W_1 \rightarrow E_{01}$ and $b_\delta : W_1^\perp \oplus W_2 \rightarrow E_{12}$. Define

$$(3.18) \quad \kappa_1 : \text{Det}(W_0^\perp \oplus W_1, E_{01}) \times \text{DET}(W_1^\perp \oplus W_2, E_{12}) \longrightarrow$$

$$\text{DET}(W_0^\perp \oplus H_1 \oplus W_2, E_{01} \oplus E_{12}),$$

$$\kappa_1(\epsilon, \delta) = \det(a_\epsilon \oplus b_\delta).$$

On the other hand, from [22], conditions (i) and (ii) mean that there is an exact sequence

$$(3.19) \quad 0 \longrightarrow E_{01} * E_{12} \longrightarrow E_{01} \oplus E_{12} \longrightarrow H_1 \longrightarrow 0,$$

and this fits into the commutative diagram with Fredholm columns

$$\begin{array}{ccccccc}
 0 & \longrightarrow & E_{01} * E_{12} & \longrightarrow & E_{01} \oplus E_{12} & \longrightarrow & H_1 \longrightarrow 0 \\
 & & \downarrow P_{W_0^\perp \oplus W_2} P_{E_{01} * E_{12}} & & \downarrow G & & \downarrow id \\
 0 & \longrightarrow & W_0^\perp \oplus W_2 & \longrightarrow & W_0^\perp \oplus H_1 \oplus W_2 & \longrightarrow & H_1 \longrightarrow 0
 \end{array}$$

where we modify (3.19) by composing the injection $E_{01} * E_{12} \longrightarrow E_{01} \oplus E_{12}$ with the involution $((u, w), (w', v) \rightarrow ((u, w), (-w', v)$, and the following surjection to $((u, w), (w', v) \mapsto (u, w + w', v)$, while the lower maps are again the obvious ones. The central column is

$$G(\xi, \eta) = (P_{W_0^\perp} P_{H_0} \xi, P_{H_1} \xi + P_{H_1} \eta, P_{W_2} P_{H_1} \eta).$$

Because $P_{H_1} P_{E_{01}} - P_{W_1} P_{H_1} P_{E_{01}} = P_{H_1} (P_{E_{01}} - P_{W_0^\perp \oplus W_1}) P_{E_{01}}$ and $P_{H_1} P_{E_{12}} - P_{W_1^\perp} P_{H_1} P_{E_{12}} = P_{H_1} (P_{E_{12}} - P_{W_1^\perp \oplus W_2}) P_{E_{12}}$ are trace-class, the operators G and G_{W_1} , where

$$G_{W_1}(\xi, \eta) = (P_{W_0^\perp} P_{H_0} \xi, P_{W_1} P_{H_1} \xi + P_{W_1^\perp} P_{H_1} \eta, P_{W_2} P_{H_1} \eta),$$

differ by only trace-class operators and so

$$\text{Det}(G) = \text{Det}(G_W) = \text{DET}(E_{01} \oplus E_{12}, W_0^\perp \oplus H_1 \oplus W_2),$$

while from the diagram we have

$$\text{Det}(G) \cong \text{Det}(E_{01} * E_{12}, W_0^\perp \oplus W_2).$$

Thus by duality (i.e. take adjoints in the above diagrams, reversing the order of the columns and rows and the direction of the arrows) we have a canonical isomorphism $\text{Det}(W_0^\perp \oplus W_2, E_{01} * E_{12}) \cong \text{DET}(W_0^\perp \oplus H_1 \oplus W_2, E_{01} \oplus E_{12})$, and so composition with κ_1 completes the proof of (3.12) in the case $W_1 = \widetilde{W}_1$. In the general case, replace H_1 in (3.18) and the lower row of the commutative diagram by $W_1 \oplus \widetilde{W}_1^\perp$ and repeat the argument used in the proof of (3.10). Finally, we note for later reference that in the ‘vacuum case’ $E_{01} = W_0^\perp \oplus W_1$ and $E_{12} = W_1^\perp \oplus W_2$ one has $E_{01} * E_{12} = W_0^\perp \oplus W_2$ and

$$(3.20) \quad \kappa(\det(id_{E_{01}}), \det(id_{E_{12}})) = \det(id_{E_{01} * E_{12}}).$$

□

From (3.14) and the identification (3.7) we now have a canonical multiplication

$$\begin{aligned}
 (3.21) \quad \text{Mor}_{\mathcal{C}_{Gr}}((\overline{H}_0, W_0^\perp), (H_1, W_1)) &\times \text{Mor}_{\mathcal{C}_{Gr}}((\overline{H}_1, W_1^\perp), (H_2, W_2)) \\
 &\longrightarrow \text{Mor}_{\mathcal{C}_{Gr}}((\overline{H}_0, W_0^\perp), (H_2, W_2)),
 \end{aligned}$$

$$(E_{0,1}, \epsilon), (E_{1,2}, \delta) \longmapsto (E_{0,1} * E_{1,2}, \epsilon * \delta),$$

where

$$\epsilon * \delta := \begin{cases} \kappa(\epsilon, \delta) & \text{if (i) and (ii) hold,} \\ 0 & \text{otherwise .} \end{cases}$$

In particular,

$$(3.22) \quad \text{Mor}_{\mathcal{C}_{Gr}}(\emptyset, (H_1, W_1)) \times \text{Mor}_{\mathcal{C}_{Gr}}((H_1, W_1^\perp), \emptyset) \longrightarrow \text{Mor}_{\mathcal{C}_{Gr}}(\emptyset, \emptyset),$$

is precisely equation (3.11).

3.5. The projective functor $\mathcal{C}_d \rightarrow \mathcal{C}_{Gr}$. Define

$$(3.23) \quad \Psi : \text{Ob}(\mathcal{C}_d) \longrightarrow \text{Ob}(\mathcal{C}_{Gr}), \quad (\mathcal{Y}, W) \longmapsto (H_Y, W),$$

where as before $H_Y = L^2(Y, S_Y \otimes \xi_Y)$ and W is an admissible polarization.

While for $(\mathcal{Y}_1, W_1), (\mathcal{Y}_2, W_2) \in \text{Ob}(\mathcal{C}_d)$

$$(3.24) \quad \Psi : \text{Mor}_{\widehat{\mathcal{C}}_d}((\mathcal{Y}_1, W_1), (\mathcal{Y}_2, W_2)) \longrightarrow \text{Mor}_{\mathcal{C}_{Gr}}((\overline{H}_{Y_1}, W_1^\perp), (H_{Y_2}, W_2)) \\ \mathcal{X} \longmapsto (K_{12}, \epsilon),$$

where $K_{12} \subset \overline{H}_{Y_1} \oplus H_{Y_2}$ is the Calderon subspace of boundary ‘traces’ of solutions to the Dirac operator $D^{1,2}$ over X defined by the geometric data in \mathcal{X} , and $\epsilon \in \text{Det}(W_1^\perp \oplus W_2, K_{12}) \cong \text{Det}(D_{P_{W_1^\perp, W_2}}^{1,2})^*$. Taking into account that Y_1 is an incoming boundary, we have $K_{12} \in Gr(\overline{H}_{Y_1} \oplus H_{Y_2}; W_1^\perp \oplus W_2)$ (in fact, an element of the ‘smooth Grassmannian’). The choice needed of the element ϵ means that Ψ is a true functor $\widehat{\mathcal{C}}_d \rightarrow \mathcal{C}_{Gr}$, where $\widehat{\mathcal{C}}_d$ is the extension category of \mathcal{C}_d whose objects are the same as \mathcal{C}_d , and

(3.25)

$$\text{Mor}_{\widehat{\mathcal{C}}_d}(\mathcal{Y}_1, \mathcal{Y}_2) = \{(\mathcal{X}, z) \mid \mathcal{X} \in \text{Mor}_{\mathcal{C}_d}(\mathcal{Y}_1, \mathcal{Y}_2), \epsilon \in \text{Det}(W_1^\perp \oplus W_2, K_{12})\}.$$

For a closed geometric cobordism $\mathcal{X} \in \text{Mor}_{\mathcal{C}_d}(\emptyset, \emptyset)$ we set

$$(3.26) \quad \text{Mor}_{\widehat{\mathcal{C}}_d}(\emptyset, \emptyset) = \text{Det}(D_X),$$

the projectivity of the functor in this case corresponds to a choice of generator for $\text{Det}(D_X)$, defining $\text{Det}(D_X) \cong \mathbb{C} = \text{Mor}_{\mathcal{C}_{Gr}}(\emptyset, \emptyset)$.

To see the functor respects the product rules in each category, it is enough to show that $K_{01} * K_{12}$ is the Calderon subspace of the operator $D^{0,1} \cup D^{1,2}$, i.e. $K(D^{0,1} \cup D^{1,2}) = K(D^{0,1}) * K(D^{1,2})$ defined by morphisms $\mathcal{X}_{0,1}, \mathcal{X}_{1,2}$. This, however, is immediate from the definition of $D^{0,1} \cup D^{1,2}$, and the fact that given $\psi \in \text{Ker } D^{0,1}, \phi \in \text{Ker } D^{1,2}$ it is enough for their boundary values to match up in order to get an element of $\text{Ker}(D^{0,1} \cup D^{1,2})$. That in turn follows because the product geometry in the collar neighbourhood U of the outgoing boundary Y_1 of $X_{0,1}$ implies that ψ has the form $\psi(u, y) = \sum_k e^{-\lambda_k u} \psi_k(0) e_k(y)$, where $\{\lambda_k, e_k\}$ is a spectral resolution of H_{Y_1} defined by the boundary Dirac operator. (To be quite correct, we should also include the identification by the boundary isomorphism $\sigma(y)$ in the definition of

the join $K_{01} * K_{12}$, but this introduces no new phenomena.) Thus the requirement (3.3) for the rule $\mathcal{X} \mapsto \text{Det}(W_1^\perp \oplus W_2, K_{12})$ to define a projective extension of \mathcal{C}_{Gr} , is (3.14) of Proposition 3.1.

3.6. The functor $\mathcal{C}_{Gr} \rightarrow \mathcal{C}_{vect}$. The functor Φ from the category \mathcal{C}_{Gr} to the category \mathcal{C}_{vect} of (\mathbb{Z} -graded) vector spaces and linear maps, is defined on objects of \mathcal{C}_{Gr} by

$$(3.27) \quad (H, W) \longrightarrow \mathcal{F}_W = \mathcal{F}_W(H) \quad (\overline{H}, W^\perp) \longrightarrow \mathcal{F}_{W^\perp} = \mathcal{F}_{W^\perp}(\overline{H}) \\ \emptyset \longrightarrow \mathbb{C},$$

Thus Φ takes a polarized vector space to the Fock space defined by the polarization, and $\mathcal{F}_\emptyset = \mathbb{C}$ is by *fiat*. Here $\mathcal{F}_{W^\perp} = \mathcal{F}_{W^\perp}(\overline{H}) := \Gamma_{hol}(Gr(\overline{H}), \text{DET}_{W^\perp})$ is the Fock space associated with the reverse polarization.

Φ is defined on morphisms as follows. From (3.7), a morphism in

$$\text{Mor}_{\mathcal{C}_{Gr}}((H_1, W_1^\perp), (H_2, W_2))$$

is the same thing as an element $\epsilon \in \text{DET}_{W_1^\perp \oplus W_2}$, which we may think of as the pair (E, ϵ) where $\epsilon \in \text{Det}(W_1^\perp \oplus W_2, E)$. By the Plücker embedding (2.22) this gives us a canonical vector

$$(3.28) \quad \phi_\epsilon \in \mathcal{F}_{W_1^\perp \oplus W_2}(\overline{H}_1 \oplus H_2) \cong \mathcal{F}_{W_1^\perp}(\overline{H}_1) \otimes \mathcal{F}_{W_2}(H_2).$$

The isomorphism is immediate from (2.31) since $\mathcal{F}_W(H)$ is the completion of $\mathcal{F}(H, W)$.

To proceed we need the following facts, generalizing eqn.(8.10) of [22]:

Proposition 3.2. *The determinant bundle pairing κ of Proposition 3.1 defines a canonical Fock space pairing*

$$(3.29) \quad () : \mathcal{F}_{W_1} \times \mathcal{F}_{\widetilde{W}_1^\perp} \longrightarrow \text{Det}(W_1, \widetilde{W}_1),$$

with

$$(3.30) \quad (\nu_{W_1}, \nu_{\widetilde{W}_1^\perp}) = \det(P_{\widetilde{W}_1, W_1}).$$

If $W_1 = \widetilde{W}_1$, this becomes

$$(3.31) \quad () : \mathcal{F}_{W_1} \times \mathcal{F}_{\widetilde{W}_1^\perp} \longrightarrow \mathbb{C}, \quad (\nu_{W_1}, \nu_{W_1^\perp}) = 1.$$

More generally, κ defines a pairing

$$(3.32) \quad \mathcal{F}_{W_0^\perp \oplus W_1}(\overline{H}_0 \oplus H_1) \times \mathcal{F}_{\widetilde{W}_1^\perp \oplus W_2}(\overline{H}_1 \oplus H_2) \\ \longrightarrow \mathcal{F}_{W_0^\perp \oplus W_2}(\overline{H}_0 \oplus H_2) \otimes \text{Det}(W_1, \widetilde{W}_1),$$

with

$$(3.33) \quad (\phi_\epsilon, \phi_\delta) = \phi_{\kappa(\epsilon, \delta)}.$$

In particular,

$$(3.34) \quad (\nu_{W_0^\perp \oplus W_1}, \nu_{\widetilde{W}_1^\perp \oplus W_2}) = \nu_{W_0^\perp \oplus W_2} \otimes \det(P_{W_2, W_0}).$$

Proof. First notice that in the finite-dimensional case there is a natural isomorphism between the Fock space (the exterior algebra) and its dual defined by the pairing $\wedge^k H \times \wedge^{n-k} H \rightarrow \text{Det}(H)$, $(\lambda_1, \lambda_2) \mapsto \lambda_1 \wedge \lambda_2$, while in the infinite-dimensional case the pairing using the CAR construction follows directly from the definition $\mathcal{F}(H, W) = \wedge(W) \otimes \wedge((W^\perp)^*)$. For the geometric Fock space \mathcal{F}_W , the construction of the pairing from the determinant bundle pairing κ on $\text{DET}_W \times \text{DET}_{W^\perp}$ is entirely analogous to the construction of the inner-product $\langle \cdot, \cdot \rangle_W$ on \mathcal{F}_W from the determinant bundle pairing g_ϕ on $\text{DET}_W \times \text{DET}_W$ in equation (2.21). Indeed, in the case of the vacuum elements the two pairings are canonically identified (see (3.36) below and Section 5).

Let us deal first with the case (3.29). We give first the invariant definition, and then the ‘constructive’ definition along the lines of $\langle \cdot, \cdot \rangle_W$ in Section 2. Invariantly, in the case $W_1 = \widetilde{W}_1$, the pairing $\kappa : \text{DET}_{W_1} \times \text{DET}_{W_1^\perp} \rightarrow \mathbb{C}$ defines an embedding $\gamma : \text{DET}_{W_1} - 0 \rightarrow \mathcal{F}_{W_1^\perp}$ by $\gamma(a)(\cdot) = f(a, \cdot)$, and hence a map $\rho : \mathcal{F}_{W_1^\perp}^* \rightarrow \mathcal{F}_{W_1}$, $\rho(f)(\cdot) = f(\gamma(\cdot))$. This gives us a pairing $\mathcal{F}_W^* \times \mathcal{F}_{W_1^\perp}^* \rightarrow \mathbb{C}$ with $(f, g) = f(\gamma(g))$, and by duality the asserted pairing, since $\mathcal{F}_W^{**} \cong \mathcal{F}_W$ in the topology of uniform convergence on compact subsets of $Gr(H)$ ($\psi_S \leftrightarrow \text{evaluation at } \xi(S)$, cf. [17] Sect. 10.2, [12] Sect. 6.2). The general case follows in the same way with \mathbb{C} replaced by $\text{Det}(\widetilde{W}_1, W_1)$.

Constructively, recall that any section in \mathcal{F}_W can be written as a linear combination of the $\psi_{[\alpha, \lambda]}$, with $[\alpha, \lambda] \in \text{DET}_W$. Hence for $[\alpha, \lambda] \in \text{DET}_{W_1}$, $[\beta, \mu] \in \text{DET}_{\widetilde{W}_1^\perp}$ we can define the Fock pairing by setting

$$(3.35) \quad (\psi_{[\alpha, \lambda]}, \psi_{[\beta, \mu]}) = \kappa([\alpha, \lambda], [\beta, \mu]) \in \text{Det}(\widetilde{W}_1, W_1),$$

and then extending by linearity. In particular, from (2.25) we have $\nu_{W_1} = \psi_{[id_{W_1}, 1]}$ and $\nu_{\widetilde{W}_1^\perp} = \psi_{[id_{\widetilde{W}_1^\perp}, 1]}$, and so

$$(\nu_{W_1}, \nu_{\widetilde{W}_1^\perp}) = \kappa(\det(id_{W_1}), \det(id_{\widetilde{W}_1^\perp})) = \det(P_{\widetilde{W}_1, W_1}),$$

where the final equality is equation (3.17). Notice further that if we extend g_ϕ in (2.21) to a map $g_\phi : \text{DET}_{W_1} \times \text{DET}_{\widetilde{W}_1^\perp} \rightarrow \text{Det}(\widetilde{W}_1, W_1)$ by $g_\phi([\alpha, \lambda], [\beta, \mu]) = \bar{\lambda} \mu \det(\alpha^* P_W \beta)$, then the Fock space inner-product becomes a Hermitian pairing $\langle \cdot, \cdot \rangle_{W_1} : \mathcal{F}_{W_1} \times \mathcal{F}_{\widetilde{W}_1^\perp} \rightarrow \text{Det}(\widetilde{W}_1, W_1)$ and with respect to the identification $Gr(H, W_1) \leftrightarrow Gr(\overline{H}_1, W_1^\perp)$, $W \leftrightarrow W^\perp$ we have $\nu_{\widetilde{W}_1^\perp} \leftrightarrow \nu_{\widetilde{W}_1}$ and

$$(3.36) \quad (\nu_{W_1}, \nu_{\widetilde{W}_1^\perp}) = \langle \nu_{W_1}, \nu_{\widetilde{W}_1} \rangle_{W_1} = \det(P_{\widetilde{W}_1, W_1}).$$

The pairing (3.32) now follows from (3.29) and (3.28). Alternatively we can define it directly as $(\psi_{[\alpha,\lambda]}, \psi_{[\beta,\mu]}) = \psi_{\kappa([\alpha,\lambda],[\beta,\mu])}$, where κ is the pairing (3.12). Note that if conditions (i) and (ii) do not hold then $\kappa([\alpha,\lambda],[\beta,\mu]) = 0$. Equation (3.33) is now just by construction, and equation (3.34) follows easily from (3.20). \square

The Fock space pairing (3.31) defines an isomorphism $\mathcal{F}_{W_1^\perp} \cong \mathcal{F}_{W_1}^*$ and hence the vector $\phi_\epsilon \in \mathcal{F}_{W_1^\perp}(\overline{H}_1) \otimes \mathcal{F}_{W_2}(H_2)$ defined by

$$\epsilon \in \text{Mor}_{\mathcal{C}_{Gr}}((H_1, W_1^\perp), (H_2, W_2))$$

is canonically an element of $\text{Hom}(\mathcal{F}_{W_1}(H_1), \mathcal{F}_{W_2}(H_2))$ which is a morphism of \mathcal{C}_{vect} , as required. In the case $(\overline{H}_0, W_0^\perp) = \emptyset$ the map

$$\text{Mor}_{\mathcal{C}_{Gr}}(\emptyset, (H_1, W_1)) \rightarrow \text{Hom}(\mathbb{C}, \mathcal{F}_{W_1})$$

is defined by $1 \mapsto \nu_W$, and similarly when $(H_2, W_2) = \emptyset$. The functoriality of the composition of linear maps with respect to the multiplication in \mathcal{C}_{Gr} is precisely (3.33). We may state this as:

Theorem 3.3. *The category multiplication in \mathcal{C}_{Gr} induces through the Fock space functor a canonical multiplication in the category \mathcal{C}_{vect} .*

This can be conveniently summarized in the statement that the following diagram commutes:

$$\begin{array}{ccc} Gr(\overline{H}_0 \oplus H_1, W_0^\perp \oplus W_1) \times Gr(\overline{H}_1 \oplus H_2, W_1^\perp \oplus W_2) & \xrightarrow{\quad * \quad} & Gr(\overline{H}_0 \oplus H_2, W_0^\perp \oplus W_2) \\ \downarrow (\epsilon, \delta) & & \downarrow \epsilon * \delta \\ \text{DET}_{W_0^\perp \oplus W_1} \times \text{DET}_{W_1^\perp \oplus W_2} & \xrightarrow{\quad \kappa \quad} & \text{DET}_{W_0^\perp \oplus W_2} \\ \downarrow \text{Plucker} & & \downarrow \text{Plucker} \\ \mathcal{F}_{W_0^\perp} \otimes \mathcal{F}_{W_1} \otimes \mathcal{F}_{W_1^\perp} \otimes \mathcal{F}_{W_2} & \xrightarrow{\quad (,) \quad} & \mathcal{F}_{W_0^\perp} \otimes \mathcal{F}_{W_2} \end{array} ,$$

where ϵ, δ are, respectively, a choice of section of the bundles $\text{DET}_{W_0^\perp \oplus W_1}$ and $\text{DET}_{W_1^\perp \oplus W_2}$.

3.7. The Fock Functor. The Fock functor $Z : \mathcal{C}_d \rightarrow \mathcal{C}_{vect}$ is the projective representation of \mathcal{C}_d defined by the composition of the functors Ψ and Φ , thus Z is the functor

$$(3.37) \quad Z = \Phi \circ \Psi : \tilde{\mathcal{C}}_d \longrightarrow \mathcal{C}_{vect}.$$

Z acts on objects of $\tilde{\mathcal{C}}_d$ by

$$(3.38) \quad \begin{aligned} Z : \text{Ob}(\tilde{\mathcal{C}}_d) &\longrightarrow \text{Ob}(\mathcal{C}_{vect}), \\ Z((\mathcal{Y}, W)) &= \mathcal{F}_W(H_Y), \quad Z(\emptyset) = \mathbb{C}, \end{aligned}$$

and on morphisms by

$$(3.39) \quad Z : \text{Mor}_{\bar{\mathcal{C}}_d}((\mathcal{Y}_1, W_1), (\mathcal{Y}_2, W_2)) \longrightarrow \text{Mor}_{\bar{\mathcal{C}}_d}(\mathcal{F}_{W_1}(H_{Y_1}), \mathcal{F}_{W_2}(H_{Y_2})),$$

$$Z((\mathcal{X}, \epsilon)) = \phi_\epsilon, \quad \epsilon \in \text{Det}(W_1^\perp \oplus W_2, K(D_X)),$$

where $(\mathcal{Y}_1, W_1), (\mathcal{Y}_2, W_2)$ are not both empty, and ϕ_ϵ is defined as in Section 3.6 by the Fock space pairing. If $(\mathcal{Y}_1, W_1) = \emptyset = (\mathcal{Y}_2, W_2)$, so X is a closed manifold, then

$$Z(\mathcal{X}) = \det(D_X) \in \text{Det}(D_X) \cong \mathbb{C},$$

where the trivialization requires a choice.

The ‘sewing property’ of the FQFT is precisely the functorial Fock space pairing of Proposition 3.2. Note that if both $W_i \neq \emptyset$ it is not possible to choose ϕ_ϵ to be the vacuum vector $\nu_{W_1^\perp \oplus W_2} \in \mathcal{F}_{W_1^\perp \oplus W_2}$, since $K(D_X)$, depending on global data, is always transverse to the pure boundary data $W_1^\perp \oplus W_2$. Consider though the case $W_1 = \emptyset$. Let X be a closed connected manifold partitioned by an embedded codimension 1 submanifold Y , so that $X = X^0 \cup_Y X^1$. Here X^0, X^1 are manifolds with boundary Y , where $\partial X^0 = \bar{Y}$ has outgoing orientation and $\partial X^1 = Y$ has incoming orientation. X^0 is assumed to be associated to a morphism \mathcal{X}^0 in $\text{Mor}_{\mathcal{C}_d}(\emptyset, (\mathcal{Y}, W))$ for a choice of admissible polarization $W \in \text{Gr}(H_Y)$. In this case we *can* choose $W = K(D^0)$ and $\phi_\epsilon = \nu_{K(D^0)}$. Similarly, we have $\mathcal{X}^1 \in \text{Mor}_{\mathcal{C}_d}((\mathcal{Y}, W^\perp), \emptyset)$, and we may choose $W^\perp = K(D^1)$. As a Corollary of the properties of Z we then have the following algebraic sewing law for the determinant with respect to a partitioned closed manifold.

Theorem 3.4. *There are functorial bilinear pairings*

$$(3.40) \quad (,) : \mathcal{F}_{K(D^0)}(H_Y) \times \mathcal{F}_{W^\perp}(\bar{H}_Y) \longrightarrow \text{Det}(D_{P_W}^0),$$

where the right-side is the determinant line of the EBVP D_P with

$$(3.41) \quad (\nu_{K(D^0)}, \nu_{W^\perp}) = \det(D_P^0),$$

and

$$(3.42) \quad (,) : \mathcal{F}_{K(D^0)}(H_Y) \times \mathcal{F}_{K(D^1)}(\bar{H}_Y) \longrightarrow \text{Det}(D_X),$$

where the right-side is the determinant line of the Dirac operator D_X over the closed manifold X , with

$$(3.43) \quad (\nu_{K(D^0)}, \nu_{K(D^1)}) = \det(D_X).$$

Proof. We just need to recall a couple of facts. From equations (3.29) and (3.30) we have a pairing $\mathcal{F}_{K(D^0)}(H_Y) \times \mathcal{F}_{W^\perp}(\bar{H}_Y) \longrightarrow \text{Det}(K(D^0), W)$ with $(\nu_{K(D^0)}, \nu_{W^\perp}) = \det(S(P_W))$, where $S(P_W) : K(D^0) \rightarrow W$ is the operator of Section 2. But from (2.9) there is a canonical isomorphism $\text{Det}(S(P_W)) \cong \text{Det}(D_{P_W})$, with $\det(S(P_W)) \mapsto \det(D_{P_b})$. This proves the first statement.

The second statement follows similarly upon recalling from [20] (Theorem 3.2) that there is a canonical isomorphism

$$\text{DET}((I - P(D^1)) \circ P(D^0)) \cong \text{Det}(D_X),$$

again preserving the determinant elements. \square

Thus one may think of the determinant $\det(D_P)$ as an object in the complex line $\text{Det}(K(D), P)$ depending on a choice of boundary condition P , or absolutely the ‘quantum determinant’ of D as a ray in the Fock space $\mathcal{F}_{K(D)}$ defined by the vacuum vector that does not depend on a choice of P . The two view points being related by (3.41).

Finally, we point out that, in particular, the Fock functor naturally defines a map from geometric fibrations to vector bundles. To a geometric fibration \mathbf{N} of closed d -dimensional manifolds endowed with a spectral section \mathbb{P} it assigns the corresponding Fock bundle $\mathcal{F}_{\mathbb{P}}$. A ‘projective’ morphism between objects $(\mathbf{N}_1, \mathbb{P}_1)$ and $(\mathbf{N}_2, \mathbb{P}_2)$ is a geometric fibration of \mathbf{M} of $d+1$ -dimensional manifolds with boundary $\mathbf{N}_1 \sqcup \mathbf{N}_2$ along with a section of the determinant bundle $\text{DET}(\mathbb{P}_1^\perp \oplus \mathbb{P}_2, K(\mathbb{D}))$, where \mathbb{D} is the family of Dirac operators defined by \mathbf{M} . This defines a bundle map $\mathcal{F}_{\mathbb{P}_1} \rightarrow \mathcal{F}_{\mathbb{P}_2}$ using the generalized Plucker embedding (2.34) and the Fock space pairing. For a partition of a closed geometric fibration $M = M^0 \cup_N M^1$ over a parameter manifold B by an embedded fibration of codimension 1 manifolds, the analogue of Theorem 3.4 then states that there are functorial Fock bundle pairings:

$$(3.44) \quad (,) : \mathcal{F}_{\mathbb{D}^0} \times \mathcal{F}_{\mathbb{P}^\perp} \longrightarrow \text{DET}(\mathbb{D}^0, \mathbb{P}), \quad (\nu_{\mathbb{D}^0}, \nu_{\mathbb{P}^\perp}) = \det(D_{\mathbb{P}}^0),$$

where the right-side is the determinant line bundle of the family of EBVPs $(\mathbb{D}^0, \mathbb{P})$, while $\nu_{\mathbb{D}^0}, \nu_{\mathbb{P}}$ are the vacuum sections of the Fock bundles $\mathcal{F}_{\mathbb{D}^0}, \mathcal{F}_{\mathbb{P}}$ (see (2.15)), and $\det(D_{\mathbb{P}}^0)$ the determinant section of $\text{DET}(\mathbb{D}^0, \mathbb{P})$; and

$$(3.45) \quad (,) : \mathcal{F}_{\mathbb{D}^0} \times \mathcal{F}_{\mathbb{D}^1} \longrightarrow \text{Det}(\mathbb{D}_{\mathbf{M}}), \quad (\nu_{\mathbb{D}^0}, \nu_{\mathbb{D}^1}) = \det(D_{\mathbf{M}}).$$

The proof again requires only the properties of the Fock bundle pairing and the determinant bundle identifications of Section 2 and [20].

Notice that there is no regularization here of the determinant, only a pairing between bundle sections.

4. GAUGE ANOMALIES AND THE FOCK FUNCTOR

In this Section we give a physical application of these ideas with a Fock functor description of the chiral and commutator anomalies for an even-dimensional manifold with (odd-dimensional) boundary.

The Fock functor assigns vector spaces to all odd-dimensional compact oriented spin manifolds Y and polarizations. There is no further restriction

on the topology of Y . However, in this section we shall restrict to a fixed topological type for Y . For our purposes this is no real restriction since our principal aim is to understand the action of continuous symmetries, diffeomorphisms and gauge transformations, on the family of Fock spaces and on the morphisms between the Fock spaces; the action of the symmetry group cannot change the topological type of Y . To be more concrete, we shall consider the case of the parameter space $B = \mathcal{A}$ of smooth vector potentials labeling the geometries over Y .

Thus we are lead to consider the action of the group of gauge transformations on the bundle \mathcal{F} of Fock spaces over the base \mathcal{A} . The gauge transformations act naturally on the base \mathcal{A} and thus we have a lifting problem: Construct a (projective) action of the gauge group in the total space of \mathcal{F} intertwining with the family of quantized Dirac Hamiltonians in the fibers. We want to stress that we are not going to construct a representation of the gauge group in a single Fock space but we have have a linear isometric action between different fibers of the Fock bundle.

First, we recall some known facts about gauge anomalies in even dimensions. Let M be a closed even-dimensional Riemannian spin manifold and let \mathcal{A} be the space of vector potentials on a trivial complex G -bundle over M . For each $A \in \mathcal{A}$ we have a coupled Dirac operator $D_A : C^\infty(M; S \otimes E) \rightarrow C^\infty(M; S \otimes E)$ given locally by

$$D_A = \sum_{i=1}^n \sigma_i (\partial_i + \Gamma_i + A_i),$$

where Γ_i and A_i are respectively the components of the local spin connection and and G -connection A , and σ_i the Clifford matrices. Since M is even-dimensional, then D_A splits into positive and negative chirality components, and the object of interest is the Chiral Dirac operator

$$D_A^+ = D_A \left(\frac{1 + \gamma_{n+1}}{2} \right) : C^\infty(M; S^+ \otimes E) \rightarrow C^\infty(M; S^- \otimes E).$$

Acting on \mathcal{A} we have the group of based gauge transformations \mathcal{G} , which acts covariantly on the Dirac operators $D_{g.A}^+ = g^{-1} D_A^+ g$, so that $\text{Ker} D_{g.A}^\pm = g(\text{Ker} D_A^\pm)$. We are interested in the Fermionic path integral:

$$(4.1) \quad Z(A) = \int_{C^\infty(M; S^+ \otimes E)} e^{\int_M \psi^* D_A^+ \psi \, dm} \mathcal{D}\psi \mathcal{D}\psi^*,$$

and a formal extension of finite-dimensional functional calculus gives

$$Z(A) := \det(D_A^+).$$

To obtain an unambiguous regularization of (4.1) we therefore require a gauge covariant regularized determinant varying smoothly with A in order that $Z(A)$ pushes down smoothly to the moduli space $\mathcal{M} = \mathcal{A}/\mathcal{G}$. In the

case of Dirac fermions (both chirality sectors) this can be done and there is a gauge invariant regularized determinant $\det_{reg}(D_A)$. For chiral Fermions on the other hand, there is an obstruction due to the presence of zero modes of the Dirac operator. The covariance of the kernels means that the determinant line bundle descends to \mathcal{M} and the obstruction to the existence of a covariant $Z(A)$ varying smoothly with $A \in \mathcal{A}$ is the first Chern class of the determinant bundle on \mathcal{M} , which is the topological chiral anomaly. A 2-form representative for the Chern class $\text{DET } \mathbb{D}^+$ can be constructed as the transgression of the 1-form $\omega_1 \in \Omega(\mathcal{G})$

$$\omega_1(g) = \frac{1}{2\pi i} \frac{d(\det_r(D_{gA}^+))}{(\det_r(D_{gA}^+))},$$

measuring the obstruction to gauge covariance of a choice of regularized determinant $\det_r(D_A^+)$. For details see [2, 14].

In the case of a manifold X with boundary Y new complications arise. Fixing an elliptic boundary condition (spectral section) \mathbb{P} for the family of chiral Dirac operators $\mathbb{D}^+ = \{D_A^+ : A \in \mathcal{A}\}$, we obtain a Fock bundle $\mathcal{F}_{\mathbb{P}}$ over \mathcal{A} to which we aim to lift the \mathcal{G} action. It is natural to look first at gauge transformations (or diffeomorphisms) which are trivial on the boundary. In fact, the calculation of the Chern class in [2] can be extended to this case using a version of the families index theory for a manifold with boundary, [5, 16]. The gauge variation of the chiral determinant can be written as

$$(4.2) \quad \det_r(D_{gA}^+) = \det_r(D_A^+) \omega(g; A)$$

where $\log \omega$ is an integral over X of a local differential polynomial in g , A and the metric on X ; ω is the integrated version of the ‘infinitesimal’ anomaly form ω_1 . The important point is that the formula applies both to the case of a manifold with/without boundary. In fact, in the latter case this gives a direct way to define the determinant bundle over \mathcal{A}/\mathcal{G} , [13].

The locality of the anomaly (4.2) is compatible with the formal sewing formula (5.2). Applying a gauge transformation which is trivial on Y to the right-hand-side of the equation gives a gauge variation which is a product of gauge variations on the two halves X_0, X_1 of M . This product is equal, by locality of the logarithm, to the gauge variation on M of the path integral on the left-hand-side. Since the cutting surface Y is arbitrary, one can drop the requirement that g is trivial on Y .

The gauge transformations (and diffeomorphisms) which are not trivial on the boundary need a different treatment. This is because they act non-trivially on the boundary Fock spaces \mathcal{F}_Y . We shall concentrate on the case when Y is odd dimensional. The first question to ask is how the action of the gauge group on the parameter space B of boundary geometries on Y is lifted to the total space of the bundle of Fock spaces $\mathcal{F} \rightarrow B$. This problem

has already been analyzed (leading to Schwinger terms in the Lie algebra of the group \mathcal{G}) in the literature, but in the present article we want to clarify how the boundary action intertwines with the Fock functor construction.

4.1. Commutator Anomaly on the Boundary. Let $b \in B$ and $W \in Gr_b$. In the rest of this section B denotes the space of metrics and vector potentials on a fixed manifold Y and Y_b is the manifold Y equipped with the geometric data b . The pair (b, W) is mapped to $(g.b, g.W)$ by a gauge transformation (or a diffeomorphism) g , acting on both potentials, metrics and spinor fields. This induces an unitary map from the Fock space $\mathcal{F}(H_b, W)$ to $\mathcal{F}(H_{g.b}, g.W)$, by $a^*(u) \mapsto a^*(g \cdot u)$ and similarly for the annihilation operators.

However, sometimes (b, W) do not appear independently, but W is given as a function of the boundary geometry; $b \rightarrow P_{W=W_b}$ is a Grassmann section; this leads to the construction of the bundle of Fock spaces \mathcal{F}_b parameterized by $b \in B$, as already mentioned above. An example of this situation is the following. Suppose the Dirac operators on the boundary do not have zero eigenvalue (this happens when massive Fermions are coupled to vector potentials). Then it is natural to take $W_b = H_b^+$ as the space of positive energy states. Still this case does not lead to any complications because of the equivariance property $W_{g.b} = g.W_b$. However, there are cases when no equivariant choices for W_b exist. This happens when we have massless chiral Fermions coupled to gauge potentials. For some potentials there are always zero modes and one cannot take W_b as the positive energy subspace without introducing discontinuities into the construction.

Let us assume that a Grassmann section W_b is given. For each boundary geometry b we have a Fermionic Fock space $\mathcal{F}_b = \mathcal{F}(H_b, W_b)$ determined by the polarization $H_{Y_b} = W_b \oplus W_b^\perp$. In order to determine the obstruction to lifting the gauge group action on Y to the bundle of Fock spaces such that $g^{-1}D_{Y_b}g = D_{Y_{g.b}}$ we compare the action on \mathcal{F} to the natural action in the case of polarizations W'_b defined by the positive energy subspaces of Dirac operators $D_{Y_b} - \lambda$. We have fixed a real parameter λ and we consider only those boundary geometries $b \in B$ for which λ is not an eigenvalue. Since the choice of polarizations W' is equivariant, the gauge action lifts to the (local) Fock bundle \mathcal{F}' . Relative to W the \mathcal{F}' vacua form a complex line bundle $\text{DET}(W', W)$; again, this is defined only locally in the parameter space.

Example 4.1 Let Y be a unit circle with standard metric but varying gauge potentials. We can choose $H_Y = W \oplus W^\perp$ as the fixed polarization defined by the decomposition to positive and negative Fourier modes.

If the gauge group is $SU(n)$ and Fermions are in the fundamental representation of $SU(n)$ then the mapping $g \mapsto g \cdot W$ defines an embedding of the loop group $LSU(n)$ to the Hilbert-Schmidt Grassmannian $Gr_1(H_Y, W)$. The pull-back of the Quillen determinant bundle over the Grassmannian to $LSU(n)$ defines the central extension of the loop group with level $k = 1$, [17].

There is a general method to describe the relative determinant bundle in terms of index theory on X for $\partial X = Y$. We assume that the spin and gauge vector bundle on Y can be smoothly continued to bundles on X . This is the case for example when $X = S^{2n-1}$ and Y is chosen such that it has the topology of a solid ball, with a product metric near the boundary. Any vector potential can be smoothly continued to a potential on X for example as $A(x, r) = f(r)A(x)$ with f increasing smoothly from zero to the value one at $r = 1$; all derivatives of f vanishing at $r = 0, 1$. We can now define a spectral section $b \mapsto W_b$ as the Calderon subspace associated to the continued metric and vector potential in the bulk; we denote the Dirac operator defined by this geometric data in X by $D_{X,b}$. The determinant line for a Dirac operator $D_{X,b}$ subject to the boundary condition W is canonically the tensor product of the line $\text{DET}(W', W)$ and the determinant line of the same operator $D_{X,b}$ but subject to another choice of boundary conditions W' , (2.7)

Since the spectral section W_b and the Dirac operator $D_{X,b}$ is parameterized by the affine space of geometric data (metrics and potentials) on the boundary, the corresponding Dirac determinant bundle is topologically trivial. Let U_λ be the set of $b \in B$ such that the real number λ is not in the spectrum of the corresponding Dirac operator D_{Y_b} . On U_λ we can define the boundary conditions $W'_{b,\lambda}$ as the spectral subspace $D_{Y_b} > \lambda$ of the boundary Dirac operator. The set U_λ is in general non-contractible and the Dirac determinant line bundle defined by the boundary conditions W' can be nontrivial. The curvature of this bundle is given by the families index theorem [5, 16]. It can be written in terms of characteristic classes in the bulk and the so-called η -form on the boundary; the latter depends on spectral information about the family of Dirac operators. The curvature Ω when evaluated along gauge and diffeomorphism directions on the boundary data has a simplified expression; in particular, the η -form drops out since it is a spectral invariant and the contribution from the characteristic classes in the bulk reduces to a boundary integral involving the (gauge and metric) Chern-Simons forms, [7, 8]:

$$(4.3) \quad \frac{1}{2\pi} \Omega = \int_Y CS_{[2]}(A + v, \Gamma + w)$$

with

$$dCS(A, \Gamma) = \hat{A}(R)\text{ch}(F)$$

where [2] denotes the part that is a 2-form along parameter directions. The symbol $A+v$ means a connection form on $Y \times B$ such that in the Y directions it is given by a vector potential A and in the gauge directions \mathcal{L}_u on B it is equal to the Lie algebra valued function u . In a similar way, $\Gamma + w$ is the sum of the Levi-Civita connection (on Y) and a metric connection w such that the value of w along a vector field \mathcal{L}_u on B , generated by a vector field u on Y , is equal to the matrix valued function on Y given by the Jacobian of the vector field u .

The characteristic classes are

$$\begin{aligned} \hat{A}(X) &= \det^{1/2} \left(\frac{iR/4\pi}{\sinh(iR/4\pi)} \right) \\ \text{ch}(X) &= \text{tr}(\exp(iF/2\pi)) \end{aligned}$$

where R is the Riemann curvature tensor associated to a metric g in the bulk, F is the curvature of a gauge connection A .

From the previous discussion it follows that the topological information (de Rham cohomology class of the curvature) in the relative determinant bundle $\text{DET}(W', W)$ is given by the curvature formula for the Dirac determinant bundle for boundary polarization W' . This leads to the explicit formula for lifting the gauge and diffeomorphism group action from the base $B = \mathcal{A} \times \mathcal{M}$ to the Fock bundle \mathcal{F} , [7], [8]. Infinitesimally, the lifting leads to an extension of the Lie algebras $\text{Lie}(\mathcal{G})$ and $\text{Vect}(Y)$ by an abelian ideal J consisting of complex valued functions on $\mathcal{A} \times \mathcal{M}$. The commutator of two pairs of elements (u, f) and (v, g) (where f, g are in the extension part J and u, v are infinitesimal gauge transformations or vector fields) is given as

$$(4.4) \quad [(u, f), (v, g)] = ([u, v], \mathcal{L}_u \cdot g - \mathcal{L}_v \cdot f + c(u, v))$$

where $c(u, v)$ is an anti-symmetric bilinear function of the arguments u, v taking values in the ideal J . It satisfies the cocycle condition

$$(4.5) \quad c(u_1, [u_2, u_3]) + \mathcal{L}_{u_1} \cdot c(u_2, u_3) + \text{cyclic permutations} = 0.$$

The cocycle c is just the curvature form evaluated along gauge (or diffeomorphism group) directions,

$$c(u, v) = \Omega(\mathcal{L}_u, \mathcal{L}_v)$$

where \mathcal{L}_u is the vector field on \mathcal{A} (resp. \mathcal{M}) generated by the gauge (diffeomorphism) group action. When Y is one-dimensional, the cocycle reduces to the central term in an affine Lie algebra or in the Virasoro algebra; in

this case the cocycle does not depend on the vector potential or the metric on Y . In dimension 3 the cocycle (Schwinger term) is given as [12, 9]

$$(4.6) \quad c(u, v) = \frac{i}{24\pi^2} \int_Y \text{tr} A[du, dv]$$

when the Fermions are in the fundamental representation of the gauge group; here $u, v : Y \rightarrow G$ are smooth infinitesimal gauge transformations. In dimension 3 the cocycle is trivial in case of vector fields and metrics. Also in higher dimensions explicit expressions can be worked out starting from (4.3), [8].

In the case when the projections P_b, P'_b on the spaces W_b, W'_b of the Grassmann sections differ by trace-class operators, we have a general formula for the curvature of the relative determinant bundle as

$$(4.7) \quad \Omega(\mathcal{L}_u, \mathcal{L}_v) = \frac{1}{8\pi i} \text{tr} [F'_b(\mathcal{L}_u F'_b)(\mathcal{L}_v F'_b) - F_b(\mathcal{L}_u F_b)(\mathcal{L}_v F_b)],$$

where $F_b = P_b - P_b^\perp$ and P_b^\perp is the projection on to the orthogonal complement W_b^\perp . Note that neither of the two terms on the right have a finite trace but the difference is trace class by the relative trace-class property of P_b, P'_b . Note also that in the case when all the projections P' are in a single restricted Grassmannian, the first term is the standard formula for the curvature of the Grassmannian. The second term can be viewed as a renormalization; it is in fact a background field dependent vacuum energy subtraction.

The proof of the curvature formula (4.7) is as follows. First, one notices that this gives the curvature of the relative determinant bundle when both variables W_b, W'_b lie in the same restricted Grassmannian relative to a fixed base point P_0 . Then one has to show that the difference actually makes sense when dropping the existence of common base point. For that purpose one writes

$$\begin{aligned} \Omega(\mathcal{L}_u, \mathcal{L}_v) = \frac{1}{8\pi i} \text{tr} [& (F'_b - F_b)(\mathcal{L}_u F'_b)(\mathcal{L}_v F'_b) + F_b(\mathcal{L}_u F'_b - \mathcal{L}_u F_b)(\mathcal{L}_v F'_b) \\ & + F_b(\mathcal{L}_u F_b)(\mathcal{L}_v F'_b - \mathcal{L}_v F_b)], \end{aligned}$$

which is manifestly a trace of a sum of trace-class operators.

4.2. Chiral Anomaly in the Bulk. In the construction of the Fock functor we took as independent parameter a choice of an element $\epsilon \in \text{DET}(K(D_b^+), W_{Y_b})$ in the boundary determinant bundle; recall that $K(D_b^+)$ is the range of the Calderon projection. A choice of this element, as a function of the geometric data in the bulk, is a section of the determinant bundle. In quantum field theory such a choice is provided by a choice of the regularized determinant of the chiral Dirac operator D_b^+ . The determinant vanishes if and only if the

orthogonal projection $\pi : K(D_b^+) \rightarrow W_{Y_b}$ is singular and therefore it makes sense to choose $\epsilon = \epsilon_b \in \text{DET}(K(D_b^+), W_{Y_b})$ (represented as an admissible linear map $\epsilon_b : W_{Y_b} \rightarrow K(D_b^+)$) such that $\det_r(D_b^+)$, defined subject to the boundary conditions W_b , is equal to $\det_F(\pi \circ \epsilon_b)$.

In the case of chiral Fermions the determinant $\det_r(D_b^+)$ is anomalous with respect to diffeomorphisms and gauge transformations on X and the variation of the determinant is given by the factor $\omega(g; b)$ in (4.2). This implies the transformation rule

$$(4.8) \quad \epsilon_{g,b} = \epsilon_b \cdot \omega(g; b)$$

where g is either a gauge transformation or a diffeomorphism and b stands for both the metric and gauge potential on X . ω is a non-vanishing complex function, satisfying the cocycle condition

$$(4.9) \quad \omega(g_1 g_2; b) = \omega(g_1; g_2 \cdot b) \omega(g_2; b).$$

Here the boundary conditions should be invariant under g , meaning that the gauge transformations (and diffeomorphism) approach smoothly the identity at the boundary.

If the cocycle ω is nontrivial (and this is the generic case for chiral Fermions) in cohomology, then the relation (4.2) above tells us that the Fock functor is determined by the family of Calderon subspaces $K(D_b^+)$ and a choice of a section (the regularized determinant) of a nontrivial line bundle over the quotient space \mathcal{B} of B modulo diffeomorphisms and gauge transformations.

Example 4.2 Let $\mathcal{A}(D)$ be the space of smooth potentials in a unit disk D . Let $\mathcal{G}(D, \partial D)$ be the group of gauge transformations which are trivial on the boundary $\partial D = S^1$. For each $A \in \mathcal{A}(D)$ there is a unique $g = g_A : D \rightarrow G$ such that $A' = g^{-1} A g + g^{-1} dg$ is in the radial gauge, $A_r = 0$, and $g(p) = 1$, where $p \in S^1$ is a fixed point on the boundary. It follows that $\mathcal{B} = \mathcal{A}(D)/\mathcal{G}(D, \partial D)$ can be identified as $\mathcal{A}_{rad}(D) \times \mathcal{G}(D)/\mathcal{G}(D, \partial D) = \mathcal{A}_{rad} \times \Omega G$, where $\mathcal{A}_{rad}(D)$ is the set of potentials in the radial gauge and ΩG is the group of based loops, i.e., those loops in G which take the value 1 at the point p . The first factor $\mathcal{A}_{rad}(D)$ is topologically trivial as a vector space. Thus in this case the topology of the Dirac determinant bundle over the moduli space \mathcal{B} is given by the pull-back of the canonical line bundle over ΩG , [17], with respect to the map $A \mapsto g_A|_{\partial D}$. The sections of $\text{DET} \rightarrow \mathcal{B}$ are by definition complex functions $\lambda : \mathcal{A}(D) \rightarrow \mathbb{C}$ which obey the anomaly condition (4.2), [13].

4.3. Relation of the Chiral Anomaly to the Commutator Anomaly. The bulk anomaly and the extension (Schwinger terms) of the gauge group on the boundary are closely related, [13]. As we saw above, the Fock functor is determined by a choice of a section $b \rightarrow \epsilon_b$ of the relative line bundle $\text{DET}(K(D_b^+), W_{Y_b})$. The section transforms according to the chiral transformation law for regularized determinants,

$$\epsilon_{g \cdot b} = \omega(g; b)\epsilon_b,$$

for transformations g which are equal to the identity on the boundary. If now h is a transformation which is not equal to the identity on the boundary, we can define an operator $T(h)$ acting on sections by

$$(4.10) \quad (T(h)\xi)(b) = \gamma(h; b)\xi(h^{-1} \cdot b),$$

where γ is a complex function of modulus one and must be chosen in such a way that $\xi'(b) = (T(h)\xi)(b)$ satisfies the condition (4.2). Explicit expressions for γ have been worked out in several cases, [14]. For example, if $\dim X = 2$ and g is a gauge transformation then

$$(4.11) \quad \gamma(h; A) = \exp\left(\frac{i}{2\pi} \int_X \text{tr} A dh h^{-1}\right),$$

where tr is the trace in the representation of the gauge group determined by the action on Fermions. In general, γ must satisfy the consistency condition,

$$(4.12) \quad \gamma(h; g \cdot b)\omega(hgh^{-1}; h^{-1} \cdot b) = \gamma(h; b)\omega(g; b).$$

In the two-dimensional gauge theory example,

$$(4.13) \quad \omega(g; A) = \exp\left(\frac{i}{2\pi} \int_X \text{tr} Adgg^{-1} + \frac{i}{24\pi} \int \text{tr} (dgg^{-1})^3\right).$$

The latter integral is evaluated over a 3-manifold M such that its boundary is the closed 2-manifold obtained from X by shrinking all its boundary components to a point.

The introduction of the factor γ in (4.10) has the consequence that the composition law for the group elements is modified,

$$(4.14) \quad T(g_1)T(g_2) = \theta(g_1, g_2; z)T(g_1g_2),$$

where θ is a S^1 valued function, defined by

$$(4.15) \quad \theta(g_1, g_2; b) = \gamma(g_1g_2; b)\gamma(g_1; b)^{-1}\gamma(g_2; g_1^{-1} \cdot b).$$

Thus we have extended the original group of gauge transformations (diffeomorphisms) by the abelian group of circle valued functions on the parameter space B .

At the Lie algebra level, the relation (4.14) leads to a modified commutator (by Schwinger terms discussed above) of the ‘naive’ commutation

relations of the algebra of infinitesimal gauge transformations or the algebra of vector fields on the manifold X . Actually, the modification is ‘sitting on the boundary’; the action of $T(g)$ was defined in such a way that the (normal) subgroup of gauge transformations which are equal to the identity on the boundary acts trivially on the sections $\xi(b)$. There is an additional slight twist to this statement. Actually, the normal subgroup is embedded in the extended group as the set of pairs $(g, c(g))$, where $c(g)$ is the circle valued function defined by

$$(4.16) \quad c(g) = \gamma(g; b)^{-1} \omega(g; b).$$

The consistency condition (4.12) guarantees that the multiplication rule

$$(g_1, c(g_1))(g_2, c(g_2)) = (g_1 g_2, c(g_1 g_2)),$$

holds in the extended group with the multiplication law

$$(g_1, \mu_1)(g_2, \mu_2) = (g_1 g_2, \theta(g_1, g_2) \mu_1 \mu_2^{g_1}),$$

where $\mu^g(b) = \mu(g^{-1}b)$.

4.4. Summary. Let us summarize the above discussion on Fock functors and group extensions. On the boundary manifold $Y = \partial X$ a choice of boundary conditions W_b (labeled by a parameter space B_Y of boundary geometries) defines a fermionic Fock space \mathcal{F}_{Y_b} . The group of gauge transformations (or diffeomorphisms) on Y acts in the bundle of Fock spaces (parameterized by geometric data on the boundary) through an abelian extension; the Lie algebra of the extension is determined by a 2-cocycle (Schwinger terms) which are computed via index theory from the curvature of the relative determinant bundles $\text{DET}(W'_b, W_b)$ where W'_b is the positive energy subspace defined by the boundary Dirac operator. If the boundary is written as a union $Y = Y_{in} \cup Y_{out}$ of the ingoing and outgoing components then the Fock functor assigns to the geometric data on X a linear operator $Z_X : \mathcal{F}_{in} \rightarrow \mathcal{F}_{out}$. A gauge transformation in the bulk X sends Z_X to $\gamma(g; X) Z_{g^{-1}X}$. This action defines an abelian extension of the gauge group. There is a normal subgroup isomorphic to the group of gauge transformations which are equal to the identity on the boundary. This subgroup acts trivially, therefore giving an action of (the abelian extension of) the quotient group on the boundary. The latter group is isomorphic to the group acting in the Fock bundle over boundary geometries.

5. PATH INTEGRAL FORMULAE AND A 0+1-DIMENSIONAL EXAMPLE

In this section we outline the fermionic path integral formalism for an EBVP and explain how the Fock functor models this algebraically.

5.1. **Path integral formulae.** The analogue of (4.1) for an EBVP is

$$(5.1) \quad Z_X(P) := \det(D_P) = \int_{\mathcal{E}_P} e^{\int_X \psi^* D \psi \, dx} \mathcal{D}\psi \mathcal{D}\psi^*,$$

where $\mathcal{E}_P = \text{dom}(D_P)$. This is equation (1.4) for the case $S(\psi) = \int_X \psi^* D \psi \, dx$ and where the local boundary condition f has been replaced by the global boundary condition P . If we consider a partition of the closed manifold $M = X_0 \cup_Y X_1$. The Dirac operator over M restricts to Dirac operators D^0 over X^0 and D^1 over X^1 . We assume that the geometry is tubular in a neighbourhood of the splitting manifold Y , then we have Grassmannians Gr_{Y^i} of boundary conditions associated to D^i , where $Y^1 = Y = \overline{Y^0}$. The reversal of orientation means that there is a diffeomorphism $Gr_{Y^0} \equiv Gr_{Y^1}$ given by $P \leftrightarrow I - P$, so that each $P \in Gr_{Y^0}$ defines the boundary value problems D_P^0 and D_{I-P}^1 . According to (4.1) and (5.1), the analogue of the sewing formula (5.2) is

$$(5.2) \quad \int_{\mathcal{E}(M)} e^{\int_M \psi^* D_A^+ \psi \, dx} \mathcal{D}\psi \mathcal{D}\psi^* = \int_{Gr_Y} \mathcal{D}P \left\{ \int_{\mathcal{E}_P(X^0)} e^{\int_{X^0} \psi_0^* D^0 \psi_0 \, dx_0} \mathcal{D}\psi_0 \mathcal{D}\psi_0^* \right. \\ \left. \times \int_{\mathcal{E}_{I-P}(X^1)} e^{\int_{X^1} \psi_1^* D^1 \psi_1 \, dx_1} \mathcal{D}\psi_1 \mathcal{D}\psi_1^* \right\}.$$

That is,

$$(5.3) \quad Z_M = \int_{Gr_Y} Z_{X^0}(P) Z_{X^1}(I - P) \mathcal{D}P$$

or

$$(5.4) \quad \det(D) = \int_{Gr_Y} \det D_P^0 \cdot \det D_{I-P}^1 \mathcal{D}P.$$

Notice there is no regularization involved here of the determinant. Without further choices, the path integral formulae express a relation between sections of the corresponding determinant bundles, and (5.4) is a pairing on the spaces of sections of those line bundles. More precisely, according to the properties of the Fock functor (see Section 3), the Fermionic integral may be rigorously understood as a linear functional $\wedge(H_Y \oplus \overline{H}_Y) \longrightarrow \text{Det}(D_P^0)$, while (5.4) is replaced by the evaluation of the Fock space bilinear pairing on vacuum elements (3.42):

$$(5.5) \quad \det(D) = (\nu_{K(D^0)}, \nu_{K(D^1)}).$$

However, adopting a slightly different point of view gives a more precise meaning to the integral formulae above. With a given boundary condition

P the determinants of the (chiral) Dirac operators on the manifolds X_0 and X_1 should be interpreted as elements of the determinant line bundle DET over the Grassmannian Gr_Y , with base point H^+ . The actual numerical value of the Dirac determinant depends on the choice of a (local) trivialization. For example, one could define $\det(D)$ as the zeta function regularized determinant $\det_{\zeta}((D_B)^*D_A)$, where D_B is a background Dirac operator chosen in such a way that $(D_B)^*D_A$ has a spectral cut, i.e., there is a cone in the complex plane with vertex at the origin and no eigenvalues of the operator lie inside of the cone. The value of the zeta determinant will depend on the choice of the background field B .

A choice of an element in the line in DET over $P^0 \in Gr_Y$ is given by a choice of a pair (α, λ) , where $\alpha : H_+ \rightarrow P^0$ is a unitary map and $\lambda \in \mathbb{C}$. It can be viewed as a holomorphic section of the dual determinant bundle DET^* according to (2.24),

$$\psi_{[\alpha, \lambda]}(\xi) = \lambda \det_F(\alpha^* \circ \pi \circ \xi),$$

where $\pi : \xi(H_+) \rightarrow P^0$ is the orthogonal projection. We can think of the variable ξ as the parameter for different elements $W = \xi(H_+) \in Gr_Y$. We want to replace the (ill-defined) integral $\int_{Gr_Y} \det(D_P^0) \det(D_P^1) dP$ by a (so far ill-defined) integral of the form

$$(5.6) \quad \int_{\xi} \psi_{[\alpha, \lambda]}(\xi)^* \psi_{[\beta, \mu]}(\xi) d\xi.$$

But this integral looks like the functional integral defining the inner product between a pair of fermionic wave functions (vectors in the Fock space) defined in equation (2.33):

$$(5.7) \quad \det_F(\alpha^* \beta) = \langle \psi_{[\alpha, \lambda]}, \psi_{[\beta, \mu]} \rangle = \sum_{S \in \mathcal{S}} \psi_{\alpha, \lambda}(\xi(S))^* \psi_{[\beta, \mu]}(\xi(S)).$$

The relation with (5.5) is given by the identity (3.36) which tells us that

$$(5.8) \quad (\nu_{K(D^0)}, \nu_{K(D^1)}) = \langle \nu_{K(D^0)}, \nu_{K(D^1)^\perp} \rangle,$$

so here $[\alpha, \lambda] = \det(P_{K^0} P_{K^0})$, $[\beta, \mu] = \det(P_{K^1}^\perp P_{K^1}^\perp)$.

To illustrate this consider the case of the Dirac operator over a closed odd-dimensional spin manifold X partitioned by Y . According to (3.16), (3.35), the Fock pairing

$$(\cdot, \cdot) : \mathcal{F}_{K(D^0)}(H_Y) \otimes \mathcal{F}_{K(D^1)}(\overline{H}_Y) \longrightarrow \text{Det}(K(D^0), K(D^1)^\perp) \cong \text{Det}(D_X)$$

evaluated on vacuum elements is the abstract determinant of the operator

$$(5.9) \quad P(D^0) + P(D^1) : K(D^0) \oplus K(D^1) \longrightarrow H_Y, \quad (\xi, \eta) \longmapsto \xi + \eta.$$

To realize this canonically as a number we can compare it to the Fock pairing coming from forming the doubled elliptic operator $D_{\text{double}} = D^0 \cup -D^0$ over the closed double manifold $X_0 U_Y X_0$ [6].

To do that we use a canonical trivialization of the determinant lines resulting from the identification $K(D^0) = \text{graph}(h_0 : F^+ \rightarrow F^-)$ proved in [19], where F^\pm are the spaces of positive and negative spinor fields over the even-dimensional boundary Y , h_0 is a uniquely determined unitary isomorphism differing from $g_+ = (D_Y^- D_Y^+)^{-1/2} D_Y^+$ by a smoothing operator, and D_Y^\pm are the boundary chiral Dirac operators which we assume to be invertible. In particular, $H^+ = \text{graph}(g_+ : F^+ \rightarrow F^-)$. Similarly $K(D^1) = \text{graph}(h_1 : F^- \rightarrow F^+)$, where h_1 differs from $-g_+^{-1}$ by a smoothing operator $F^- \rightarrow F^+$. In this trivialization one has

$$P(D^0) = \frac{1}{2} \begin{pmatrix} I & h_0^{-1} \\ h_0 & I \end{pmatrix} \quad P(D^1) = \frac{1}{2} \begin{pmatrix} I & h_1 \\ h_1^{-1} & I \end{pmatrix}.$$

Hence the composition of (5.9) with the graph trivialization

$$H_Y \longrightarrow K(D^0) \oplus K(D^1), \quad (\xi, \eta) \longmapsto ((\xi, h_0 \xi), (h_1 \eta, \eta)),$$

gives us the automorphism of $H_Y = F^+ \oplus F^-$

$$(5.10) \quad \begin{pmatrix} \xi \\ \eta \end{pmatrix} \longmapsto \begin{pmatrix} I & h_1 \\ h_0 & I \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

In the case of D_{double} one has $K(D^1) = K(D^0)^\perp$ and so h_1 in (5.10) is then replaced by $h_1 = -h_0^{-1}$. Hence the composition of (5.10) with the inverse of (5.10) for D_{double} is the operator

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \longmapsto \begin{pmatrix} I & \frac{1}{2}(h_1 + h_0^{-1}) \\ 0 & \frac{1}{2}(I - h_0 h_1) \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix},$$

which is of (Fredholm) determinant class and this gives the realization of the left-side of (5.8) in the graph trivialization as

$$(\nu_{K(D^0)}, \nu_{K(D^1)})_{graph} = \det_F \frac{1}{2} (I - h_0 h_1).$$

On the other hand, a similar computation of the right-side of (5.8) in the graph trivialization, using (5.7) yields

$$\langle \nu_{K(D^0)}, \nu_{K(D^1)^\perp} \rangle_{graph} = \det_F \frac{1}{2} (\alpha_{-h_1^{-1}}^* \alpha_{h_0}) = \det_F \left(\frac{1}{2} (I - h_1 h_2) \right),$$

the $1/2$ arising from the relative term $\alpha_{h_0^*} \alpha_{h_0}$. Here $\alpha_T = \begin{pmatrix} \alpha_+ \\ T \alpha_+ \end{pmatrix}$ where the column index labels the different vectors of the canonical basis for the the graph of $T : F^+ \rightarrow F^-$, and the row labels of α_+ label the different coordinates of a basis for F^+ .

5.2. A (0+1)-dimensional example. The motivation for replacing the integration formula (5.6) by the sum in (2.33) comes from finite dimensions. If $H = H_- \oplus H_+$ is a decomposition of a $2N$ dimensional vector space into a pair of orthogonal N dimensional subspaces then the maps α, β, ξ above become (with respect to the basis $\{e_i\}$ with $i = \pm 1, \pm 2, \dots, \pm N$) $2N \times N$ matrices and we have the matrix identity

$$\det(\alpha^* \beta) = \sum_{(i)} \det(\alpha^* \xi(i)) \det(\xi(i)^* \beta),$$

the sum being over all sequences $-N \leq i_1 < i_2 \dots i_N \leq N$ (with $i_\nu \neq 0$). On the other hand, it follows from eq. (3.48) in [10], that the following integration formula holds in this situation:

$$(5.11) \quad \det(\alpha^* \beta) = a_N \int d\xi d\xi^* \det(\alpha^* \xi) \det(\xi^* \beta) \cdot \det(\xi^* \xi)^{-2N-1},$$

where a_N is a numerical factor and the last factor under the integral sign can be incorporated to the definition of the integration measure. If we consider the basis elements $\alpha_{-h_1^{-1}}, \beta_{h_0}$ for linear maps $h_i : H^+ \rightarrow H^-$ and integrate over the dense subspace U_{graph} parameterizing the elements $\xi_T = (\xi^+, T\xi^-)$ with $T \in \text{Hom}(H^+, H^-)$, the integral (5.11) becomes

$$(5.12) \quad \det(1 - h_2 h_1) = a_N \int dT dT^* \det_{H^+}(1 - h_2 T) \det_{H^-}(1 + T^* h_2) \cdot \det(1 + T^* T)^{-2N-1},$$

This has consequences for determinants in dimension one, where we work with the compact Grassmannian. Let $X = [a_0, a_1]$ and let E be a complex Hermitian bundle over X with unitary connection ∇ . Then the associated generalized Dirac operator is simply $D = i\nabla_{d/dx} : C^\infty(X; E) \rightarrow C^\infty(X; E)$. Choosing a trivialization of E , so that $E_{a_0} \oplus E_{a_1} = \mathbf{C}^n \oplus \mathbf{C}^n$, a global boundary condition D is specified by an element $P \in \text{Gr}(\mathbf{C}^n \oplus \mathbf{C}^n)$, defining the elliptic boundary value problem: $D_P = i\nabla_{d/dx} : \text{dom}(D_P) \rightarrow L^2([a_0, a_1]; E)$.

The Fock functor is here a topological 0+1-dimensional FQFT from the category \mathcal{C}_1 , whose objects are points endowed with a complex finite-dimensional Hermitian vector space V (we do not need to give a polarization in this finite-dimensional situation), and whose morphisms are compact 1-dimensional manifolds with boundary with Hermitian bundle with unitary connection. The Fock functor Z takes an object $(p, V) \in \mathcal{C}_1$ to the Fock space $Z(p, V) := \Gamma_{\text{hol}}(\text{Gr}(V); (\text{Det}(\mathcal{E}))^*)^* \cong \wedge V$, where \mathcal{E} is the usual canonical vector bundle over the Grassmannian. Consider two compatible morphisms $\Lambda_{01} = ([a_0, a_1], E^{01}, \nabla^{01})$ and $\Lambda_{12} = ([a_1, a_2], E^{12}, \nabla^{12})$ in \mathcal{C}_1 , so that

$$\Lambda_{02} = \Lambda_{12} \Lambda_{01} = ([a_0, a_2], E^{02}, \nabla^{02}),$$

with $E^{02}|_{[a_0, a_1]} = E_{01}$ etc. Let V_i be the fibre over a_i , and in $[a_i, a_j]$ we assign a_i to be ‘incoming’ and a_j to be ‘outgoing’. For incoming boundary components a_i the associated object in \mathcal{C}_1 is (a_i, \overline{V}_i) . Then we define $Z(\Lambda_{ij}) = \nu_{K_{ij}}$, where $K_{ij} \in Gr(V_0 \oplus V_j)$ is the Calderon subspace of boundary values of solutions to the ‘Dirac’ operator $D = i\nabla_{d/dx}^{ij}$. We have

$$\begin{aligned} Z(\Lambda_{ij}) &\in Z((p_i, \overline{V}_i) \sqcup (p_j, V_j)) = Z(p_i, \overline{V}_i) \otimes Z(p_j, V_j) \\ &\cong (\wedge \overline{V}_i) \otimes (\wedge V_j) \cong \text{Hom}(\wedge V_i, \wedge V_j) := \mathcal{F}_{K_{ij}}. \end{aligned}$$

Because $K_{ij} = \text{graph}(h_{ij} : V_i \rightarrow V_j)$ with h_{ij} the parallel-transport of the connection on E_{ij} between a_i and a_j , a simple computation gives under the above identification $Z(\Lambda_{ij}) \longleftrightarrow \wedge h_{ij} \in \text{Hom}(\wedge V_i, \wedge V_j)$. Next we have a canonical pairing

$$(5.13) \quad Z(a_0, \overline{V}_0) \otimes Z(a_1, V_1) \otimes Z(a_1, \overline{V}_1) \otimes Z(a_2, V_2) \longrightarrow Z(a_0, \overline{V}_0) \otimes Z(a_2, V_2),$$

induced by subtraction $V_1 \oplus V_1 \rightarrow V_1$, with $\wedge h_{01} \otimes \wedge h_{12} \longrightarrow \wedge h_{01} h_{12}$. If we take the case where $a_2 = a_0$, so that $\Lambda_{02} = \Lambda_{12} \Lambda_{01} = (S^1 = [a_0, a_2], E^{02}, \nabla^{02})$, corresponding to morphisms in $Gr(\overline{V}_0 \oplus V_1)$ and $Gr(\overline{V}_1 \oplus V_0)$ respectively, then $Z(\Lambda_{01}) \in \mathcal{F}_{K_{01}}$ and $Z(\Lambda_{10}) \in \mathcal{F}_{K_{10}^\perp}$, and the induced pairing $\mathcal{F}_{K_{01}} \otimes \mathcal{F}_{K_{10}^\perp} \longrightarrow \mathbb{C}$, under the above identifications is just the supertrace

$$\begin{aligned} (,) &: \text{Hom}(\wedge V_0, \wedge V_1) \otimes \text{Hom}(\wedge V_1, \wedge V_0) \longrightarrow \mathbb{C}, \\ (a, b) &\mapsto \text{tr}_s(ab) := \sum_k (-1)^k \text{tr}(ab|_{\wedge^k}). \end{aligned}$$

Applied to the vacuum elements $\nu_{W_{T_{01}}} \leftrightarrow \wedge T_{01} \in \text{Hom}(\wedge V_0, \wedge V_1)$ and $\nu_{W_{T_{10}^\perp}} \leftrightarrow \wedge(-T_{10}^*) \in \text{Hom}(\wedge V_1, \wedge V_0)$ we have

$$(5.14) \quad (\nu_{W_{T_{01}}}, \nu_{W_{T_{10}^\perp}}) = \text{tr}_s(\wedge -T_{10}^* \wedge T_{01}) = \sum_k \text{tr}(\wedge^k(T_{10}^* T_{01})) = \det(I + T_{10}^* T_{01}).$$

Hence we have $(Z(\Lambda_{01}), Z(\Lambda_{10})) = \det(I - h_{10} h_{01})$, (since h_{ij} is unitary), and $(Z(\Lambda_{01}), \nu_{W_T^\perp}) = \det(I + T^* h_{01})$. On the other hand it well-known that $\det(I + T^* h_{01}) = \det_\zeta(D_{P_T})$. So from eq. (5.12) we have

$$(5.15) \quad (Z(\Lambda_{01}), Z(\Lambda_{10})) = a_N \int dT dT^* \det_\zeta(D_{P_T}^{10}) \det_\zeta(D_{P_{-T^*}}^{01}) \cdot \det(1 + T^* T)^{-2N-1},$$

where $P_{-T^*} = I - P_T$, expressing the relation of the algebraic Fock space pairing to the path integral sewing formula eq. (5.4).

Notice that the gauge group of a boundary component of $[a_0, a_1]$ is just a copy of the unitary group $U(n)$ and under the embedding $g \rightarrow \text{graph}(g) := W_g$, the Fock functor maps g to $\wedge g$ on $\wedge V$. Thus in the case of $0 + 1$ -dimensions the FQFT representation of the boundary gauge group is the fundamental $U(n)$ -representation, which is a restatement of the Borel-Weil Theorem for $U(n)$. This means that the invariant output by the FQFT,

which in fact here is a TQFT, is the character of the fundamental representation π of $U(n)$. This is what we would expect. We are dealing with a single particle evolving through time, and so its only invariants are the representations of its internal symmetry group, which is the symmetry group of the bundle E over $[a, b]$. In this sense we are dealing with quantum mechanics, rather than QFT, and because it is a topological field theory the Hilbert space is finite-dimensional.

5.3. Relation to the Berezin integral. The above pairing can also be described by a Fermionic integral. Let $\wedge V$ denote the exterior algebra of the complex vector space V with odd generators ξ_1, \dots, ξ_n . It has basis the monomials $\xi_I = \xi_{i_1} \dots \xi_{i_p}$, $I = \{i_1, \dots, i_p\}$, $i_1 < \dots < i_p$, where I runs over subsets of $\{1, \dots, n\}$, and we set $|I| := p$. The Fermionic (or Berezin) integral is the linear functional

$$\int : \wedge V \longrightarrow \mathbb{C}, \quad f(\xi) \longmapsto \int f(\xi) \mathcal{D}\xi$$

which picks out the the top degree coefficient of $f(\xi)$ (a polynomial in the generators) relative to the generator $\xi = \xi_1 \dots \xi_n$ of $\text{Det } V = \wedge^n V$. This extends to a functional

$$\int : \wedge \bar{V} \otimes \wedge V \longrightarrow \mathbb{C}, \quad f(\bar{\xi}, \xi) \longmapsto \int f(\bar{\xi}, \xi) \mathcal{D}\xi \mathcal{D}\bar{\xi},$$

defined relative to the generator $\bar{\xi}\xi := \xi_1 \bar{\xi}_1 \dots \xi_n \bar{\xi}_n$ of $\text{Det } V \otimes \text{Det } \bar{V}$. Given an element $T \in \text{End}(V)$ we associate to the quadratic element $\bar{\xi}T\xi := \sum_{i,j} t_{ij} \bar{\xi}_i \xi_j$. We then have $\frac{1}{n!} (\bar{\xi}T\xi)^n = \det(T) \bar{\xi}\xi$, and more generally the Gaussian expression

$$e^{\bar{\xi}T\xi} = \sum_I \det(T_I) \xi_I \bar{\xi}_I,$$

where T_I denotes the submatrix (t_{ij}) with $i, j \in I$, so that we can write

$$(5.16) \quad \int e^{\bar{\xi}T\xi} \mathcal{D}\xi \mathcal{D}\bar{\xi} = \det(T),$$

so determinants are expressible as complex Fermion Gaussian integrals.

Next, we have a bilinear form on $\wedge \bar{V} \otimes \wedge V$ defined by

$$(5.17) \quad \langle f, g \rangle = \int g(\bar{\xi}, \xi)^\sigma f(\bar{\xi}, \xi) \mathcal{D}\mu[(\xi, \xi), (\bar{\xi}, \bar{\xi})],$$

where $f(\xi)^\sigma$ is $f(\xi)$ with the order of the generators reversed, and

$$\int f(\bar{\xi}, \xi) \mathcal{D}\mu[(\xi, \xi), (\bar{\xi}, \bar{\xi})] := \int f(\bar{\xi}, \xi) e^{2\bar{\xi}\xi} \mathcal{D}(\xi, \xi) \mathcal{D}(\bar{\xi}, \bar{\xi}),$$

is the Fermionic integral with respect to a Gaussian measure. The 2 arises in the exponent because we are dealing with $\wedge \bar{V} \otimes \wedge V$ rather than $\wedge V$.

Applied to quadratic elements $e^{\bar{\xi}T\xi}$ and $e^{\bar{\xi}S\xi}$ defined for $T, S \in \text{End}(V)$ we then have

$$\begin{aligned} \langle e^{\bar{\xi}T\xi}, e^{\bar{\xi}S\xi} \rangle &= \int e^{\bar{\xi}S\xi + \bar{\xi}T\xi + 2\bar{\xi}\xi} \mathcal{D}\xi \mathcal{D}\bar{\xi} \\ &= \int e^{\binom{\bar{\xi} \ \bar{\xi}}{\xi \ \xi}} \begin{pmatrix} I & T \\ S & I \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \mathcal{D}\xi \mathcal{D}\bar{\xi} \\ &= \det_{V \oplus V} \begin{pmatrix} I & T \\ S & I \end{pmatrix} \\ &= \det_V (I - ST). \end{aligned}$$

Here we use (5.16) applied to $\begin{pmatrix} I & T \\ S & I \end{pmatrix} : V \oplus V \rightarrow V \oplus V$ and the general formula $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det(d) \det(a - bd^{-1}c)$, valid provided $d : V \rightarrow V$ is invertible.

We can repeat the process for a pair of complex vector spaces $V_0 \neq V_1$ of the same dimension and $T \in \text{Hom}(V_0, V_1)$ and $S \in \text{Hom}(V_1, V_0)$. Now define the Fermionic integral just to be the projection onto the form of top degree $\int : \wedge \bar{V}_0 \otimes \wedge V_1 \rightarrow \text{Det}(V_0, V_1)$. Associated to T we have $e^T \in \bar{V}_0 \otimes \wedge V_1 \cong \text{Hom}(\wedge V_0, \wedge V_1)$ we may regard as an element of $\bar{V}_0 \otimes \wedge V_1$ via the Hermitian isomorphism $\bar{V}_0 \cong V_0^*$, and then $\int e^T = \det(T) \in \text{Det}(V_0, V_1)$. Here $\det(T)$ is the element $\det(T)(\xi_1 \dots \xi_n) = T\xi_1 \dots T\xi_n$, for a basis ξ_i of V_0 , which is canonically identified with $\det(T) \in \mathbb{C}$ when $V_0 = V_1$, and the Gaussian element is then $e^T = e^{\bar{\xi}T\xi}$. The bilinear pairing goes through as before, with $\langle e^T, e^S \rangle = \det_{V_0}(I - ST)$, which gives an alternative formulation of the Fock pairing $\langle, \rangle : \mathcal{F}_{W_{T_0}} \times \mathcal{F}_{W_{T_1}^\perp} \rightarrow \mathbb{C}$.

REFERENCES

- [1] Atiyah, M.F.: Topological quantum field theories. Inst. Hautes Etudes Sci. Publ. Math. **68**, 175 (1989).
- [2] Atiyah, M.F. and Singer, I.M.: Dirac operators coupled to vector potentials. Proc. Nat. Acad. Sci. USA **81**, 2597 (1984).
- [3] Atiyah, M.F., Patodi, V.K., and Singer, I.M.: Spectral asymmetry and Riemannian geometry. I. Math. Proc. Cambridge Phil. Soc. **77**, 43 (1975).
- [4] Berline, N., E. Getzler, and M. Vergne: *Heat Kernels and Dirac Operators*. Grundlehren der Mathematischen Wissenschaften **298**, Springer-Verlag, Berlin, (1992).
- [5] Bismut, J-M and Freed, D.S.: The analysis of elliptic families.II. Dirac operators, eta invariants, and the holonomy theorem. Comm. Math. Phys. **107**, 103 (1986).
- [6] Booß-Bavnbek, B., and Wojciechowski, K.P.: *Elliptic Boundary Problems for Dirac Operators*. Boston: Birkhäuser, 1993.

- [7] Carey, A., Mickelsson, J., and Murray, M.: Index theory, gerbes and Hamiltonian quantization. *Comm. Math. Phys.* **183**, 707 (1997).
- [8] Ekstrand, C., and Mickelsson, J.: Gravitational anomalies, gerbes, and hamiltonian quantization. [hep-th/9904189](https://arxiv.org/abs/hep-th/9904189)
- [9] Faddeev, L., Shatasvili, S.: Algebraic and Hamiltonian methods in the theory of nonabelian anomalies. *Theoret. Math. Phys.* **60**, 770 (1985).
- [10] Fujii, K., Kashiwa, T., and Sakoda, S.: Coherent states over Grassmann manifolds and the WKB exactness in path integral. *J. Math. Phys.* **37**, 567 (1996).
- [11] Grubb, G.: Trace expansions for pseudodifferential boundary problems for Dirac-type operators and more general systems. *Ark. Mat.* **37**, 45 (1999).
- [12] Mickelsson, J.: Chiral anomalies in even and odd dimensions. *Comm. Math. Phys.* **97**, 361 (1985).
- [13] Mickelsson, J.: Kac-Moody groups, topology of the Dirac determinant bundle, and fermionization. *Comm. Math. Phys.* **110**, 173 (1987).
- [14] Mickelsson, J.: *Current algebras and groups*. London and New York: Plenum Press, 1989.
- [15] Mickelsson, J.: On the hamiltonian approach to commutator anomalies in 3+1 dimensions. *Phys. Lett.* **B 241**, 70 (1990).
- [16] Piazza, P.: Determinant bundles, manifolds with boundary and surgery. *Comm. Math. Phys.* **178**, 597 (1996).
- [17] Pressley, A. and Segal, G.B.: *Loop Groups*. Oxford: Clarendon Press, 1986.
- [18] Quillen, D. G.: Determinants of Cauchy-Riemann operators over a Riemann surface. *Funkcionalnyi Analiz i ego Prilozheniya* **19**, 37 (1985).
- [19] Scott, S.G.: Determinants of Dirac boundary value problems over odd-dimensional manifolds. *Comm. Math. Phys.* **173**, 43 (1995).
- [20] Scott, S.G.: 2000, ‘Splitting the curvature of the determinant line bundle’, *Proc. Am. Math. Soc.* **128**, 2763–2775.
- [21] Scott, S.G., and Wojciechowski, K.P.: 2000, ‘The ζ -Determinant and Quillen’s determinant for a Dirac operator on a manifold with boundary’, *Geom. Funct. Anal.* **10**, 1202–1236, 2000.
- [22] Segal, G.B.: The definition of conformal field theory. Oxford preprint (1990).
- [23] Segal, G.B.: *Geometric aspects of quantum field theory*. Proc. Int. Cong. Math., Tokyo, (1990).
- [24] Segal, G.B, and Wilson, G.: Loop groups and equations of the KdV type. *Inst. Hautes Etudes Sci. Publ. Math.* **61**, 5 (1985).
- [25] Witten, E: Topological quantum field theory, *Comm. Math. Phys.* **117**, 353 (1988).
- [26] Witten, E: *Geometry and physics*. Proc. Int. Cong. Math., Tokyo (1990).

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