ANALYSIS OF ELLIPTIC FAMILIES IN DIMENSION ONE

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Abstract. We investigate the spectral zeta metric and and connection on the determinant line bundle of a family of elliptic boundary value problems over an interval.

1. Introduction

The purpose of this paper is to study the spectral zeta function on a parameter space of global boundary value problems over an interval. In the case of closed manifolds the meromorphic continuation of a zeta-function on families of elliptic operators trace leads to the construction of a natural Hermitian geometry on a complex line bundle over the parameter space, called the determinant line bundle, which has provided a crucial mechanism for detecting gauge and gravitational anomalies in quantum field theory [7, 2]. In the case of elliptic boundary problems we find that the analysis is reduced to a boundary integral problem. The essential feature is that the *relative* ζ-function geometry coincides with the standard differential geometry defined by the boundary integral.

1.1. **Relative Zeta-Determinants.** Let A be an operator on a Hilbert space $\mathcal{H}$. If A is bounded of the form $I+\alpha$ with α of trace class, then it has a Fredholm determinant $\det_F A = \sum \mathrm{Tr}\,(\wedge^k \alpha)$, equal by Lidskii's Theorem to the product of its eigenvalues, and satisfying the characteristic properties of the determinant in finite-dimensions. If A is an unbounded operator, then to make sense of its determinant a choice of regularization procedure is needed. To define the zeta-determinant regularization we assume that A has principal angle θ, meaning there is a neighbourhood of the ray $R_\theta = \{re^{i\theta} \mid r \geq 0\}$ disjoint from $\mathrm{spec}(A)$, and that the operator norm of $(A-\lambda)^{-1}$ decays like $1/|\lambda|$ as $\lambda \to \infty$ along R_θ. For $\mathrm{Re}(s) > 0$ one then has the convergent integral

$$A_\theta^{-s} = \frac{i}{2\pi}\int_{\Gamma_\theta} \lambda_\theta^{-s}(A-\lambda I)^{-1}\, d\lambda,$$

where $\lambda_\theta^{-s} = |\lambda|^{-s}e^{-is\arg(\lambda)}$, $\theta \leq \arg\lambda \leq \theta + 2\pi$, is the branch of λ^{-s} defined by R_θ, and Γ_θ is the contour beginning at ∞, traversing R_θ to a small circle around the origin, and then back along R_θ to ∞.

We assume $(A-\lambda)^{-m}$ is trace class for $m > -\alpha_0 > 0$ and that as $\lambda \to \infty$ along R_θ there is an asymptotic expansion of the form

$$\mathrm{Tr}\,(A-\lambda)^{-m} \sim \sum_{i=0}^{\infty}\sum_{k=0}^{k_i} a_{ik}\lambda^{-m-\alpha_i}\log^k \lambda, \tag{1.1}$$

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where $-\infty < \alpha_0 < \alpha_1 < \dots$ and $\alpha_i \to \infty$. This means that the spectral zeta function

$$\zeta_\theta(s, A) = \operatorname{Tr} A_\theta^{-s}$$

defined in the standard way for $\mathrm{Re}(s) > -\alpha_0$ extends meromorphically to all of $\mathbb{C}$. If furthermore for $\alpha_J = 1$ one has $a_{J,k} = 0$ for $k > 0$, A is said to be *admissible.* $\zeta_\theta(s, A)$ is then holomorphic near $s = 0$, and the ζ-determinant is defined by

$$\det{}_{\zeta,\theta} A := \exp\left(-\frac{d}{ds}_{|s=0} \zeta_\theta(s, A)\right). \tag{1.2}$$

Seeley [11] showed that any elliptic (pseudo-)differential operator D of positive order over a closed manifold is admissible, and $\zeta_\theta(0, D)$ is then a local invariant depending only on the leading symbol of D which can be read off from (1.1). In contrast $\log\det_{\zeta,\theta} D = -\zeta_\theta^{'}(0, D)$ is highly non-local (its first variation is also non-local) and this makes it a hard invariant to compute, except on certain symmetric spaces where an exact identification of the eigenvalues is possible. As a simple example of the latter, the Laplacian $\Delta_a = D_a^* D_a$ on the circle S^1, where $D_a = id/dx + a$ and $0 < a \leq 1$, has eigenvalues $\{(n + a)^2 \mid n \in \mathbb{Z}\}$ and so in terms of the classical Riemann-Hurwitz zeta function $\zeta(s, a) = \sum_{n=0}^{\infty} 1/(n + a)^s$, one has $\zeta_\pi(\Delta, s) = \zeta(2s, a) + \zeta(2s, 1 - a)$ for $\mathrm{Re}(s) > 1/2$. The meromorphically continued value $\zeta^{'}(0, a) = \log(\Gamma(a)/\sqrt{2\pi})$ then yields $\det_\zeta \Delta_a = 4\sin^2 \pi a$. On the other hand, one can completely ignore zeta functions and formally compute

$$\begin{aligned}\det \Delta_a = \prod_{n\in\mathbb{Z}} (n + a)^2 &= \left(\prod_{n\in\mathbb{Z}\backslash 0} n^2\right) .a^2 . \prod_{n\in\mathbb{Z}\backslash 0} (1 - \frac{a^2}{n^2})^2 \\ &= \left(\prod_{n\in\mathbb{Z}\backslash 0} n^2\right) \frac{\sin^2 \pi a}{\pi^2}.\end{aligned}$$

Hence for $a_1, a_2 \in (0, 1)$

$$\frac{\det_\zeta \Delta_{a_1}}{\det_\zeta \Delta_{a_2}} = \frac{\sin^2 \pi a_1}{\sin^2 \pi a_2} = \frac{\det \Delta_{a_1}}{\det \Delta_{a_2}}, \tag{1.3}$$

identifying the ratio of the rigorous ζ-function regularized determinant for Δ_a with the ratio of the 'ad-hoc' determinant $\det \Delta_a$.

Equation (1.3) portrays a certain 'relativity principle' for determinants of admissible operators, which asserts that for comparable operators A_1, A_2 the relative ζ-determinant $\det_{\zeta,\theta}(A_1, A_2)$ can be computed as a Fredholm determinant of an operator canonically determined by the relative resolvent. By 'comparable' we mean that A_1, A_2 have the same principal angle θ and that the relative resolvent $(A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1}$ is trace class with

$$\operatorname{Tr}\left((A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1}\right) = -\partial_\lambda \log\det{}_F \mathcal{S}_\lambda, \tag{1.4}$$

where the *scattering matrix* $\mathcal{S}_\lambda = \mathcal{S}_\lambda(A_1, A_2)$ is an operator with a Fredholm determinant $\det_F \mathcal{S}_\lambda$. For large enough real s one has the relative zeta function

$$\zeta(s, A_1, A_2) = \frac{i}{2\pi} \int_\Gamma \lambda^{-s} \operatorname{Tr}\left((A_1 - \lambda)^{-1} - (A_2 - \lambda)^{-1}\right)\, d\lambda.$$

Further, we assume the relative resolvent trace has an expansion as $\lambda \to \infty$

$$\mathrm{Tr}\,((A_1-\lambda)^{-1}-(A_2-\lambda)^{-1}) \sim \sum_{j=1}^{\infty}\sum_{k=0}^{1} b_{j,k}(-\lambda)^{-\alpha_j}\log^k(-\lambda), \tag{1.5}$$

where $\alpha_j \nearrow +\infty$. This defines the meromorphic continuation of $\zeta(s, A_1, A_2)$ to all of $\mathbb{C}$.

For a function with an asymptotic expansion $f(\lambda) \sim \sum_{j=0}^{\infty}\sum_{k=0}^{1} c_{jk}(-\lambda)^{-\beta_j}\log^k(-\lambda) + c_0\log(-\lambda) + c_1$ as $\lambda \to \infty$ in $\Lambda_{\theta,\varepsilon}$, where $\beta_j \nearrow +\infty$ and $\beta_j \neq 0$, its regularized limit is defined to be the constant term in the expansion: $\mathrm{LIM}^{\theta}_{\lambda\to\infty} f(\lambda) = c_1$. Then (with $\mathcal{S} := \mathcal{S}_0$) there is the following precise form of the relativity principle for determinants:

Theorem [9] *For operators A_1, A_2 which are ζ-comparable and ζ-admissible, one has*

$$\frac{\det_{\zeta,\theta}A_1}{\det_{\zeta,\theta}A_2} = \det_F\mathcal{S}\,.\,e^{-\mathrm{LIM}^{\theta}_{\lambda\to\infty}\log\det_F\mathcal{S}_\lambda}\ . \tag{1.6}$$

Here, we show that for a family of boundary problems over an interval this principle is also central in understanding the Hermitian structure on the determinant line bundle defined by zeta-function.

1.2. **Families of Global Boundary Problems.** Let $X = [0, \beta]$ where β is a positive real number and let E be a complex Hermitian vector bundle over X of rank n. Relative to a choice of trivialization of E, a first-order elliptic differential operator $\mathcal{D}$ acting on $C^\infty(X; E)$ has the form $A(x)d/dx + B(x)$, where A, B are complex $n \times n$ matrices and $\det A(x) \neq 0$. The space $\mathrm{Ell}_{1,n}$ of all first-order elliptic operators on E is therefore identified with $C^\infty(X, \mathrm{Gl}_n(\mathbb{C})) \times C^\infty(X, \mathrm{End}(\mathbb{C}^n))$. The operator $\mathcal{D}$ extends to a continuous map $\mathcal{D} : H^1(X; E) \to L^2(X; E)$ on the first Sobolev completion. The bundle over the boundary for the Cauchy data is the $2n$-dimensional graded vector space $C^0(\partial X, (E_0 \oplus E_\beta)) \cong E_0 \oplus E_\beta$, and we have the restriction map to the boundary

$$\gamma : H^1(X; E) \longrightarrow E_0 \oplus E_\beta,\ \gamma(\psi) = (\psi(0), \psi(\beta)).$$

A *global boundary condition* for $\mathcal{D} \in Ell_{1,n}$ is specified by a projection P on $E_0 \oplus E_\beta$, where by projection we mean self-adjoint indempotent. The pair $(\mathcal{D}, P)$ combine to define the elliptic boundary value problem:

$$\mathcal{D}_P = \mathcal{D} : \mathrm{dom}(\mathcal{D}_P) \longrightarrow L^2(X; E),$$

where

$$\mathrm{dom}(\mathcal{D}_P) = \{\psi \in H^1(X; E) \mid P\gamma\psi = 0\}\ .$$

The parameter space of global boundary conditions for $\mathcal{D}$ is therefore the complex Grassmannian $Gr(E_0 \oplus E_\beta)$, consisting of one component

$$\mathrm{Gr}_k(E_0 \oplus E_\beta) = \{P \in \mathrm{End}(E_0 \oplus E_\beta) \mid P^2 = P,\ P^* = P,\ \mathrm{tr}\,(P) = k\},$$

for each integer $k = 0, \dots, 2n$. A point P of the $k(2n-k)$-dimensional complex manifold $\mathrm{Gr}_k(E_0 \oplus E_\beta)$ is equivalently specified by $W = \mathrm{range}\,(P) \subset E_0 \oplus E_\beta$. In particular, the *Calderon projection* $P(\mathcal{D})$ onto the Cauchy data subspace $K(\mathcal{D}) = \{v \in E_0 \oplus E_\beta \mid \exists\,\psi \in C^\infty(X; E),\ \mathcal{D}\psi = 0, \gamma\psi = v\}$ defines a distinguished element of $Gr_n(E_0 \oplus E_\beta)$. More precisely, any element of $\mathrm{Ker}(\mathcal{D})$ is of the form $h(x)v$ for some $v \in E_0$, where $h(x) \in$

$\mathrm{End}(E_0, E_x)$ is the fundamental solution matrix uniquely solving $\mathcal{D}h(x) = 0$ subject to $h(0) = I$. Hence there is a canonical isomorphism $\gamma : \mathrm{Ker}(\mathcal{D}) \to K(\mathcal{D})$ and

$$K(\mathcal{D}) = \mathrm{graph}(h := h(\beta) : E_0 \to E_\beta) \subset E_0 \oplus E_\beta. \tag{1.7}$$

For any two $P_1, P_2 \in Gr(E_0 \oplus E_\beta)$ we have a finite-rank operator

$$(P_1, P_2) := P_2 \circ P_1 : W_1 \to W_2,$$

where $W_i = \mathrm{range}\,(P_i)$, and $\mathrm{index}\,(P_1, P_2) = \dim W_1 - \dim W_2$. The pivotal fact is that the unbounded operator $\mathcal{D}_P$ is modeled by the finite-rank operator on boundary data

$$\mathcal{S}(P) := (P(\mathcal{D}), P) : K(\mathcal{D}) \to W\ .$$

The operator $\mathcal{D}_P$ is a Fredholm operator with kernel and cokernel consisting of smooth sections, and $Gr_k(E_0 \oplus E_\beta)$ parameterizes EBVPs of index

$$\mathrm{ind}\,\mathcal{D}_P = \mathrm{ind}\,\mathcal{S}(P) = n - k. \tag{1.8}$$

This implies the relative index formula $\mathrm{ind}\,\mathcal{D}_{P_1} - \mathrm{ind}\,\mathcal{D}_{P_2} = \mathrm{ind}\,(P_2, P_1)$ (see [3, 8] for the general dimensional case.)

Definition 1.1. *By a smooth family of first-order elliptic differential operators over* $[0, \beta]$ *parameterized a manifold* B *we shall mean an element* $\mathbb{D} \in C^\infty(B, \mathrm{Ell}_{1,n})$. *A Grassmann section means an element* $\mathbb{P} \in C^\infty(B, \mathrm{Gr}_k(E_0 \oplus E_\beta)) := Gr(B, E, k)$. *A smooth family of elliptic boundary value problems means a pair* $(\mathbb{D}, \mathbb{P})$.

For each $b \in B$, $(\mathbb{D}, \mathbb{P})$ parameterizes $\mathcal{D}_b \in \mathrm{Ell}_{1,n}$ and $P_b \in \mathrm{Gr}_k(E_0 \oplus E_\beta)$ and hence the EBVP $\mathcal{D}_{P_b} = \mathcal{D}_b : \mathrm{dom}(\mathcal{D}_{P_b}) \longrightarrow L^2(X; E)$. Equivalently [8] one may think of $(\mathbb{D}, \mathbb{P})$ as a bundle homomorphism

$$(\mathbb{D}, \mathbb{P}) : \mathcal{H}_{\mathbb{P}} \longrightarrow \mathcal{H},$$

where $\mathcal{H}$ is the trivial bundle $\mathcal{H} = B \times C^\infty(X; \mathbb{C}^n)$ and $\mathcal{H}_{\mathbb{P}}$ is the weak vector bundle with fibre $\mathrm{dom}_\infty(\mathcal{D}_{P_b}) = \{\psi \in C^\infty(X; E) \mid P_b \gamma \psi = 0\}$.

Proposition 1.2. *A Grassmann section* $\mathbb{P} \in Gr(B, E, k)$ *is equivalent to a smooth rank* k *complex bundle* $\mathcal{W} \longrightarrow B$ *with fibre* $W_b := \mathrm{range}\,(P_b)$. *The bundle* $\mathcal{W}$ *is endowed with a natural Hermitian metric* $g^{\mathcal{W}}$ *and compatible connection* $\nabla^{\mathcal{W}} = \mathbb{P} \cdot d \cdot \mathbb{P}$ *with curvature 2-form* $\mathbf{R}^{\mathcal{W}} = \mathbb{P} d\mathbb{P} d\mathbb{P} \in \Omega^2(B; \mathrm{End}(\mathcal{W}))$. *The induced connection on the complex line bundle* $\mathrm{Det}\,(\mathcal{W})$ *has curvature* $\mathrm{tr}\,(\mathbf{R}^{\mathcal{W}}) = \mathrm{tr}\,(\mathbb{P} d\mathbb{P} d\mathbb{P}) \in \Omega^2(B)$.

We omit the straightforward proof.

Therefore to each pair of Grassmann sections $\mathbb{P}^0, \mathbb{P}^1$ there is the smooth finite-rank family

$$(\mathbb{P}^0, \mathbb{P}^1) \in C^\infty(B; \mathrm{Hom}(\mathcal{W}^0, \mathcal{W}^1)), \quad (\mathbb{P}^0, \mathbb{P}^1)_b \equiv P_b^1 P_b^0 : W_{0,b} \longrightarrow W_{1,b},$$

where $\mathcal{W}^i$ are the bundles of Proposition 1.2. In particular, associated to $\mathbb{D} \in C^\infty(B, \mathrm{Ell}_{1,n})$ is a preferred *Calderon section* $P(\mathbb{D}) \in Gr(B, E, n)$, defined by $b \mapsto P(\mathcal{D}_b)$ and constructed from global data. The bundle $\mathcal{K}(\mathbb{D}) \to B$ associated to $P(\mathbb{D})$ is canonically trivial by (1.7).

Abstractly, the determinant associated to a smooth family of Fredholm operators $\mathcal{A} = \{A_b : H_b^1 \longrightarrow \mathcal{H}_b^2 \mid b \in B\}$ arises as a canonical section $b \mapsto \det(A_b)$ of the determinant line bundle $\mathrm{DET}\,\mathcal{A} = \cup_{b \in B} \mathrm{Det}\,A_b$. the fibre $\mathrm{Det}\,A_b$ of the complex line bundle $\mathrm{DET}\,\mathcal{A}$

is canonically isomorphic to $\mathrm{Det}\,\mathrm{Ker}(A_b)^* \otimes \mathrm{Det}\,\mathrm{Coker}(A_b)$, where $\mathrm{Det}\, V := \wedge^{max} V$. See [7, 2, 8] for details.

For each smooth family of EBVPs $(\mathbb{D}, \mathbb{P})$ we have a determinant line bundle $\mathrm{DET}\,(\mathbb{D}, \mathbb{P})$ equipped with its determinant section $b \mapsto \det(\mathcal{D}_{P_b})$, and for $(\mathbb{P}^1, \mathbb{P}^2) : \mathcal{W}_1 \to \mathcal{W}_2$, we have a determinant bundle $\mathrm{DET}\,(\mathbb{P}^0, \mathbb{P}^1)$. In particular, associated to $(\mathbb{D}, \mathbb{P})$ is the finite rank family $\mathbb{S}(\mathbb{P}) = (P(\mathbb{D}), \mathbb{P})$ with determinant bundle $\mathrm{DET}\,(\mathbb{S}(\mathbb{P}))$ and canonical section $b \mapsto \det(\mathcal{S}(P_b))$. From [8] there is a canonical line bundle isomorphism

$$\mathrm{DET}\,(\mathbb{D}, \mathbb{P}) \cong \mathrm{DET}\,(\mathbb{S}(\mathbb{P})) = \mathrm{Det}\ \mathcal{K}(\mathbb{D})^* \otimes \mathrm{Det}\,\mathcal{W}, \tag{1.9}$$

preserving the determinant sections $\det(\mathcal{D}_{P_b}) \longleftrightarrow \det(\mathcal{S}(P_b))$. $\mathrm{DET}\,(\mathbb{D}, \mathbb{P})$ is therefore classified by the isomorphism class of the complex line bundle $\mathrm{Det}\,\mathcal{W}$: in terms of Chern classes, $c_1(\mathrm{DET}\,(\mathbb{D}, \mathbb{P})) = c_1(\mathrm{Det}\,\mathcal{W})$.

1.3. **Statement of Results.**

The identification (1.9) means that by pull-back $\mathrm{DET}\,(\mathbb{D}, \mathbb{P})$ inherits a metric $\|\ .\ \|_{\mathcal{C}}$ and compatible connection $\nabla^{\mathcal{C}}$ from $\mathrm{DET}\,(\mathbb{S}(\mathbb{P}))$. We recall from [8] that over the open subset U of B where the operators $\mathcal{D}_{P_b}$ are invertible the *canonical metric* on $\mathrm{DET}\,(\mathbb{D}, \mathbb{P})$ is defined by

$$\| \det \mathcal{D}_{P_b} \|^2_{\mathcal{C}} = \det\nolimits_{\mathcal{C}}(\Delta_{P_b})$$

where $\Delta_P = (\mathcal{D}_P)^* \mathcal{D}_P$, and the canonical regularization of $\det(\Delta_P)$ is the finite-rank determinant on $K(\mathcal{D})$, $\det_{\mathcal{C}}(\Delta_P) := \det(\mathcal{S}(P)^* \mathcal{S}(P))$.

On the other hand, $\mathrm{DET}\,(\mathbb{D}, \mathbb{P})$ has a Quillen metric defined over U by the zeta determinant $\| \det \mathcal{D}_P \|^2_{\zeta} = \det_{\zeta}(\gneqq_P)$.

Theorem 1 *Let $\mathbb{P}^1$, $\mathbb{P}^2$ be Grassmann sections for $\mathbb{D}$ and let $\mathcal{D}_{P_1} \in (\mathbb{D}, \mathbb{P}^1), \mathcal{D}_{P_2} \in (\mathbb{D}, \mathbb{P}^2)$ be invertible at $b \in B$. Then*

$$\frac{\|\det(\mathcal{D}_{P_1})\|_{\zeta}}{\|\det(\mathcal{D}_{P_2})\|_{\zeta}} = \frac{\|\det(\mathcal{D}_{P_1})\|_{\mathcal{C}}}{\|\det(\mathcal{D}_{P_2})\|_{\mathcal{C}}}. \tag{1.10}$$

That is,

$$\frac{\det_{\zeta}(\gneqq_{P_1})}{\det_{\zeta}(\gneqq_{P_2})} = \frac{\det_{\mathcal{C}}(\gneqq_{P_1})}{\det_{\mathcal{C}}(\gneqq_{P_2})}. \tag{1.11}$$

Equivalently, since $\mathcal{S}(P(\mathcal{D})) = Id$,

$$\det\nolimits_{\zeta}(\gneqq_P) = \det\nolimits_{\zeta}(\gneqq_{P(\mathcal{D})}) \det\nolimits_{\mathcal{C}}(\gneqq_P). \tag{1.12}$$

Example: Consider the simplest case: $n = 1$, $\mathcal{D} = id/dx$ over $[0, 2\pi]$, with Laplacian $\gneqq = -d^2/dx^2$. Boundary conditions for $\mathcal{D}$ are parameterized by $\mathbb{C}P^1 \cong S^2$. In this case U_{graph} is the dense open subset of $\mathbb{C}P^1$ parameterizing complex lines given by the homogeneous coordinates $[1, z]$ for $z \in \mathbb{C}$. Hence $P_z = \frac{1}{1+|z|^2} \begin{pmatrix} 1 & \bar{z} \\ z & |z|^2 \end{pmatrix}$ defines the boundary condition $\psi(0) = -\bar{z}\psi(2\pi)$, and $P_z^* = \frac{1}{1+|z|^2} \begin{pmatrix} |z|^2 & -\bar{z} \\ -z & -1 \end{pmatrix}$ the condition $-z\psi'(0) = \psi'(2\pi)$. Then $\gneqq_{P_z}$

has discrete spectrum consisting of the eigenvalues $\{(n+\alpha)^2, (n-\alpha)^2 : n \in \mathbb{Z}\}$, where $u = e^{2\pi i\alpha}$ satisfies

$$u^2(1+|z|^2) + 2u(z+\overline{z}) + (1+|z|^2) = 0.$$

Hence, using the ζ-function $\zeta(s,a)$, we compute

$$\begin{aligned}\det{}_\zeta \underset{\mp}{>}_{P_z} = 4\sin^2\pi\alpha = (u-\overline{u})^2 &= \frac{2|1+\overline{z}|^2}{1+|z|^2} \\ &= 2Q_z^{-1}|\det{}_{\zeta,-\pi/2}\mathcal{D}_z|^2 \quad \text{if } z>0.\end{aligned}$$

Note that $\det_\zeta \underset{\mp}{>}_{P_z}$ is globally defined, while $\det_{\zeta,-\pi/2}\mathcal{D}_{P_z}$ is defined only for $z > 0$, corresponding to the non-triviality of the canonical line bundle over $\mathbb{C}P^1$ (note $\mathcal{D}_{P_0}$ has empty spectrum). On the other hand (see (2.18))

$$\det{}_{\mathcal{C}} \underset{\mp}{>}_{P_z} = \frac{1}{2}.\frac{|1+\overline{z}|^2}{1+|z|^2}.$$

Hence $\det_\zeta \underset{\mp}{>}_{P_z}$ and $\det_{\mathcal{C}} \underset{\mp}{>}_{P_z}$ differ by a factor of 4; as we expect from (1.12), since here $\det_\zeta \underset{\mp}{>}_{P(\mathcal{D})} = \det_\zeta \underset{\mp}{>}_{P_1} = 4$.

The canonical metric is the natural metric on DET $(\mathbb{D}, \mathbb{P})$ induced from the Hermitian metrics on the bundles $\mathcal{K}(\mathbb{D})$ and $\mathcal{W}$. We therefore obtain by functoriality a canonical connection on DET $(\mathbb{D}, \mathbb{P})$ compatible with $\|.\|_{\mathcal{C}}$, defined over U by

$$\nabla^{\mathcal{C},\mathbb{P}}\det(\mathcal{D}_P) = \mathrm{Tr}_{\,\mathcal{C}}(\mathcal{D}_P^{-1}\nabla\mathcal{D}_P)\det(\mathcal{D}_P),$$

where[1] $\mathrm{Tr}_{\,\mathcal{C}}(\mathcal{D}_P^{-1}\nabla\mathcal{D}_P) := \mathrm{tr}_{\,K}\left(\mathcal{S}(P)^{-1}\nabla^{K,\mathcal{W}}\mathcal{S}(P)\right)$. Here $\nabla^{\mathcal{K},\mathcal{W}}$ is the induced connection on $\mathrm{Hom}(\mathcal{K}(\mathbb{D}), \mathcal{W})$,

$$\nabla^{\mathcal{K},\mathcal{W}}(B)(\xi) = \nabla^{\mathcal{W}}(B(\xi)) - B\nabla^{\mathcal{K}}\xi, \tag{1.13}$$

for $B \in \mathrm{Hom}(\mathcal{K}(\mathbb{D}), \mathcal{W})$, where $\nabla^{\mathcal{K}}, \nabla^{\mathcal{W}}$ are the connections of Proposition 1.2.

On the other hand, we can use $\mathbb{P}$ to define a modified Bismut connection $\widetilde{\nabla}^{\mathbb{P}}$ on $\mathrm{Hom}(\mathcal{H}, \mathcal{H}_{\mathbb{P}})$. A ζ-function connection on DET $(\mathbb{D}, \mathbb{P})$ can then be defined over U analogously to [2, 7] by setting

$$\frac{\nabla^{\zeta,\mathbb{P}}\det\mathcal{D}_P}{\det\mathcal{D}_P} = \mathrm{Tr}_{\,\zeta}(\mathcal{D}_P^{-1}\nabla\mathcal{D}_P) := \frac{d}{ds}_{|s=0}(s\theta_{\mathbb{P}}(s)) \tag{1.14}$$

where, with $\widetilde{\underset{\mp}{>}}_P = \mathcal{D}_P\mathcal{D}_{P^*}^*$,

$$\theta_{\mathbb{P}}(s) = -\mathrm{Tr}\,(\widetilde{\underset{\mp}{>}}_P^{-s}\mathcal{D}\widetilde{\nabla}^{\mathbb{P}}\mathcal{D}_P^{-1}),$$

is defined around zero by analytic continuation[2] .

Theorem 2 *Let $\mathbb{P}^1$, $\mathbb{P}^2$ be choices of Grassmann sections. Let $\Omega_{\mathcal{C}}^{\mathbb{P}^1}, \Omega_\zeta^{\mathbb{P}^1}$ be the curvature 2-forms of the canonical and zeta connection on* DET $(\mathbb{D}, \mathbb{P}^1)$*, and let $\Omega_{\mathcal{C}}^{\mathbb{P}^2}, \Omega_\zeta^{\mathbb{P}^2}$ be curvature forms on* DET $(\mathbb{D}, \mathbb{P}^2)$*. Then one has*

$$\Omega_\zeta^{\mathbb{P}^1} - \Omega_\zeta^{\mathbb{P}^2} = \Omega_{\mathcal{C}}^{\mathbb{P}^1} - \Omega_{\mathcal{C}}^{\mathbb{P}^2}. \tag{1.15}$$

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[1] Throughout, tr $= \mathrm{tr}_{\,V}$ denotes the trace on a finite-dimensional vector space V, Tr an operator trace, and $\mathrm{Tr}_{\,\mathcal{C}}, \mathrm{Tr}_{\,\zeta}$ the canonical and zeta regularized traces

[2] We differ from [2] by a sign since we use the form on the dual bundle

Equivalently,

$$\begin{aligned} \Omega_\zeta^{\mathbb{P}} &= \Omega_\zeta^{P(\mathbb{D})} + \Omega_{\mathcal{C}}^{\mathbb{P}} && (1.16)\\ &= \Omega_\zeta^{P(\mathbb{D})} + \mathrm{tr}\,(\mathbf{R}^{\mathcal{W}}) - \mathrm{tr}\,(\mathbf{R}^{\mathcal{K}(\mathbb{D})}). && (1.17) \end{aligned}$$

The second identity (1.16), which says that $\Omega_\zeta^{\mathbb{P}}$ consists of an interior part plus a boundary correction term, follows from (1.15) because $\nabla^{\mathcal{C},P(\mathbb{D})}$ is the trivial connection. (1.17) then follows from Proposition 1.2 and the definition of $\nabla^{\mathcal{C},\mathbb{P}}$.

As an example, consider the case of the 'universal' family of EBVPs

$$(\mathbb{D},\mathbb{P}) = \{\mathcal{D}_P : P \in Gr(E_0 \oplus E_\beta)\}$$

relative to a fixed operator $\mathcal{D}$. Let Ω_ζ be the ζ curvature of the corresponding determinant bundle. Then the first and third terms in (1.17) vanish, and we obtain:

Corollary 1.3.

$$\Omega_\zeta = i\omega_{Gr},$$

where ω_{Gr} *is the Kahler form on the Grassmannian.*

In this paper for brevity we restrict our attention to the open subset of B where the operators are invertible. The patching methods required to extend the results globally are well-known [2, 7, 8].

2. Relative Zeta-Function Metric: Proof of Theorem 1

For smooth sections ψ, ϕ of E one has the Green's form

$$< \mathcal{D}\psi, \phi >_X - < \psi, \mathcal{D}^*\phi >_X = < \sigma\gamma\psi, \gamma\phi >, \tag{2.1}$$

where, if $A(x)$ is the leading coefficient of $\mathcal{D}$, $\sigma = -A(0) \oplus A(\beta) \in \mathrm{Gl}(E_0 \oplus E_\beta)$. By definition, (2.1) vanishes for all $\psi \in \mathrm{dom}\,(\mathcal{D}_P)$ if and only if $\phi \in \mathrm{dom}\,(\mathcal{D}^*_{P^*})$, where P^* denotes the adjoint boundary problem. If $A(x)$ is unitary then

$$P^* = \sigma(I-P)\sigma^{-1} \tag{2.2}$$

(cf.[3]). In order to simplify some of the formulas, we will assume this to be the case, so that (2.2) holds, but this assumption is easily removed.

To study the Laplacian boundary problem

$$\begin{cases} \underset{\mp}{\geq}_P &= \mathcal{D}^*\mathcal{D} : \mathrm{dom}\,(\underset{\mp}{\geq}_P) \longrightarrow L^2(X;E) \\ & \\ \mathrm{dom}(\underset{\mp}{\geq}_P) &= \{\psi \in H^2(X;E) \mid P^*\gamma\mathcal{D}\psi = 0,\ P\gamma\psi = 0\}\ , \end{cases}$$

observe that $\mathrm{dom}(\underset{\mp}{\geq}_P)$ is a subspace of the domain of the first-order EBVP

$$\widehat{\underset{\mp}{\geq}}_P = \widehat{\underset{\mp}{\geq}} : \mathrm{dom}(\widehat{\underset{\mp}{\geq}}_P) \to L^2(X;E \oplus E)\ .$$

Here

$$\widehat{\underset{\mp}{\geq}} := \begin{pmatrix} \mathcal{D} & -I \\ 0 & \mathcal{D}^* \end{pmatrix} : H^1(X;E\oplus E) \longrightarrow L^2(X;E\oplus E),$$

with dom($\widehat{\gvertneqq}_P$) $= \{(\psi,\phi) \in H^1(X;E\oplus E) \mid \widehat{P}\widehat{\gamma}(\psi,\phi) = 0\}$, where $\widehat{P} := P \oplus P^*$ and $\widehat{\gamma}(\psi,\phi) := (\gamma\psi,\gamma\phi)$. The map $\psi \longmapsto \widehat{\psi} = (\psi,\mathcal{D}\psi)$ defines a canonical embedding $H^2(X;E) \longrightarrow H^1(X;E\oplus(X;E)$ and we have

$$\widehat{\gvertneqq}\widehat{\psi} = \begin{pmatrix} 0 \\ \gvertneqq\psi \end{pmatrix}, \tag{2.3}$$

identifying the solution spaces of the operators $\gvertneqq$ and $\widehat{\gvertneqq}$: if $\{\psi_1,\dots,\psi_k\}$ is a basis for Ker($\gvertneqq$) then $\{\widehat{\psi}_1,\dots,\widehat{\psi}_k\}$ is a basis for Ker($\widehat{\gvertneqq}$). Moreover, there is a preferred such basis formed by the columns of the fundamental solution matrix $\widehat{h}(x) : E_0\oplus E_0 \to E_x \oplus E_x$ for $\widehat{\gvertneqq}$, solving uniquely $\widehat{\gvertneqq}\widehat{h}(x) = 0,\ \widehat{h}(0) = I$. We define

$$\mathcal{S}(\widehat{P}) := \widehat{P}\circ P(\widehat{\gvertneqq}) : K(\widehat{\gvertneqq}) \longrightarrow \widehat{W} = \operatorname{range}(\widehat{P})\ , \tag{2.4}$$

where $K(\widehat{\gvertneqq}) = \operatorname{graph}(\widehat{h} = \widehat{h}(\beta) : E_0\oplus E_0 \to E_\beta \oplus E_\beta)$ is the Cauchy space for $\widehat{\gvertneqq}$.

For a linear operator $A : E_0 \oplus E_1 \to F_0 \oplus F_1$ considered as a block 2×2 matrix relative to the direct sums, $[A]_{(1,2)} : E_1 \to F_0$ refers to the top-right entry in the $(1,2)$ position. From (2.3)

$$(\gvertneqq_P - \lambda)^{-1} = \left[\widehat{\gvertneqq}_{P,\lambda}^{-1}\right]_{(1,2)}, \tag{2.5}$$

where $\widehat{\gvertneqq}_{P,\lambda} = \widehat{\gvertneqq}_\lambda = \begin{pmatrix} \mathcal{D} & -I \\ -\lambda & \mathcal{D}^* \end{pmatrix}$, with domain dom$(\widehat{\gvertneqq}_P)$. Indeed, we compute

$$\widehat{\gvertneqq}_{P,\lambda}^{-1} = \begin{pmatrix} \mathcal{D}_P^*(\tilde{\gvertneqq}_P - \lambda)^{-1} & (\gvertneqq_P - \lambda)^{-1} \\ \lambda(\tilde{\gvertneqq}_P - \lambda)^{-1} & \mathcal{D}_P(\gvertneqq_P - \lambda)^{-1} \end{pmatrix}, \tag{2.6}$$

where $\tilde{\gvertneqq} = \mathcal{D}^*\mathcal{D}$, $\tilde{\gvertneqq}_P = \mathcal{D}_P\mathcal{D}_P^*$.

The Poisson operator of $\gvertneqq$ is the operator

$$\widehat{\mathcal{K}} : (E_0\oplus E_0)\oplus(E_\beta\oplus E_\beta) \to C^\infty(X,E), \tag{2.7}$$

$$\widehat{\mathcal{K}}(v)(x) = \widehat{h}(x)p_0P(\widehat{\gvertneqq})v,$$

where p_0 is the projection map $(E_0\oplus E_0)\oplus(E_\beta\oplus E_\beta) \to (E_0\oplus E_0)$. The restriction of $\widehat{\mathcal{K}}$ to $K(\widehat{\gvertneqq})$ is an isomorphism $\widehat{\mathcal{K}} : K(\widehat{\gvertneqq}) \longrightarrow \operatorname{Ker}\widehat{\gvertneqq} \cong \operatorname{Ker}\gvertneqq$ while

$$\widehat{\gamma}\circ\widehat{\mathcal{K}} = P(\widehat{\gvertneqq}) \tag{2.8}$$

as operators on $(E_0\oplus E_0)\oplus(E_\beta\oplus E_\beta)$.

The invertibility of the operators $\mathcal{D}_P, \gvertneqq_P, \widehat{\gvertneqq}_P, \mathcal{S}(\widehat{P})$ are equivalent statements and in this case we can define the *Poisson operator* of $\gvertneqq_P$ by

$$\widehat{\mathcal{K}}(\widehat{P}) = \widehat{\mathcal{K}}\mathcal{S}(\widehat{P})^{-1}\widehat{P} : (E_0\oplus E_0)\oplus(E_\beta\oplus E_\beta) \to C^\infty(X,E). \tag{2.9}$$

The restriction $\widehat{\mathcal{K}}(\widehat{P}) : \operatorname{range}(\widehat{P}) \to \operatorname{Ker}(\widehat{\gvertneqq})$ is an isomorphism with left-inverse $\widehat{P}\gamma_{|Ker(\widehat{\gvertneqq})}$, for

$$\widehat{P}\gamma\widehat{\mathcal{K}}(\widehat{P}) = \widehat{P}\gamma\widehat{\mathcal{K}}\mathcal{S}(\widehat{P})^{-1}\widehat{P} = P(\widehat{\gvertneqq})\mathcal{S}(\widehat{P})^{-1}\widehat{P} = \widehat{P}\ . \tag{2.10}$$

The following relative inverse formula holds:

Proposition 2.1. *If* $\gneqq_{P_1}, \gneqq_{P_2}$ *are invertible, then*

$$\gneqq_{P_1}^{-1} = \gneqq_{P_2}^{-1} - [\widehat{\mathcal{K}}(\widehat{P}_1)\widehat{\gamma}\widehat{\gneqq}_{P_2}^{-1}]_{(1,2)}. \tag{2.11}$$

In particular, $\gneqq_{P_1}^{-1} - \gneqq_{P_2}^{-1}$ *is a smoothing operator.*

Proof. We have

$$\widehat{\gneqq}_{P}^{-1}\widehat{\gneqq} = I - \widehat{\mathcal{K}}(\widehat{P})\gamma \tag{2.12}$$

and hence

$$\gneqq_{P_1}^{-1} = [\widehat{\gneqq}_{P_1}^{-1}]_{(1,2)} = [\widehat{\gneqq}_{P_1}^{-1}\widehat{\gneqq}\widehat{\gneqq}_{P_2}^{-1}]_{(1,2)} = [(I - \widehat{\mathcal{K}}(\widehat{P}_1)\gamma)\widehat{\gneqq}_{P_2}^{-1}]_{(1,2)} = \gneqq_{P_2}^{-1} - [\widehat{\mathcal{K}}(\widehat{P}_1)\widehat{\gamma}\widehat{\gneqq}_{P_2}^{-1}]_{(1,2)} \ .$$

To see (2.12), one can either check it directly using (2.24), or invariantly as in [10, 9]. ☐

For later use, note that there is a similar relative inverse for the EBVP $\mathcal{D}_P$. $\mathcal{D}$ has Poisson operator

$$\mathcal{K} : E_0 \oplus E_\beta \longrightarrow C^\infty(X, E), \qquad \mathcal{K}(u)(x) = h(x)p_0 P(\mathcal{D})u,$$

with p_0 the projection map $E_0 \oplus E_\beta \to E_0$, which restricts to an isomorphism $\mathcal{K} : K(\mathcal{D}) \to \mathrm{Ker}(\mathcal{D})$. If $\mathcal{D}_P$ is invertible

$$\mathcal{D}_P^{-1}\mathcal{D} = I - \mathcal{K}(\widehat{P})\gamma \ , \tag{2.13}$$

where $\mathcal{K}(P) = \mathcal{K}\mathcal{S}(P)^{-1}P : E_0 \oplus E_\beta \longrightarrow C^\infty(X, E)$, and by a similar argument to Proposition 2.1

$$\mathcal{D}_{P_1}^{-1} = \mathcal{D}_{P_2}^{-1} - \mathcal{K}(P_1)\gamma\mathcal{D}_{P_2}^{-1} \ . \tag{2.14}$$

2.1. **Stiefel coordinates.** An element of $\mathrm{Hom}(E_0 \oplus E_\beta, E_0)$ can be written $M = [M_0 \ M_\beta]$ where $M_0 \in \mathrm{Hom}(E_0, E_0)$ and $M_\beta \in \mathrm{Hom}(E_\beta, E_0)$. The complex Stiefel manifold St_k parameterizes elements of $\mathrm{Hom}(E_0 \oplus E_\beta, E_0)$ of rank k (at least one invertible $k \times k$ minor), and the projection map

$$\pi : St_k \longrightarrow \mathrm{Gr}_k(E_0 \oplus E_\beta), \quad M \longmapsto P_{[M_0, M_\beta]} = \begin{pmatrix} M_0^* M_{0,\beta}^{-1} M_0 & M_0^* M_{0,\beta}^{-1} M_\beta \\ M_\beta^* M_{0,\beta}^{-1} M_0 & M_\beta^* M_{0,\beta}^{-1} M_\beta \end{pmatrix}, \tag{2.15}$$

where $M_{0,\beta} := MM^* = M_0 M_0^* + M_\beta M_\beta^*$, defines St_k as a principal $Gl(\mathbb{C}^k)$ bundle over $\mathrm{Gr}_k(E_0 \oplus E_\beta)$, the Stiefel frame bundle. Over the index zero component of the Grassmannian dom$\,(\mathcal{D}_P)$ has the following description in Stiefel coordinates $[M_0, M_\beta]$:

Lemma 2.2. *For* $P = P_{[M_0, M_\beta]} \in \mathrm{Gr}_n(E_0 \oplus E_\beta)$ *one has*

$$\mathrm{dom}(\mathcal{D}_P) = \left\{ \psi \in H^1(X; E) \mid M_0\psi(0) + M_\beta\psi(\beta) = 0 \right\}. \tag{2.16}$$

Proof. The Lemma states that

$$P \begin{pmatrix} \psi(0) \\ \psi(\beta) \end{pmatrix} = 0 \quad \text{and} \quad M_0\psi(0) + M_\beta\psi(\beta) = 0 \tag{2.17}$$

are the same statement. But from (2.15) the first equality gives

$$\begin{cases} M_0^* M_{0,\beta}^{-1}(M_0\psi(0) + M_\beta\psi(\beta)) = 0 \\ M_\beta^* M_{0,\beta}^{-1}(M_0\psi(0) + M_\beta\psi(\beta)) = 0, \end{cases},$$

while multiplying these equations respectively by M_0 and M_β and summing them is the second equation in (2.17). The reverse implication is obvious. ☐

Note that (2.16) has a $\mathrm{Gl}(\mathbb{C}^n)$'s worth of ambiguity in describing $\mathrm{dom}(\mathcal{D}_P)$ corresponding to a choice of generator $[M_0 \;\; M_\beta]$ in the fibre of St_n over P.

We have the following Stiefel coordinate formula for the canonical metric:

Proposition 2.3. *Let* $P = P_{[M_0, M_\beta]} \in \mathrm{Gr}_n(E_0 \oplus E_\beta)$ *and let* $\mathcal{M} = M_0 + M_\beta h \in \mathrm{End}(E_0)$. *Then*

$$\det{}_{\mathcal{C}}\Delta_P = \det Q_h^{-1} \det M_{0,\beta}^{-1} \, |\det \mathcal{M}|^2. \tag{2.18}$$

Proof. We have $\mathcal{S}(P)^*\mathcal{S}(P) = P(\mathcal{D})PP(\mathcal{D}) : K(\mathcal{D}) \to K(\mathcal{D})$ and $K(\mathcal{D}) = \{(\xi, h\xi) : \xi \in E_0\} \subset E_0 \oplus E_\beta$. $\mathrm{End}(E_0)$ acts on $K(\mathcal{D})$ by $q.(\xi, h\xi) = (q\xi, hq\xi)$. So, using (2.15),

$$P(\mathcal{D})PP(\mathcal{D}) \begin{pmatrix} \xi \\ h\xi \end{pmatrix} = \begin{pmatrix} Q_h^{-1} & Q_h^{-1}h^* \\ hQ_h^{-1} & hQ_h^{-1}h^* \end{pmatrix} \begin{pmatrix} M_0^* M_{0,\beta}^{-1} \mathcal{M}\,\xi \\ M_\beta^* M_{0,\beta}^{-1} \mathcal{M}\,\xi \end{pmatrix} = Q_h^{-1} \mathcal{M}^* M_{0,\beta}^{-1} \mathcal{M} \begin{pmatrix} \xi \\ h\xi \end{pmatrix}.$$

Hence $\det_{\mathcal{C}}\Delta = \det(Q_h^{-1} M_{0,\beta}^{-1} \mathcal{M}^* \mathcal{M})$, and we reach the conclusion. □

We also need a Stiefel coordinate formula for $\widehat{\gneqq}_P^{-1}$. First:

Lemma 2.4. *Let* $P = P_{[M_0, M_\beta]}$. *Then*

$$\mathrm{dom}\,(\gneqq_P) = \left\{ \psi \in H^2(X; E) \mid \widehat{M_0} \begin{pmatrix} \psi(0) \\ \mathcal{D}\psi(0) \end{pmatrix} + \widehat{M_\beta} \begin{pmatrix} \psi(\beta) \\ \mathcal{D}\psi(\beta) \end{pmatrix} = 0 \right\}, \tag{2.19}$$

where

(2.20)
$$\widehat{M_0} = \begin{pmatrix} M_0^* M_{0,\beta}^{-1} M_0 & M_0^* M_{0,\beta}^{-1} M_0 A_0^{-1} - A_0^{-1} \\ M_\beta^* M_{0,\beta}^{-1} M_0 & M_\beta^* M_{0,\beta}^{-1} M_0 A_0^{-1} \end{pmatrix}, \quad \widehat{M_\beta} = \begin{pmatrix} M_0^* M_{0,\beta}^{-1} M_\beta & -M_0^* M_{0,\beta}^{-1} M_\beta A_\beta^{-1} \\ M_\beta^* M_{0,\beta}^{-1} M_\beta & A_\beta^{-1} - M_\beta^* M_{0,\beta}^{-1} M_\beta A_\beta^{-1} \end{pmatrix},$$

are canonically defined by $\widehat{P}$. *Here* $A_0 := A(0), A_\beta := A(\beta)$. *With respect to the decomposition* $(E_0 \oplus E_0) \oplus (E_\beta \oplus E_\beta)$ *of the space of boundary data, one has*

$$\widehat{P} = P_{[\widehat{M_0}, \widehat{M_\beta}]}. \tag{2.21}$$

Proof. From (2.2) we have $P\gamma\psi = 0,\ P^*\gamma\mathcal{D}\psi = 0$ is equivalent to

$$P \begin{pmatrix} \psi(0) \\ \psi(\beta) \end{pmatrix} + \sigma^{-1} P^* \begin{pmatrix} \mathcal{D}\psi(0) \\ \mathcal{D}\psi(\beta) \end{pmatrix} = 0. \tag{2.22}$$

But $\sigma = -A_0 \oplus A_\beta$, and from (2.15) we obtain (2.20) by substituting in (2.22). The identity (2.21) follows as in Lemma 2.2. □

Let $\widehat{h}_\lambda(x) : E_0 \oplus E_0 \to E_x \oplus E_x$ be the fundamental solution matrix for $\widehat{\gneqq}_\lambda$,

$$\widehat{\gneqq}_\lambda \widehat{h}_\lambda(x) = 0, \quad \widehat{h}_\lambda(0) = I. \tag{2.23}$$

Then we have:

Proposition 2.5. *Let* $P = P_{[M_0, M_\beta]}$. *Then* $\gneqq_{P,\lambda}$ *is invertible if and only if*

$$\widehat{\mathcal{M}_\lambda} = \widehat{M_0} + \widehat{M_\beta} \widehat{h}_\lambda,$$

is invertible, and in that case $\gneqq_{P,\lambda}^{-1}$ *has kernel*

$$
k_{P,\lambda}(x,y) = \begin{cases} -\left[\widehat{h}_\lambda(x)(\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M}_\beta\widehat{h}_\lambda)\widehat{h}_\lambda(y)^{-1}\widehat{A}(y)^{-1}\right]_{(1,2)} & x<y \\ \\ \left[\widehat{h}_\lambda(x)(I-\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M}_\beta\widehat{h}_\lambda)\widehat{h}_\lambda(y)^{-1}\widehat{A}(y)^{-1}\right]_{(1,2)} & x>y, \end{cases} \tag{2.24}
$$

where $\widehat{A}(x) = A(x)\oplus -A^*(x)$. *In particular, if* $P_1 = P_{[M_0,M_\beta]}$, $P_2 = P_{[N_0,N_\beta]}$ *then* $\gvertneqq_{P_1,\lambda}^{-1} - \gvertneqq_{P_2,\lambda}^{-1}$ *has smooth kernel*

$$
-\left[\widehat{h}_\lambda(x)(\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M}_\beta - \widehat{\mathcal{N}}_\lambda^{-1}\widehat{N}_\beta)\widehat{h}_\lambda\widehat{h}_\lambda(y)^{-1}\widehat{A}(y)^{-1}\right]_{(1,2)}.
$$

Proof. For each fixed y, $\widehat{k}_P(x,y)$ must satisfy $\widehat{P}\begin{pmatrix}\widehat{k}_P(0,y)v\\ \widehat{k}_P(\beta,y)v\end{pmatrix} = 0$, for all $v\in E_y$. By Lemma 2.4 this is equivalent to $\widehat{M_0}\widehat{k}_P(0,y) + \widehat{M_\beta}\widehat{k}_P(\beta,y) = 0$. On the other hand, from (2.23) we have $\widehat{\gvertneqq}_{P,\lambda}^{-1} = \widehat{h}_\lambda(x)\left(\frac{d}{dx}\right)_{P'}^{-1}\widehat{h}_\lambda(x)^{-1}\widehat{A}(x)^{-1}$, where P' is the gauge transformed boundary condition with respect to $\widehat{h}_\lambda^{-1}$. Since the derivative of the Heaviside function is the Dirac delta distribution (2.24) now follows.

For the first statement, note that $\gvertneqq_{P,\lambda}$ is invertible if and only if $\mathcal{S}_\lambda(\widehat{P}) = \widehat{P}\circ P(\widehat{\gvertneqq}_\lambda)$ is invertible, while by a direct computation

$$
\mathcal{S}_\lambda(\widehat{P})^{-1} = \begin{pmatrix}\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M_0} & \widehat{\mathcal{M}}_\lambda^{-1}\widehat{M_\beta}\\ \widehat{h}\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M_0} & \widehat{h}\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M_\beta}\end{pmatrix}. \tag{2.25}
$$

□

Note that the Stiefel coordinate formula (2.24) also follows from (2.11) by a direct substitution using (2.25).

2.2. **Proof of Theorem 1.** We know that $(\gvertneqq_P - \lambda)^{-1}$ is trace class. In fact, there is the following precise formula:

Proposition 2.6.

$$
\operatorname{Tr}(\gvertneqq_P - \lambda)^{-1} = -\frac{\partial}{\partial\lambda}\log\det\widehat{\mathcal{M}}_\lambda, \tag{2.26}
$$

Proof. For $C(x) : \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n$ we have tr $[C(x)]_{1,2} = -\mathrm{tr}\,(\frac{\partial}{\partial\lambda}(\widehat{\gneqq}_\lambda)C(x))$, while from (2.23), $\frac{\partial}{\partial\lambda}\widehat{\gneqq}_\lambda\widehat{h}_\lambda(x) = -\widehat{\gneqq}_\lambda\frac{\partial}{\partial\lambda}\widehat{h}_\lambda(x)$. We therefore have

$$\begin{aligned}
\mathrm{Tr}\ (\gneqq_P - \lambda)^{-1} &= \int_0^\beta \mathrm{tr}\ \{k_{P,\lambda}(x,x)\}\ dx \\
&= \int_0^\beta \mathrm{tr}\ \left\{\left[\widehat{h}_\lambda(x)\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M}_\beta\widehat{h}_\lambda\widehat{h}_\lambda(x)^{-1}\widehat{A}(x)^{-1}\right]_{(1,2)}\right\}\ dx \\
&= -\int_0^\beta \mathrm{tr}\ \left\{\frac{\partial}{\partial\lambda}\widehat{\gneqq}_\lambda\widehat{h}_\lambda(x)\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M}_\beta\widehat{h}_\lambda\widehat{h}_\lambda(x)^{-1}\widehat{A}(x)^{-1}\right\}\ dx \\
&= -\int_0^\beta \mathrm{tr}\ \left\{\widehat{\gneqq}_\lambda\frac{\partial}{\partial\lambda}(\widehat{h}_\lambda(x))\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M}_\beta\widehat{h}_\lambda\widehat{h}_\lambda(x)^{-1}\widehat{A}(x)^{-1}\right\}\ dx \\
&= -\int_0^\beta \mathrm{tr}\ \left\{(\widehat{h}_\lambda(x)^{-1}\widehat{A}(x)^{-1}\widehat{\gneqq}_\lambda\widehat{h}_\lambda(x)).\widehat{h}_\lambda(x)^{-1}\frac{\partial}{\partial\lambda}(\widehat{h}_\lambda(x))\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M}_\beta\widehat{h}_\lambda\right\}\ dx \\
&= -\int_0^\beta \mathrm{tr}\ \left\{\frac{d}{dx}\left(\widehat{h}_\lambda(x)^{-1}\frac{\partial}{\partial\lambda}(\widehat{h}_\lambda(x))\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M}_\beta\widehat{h}_\lambda\right)\right\}\ dx \\
&= -\left[\mathrm{tr}\ \left\{\widehat{h}_\lambda(x)^{-1}\frac{\partial}{\partial\lambda}(\widehat{h}_\lambda(x))\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M}_\beta\widehat{h}_\lambda\right\}\right]_{x=0}^{\beta} \\
&= -\mathrm{tr}\ \left\{\widehat{h}_\lambda^{-1}\frac{\partial}{\partial\lambda}(\widehat{h}_\lambda)\widehat{\mathcal{M}}_\lambda^{-1}\widehat{M}_\beta\widehat{h}_\lambda\right\} \qquad\qquad \text{since } \frac{\partial}{\partial\lambda}\widehat{h}_\lambda(0) = 0 \\
&= -\frac{\partial}{\partial\lambda}\log\det\ \widehat{\mathcal{M}}_\lambda.
\end{aligned}$$

□

If $\gneqq_{P_1}, \gneqq_{P_2}$ are invertible, with $P_1 = P_{[M_0,M_\beta]}, P_2 = P_{[N_0,N_\beta]}$, then from (2.26) we have

$$\mathrm{Tr}\ \left((\gneqq_{P_1} - \lambda)^{-1} - (\gneqq_{P_2} - \lambda)^{-1}\right) = -\frac{\partial}{\partial\lambda}\log\frac{\det\widehat{\mathcal{M}}_\lambda}{\det\widehat{\mathcal{N}}_\lambda}, \tag{2.27}$$

so the scattering matrix is $\mathcal{S}_\lambda = \widehat{\mathcal{N}}_\lambda^{-1}\widehat{\mathcal{M}}_\lambda$. Because the boundary problems $\gneqq_{P_i}$ are elliptic in the classical sense of Seeley [11], they have asymptotic expansions as $\lambda \to \infty$ in $\Lambda_{\pi,\varepsilon}$

$$\mathrm{Tr}\,(\gneqq_{P_i} - \lambda)^{-1} \sim c_{-1}^{(i)}(-\lambda)^{1/2} + \sum_{k\geq 1} c_k^{(i)}(-\lambda)^{-k/2}\ . \tag{2.28}$$

Hence we find that $\gneqq_{P_1}, \gneqq_{P_2}$ are ζ-comparable and ζ-admissible, and so by equation (1.6) (or by [5]) we have

$$\frac{\det_\zeta(\gneqq_{P_1})}{\det_\zeta(\gneqq_{P_2})} = \frac{\det\widehat{\mathcal{M}}}{\det\widehat{\mathcal{N}}}.\ \exp\left[-\mathrm{LIM}_{\lambda\to\infty}\log\det\frac{\det\widehat{\mathcal{M}}_{-\lambda}}{\det\widehat{\mathcal{N}}_{-\lambda}}\right]\ . \tag{2.29}$$

Proposition 2.7.

$$\det\widehat{\mathcal{M}} = \det(A_0)^{-1}\det(h^*)^{-1}\det M_{0,\beta}^{-1}\,|\det\mathcal{M}|^2 \tag{2.30}$$

Proof. We compute that

$$\widehat{h}(x) = \begin{pmatrix} h(x) & J(x) \\ 0 & A(x)(h(x)^*)^{-1}A_0^{-1} \end{pmatrix},$$

where $h(x)$ is the parallel transport for $\mathcal{D}$, and $J(x)$ is the unique solution to

$$\mathcal{D}J(x) = A(x)(h(x)^*)^{-1}A_0^{-1}, \qquad J(0) = 0.$$

Hence since $\widehat{h} = \widehat{h}(\beta)$, and setting $J := J(\beta)$, we have from Lemma 2.4

$$\widehat{\mathcal{M}} = \begin{pmatrix} M_0^* M_{0,\beta}^{-1}\mathcal{M} & M_0^* M_{0,\beta}^{-1} M_\beta J + M_0^* M_{0,\beta}^{-1}(M_0 - M_\beta(h^*)^{-1})A_0^{-1} - A_0^{-1} \\ M_\beta^* M_{0,\beta}^{-1}\mathcal{M} & M_0^* M_{0,\beta}^{-1} M_\beta J + M_\beta^* M_{0,\beta}^{-1}(M_0 - M_\beta(h^*)^{-1}) + (h^*)^{-1}A_0^{-1} \end{pmatrix}$$

(2.31)

$$= \begin{pmatrix} 0 & I \\ (h^*)^{-1}\mathcal{M}^* & -(h^*)^{-1} \end{pmatrix} \begin{pmatrix} I & M_{0,\beta}^{-1}(M_\beta J + M_0 - M_\beta(h^*)^{-1}) \\ M_0^* & M_0^* M_{0,\beta}^{-1}(M_\beta J + M_0 - M_\beta(h^*)^{-1}) - I \end{pmatrix} \begin{pmatrix} M_{0,\beta}^{-1}\mathcal{M} & 0 \\ 0 & A_0^{-1} \end{pmatrix}.$$

Since a 2×2 block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} : E_0 \oplus E_1 \longrightarrow E_0 \oplus E_1$ where $A : E_0 \to E_0$, $B : E_1 \to E_0$ etc, has determinant $\det(A).\det(D - CA^{-1}B)$ if A is invertible, and determinant $\det(D).\det(A - BD^{-1}C)$ if D is invertible, we find that the determinant of the second matrix in (2.31) reduces to $\det(-I)$, in particular the J term disappears, and term by term we obtain

$$\det \widehat{\mathcal{M}} = \det(-(h^*)^{-1}\mathcal{M}^*).\det(-I).\left(\det(M_{0,\beta}^{-1}\mathcal{M})\det(A_0)^{-1}\right),$$

and this is (2.30). □

In view of (2.18) and (2.30), we can rewrite (2.29) as

$$\frac{\det_\zeta(\gvertneqq_{P_1})}{\det_\zeta(\gvertneqq_{P_2})} = \frac{\det_{\mathcal{C}}(\gvertneqq_{P_1})}{\det_{\mathcal{C}}(\gvertneqq_{P_2})} \exp\left(-\mathrm{LIM}_{\lambda\to\infty} \log\det \frac{\det \widehat{\mathcal{M}}_{-\lambda}}{\det \widehat{\mathcal{N}}_{-\lambda}}\right). \tag{2.32}$$

It remains to show that the LIM term vanishes. Consider first the case where $\mathcal{D}$ is self-adjoint. Then $h(x) \in U(n)$ and we can gauge transform $\gvertneqq_P$ to $\gvertneqq^0_{U^{-1}PU}$, where $\gvertneqq^0$ is a flat Laplacian and $U = \gamma h(x) = I \oplus h$. By continuity it is enough to work over the dense open subset $U_{Gl} \subset Gr_{2n}(E_0 \oplus E_\beta)$, parameterizing graphs of invertible $T : E_0 \to E_\beta$, defining Stiefel coordinates $M_0 = I$ and $M_\beta := T^*$. Set $P_1 := P_T$. Then $\det \widehat{M_\beta} = \det(Q_T^{-1}T^*)$ and so $\widehat{M_\beta}$ is invertible, and we have

$$\log\det \widehat{\mathcal{M}_\lambda} = \log\det \widehat{M_\beta} + \log\det(\widehat{M_\beta}^{-1}\widehat{M_0} + \widehat{h}_\lambda).$$

A similar computation to [5], Prop. 3.4, yields for $\lambda \longrightarrow \infty$

$$\log\det(\widehat{M_\beta}^{-1}\widehat{M_0} + \widehat{h}_\lambda) = n\beta\sqrt{\lambda} + \log 2^{-n} - \log\det \widehat{M_\beta} + O(\frac{1}{\sqrt{\lambda}}),$$

and hence that $\log\det \widehat{\mathcal{M}_\lambda} = n\beta\sqrt{\lambda} + \log 2^{-n} + O(1/\sqrt{\lambda})$. Repeating the argument for $\widehat{\mathcal{N}_\lambda}$ we therefore have

$$\log\det \frac{\det \widehat{\mathcal{M}}_{-\lambda}}{\det \widehat{\mathcal{N}}_{-\lambda}} = O(\frac{1}{\sqrt{\lambda}}),$$

as $\lambda \longrightarrow \infty$, and hence (2.32) reduces to (1.11).

We extend this to general $\mathcal{D}$ through a variational method:

Proposition 2.8. *Let $\mathcal{D}^r$ be a one parameter family of Dirac operators. Let $P_1, P_2 \in Gr_n(E_0 \oplus E_\beta)$ with $\mathcal{D}^r_{P_i}$ invertible. Then the $\gneqq_{P_i} = \gneqq^r_{P_i}$ are invertible and*

$$\frac{d}{dr}\log\frac{\det_\zeta(\gneqq_{P_1})}{\det_\zeta(\gneqq_{P_2})} = \frac{d}{dr}\log\frac{\det_{\mathcal{C}}(\gneqq_{P_1})}{\det_{\mathcal{C}}(\gneqq_{P_2})}. \tag{2.33}$$

Proof. Let $h_r, \widehat{h}_r$ denote the respective paths of parallel transport operators of the first-order elliptic operators $\mathcal{D}^r, \widehat{\gneqq}^r$. Let $\mathcal{M}_r = M_0 + M_\beta h_r$, $\mathcal{N}_r = N_0 + N_\beta h_r$, and let $\widehat{\mathcal{M}}_r = \widehat{M}_0 + \widehat{M}_\beta \widehat{h}_r$, $\widehat{\mathcal{N}}_r = \widehat{N}_0 + \widehat{N}_\beta \widehat{h}_r$. Then, from (2.30),

$$\frac{d}{dr}\log\frac{\det_{\mathcal{C}}(\gneqq_{P_1})}{\det_{\mathcal{C}}(\gneqq_{P_2})} = \frac{d}{dr}\log\frac{\det(\mathcal{M}_r^*\mathcal{M}_r)}{\det(\mathcal{N}_r^*\mathcal{N}_r)}. \tag{2.34}$$

On the other hand, a straightforward application of Duhamel's Principle yields

$$\frac{d}{dr}\log\frac{\det_\zeta(\gneqq_{P_1})}{\det_\zeta(\gneqq_{P_2})} = \operatorname{Tr}\left\{\dot{\mathcal{D}}^*\left((\mathcal{D}^*_{P_1^*})^{-1} - (\mathcal{D}^*_{P_2^*})^{-1}\right)\right\} + \operatorname{Tr}\left\{\dot{\mathcal{D}}\left\{\mathcal{D}^{-1}_{P_1} - \mathcal{D}^{-1}_{P_2}\right)\right\},$$

while from (2.6) and $\dot{\widehat{\gneqq}} = \dot{\mathcal{D}} \oplus \dot{\mathcal{D}}^*$ we find

$$\widehat{\gneqq}\left(\widehat{\gneqq}^{-1}_{P_1} - \widehat{\gneqq}^{-1}_{P_2}\right) = \begin{pmatrix} \dot{\mathcal{D}}\left(\mathcal{D}^{-1}_{P_1} - \mathcal{D}^{-1}_{P_2}\right) & \gneqq^{-1}_{P_1} - \gneqq^{-1}_{P_2} \\ 0 & \dot{\mathcal{D}}^*\left((\mathcal{D}^*_{P_1^*})^{-1} - (\mathcal{D}^*_{P_2^*})^{-1}\right) \end{pmatrix},$$

and hence that

$$\frac{d}{dr}\log\frac{\det_\zeta(\gneqq_{P_1})}{\det_\zeta(\gneqq_{P_2})} = \operatorname{Tr}\left\{\dot{\widehat{\gneqq}}\left(\widehat{\gneqq}^{-1}_{P_1} - \widehat{\gneqq}^{-1}_{P_2}\right)\right\}. \tag{2.35}$$

We have $\widehat{\gneqq}^{-1}_{P_1} = \widehat{\gneqq}^{-1}_{P_2} - \widehat{\mathcal{K}}_r(\widehat{P}_1)\widehat{\gamma}\widehat{\gneqq}^{-1}_{P_2}$, where $\widehat{\mathcal{K}}_r(\widehat{P}_1)\widehat{\gamma} = \widehat{H}_r p_0 P(\widehat{\gneqq})\mathcal{S}_r(\widehat{P}_1)^{-1}\widehat{P}_1\gamma$, and $(\widehat{H}_r v)(x) = \widehat{h}_r(x)v$, and since $\widehat{\gneqq}^r\widehat{H}_r = 0$, then $\dot{\widehat{\gneqq}}\widehat{H}_r = -\widehat{\gneqq}\dot{\widehat{H}}_r$. Therefore using (2.12) we have

$$\begin{aligned}
\operatorname{Tr}\left\{\dot{\widehat{\gneqq}}\left(\widehat{\gneqq}^{-1}_{P_1} - \widehat{\gneqq}^{-1}_{P_2}\right)\right\} &= -\operatorname{Tr}\left\{\dot{\widehat{\gneqq}}\widehat{H}_r p_0\mathcal{S}_r(\widehat{P}_1)^{-1}\widehat{P}_1\gamma\widehat{\gneqq}^{-1}_{P_2}\right\} \\
&= \operatorname{Tr}\left\{\widehat{\gneqq}\dot{\widehat{H}}_r p_0\mathcal{S}_r(\widehat{P}_1)^{-1}\widehat{P}_1\gamma\widehat{\gneqq}^{-1}_{P_2}\right\} \\
&= \operatorname{Tr}\left\{\widehat{P}_1\gamma\widehat{\gneqq}^{-1}_{P_2}\widehat{\gneqq}\dot{\widehat{H}}_r p_0\mathcal{S}_r(\widehat{P}_1)^{-1}\widehat{P}_1\right\} \\
&= \operatorname{Tr}\left\{\widehat{P}_1\gamma\left(I - \widehat{\mathcal{K}}_r(\widehat{P}_1)\widehat{\gamma}\right)\dot{\widehat{H}}_r p_0\mathcal{S}_r(\widehat{P}_1)^{-1}\widehat{P}_1\right\} \\
&= \operatorname{Tr}\left\{\widehat{P}_1\gamma\dot{\widehat{H}}_r p_0\mathcal{S}_r(\widehat{P}_1)^{-1}\widehat{P}_1\right\} - \operatorname{Tr}\left\{\widehat{P}_1\gamma\widehat{\mathcal{K}}_r(\widehat{P}_1)\widehat{\gamma}\dot{\widehat{H}}_r p_0\mathcal{S}_r(\widehat{P}_1)^{-1}\widehat{P}_1\right\} \\
&= \operatorname{Tr}\left\{\mathcal{S}_r(\widehat{P}_1)^{-1}\widehat{P}_1\frac{d}{dr}P(\widehat{\gneqq})\right\} - \operatorname{Tr}\left\{\mathcal{S}_r(\widehat{P}_1)\mathcal{S}_r(\widehat{P}_2)^{-1}\widehat{P}_2\frac{d}{dr}P(\widehat{\gneqq})\mathcal{S}_r(\widehat{P}_1)^{-1}\right\} \\
&= \operatorname{Tr}\left\{\mathcal{S}_r(\widehat{P}_1)^{-1}\frac{d}{dr}\mathcal{S}_r(\widehat{P}_1)\right\} - \operatorname{Tr}\left\{\mathcal{S}_r(\widehat{P}_2)^{-1}\frac{d}{dr}\mathcal{S}_r(\widehat{P}_2)\right\},
\end{aligned}$$

using the symmetry of the trace, and $\frac{d}{dr}\mathcal{S}(\widehat{P}_i) = \widehat{P}_i\frac{d}{dr}P(\widehat{\gneqq})$ and equation (2.8).

Now choose Stiefel coordinates for $P = P_{[M_0,M_\beta]}$. The finite-rank operator $\mathcal{S}_r(\widehat{P}) : K(\widehat{\underset{\mp}{>}}{}^r) \to W \oplus W^*$, where $W = \text{range}\,(P), W^* = \text{range}\,(P^*)$, has derivative

$$\frac{d}{dr}\left\{\mathcal{S}_r(\widehat{P})\right\} v_r = \frac{d}{dr}\left\{\mathcal{S}_r(\widehat{P}) v_r\right\} - \mathcal{S}_r(\widehat{P})\frac{d}{dr}v_r.$$

An element $v_r \in K(\widehat{\underset{\mp}{>}}{}^r)$ has the form $(\xi, \widehat{h}_r\xi)$, $\xi \in E_0 \oplus E_0$, and so using Stiefel coordinate representation for $\widehat{P} = \widehat{P}_{[\widehat{M}_0,\widehat{M}_\beta]}$, we have

$$\frac{d}{dr}\left\{\mathcal{S}_r(\widehat{P})\right\} v_r = \frac{d}{dr}\begin{pmatrix}\widehat{M}_0^*\widehat{M}_{0,\beta}^{-1}\widehat{\mathcal{M}}_r\xi \\ \widehat{M}_\beta^*\widehat{M}_{0,\beta}^{-1}\widehat{\mathcal{M}}_r\xi\end{pmatrix} - \mathcal{S}_r(\widehat{P})\begin{pmatrix}0 \\ \dot{\widehat{h}}_r\xi\end{pmatrix}$$

$$= \begin{pmatrix}\widehat{M}_0^*\widehat{M}_{0,\beta}^{-1}\left(\widehat{M}_\beta - \widehat{\mathcal{M}}_r Q_{\widehat{h}}^{-1}\widehat{h}^*\right)\dot{\widehat{h}}_r\xi \\ \widehat{M}_\beta^*\widehat{M}_{0,\beta}^{-1}\left(\widehat{M}_\beta - \widehat{\mathcal{M}}_r Q_{\widehat{h}}^{-1}\widehat{h}^*\right)\dot{\widehat{h}}_r\xi\end{pmatrix},$$

and so (2.25) implies $\mathcal{S}_r(\widehat{P})^{-1}\frac{d}{dr}\mathcal{S}_r(\widehat{P})v_r = \left\{\widehat{\mathcal{M}}_r^{-1}\left(\widehat{M}_\beta - \widehat{\mathcal{M}}_r Q_{\widehat{h}}^{-1}\widehat{h}^*\right)\dot{\widehat{h}}_r\right\} v_r$. Therefore

$$\begin{aligned}\text{Tr}\left\{\mathcal{S}_r(\widehat{P})^{-1}\frac{d}{dr}\mathcal{S}_r(\widehat{P})\right\} &= \text{tr}\left\{\widehat{\mathcal{M}}_r^{-1}\widehat{M}_\beta\dot{\widehat{h}}_r\right\} - \text{tr}\left\{Q_{\widehat{h}}^{-1}\widehat{h}^*\dot{\widehat{h}}_r\right\} \\ &= \frac{d}{dr}\log\det\widehat{\mathcal{M}}_r - \text{tr}\left\{Q_{\widehat{h}}^{-1}\widehat{h}^*\dot{\widehat{h}}_r\right\} \\ &= \frac{d}{dr}\log\det\left(\mathcal{M}_r^*\mathcal{M}_r\right) + \alpha(\widehat{\underset{\mp}{>}}{}^r),\end{aligned}$$

where $\alpha(\widehat{\underset{\mp}{>}}{}^r) = -\text{tr}\,\{(h_r^*A_0^r)^{-1}\frac{d}{dr}(h_r^*A_0^r\} - \text{tr}\,\{Q_{\widehat{h}}^{-1}\widehat{h}^*\dot{\widehat{h}}_r\}$, and we use (2.30). The term $\alpha(\widehat{\underset{\mp}{>}}{}^r)$ depends only on the operator $\widehat{\underset{\mp}{>}}{}^r$, not on $\widehat{P}$, and therefore

$$\text{Tr}\left\{\mathcal{S}_r(\widehat{P}_1)^{-1}\frac{d}{dr}\mathcal{S}_r(\widehat{P}_1)\right\} - \text{Tr}\left\{\mathcal{S}_r(\widehat{P}_2)^{-1}\frac{d}{dr}\mathcal{S}_r(\widehat{P}_2)\right\} = \frac{d}{dr}\log\frac{\det(\mathcal{M}_r^*\mathcal{M}_r)}{\det(\mathcal{N}_r^*\mathcal{N}_r)},$$

which completes the proof. $\square$

Next, let $\mathcal{D}^r$ be a path of operators in $Ell_{1,n}$ connecting $\mathcal{D} := \mathcal{D}^1$ with a self-adjoint first-order elliptic operator $\mathcal{D}^0$ with $\mathcal{D}^0_{P_i}$ invertible. The path can always be chosen such that $\mathcal{D}^r_{P_1}$ is invertible for each r; equivalently, such that $\mathcal{M}_r$ is invertible for each r. If it occurs that an $\mathcal{N}_r$ is not invertible at some point along the path, then the path can be perturbed slightly to remove the singularity without affecting the invertibility of $\mathcal{M}_r$, since that is an open condition. Hence we can integrate (2.33) along this path to obtain

$$\frac{\det_\zeta(\underset{\mp}{>}_{P_1})}{\det_\zeta(\underset{\mp}{>}_{P_2})}\left(\frac{\det_\zeta(\underset{\mp}{>}{}^0_{P_1})}{\det_\zeta(\underset{\mp}{>}{}^0_{P_2})}\right)^{-1} = \frac{\det_{\mathcal{C}}(\underset{\mp}{>}_{P_1})}{\det_{\mathcal{C}}(\underset{\mp}{>}_{P_2})}\left(\frac{\det_{\mathcal{C}}(\underset{\mp}{>}{}^0_{P_1})}{\det_{\mathcal{C}}(\underset{\mp}{>}{}^0_{P_2})}\right)^{-1},$$

where $\underset{\mp}{>}{}^0 = (\mathcal{D}^0)^*\mathcal{D}^0$. Since $\mathcal{D}^0$ is self-adjoint and we know that (1.10) holds for such operators, this completes the proof of Theorem 1.

3. Relative Zeta Function Curvature: Proof of Theorem 2

We define a ζ-function connection on $\mathrm{DET}\,(\mathbb{D},\mathbb{P})$ following a modified version of the prescription of Quillen-Bismut-Freed. The ζ-function connection form (1.14) is defined over $U\subset B$ by

$$\omega_\zeta := \frac{d}{ds}_{|s=0}(s\theta_{\mathbb{P}}(s)), \tag{3.1}$$

where $\theta_{\mathbb{P}}(s) = -\mathrm{Tr}\,(\widetilde{\gtrneqq}_P^{-s}\mathcal{D}\widetilde{\nabla}^{\mathbb{P}}\mathcal{D}_P^{-1})$ is defined around zero by analytic continuation. Here $\widetilde{\nabla}^{\mathbb{P}}$ is a connection on the infinite-dimensional smooth bundle $\mathrm{Hom}(\mathcal{H},\mathcal{H}_{\mathbb{P}})$ induced (1.13) from connections on $\mathcal{H}$ and $\mathcal{H}_{\mathbb{P}}$. Since $\mathcal{H}$ is the trivial bundle we can choose the trivial de-Rham connection 'd'. The bundle $\mathcal{H}_{\mathbb{P}}$, however, with fibre $\mathrm{dom}_{\,\infty}(\mathcal{D}_{P_b})$ is non-trivial whenever the finite-rank bundle $\mathcal{W}$ defined by the Grassmann section $\mathbb{P}$ is non-trivial. Indeed, a section of $\mathcal{H}$ is the same thing as a C^∞ section of the trivial finite-rank vertical bundle $E^\nu\to B\times X$ equal to E along the fibres of the trivial fibration $B\times X\to B$. A section of $\mathcal{H}_{\mathbb{P}}$, on the other hand, is a C^∞ section of a non-trivial finite-rank vertical bundle $E^\nu_{\mathbb{P}}\to B\times X$; a section of $E^\nu_{\mathbb{P}}$ is required to satisfy $P_b\begin{pmatrix} s(b,0)\\ s(b,\beta)\end{pmatrix}$ at each $b\in B$. Consequently, the trivial connection d on $\mathcal{H}$ does not descend to a connection on the subbundle $\mathcal{H}_{\mathbb{P}}$ due to the variation of P_b. Hence a modified connection $\nabla^{\mathbb{P}}$ is needed which takes sections of $\mathcal{H}_{\mathbb{P}}$ to sections of $\mathcal{H}_{\mathbb{P}}$. Defining $\nabla^{\mathbb{P}}$ is the same thing as defining an 'honest' connection on the finite-rank bundle $E^\nu_{\mathbb{P}}$ and one can work entirely in that framework. Here we shall work directly with the bundle $\mathcal{H}_{\mathbb{P}}$ and define $\nabla^{\mathbb{P}}$ as follows.

First, we define the bundle restriction map

$$\gamma:\mathcal{H}\longrightarrow\mathbf{C}^{2n},\qquad \gamma s_b = \begin{pmatrix} s_b(0)\\ s_b(\beta)\end{pmatrix} \tag{3.2}$$

for $s_b\in\mathcal{H}_b = C^\infty(X,E)$, where $\mathbf{C}^{2n}$ is the trivial complex bundle over B of rank $2n$ (with fibre $\cong E_0\oplus E_\beta$). Next, fix a smooth non-decreasing function $\phi: X=[0,\beta]\to[0,1]$ with

$$\phi(x)=0\quad\text{in}\quad[0,\beta/4],\qquad\qquad \phi(x)=1\quad\text{in}\quad[3\beta/4,\beta],$$

and define the extension operator $m_\phi:\mathbb{C}^{2n}\to C^\infty(X,\mathbb{C}^n)$ by $(m_\phi v)(x)=\phi(x)v$. Let p_0,p_β be the projection maps $\mathbb{C}^{2n}\cong E_0\oplus E_\beta$ to $E_0\cong\mathbb{C}^n$, $E_\beta\cong\mathbb{C}^n$, respectively. Then we define $\mathcal{M}_\phi:\mathbf{C}^{2n}\to\mathcal{H}$, $v\longmapsto\mathcal{M}_\phi v$ by $(\mathcal{M}_\phi v)(x) = m_{1-\phi}p_0v + m_\phi p_\beta v$. We then have

$$\gamma(\mathcal{M}_\phi v) = \begin{pmatrix}(1-\phi(0))p_0v+\phi(0)p_\beta v\\ (1-\phi(\beta))p_0v+\phi(\beta)p_\beta v\end{pmatrix} = \begin{pmatrix}p_0v\\ p_\beta v\end{pmatrix} = v\ . \tag{3.3}$$

The bundle maps $\gamma,\mathcal{M}_\phi$ induce the corresponding maps between the spaces $C^\infty(B;\mathcal{H})$ and $C^\infty(B;\mathbf{C}^{2n})$, and we also denote these by γ and $\mathcal{M}_\phi$.

We now define a connection on $\mathcal{H}_{\mathbb{P}}$ by

$$\nabla^{\mathbb{P}} = d + \mathcal{M}_\phi\mathbb{P}d\mathbb{P}\gamma, \tag{3.4}$$

or pointwise on B, $\nabla^{\mathbb{P}} = d+\mathcal{M}_\phi P_b dP_b\gamma$. We may drop the b subscript in the following.

Proposition 3.1. *$\nabla^{\mathbb{P}}$ defines a connection on the bundle $\mathcal{H}^{\mathbb{P}}$. Let $\mathbb{P}^1,\mathbb{P}^2$ be Grassmann sections, then*

$$\nabla^{\mathbb{P}^1}-\nabla^{\mathbb{P}^2} = \mathcal{M}_\phi(P^1dP^1-P^2dP^2)\gamma\ . \tag{3.5}$$

Proof. We have $\nabla^{\mathbb{P}} : \Omega^0(B, \mathcal{H}_{\mathbb{P}}) \longrightarrow \Omega^1(B, \mathcal{H})$, and one easily checks that $\nabla^{\mathbb{P}}$ satisfies the Leibnitz rule $\nabla^{\mathbb{P}}(fs) = df.s + f\nabla^{\mathbb{P}}s$, for $f \in C^\infty(B), s \in \Omega^0(B, \mathcal{H}_{\mathbb{P}})$, noting that $f : B \to \mathbb{C}$ is not affected by the restriction map γ. We need to see that $\nabla^{\mathbb{P}}$ has range in $\Omega^1(B, \mathcal{H}_{\mathbb{P}})$. But if $s \in \Omega^0(B, \mathcal{H}_{\mathbb{P}})$, then $P_b\gamma s(b) = 0$ and hence $dP_b.\gamma s(b) = -P_b\gamma ds(b)$, so that

$$P_b dP_b.\gamma s(b) = -P_b\gamma ds(b)\ . \tag{3.6}$$

Therefore using (3.3) and (3.6)

$$P_b\gamma\nabla^{\mathbb{P}}s = P_b\gamma ds + P_b\gamma\mathcal{M}_\phi P_b dP_b\gamma s = -P_b dP_b\gamma s + P_b dP_b \gamma s = 0,$$

and hence $\nabla^{\mathbb{P}}s \in \Omega^1(B, \mathcal{H}_{\mathbb{P}})$. Finally, the identity (3.5) is immediate from the definition of $\nabla^{\mathbb{P}}$. □

With the connections $\nabla^{\mathbb{P}}, \nabla^{triv} = d$ on $\mathcal{H}_{\mathbb{P}}, \mathcal{H}$ at hand we have an induced connection $\widetilde{\nabla}^{\mathbb{P}}$ (pointwise $\widetilde{\nabla}^P := \widetilde{\nabla}^{P_b}$) on $\mathrm{Hom}(\mathcal{H}, \mathcal{H}_{\mathbb{P}})$. For large $\mathrm{Re}(s) > 0$, $\widetilde{\gneqq}_P^{-s}\mathcal{D}_P\widetilde{\nabla}^P\mathcal{D}_P^{-1} = \widetilde{\gneqq}_P^{-s}\mathcal{D}\widetilde{\nabla}^P\mathcal{D}_P^{-1}$ is trace class, and for small t

$$\phi_t(s) := \mathrm{Tr}\,((I + t\mathcal{D}\widetilde{\nabla}^P_X\mathcal{D}_P^{-1})\widetilde{\gneqq}_P^{-s}),$$

where $X \in \mathrm{Vect}(B)$, has a meromorphic continuation to $\mathbb{C}$ which is regular at $s = 0$. Hence (by [1], Prop 2.9)

$$\frac{d}{dt}_{t=0}\phi_t(s) := -s\mathrm{Tr}\,(\widetilde{\gneqq}_P^{-(s+1)}\mathcal{D}\widetilde{\nabla}^P_X\mathcal{D}_P^{-1}\widetilde{\gneqq}) = s\theta_P(s)(X)$$

has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 0$.

Proposition 3.2. *Let $\mathbb{P}^1, \mathbb{P}^2$ be Grassmann sections and for $i = 1, 2$ let*

$$\theta_i(s) = \mathrm{Tr}\,(\widetilde{\gneqq}_{P^i}^{-s}\mathcal{D}\widetilde{\nabla}^i\mathcal{D}_{P^i}^{-1}),$$

where $\widetilde{\nabla}^i := \widetilde{\nabla}^{P^i}$. Then

(3.7)
$$\theta_1(s) - \theta_2(s) = \mathrm{Tr}\,\left\{\widetilde{\gneqq}_{P^1}^{-s}\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}_{P^2}^{-1})\right\} - \mathrm{Tr}\,\left\{\widetilde{\gneqq}_{P^1}^{-s}\mathcal{D}\mathcal{M}_\phi(P^1dP^1 - P^2dP^2)\gamma\mathcal{D}_{P^2}^{-1}\right\}$$
$$-\mathrm{Tr}\,\left\{(\widetilde{\gneqq}_{P^1}^{-s} - \widetilde{\gneqq}_{P^2}^{-s})\mathcal{D}\widetilde{\nabla}^2\mathcal{D}_{P^2}^{-1})\right\}\ .$$

Proof. The relative connection form is the 1-form

$$\theta_1(s) - \theta_2(s) = \mathrm{Tr}\,(\widetilde{\gneqq}_{P^1}^{-s}\mathcal{D}\widetilde{\nabla}^1\mathcal{D}_{P^1}^{-1}) - \mathrm{Tr}\,(\widetilde{\gneqq}_{P^2}^{-s}\mathcal{D}\widetilde{\nabla}^2\mathcal{D}_{P^2}^{-1})\ .$$

We have from (2.14) that

$$\mathcal{D}_{P^1}^{-1} - \mathcal{D}_{P^2}^{-1} = -\mathcal{K}(P)\gamma\mathcal{D}_{P^2}^{-1}, \qquad \widetilde{\gneqq}_{P^1}^{-s} - \widetilde{\gneqq}_{P^2}^{-s} = \frac{i}{2\pi}\int_{\Gamma_\pi} \lambda^{-s}[\widetilde{\mathcal{K}}_\lambda(\widetilde{P}^1)\gamma\widetilde{\gneqq}_{P^2,\lambda}^{-1}]_{(1,2)}\,d\lambda \tag{3.8}$$

are smoothing operators, where the terms $\widetilde{\mathcal{K}}_\lambda$ and so on, are the operators for $\widetilde{\gneqq}$ corresponding to those in (2.11) for $\gneqq$.

For a section A of $\mathrm{Hom}(\mathcal{H}, \mathcal{H}_{\mathbb{P}})$ one has $\widetilde{\nabla}^P(A)(s) = \nabla^P(A(s)) - Ads$, $s \in C^\infty(B, \mathcal{H})$, and we can extend this to $\mathrm{Hom}(\mathcal{H}, \mathcal{H})$ by the same formula. Then

$$\widetilde{\nabla}^1(\mathcal{D}_{P^2}^{-1})(s) = \nabla^1(\mathcal{D}_{P^2}^{-1}(s)) - \mathcal{D}_{P^2}^{-1}ds,$$
$$\widetilde{\nabla}^2(\mathcal{D}_{P^2}^{-1})(s) = \nabla^2(\mathcal{D}_{P^2}^{-1}(s)) - \mathcal{D}_{P^2}^{-1}ds.$$

Hence from (3.5)

$$\begin{aligned}\widetilde{\nabla}^1(\mathcal{D}_{P^2}^{-1})(s)-\widetilde{\nabla}^2(\mathcal{D}_{P^2}^{-1})(s) &= (\nabla^1-\nabla^2)(\mathcal{D}_{P^2}^{-1}(s))\\ &= (\mathcal{M}_\phi(P^1dP^1-P^2dP^2)\gamma\mathcal{D}_{P^2}^{-1})(s)\ .\end{aligned}$$

And so

$$\mathcal{D}\widetilde{\nabla}^1\mathcal{D}_{P^2}^{-1}-\mathcal{D}\widetilde{\nabla}^2\mathcal{D}_{P^2}^{-1}=\mathcal{D}\mathcal{M}_\phi(P^1dP^1-P^2dP^2)\gamma\mathcal{D}_{P^2}^{-1}\ . \tag{3.9}$$

We have

$$\begin{aligned}\theta_1(s) &= -\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^1}^{-s}\mathcal{D}\widetilde{\nabla}^1\mathcal{D}_{P^1}^{-1}\right\}\\ &= -\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^1}^{-s}\mathcal{D}\widetilde{\nabla}^1(\mathcal{D}_{P^2}^{-1}-\mathcal{K}(P)\gamma\mathcal{D}_{P^2}^{-1})\right\}\\ &= -\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^1}^{-s}\mathcal{D}\widetilde{\nabla}^1\mathcal{D}_{P^2}^{-1}\right\}+\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^1}^{-s}\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P)\gamma\mathcal{D}_{P^2}^{-1})\right\} \qquad (3.10)\end{aligned}$$

since the terms are trace class. From (3.9) we have that the first term of (3.10) is

$$\begin{aligned}&-\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^1}^{-s}\mathcal{D}\widetilde{\nabla}^2\mathcal{D}_{P^2}^{-1}\right\}-\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^1}^{-s}\mathcal{D}\mathcal{M}_\phi(P^1dP^1-P^2dP^2)\gamma\mathcal{D}_{P^2}^{-1}\right\}\\ &\quad=-\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^2}^{-s}\mathcal{D}\widetilde{\nabla}^2\mathcal{D}_{P^2}^{-1}\right\}-\mathrm{Tr}\left\{(\widetilde{\geq}_{\mp P^1}^{-s}-\widetilde{\geq}_{\mp P^2}^{-s})\mathcal{D}\widetilde{\nabla}^2\mathcal{D}_{P^2}^{-1}\right\}\\ &\qquad-\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^1}^{-s}\mathcal{D}\mathcal{M}_\phi(P^1dP^1-P^2dP^2)\gamma\mathcal{D}_{P^2}^{-1}\right\}, \qquad (3.11)\end{aligned}$$

recalling that $\widetilde{\geq}_{\mp P^1}^{-s}-\widetilde{\geq}_{\mp P^2}^{-s}$ is trace class. Substituting (3.11) into (3.10) completes the proof. □

Proposition 3.3. *Let $\omega_\zeta^1,\omega_\zeta^2$ be the ζ-function connection forms associated to the Grassmann sections $\mathbb{P}^1,\mathbb{P}^2$. Then*

$$\omega_\zeta^1-\omega_\zeta^2=\mathrm{Tr}\left\{\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}_{P^2}^{-1})\right\}+\mathrm{Tr}\left\{\mathcal{D}\mathcal{M}_\phi(P^2dP^2-P^1dP^1)\gamma\mathcal{D}_{P^2}^{-1}\right\}. \tag{3.12}$$

Proof. For the first term on the right-side of (3.7) we have

$$\Theta_1(s):=\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^1}^{-s}\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}_{P^2}^{-1})\right\}=\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^1}^{-s-1}\widetilde{\geq}_{\mp P_1}\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}_{P^2}^{-1})\right\}\ . \tag{3.13}$$

But $\widetilde{\geq}_{\mp P^1}^{-s-1}$ is norm continuous for $\mathrm{Re}(s)>-1$ and $\widetilde{\geq}_{\mp P_1}\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}_{P^2}^{-1})$ is a smoothing and so trace class operator. $\Theta_1(s)$ is therefore holomorphic for $\mathrm{Re}(s)>-1$ and this allows us to go down to $s=0$ in (3.13). Thus, $\Theta_1(s)$ is regular at at $s=0$ and is given there by

$$\Theta_1(0)=\mathrm{Tr}\left\{\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}_{P^2}^{-1})\right\}\ . \tag{3.14}$$

Similarly, $\Theta_2(s):=-\mathrm{Tr}\left\{\widetilde{\geq}_{\mp P^1}^{-s}\mathcal{D}\mathcal{M}_\phi(P^1dP^1-P^2dP^2)\gamma\mathcal{D}_{P^2}^{-1}\right\}$ is regular at $s=0$ and given there by

$$\begin{aligned}\Theta_2(0) &= -\mathrm{Tr}\left\{\mathcal{D}\mathcal{M}_\phi(P^1dP^1-P^2dP^2)\gamma\mathcal{D}_{P^2}^{-1}\right\}\\ &= \mathrm{Tr}\left\{\mathcal{D}\mathcal{M}_\phi(P^2dP^2-P^1dP^1)\gamma\mathcal{D}_{P^2}^{-1}\right\}\ . \qquad (3.15)\end{aligned}$$

For $\mathrm{Re}(s)>>0$ the remaining term is $(s\Theta_3(s))\,|_{s=0}^{\mathrm{mer}}$, where

$$\Theta_3(s):=-\mathrm{Tr}\left\{(\widetilde{\geq}_{\mp P^1}^{-s}-\widetilde{\geq}_{\mp P^2}^{-s})\mathcal{D}\widetilde{\nabla}^2\mathcal{D}_{P^2}^{-1}\right\},$$

which vanishes by a similar argument using (3.8).

Thus $\theta_1(s) - \theta_2(s) = \Theta_1(s) + \Theta_1(s) + \Theta_1(s)$ is holomorphic for $\mathrm{Re}(s) > 0$ with a meromorphic continuation to all of $\mathbb{C}$, and we have

$$\omega^1_\zeta - \omega^2_\zeta = \frac{d}{ds}_{|s=0} \{s(\theta_1(s) - \theta_2(s))\} = \Theta_1(0) + \Theta_2(0) + \frac{d}{ds}_{|s=0} (s\Theta_3(s)|^{\mathrm{mer}}) \ . \tag{3.16}$$

By (3.14),(3.15), the Proposition is proved. □

Since $P^1dP^1 - P^2dP^2$ is finite-rank and $\gamma\mathcal{D}^{-1}_{P^2}$ is bounded we have using (2.13)

$$\begin{aligned} \mathrm{Tr}\left\{\mathcal{D}\mathcal{M}_\phi(P^2dP^2 - P^1dP^1)\gamma\mathcal{D}^{-1}_{P^2}\right\} &= \mathrm{Tr}\left\{\gamma\mathcal{D}^{-1}_{P^2}\mathcal{D}\mathcal{M}_\phi(P^2dP^2 - P^1dP^1)\right\} \\ &= \mathrm{Tr}\left\{\gamma(I - \mathcal{K}(P^2)\gamma)\mathcal{M}_\phi(P^2dP^2 - P^1dP^1)\right\} \\ &= \mathrm{tr}\left\{(I_{2n} - P_K\mathcal{S}(P^2)^{-1}P^2)\gamma\mathcal{M}_\phi(P^2dP^2 - P^1dP^1)\right\} \\ &= \mathrm{tr}\left\{(I_{2n} - P_K\mathcal{S}(P^2)^{-1}P^2)(P^2dP^2 - P^1dP^1)\right\} \\ &= \mathrm{tr}\,(P^2dP^2) - \mathrm{tr}\,(P^1dP^1) \\ &\quad -\mathrm{tr}\left\{P_K\mathcal{S}(P^2)^{-1}P^2(P^2dP^2 - P^1dP^1)P_K\right\} \end{aligned} \tag{3.17}$$

where $P_K := P(\mathcal{D})$, I_{2n} denotes the identity on $\mathbb{C}^{2n}$ and we use (3.3). Note that the trace $\mathrm{tr} = \mathrm{tr}_{\mathbb{C}^{2n}}$ in the third line equals the trace $\mathrm{tr} = \mathrm{tr}_{W_2^\perp}$ over $\mathrm{range}\,(P^2)^\perp$ since $P^2(I - P_K\mathcal{S}(P^2)^{-1}P_K)) = 0$.

For the first term $\Theta_1(0)$ in (3.12) one has to work a little harder. To study the trace $\mathrm{Tr}\left\{\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}^{-1}_{P^2}\right\}$, we consider the operator $\mathcal{K}(P^1)\gamma\mathcal{D}^{-1}_{P^2} : \mathcal{H}_b \longrightarrow \mathrm{Ker}(\mathcal{D}_b)$ as the composition

$$\mathcal{K}(P^1)\gamma\mathcal{D}^{-1}_{P^2} = \mathcal{K}(P^1)P^1 \circ P^1\gamma\mathcal{D}^{-1}_{P^2} : \mathcal{H}_b \longrightarrow W^1_b \longrightarrow \mathrm{Ker}(\mathcal{D}_b) \subset \mathcal{H}_b \ ,$$

where $W^1_b = \mathrm{range}\,(P^1_b)$ is the fibre of the bundle $\mathcal{W}^1$ at $b \in B$. The bundles $\mathcal{H}$, $\mathcal{W}^1$ have the connections $\nabla^{triv} = d$ and $\nabla^{\mathcal{W}^1}$, while the bundle $\mathrm{Ker}(\mathbb{D})$ in $\Theta_1(0)$ has the induced connection $\nabla^1_{|\mathrm{Ker}(\mathbb{D})}$. Let $\nabla^{(\mathcal{W}^1,\mathrm{ker})}, \nabla^{\mathcal{H},\mathcal{W}^1}$ denote the induced connections on $\mathrm{Hom}(\mathcal{W}^1, \mathrm{Ker}(\mathbb{D}))$ and $\mathrm{Hom}(\mathcal{H}, \mathcal{W}^1)$. Then we have[3]

$$\begin{aligned} \widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}^{-1}_{P^2}) &= \widetilde{\nabla}^1((\mathcal{K}(P^1)P^1 \circ P^1\gamma\mathcal{D}^{-1}_{P^2}) \\ &= \nabla^{(\mathcal{W}^1,\mathrm{ker})}(\mathcal{K}(P^1)P^1).\ P^1\gamma\mathcal{D}^{-1}_{P^2} + \mathcal{K}(P^1)P^1.\nabla^{\mathcal{H},\mathcal{W}^1}(P^1\gamma\mathcal{D}^{-1}_{P^2}) \ . \end{aligned} \tag{3.18}$$

Now since $\mathcal{K}$ has range in $\mathrm{Ker}(\mathcal{D})$, the second term in (3.18) is killed by $\mathcal{D}$ and so

$$\begin{aligned} \mathrm{Tr}\left\{\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}^{-1}_{P^2}\right\} &= \mathrm{Tr}\left\{\mathcal{D}\nabla^{(\mathcal{W}^1,\mathrm{ker})}(\mathcal{K}(P^1)P^1).P^1\gamma\mathcal{D}^{-1}_{P^2}\right\} \\ &= \mathrm{tr}\left\{P^1\gamma\mathcal{D}^{-1}_{P^2}\mathcal{D}\nabla^{(\mathcal{W}^1,\mathrm{ker})}(\mathcal{K}(P^1)P^1)\right\} \\ &= \mathrm{tr}\left\{P^1(I_{2n} - P_K\mathcal{S}(P^2)^{-1}P^2)\gamma\nabla^{(\mathcal{W}^1,\mathrm{ker})}(\mathcal{K}(P^1)P^1)\right\} \\ &= \mathrm{tr}\left\{(P^1 - \mathcal{S}(P^1)\mathcal{S}(P^2)^{-1}P^2)\gamma\nabla^{(\mathcal{W}^1,\mathrm{ker})}(\mathcal{K}(P^1)P^1)\right\}. \end{aligned} \tag{3.19}$$

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[3] Note that for bundles ξ^i, $i = 1, 2, 3$, with connection inducing connections $\nabla^{i,j}$ on $\mathrm{Hom}(\xi^i, \xi^j)$ one has for respective sections A, B of $\mathrm{Hom}(\xi^2, \xi^3)$, $\mathrm{Hom}(\xi^1, \xi^2)$, $\nabla^{1,3}(AB) = \nabla^{1,2}(A)B + A\nabla^{2,3}(B)$, for any choice of connection ∇^2 on ξ^2.

But for $\xi \in C^\infty(B, \mathcal{W}^1)$ we have $P^1\xi = \xi$, pointwise, and $\nabla^{\mathcal{W}^1}(\xi) = P^1 d\xi$, and so

$$\begin{aligned}\nabla^{(\mathcal{W}^1,\mathrm{ker})}(\mathcal{K}(P^1)P^1)(\xi) &= \nabla^{(\mathcal{W}^1,\mathrm{ker})}(\mathcal{K}\mathcal{S}(P^1)^{-1}P^1)(\xi)\\ &= \nabla^1_{|\mathrm{Ker}(\mathbb{D})}(\mathcal{K}\mathcal{S}(P^1)^{-1}P^1\xi) - (\mathcal{K}\mathcal{S}(P^1)^{-1}P^1)\nabla^{\mathcal{W}^1}(\xi)\\ &= (d + \mathcal{M}_\phi P^1 dP^1\gamma)(\mathcal{K}\mathcal{S}(P^1)^{-1}P^1\xi) - (\mathcal{K}\mathcal{S}(P^1)^{-1}P^1 d(\xi)\\ &= d(\mathcal{K}).\mathcal{S}(P^1)^{-1}P^1\xi + \mathcal{K}d(\mathcal{S}(P^1)^{-1}P^1)\xi\\ &\quad + \ \mathcal{M}_\phi(P^1dP^1)P_K\mathcal{S}(P^1)^{-1}P^1\xi \ . \end{aligned} \tag{3.20}$$

Since $\gamma\mathcal{K} = P_K$ we have $(P^1 - \mathcal{S}(P^1)\mathcal{S}(P^2)^{-1}P^2)\gamma\mathcal{K}d(\mathcal{S}(P^2)^{-1}P^1 = 0$ (or we could use $\mathcal{D}\mathcal{K} = 0$ in the previous step), so

$$\begin{aligned}\gamma\nabla^{(\mathcal{W}^1,\mathrm{ker})}(\mathcal{K}(P^1)P^1) &= \gamma d(\mathcal{K}).\mathcal{S}(P^1)^{-1}P^1P^1dP^1P_K\mathcal{S}(P^1)^{-1}P^1\\ &= dP_K.\mathcal{S}(P^1)^{-1}P^1 + P^1dP^1P_K\mathcal{S}(P^1)^{-1}P^1 \ ,\end{aligned}$$

and we now have from (3.19)

$$\mathrm{Tr}\left\{\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}^{-1}_{P^2}\right\}$$

$$\begin{aligned}&= \mathrm{tr}_{W^1}\left\{(P^1 - \mathcal{S}(P^1)\mathcal{S}(P^2)^{-1}P^2)(dP_KP_K\mathcal{S}(P^1)^{-1}P^1 + P^1dP^1P_K\mathcal{S}(P^1)^{-1}P^1)\right\}\\ &= \mathrm{tr}_{W^1}\left\{P^1dP_KP_K\mathcal{S}(P^1)^{-1}P^1 + P^1dP^1P_K\mathcal{S}(P^1)^{-1}P^1\right\}\\ &\quad -\mathrm{tr}_{W^1}\left\{\mathcal{S}(P^1)\mathcal{S}(P^2)^{-1}P^2dP_K\mathcal{S}(P^1)^{-1}P^1 - \mathcal{S}(P^1)\mathcal{S}(P^2)^{-1}P^2P^1dP^1P_K\mathcal{S}(P^1)^{-1}P^1\right\}\\ &= \mathrm{tr}_K\left\{P_K\mathcal{S}(P^1)^{-1}P^1(P^1dP_K + P^1dP^1)P_K\right\}\\ &\quad -\mathrm{tr}_K\left\{P_K\mathcal{S}(P^2)^{-1}P^2dP_KP_K\right\} - \mathrm{tr}_K\left\{P_K\mathcal{S}(P^2)^{-1}P^2P^1dP^1P_K\right\}.\end{aligned} \tag{3.21}$$

We consider next the canonical connection $\nabla^{\mathcal{C},\mathbb{P}}$ on DET $(\mathbb{D},\mathbb{P})$, defined by the 1-form $\omega_{\mathcal{C}} := \mathrm{tr}\,\left(\mathcal{S}(P)^{-1}\nabla^{\mathcal{K},\mathcal{W}}\mathcal{S}(P)\right)$ over $U \subset B$.

Lemma 3.4. *Let $P_K = P(\mathcal{D}_b), P = P_b$. One has*

$$\omega_{\mathcal{C}} = \mathrm{tr}_K\left\{P_K\mathcal{S}(P)^{-1}(PdP + PdP_k)P_K\right\}. \tag{3.22}$$

Proof. Observe first that $P_K\xi = \xi$ for $\xi \in \Omega^0(B;\mathcal{K})$ and so $dP_K.\xi + P_kd\xi = d\xi$. Hence

$$Pd\xi - PP_Kd\xi = PdP_k\xi. \tag{3.23}$$

But

$$\begin{aligned}\nabla^{\mathcal{K},\mathcal{W}}(\mathcal{S}(P))(\xi) &= \nabla^{\mathcal{W}}(\mathcal{S}(P)(\xi)) - \mathcal{S}(P)\nabla^{\mathcal{K}}\xi\\ &= Pd(P\xi) - PP_Kd(P_K\xi)\\ &= PdP.\xi + Pd\xi - PP_KdP_K.P_K\xi - PP_Kd\xi\\ &= PdP.\xi + PdP_k.\xi = PdP.P_K\xi + PdP_k.P_K\xi,\end{aligned}$$

using (3.23) and since $P_KdP_KP_K = 0$. Equation (3.22) now follows. □

From (3.21) and (3.22)

$$\begin{aligned}\mathrm{Tr}\left\{\mathcal{D}\widetilde{\nabla}^1(\mathcal{K}(P^1)\gamma\mathcal{D}_{P^2}^{-1}\right\} &= \omega_{\mathcal{C}}^1 - \mathrm{tr}_K\left\{P_K\mathcal{S}(P^2)^{-1}P^2dP_KP_K\right\}\\ &\quad - \mathrm{tr}_K\left\{P_K\mathcal{S}(P^2)^{-1}P^2P^1dP^1P_K\right\}\\ &= \omega_{\mathcal{C}}^1 - \omega_{\mathcal{C}}^2 + \mathrm{tr}_K\left\{P_K\mathcal{S}(P^2)^{-1}P^2dP^2P_K\right\}\\ &\quad - \mathrm{tr}_K\left\{P_K\mathcal{S}(P^2)^{-1}P^2P^1dP^1P_K\right\}\\ &= \omega_{\mathcal{C}}^1 - \omega_{\mathcal{C}}^2 + \mathrm{tr}_K\left\{P_K\mathcal{S}(P^2)^{-1}(P^2dP^2 - P^2P^1dP^1)P_K\right\}\ . \end{aligned} \tag{3.24}$$

Putting together equations (3.12), (3.17) and (3.24), we have proved that the ζ and $\mathcal{C}$ connection forms are related over U by

$$\omega_\zeta^1 - \omega_\zeta^2 = \omega_{\mathcal{C}}^1 - \omega_{\mathcal{C}}^2 + \mathrm{tr}\,(P^2dP^2) - \mathrm{tr}\,(P^1dP^1)\ . \tag{3.25}$$

We have from (3.25)

$$\begin{aligned}\Omega_\zeta^1 - \Omega_\zeta^2 &= \Omega_{\mathcal{C}}^1 - \Omega_{\mathcal{C}}^2 + d\mathrm{tr}\,(P^2dP^2) - d\mathrm{tr}\,(P^1dP^1)\\ &= \Omega_{\mathcal{C}}^1 - \Omega_{\mathcal{C}}^2 + \mathrm{tr}\,(dP^2\wedge dP^2) - \mathrm{tr}\,(dP^1\wedge dP^1)\ ,\end{aligned}$$

and by the symmetry of the trace $\mathrm{tr}\,(dP^i\wedge dP^i) = 0$. This completes the proof of Theorem 2.

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