η Forms and Determinant Lines

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1 Introduction

The purpose here is to give a direct computation of the zeta-function curvature for the determinant line bundle of a family of APS-type boundary value problems.

Here is the sort of computation we have in mind.

1.0.1 Example: ζ-curvature on CP^1

Consider the simplest case, $D = id/dx$ over $[0, 2\pi]$ with Laplacian $\Delta = -d^2/dx^2$. Global boundary conditions for D are parametrised by $\mathbb{C}P^1$. Specifically, over the dense open subset of $\mathbb{C}P^1$ parametrising complex lines $l_z \subset \mathbb{C}^2$ given by the homogeneous coordinates $[1, z]$ for $z \in \mathbb{C}$ the orthogonal projection $P_z = \frac{1}{1+|z|^2}\begin{pmatrix} 1 & \overline{z} \\ z & |z|^2 \end{pmatrix}$ onto l_z parametrises the boundary condition $P_z \begin{pmatrix} \psi(0) \\ \psi(2\pi) \end{pmatrix} = 0$; that is, $\psi(0) = -\overline{z}\,\psi(2\pi)$. Let D_{P_z} denote D with domain restricted to functions satisfying this boundary condition. The adjoint boundary problem is $D_{P_z^*}$ with projection $P_z^* = \frac{1}{1+|z|^2}\begin{pmatrix} |z|^2 & -\overline{z} \\ -z & -1 \end{pmatrix}$ corresponding to $-z\phi(0) = \phi(2\pi)$. Then Δ_{P_z} has discrete spectrum $\{(n+\alpha)^2, (n-\alpha)^2 : n \in \mathbb{N}\}$, where $u = e^{2\pi i\alpha}$ satisfies $u^2(1+|z|^2) + 2u(z+\overline{z}) + (1+|z|^2) = 0$. The zeta determinant of Δ_{P_z} is therefore

$$\mathrm{det}_\zeta \Delta_{P_z} = 4\sin^2 \pi\alpha = \frac{2|1+\overline{z}|^2}{1+|z|^2}, \qquad [1, z] \in \mathbb{C}P^1. \tag{1.0.1}$$

The Quillen metric evaluated on the holomorphic section identified with the abstract determinant $z \mapsto \mathrm{det}D_{P_z}$ is $\|\mathrm{det}D_{P_z}\|^2 = \mathrm{det}_\zeta\Delta_{P_z}$ and hence the canonical curvature $(1,1)$-form of the determinant line bundle is

$$\overline{\partial}\partial \log \mathrm{det}_\zeta \Delta_{P_z} = \frac{dz \wedge d\overline{z}}{(1+|z|^2)^2} \qquad = \text{ Kahler form on } \mathbb{C}P^1. \tag{1.0.2}$$

1.0.2 c_1 of the determinant

More generally, determinant bundles arise in geometric analysis, in the representation theory of loop groups, and in the construction of conformal field theories.

In a general sense, they facilitate the construction of projective representations from the bordism category to categories of graded rings. The basic invariant of a determinant bundle which one aims to compute is its Chern class.

1.0.3 Example: closed surfaces

A well known instance of that is for a family of compact boundaryless surfaces $\{\Sigma_y \,|\, y \in Y\}$ parametrised by a smooth manifold Y. Let $M = \bigcup_{y\in Y} \Sigma_y$ and $\pi : M \to Y$ the projection map. Let T_y be the tangent bundle to Σ_y, and $T := T(M/Y) = \bigcup_{y\in Y} T_y \to M$ the tangent bundle along the fibres. The index bundle $\mathrm{Ind}\,\overline{\partial}_{(m)}$ of the family of D-bar operators $\overline{\partial}_{(m)} = \{\overline{\partial}_y \mid y \in Y\}$ acting on sections of $T^{\otimes m}$ is the element $f_!(T^{\otimes m})$ of $K(Y)$, and the Grothendieck-Riemann-Roch theorem says

$$\mathrm{ch}(f_!(T^{\otimes m})) = f_* \left(\mathrm{ch}(T^{\otimes m})\,\mathrm{Todd}(T)\right),$$

where $f_* : H^i(M) \to H^{i-2}(Y)$ is integration over the fibres. That is, with $\xi = c_1(T)$

$$\mathrm{ch}(\mathrm{Ind}\,\overline{\partial}_{(m)}) = f_* \left(e^{m\xi} \cdot \frac{\xi}{1-e^{-\xi}}\right) = f_* \left(1 + (m+\frac{1}{2})\,\xi + \frac{1}{2}(m^2+m+\frac{1}{6})\,\xi^2 + \cdots\right).$$

Hence c_1 of the determinant line bundle $\mathrm{Det}\,\overline{\partial}_{(m)}$ is the degree two component

$$\mathrm{c}_1(\mathrm{Det}\,\overline{\partial}_{(m)}) = \frac{1}{12}\,(6m^2+6m+1)\,f_*(\xi^2) \;\in H^2(Y).$$

1.0.4 Quillen on the curvature formula

More refined formulae may be sought at the level of smooth invariants. The fundamental result in this direction was obtained by Quillen in 1984 in a very beautiful four page article [17] in which the zeta function regularized curvature of the determinant line bundle $\mathrm{Det}\,\mathsf{D}_\Sigma$ of a family of Cauchy-Riemann operators $\mathsf{D}_\Sigma = \{D : \Omega(\Sigma, E) \to \Omega^{0,1}(\Sigma, E)\}$ acting on sections of a complex vector bundle E over a closed Riemann surface Σ was computed to be

$$F_\zeta(\mathsf{D}_\Sigma) \;=\; \text{Kahler form on } Y \tag{1.0.3}$$

where in this case $Y = \Omega^{0,1}(\Sigma, \mathrm{End}\,E)$.

1.0.5 Bismut on Quillen

Following Quillen's idea of constructing a superconnection on the index bundle [18], Bismut [2] proved in a tour de force a local index theorem for a general family D of Dirac-type operators associated to a geometric fibration $\pi : M \to Y$ with fibre a compact boundaryless manifold and, furthermore, with $F_\zeta(\mathsf{D}) \in \Omega^2(Y)$

the curvature of the ζ-connection on the determinant line bundle $\mathrm{Det}\,\mathsf{D}$, extended (1.0.3) to

$$F_\zeta(\mathsf{D}) \;=\; \mathsf{ind}_{[2]},$$

where $\mathsf{ind} \in \Omega^*(Y)$ is the family index density, equal to $\int_{M/Y} \widehat{A}(M/Y)\,\mathrm{ch}(V)$ in the case of a family of twisted Dirac operators, and the subscript indicates the 2-form component [4].

It is worth emphasizing here the geometric naturality of the formulae; in each of the above cases, including the example of §1.0.1, the ζ-curvature hits the index form 'on the nose' — any other connection will have curvature differing from this by an exact 2-form.

1.0.6 Melrose and Piazza on Bismut

That naturality persists to the analysis of families of APS boundary problems D_P for which the fibre of $\pi : M \to Y$ is a compact manifold with boundary and $\partial M \neq \emptyset$, and $\mathsf{P} = \{P_y\}$ is a smooth family of ψdo projections on the space of boundary sections which is pointwise (with respect to Y) commensurable with the APS projection.

The principal contribution in this direction is the Chern character formula of Melrose-Piazza [11] proved using b-calculus and generalizing Bismut-Cheeger [3]. From this Piazza [14] inferred the b zeta-curvature function formula on the b determinant bundle $\mathrm{Det}\,^b(\mathsf{D}_\mathsf{P})$ to be

$$F^b_\zeta(\mathsf{D}_\mathsf{P}) \;=\; \mathsf{ind}_{[2]} \;+\; [\,\widetilde{\eta}_\mathsf{P}\,]_{(2)}\,,$$

where $\widetilde{\eta}_\mathsf{P} := \pi^{-1/2}\int_0^\infty \mathrm{Tr}\,(\dot{\mathsf{B}}_t\, e^{-\mathsf{B}_t^2})\,dt$ is an eta-form of a t-rescaled superconnection $\mathsf{B}_t = \mathsf{B}_t(\mathsf{P})$ twisted by P for the family of Dirac operators on the boundary ∂M.

1.0.7 A direct computation

On the other hand, D_P is already a smooth family of Dirac-Fredholm operators and it is natural to seek a direct computation of the ζ-curvature formula for the determinant line bundle $\mathrm{Det}\,\mathsf{D}_\mathsf{P}$, along the lines of the example of §1.0.1, without use of b-calculus or other completions. It turns out, indeed, that there is a canonical ζ-function connection on $\mathrm{Det}\,\mathsf{D}_\mathsf{P}$ and that its curvature can be computed exactly in terms of a relative eta form coming from the boundary fibration:

Theorem 1.1 *Let $F_\zeta(\mathsf{D}_\mathsf{P})$ be the curvature 2-form of the ζ-connection on* $\mathrm{Det}\,\mathsf{D}_\mathsf{P}$*. Then*

$$F_\zeta(\mathsf{D}_\mathsf{P}) \;=\; F_\zeta(\mathsf{D}_{\mathsf{P}(\mathsf{D})}) \;+\; R^{\mathcal{K},\mathcal{W}} \qquad \text{in} \;\; \Omega^2(Y) \tag{1.0.4}$$

with $R^{\mathcal{K},\mathcal{W}}$ the 2-form component of a relative η-form depending only on boundary data; that is, on the fibration of closed boundary manifolds, on $\mathrm{ran}(\mathsf{P}) = \mathcal{W}$ *and* *on* $\mathrm{ran}(\mathsf{P}(\mathsf{D})) = \mathcal{K}$. *Here,* $\mathsf{P}(\mathsf{D})$ *is the family of Calderón projections defined by* D*, equal at* $y \in Y$ *to the projection onto the (infinite dimensional) subspace equal to the restriction of* $\mathrm{Ker}\, D_y$ *to the boundary. The determinant bundle* $\mathrm{Det}\, \mathsf{D}_{\mathsf{P}(\mathsf{D})}$ *is trivial. Its* ζ*-curvature is canonically exact; there is a preferred 1-form* $\beta_\zeta(\mathsf{D})$ *in* $\Omega^1(Y)$ *such that*

$$F_\zeta(\mathsf{D}_{\mathsf{P}(\mathsf{D})}) = d\beta_\zeta(\mathsf{D}).$$

The definition of $R^{\mathcal{K},\mathcal{W}}$, which is simple and completely canonical, and why it is a 'relative eta form', is given in §6. The formula (1.0.4) is extremely 'clean', insofar as it is the simplest relation that might exist between $F_\zeta(\mathsf{D}_\mathsf{P})$ and $R^{\mathcal{K},\mathcal{W}}$, both of which represent $\mathrm{c}_1(\mathrm{Det}\, \mathsf{D}_\mathsf{P})$. It extends to geometric families of boundary problems the principle of 'reduction to the boundary' present in the analysis of Grubb and Seeley [10], [9] and Brüning and Lesch [5] of resolvent and zeta traces of pseudodifferential boundary problems, also in Booss-Wojciechowski [7], and in the zeta determinant formulae in joint work with Krzysztof Wojciechowski [23] and in [20].

1.0.8 Example: surfaces

For a real compact surface Σ with boundary S^1 our conclusions generalize the example of §1.0.1 (and §1.0.3) as follows. A choice of conformal structure $\tau \in \mathrm{Conf}(\Sigma)$ turns Σ into a Riemann surface with a D-bar operator $\overline{\partial}_\tau : \Omega^0(\Sigma) \to \Omega^{0,1}(\Sigma)$. Since $P(\overline{\partial}_\tau)$ differs from the APS projection Π by only a smoothing operator [19] a suitable parameter space of well-posed boundary conditions is the smooth Grassmannian Gr of pseudodifferential operator (ψdo) projections P with $P - P(\overline{\partial}_\tau)$ smoothing. We obtain in this way the family of APS boundary problems

$$\overline{\partial}_P := (\overline{\partial}_\tau)_P \ : \ \mathrm{dom}(\overline{\partial}_P) = \mathrm{Ker}\,(P \circ \gamma) \to \Omega^{0,1}(\Sigma)$$

parametrised by $P \in \mathrm{Gr}$. In this case $F_\zeta(\overline{\partial}_{P(\overline{\partial})}) = 0$ and (1.0.4) is

$$F_\zeta(\overline{\partial}_\mathsf{P}) \ = \ \mathrm{Tr}\,(PdPdP) \ = \ \text{Kahler form on Gr}$$

generalizing (1.0.2); in local coordinates, defined relative to a choice of basepoint $P_0 \in \mathrm{Gr}$, on the dense open subset of projections P with $\mathrm{ran}(P) = \mathrm{graph}(T = P_0^\perp S P_0 : E \to E^\perp)$, where $E = \mathrm{ran}(P_0)$ and S is a smoothing operator, the Kahler form is $\mathrm{Tr}\,((I + T^*T)^{-1} dT^* (I + TT^*)^{-1} dT)$, with trace taken on E, while with $K(\overline{\partial}_\tau) = \mathrm{graph}(h : E \to E^\perp)$ (1.0.1) generalizes by [20] Thm(4.2) as

$$\det{}_\zeta(\overline{\partial}^*\overline{\partial}_P) = \det{}_\zeta(\overline{\partial}^*\overline{\partial}_{P(D)}) \cdot \frac{|\det_F(I + T^*h)|^2}{\det_F(I + T^*T)\det_F(I + h^*h)}.$$

The restriction of $\mathrm{Det}\,\overline{\partial}_\mathsf{P}$ to the loop group LU(n) via the embedding $\mathrm{LU(n)} \hookrightarrow \mathrm{Gr}$ based at $P(\overline{\partial}_\tau)$ is the central extension of LU(n) (Segal [26]), while $F_\zeta(\overline{\partial}_\mathsf{P})_{|\mathrm{LU(n)}}$ is the 2-cocycle of the extension. Indeed, $\mathrm{Det}\,(\overline{\partial}_\mathsf{P})_{|\mathrm{LG}}$ is canonically isomorphic to the complex line bundle with fibre at $\mathsf{g} \in \mathrm{LG}$ equal to the determinant line $\mathrm{Det}\,(P(\overline{\partial}_\tau) \circ (\mathsf{g}P(\overline{\partial}_\tau)\mathsf{g}^{-1})\,)$ (defined in §4); changing the basepoint of the embedding $\mathrm{LG} \hookrightarrow \mathrm{Gr}$ from $P(\overline{\partial}_\tau)$ to P defines an isomorphic representation, the two are related by a choice of generator for the complex line $\mathrm{Det}\,(P(\overline{\partial}_\tau), P)$. The fundamental projective representation space of LG is then the space of holomorphic sections of the dual determinant line bundle $\mathrm{Det}\,(\overline{\partial}_{\mathsf{P}})^* \to \mathrm{Gr}$ [16].

On the other hand, one may consider the opposite situation of the family of D-bar operators on Σ

$$\overline{\partial}_{\Pi,\mathrm{m}} = \{(\overline{\partial}_\tau)_{\Pi,\mathrm{m}} \mid \tau \in \mathrm{Conf}(\Sigma)\}$$

parametrised by $Y = \mathrm{Conf}(\Sigma)$ and with fixed boundary condition Π acting on sections of $T^{\otimes m}\Sigma$. $\mathrm{Det}\,\overline{\partial}_{\Pi,\mathrm{m}}$ pushes-down to the moduli space $\mathcal{M}(\Sigma) = \mathrm{Conf}(\Sigma)/\mathrm{Diff}\,(\Sigma, \partial\Sigma)$ by the group of diffeomorphisms of Σ equal to the identity on the boundary. In particular, for the unit disc D then $\mathcal{M}(D) = \mathrm{Diff}^{\,+}S^1/\mathrm{PSU}_{1,1}$. By the functoriality of our constructions and the computations of [12] we obtain that the ζ-curvature of the determinant line bundle over $\mathrm{Diff}^{\,+}S^1/\mathrm{PSU}_{1,1}$ is

$$F_\zeta(\overline{\partial}_{\Pi,\mathrm{m}}) \;=\; F_\zeta(\overline{\partial}_{P(\overline{\partial}_m)}) \;+\; \frac{1}{12}\,(6m^2 + 6m + 1)\,\pi_*(\mathrm{gv}) - \frac{1}{12}\,e,$$

where $\pi_*(\mathrm{gv})$ is integration over the fibre of a Godbillon-Vey form, e an Euler form [12], and $P(\overline{\partial}_m)$ the family of Calderón boundary conditions.

2 Fibrations of Manifolds

Let $\pi : M \xrightarrow{X} Y$ be a smooth fibration of manifolds with fibre diffeomorphic to a compact connected manifold X of dimension n with boundary $\partial X \neq \emptyset$. The total space M is itself a manifold with boundary ∂M and there is a boundary fibration $\partial\pi : \partial M \xrightarrow{\partial X} Y$ of closed manifolds of dimension $n-1$. For example, for a fibration of surfaces over $Y = S^1$ then ∂M is a disjoint union of 2-tori fibred by the circle.

We assume there exists a collar neighbourhood $\mathcal{U} \subset M$ of ∂M with a diffeomorphism

$$\mathcal{U} \cong [0,1) \times \partial M, \tag{2.0.5}$$

corresponding fibrewise to a collar neighbourhood $[0,1) \times \partial X_y$ of each fibre $X_y := \pi^{-1}(y)$.

2.1 Bundles over fibrations

A smooth family of vector bundles associated to $\pi : M \xrightarrow{X} Y$ is defined to be a finite-rank C^∞ vector bundle $E \to M$. Formally, we may then consider the infinite-dimensional bundle $\mathcal{H}(E) \to Y$ whose fibre at $y \in Y$ is the space $\mathcal{H}_y(E) :=$

$\Gamma(X_y, E_{|X_y})$ of C^∞ sections of E over X_y. Concretely, a section of $\mathcal{H}(E)$ is defined to be a section of E over M,

$$\Gamma(Y, \mathcal{H}(E)) := \Gamma(M, E). \tag{2.1.1}$$

Thus, in practise one works with the right-hand side of (2.1.1), as indicated below. $\Gamma(Y, \mathcal{H}(E))$ is then a $C^\infty(Y)$-module via

$$C^\infty(Y) \times \Gamma(Y, \mathcal{H}(E)) \to \Gamma(Y, \mathcal{H}(E)), \quad (f, s) \mapsto f \cdot s := \pi^*(f)s, \tag{2.1.2}$$

that is, $f \cdot s(m) = f(\pi(m))\, s(m)$.

The restriction map to boundary sections

$$\gamma : \Gamma(Y, \mathcal{H}(E)) \to \Gamma(Y, \mathcal{H}(E_{\partial M}))$$

is defined by the restriction map to the boundary on the total space

$$\gamma : \Gamma(M, E) \to \Gamma(\partial M, E_{\partial M}) \tag{2.1.3}$$

with $E_{\partial M} = \bigcup_{m \in \partial M} E_m$ the bundle E along ∂M. Relative to (2.0.5)

$$E_{|\mathcal{U}} = \gamma^*(E_{\partial M})$$

and $\Gamma(\mathcal{U}, E) \cong C^\infty([0,1)) \otimes \Gamma(M, E_{\partial M})$. Here, $\mathrm{rank}(E_{\partial M}) = \mathrm{rank}(E)$, so, for example, $TM_{\partial M}$ is not the same thing as $T(\partial M)$, whose sections are vector fields along the boundary, while a section of $TM_{\partial M}$ includes vector fields which point out of the boundary; one has $TM_{\partial M} \cong \mathbb{R} \oplus T(\partial M)$.

The vertical tangent bundle $T(M/Y)$ (resp. $T(\partial M/Y)$) is the subbundle of TM (resp. $T\partial M$) whose fibre at $m \in M$ (resp. $m \in \partial M$) is the tangent space to the fibre $X_{\pi(m)}$ (resp. $\partial X_{\pi(m)}$). (By equipping $T(M/Y)$ with a metric and choosing an inward pointing vector field on $T(\partial M)$ with values in TZ the inward geodesic flow identifies a tubular neighbourhood $\mathcal{U}$ of ∂M of the form (2.0.5).) $\pi^*(TY)$ is the pull-back subbundle from the base. Likewise, there is the dual bundle T^*M with subbundle $T^*(M/Y)$, whose sections are vertical forms along M, and $\pi^*(\wedge T^*Y)$. More generally, the de-Rham algebra on Y with values in $\mathcal{H}(E)$ is the direct sum of the

$$\mathcal{A}^k(Y, \mathcal{H}(E)) = \Gamma(M, \pi^*(\wedge^k T^*Y) \otimes E \otimes |\wedge_\pi|^{1/2}).$$

The line bundle of vertical densities $|\wedge_\pi|$ is included to facilitate integration along the fibre.

2.2 Connections

A connection (or covariant derivative) on $\mathcal{H}(E)$ is specified by a fibration 'connection' on M

$$TM \cong T(M/Y) \oplus T_H M, \tag{2.2.1}$$

and a vector bundle connection on E

$$\widetilde{\nabla} : \Gamma(M, E) \to \Gamma(M, E \otimes T^*M),$$

which are compatible with the induced boundary connections.

The fibration connection is a complementary subbundle to $T(M/Y)$, specifying an isomorphism $\pi^*(TY) \cong T_H M$ and hence a lift of vector fields from the base to horizontal vector fields on M

$$\Gamma(Y, TY) \overset{\cong}{\to} \Gamma(M, T_H M), \qquad \xi \mapsto \xi_H.$$

A connection

$$\nabla^M : \mathcal{A}^0(Y, \mathcal{H}(E)) \to \mathcal{A}^1(Y, \mathcal{H}(E)) \tag{2.2.2}$$

is then defined by

$$\nabla^M_\xi s = \widetilde{\nabla}_{\xi_H} s, \qquad s \in \Gamma(M, E), \; \xi \in C^\infty(Y, TY). \tag{2.2.3}$$

Compatibility with the boundary means, first, that in the collar $\mathcal{U}$

$$\widetilde{\nabla}_{|\mathcal{U}} = \gamma^* \widetilde{\nabla}^{\partial M} = \partial_u \, du + \widetilde{\nabla}^{\partial M}$$

where $u \in [0, 1)$ is the normal coordinate to ∂M and $\widetilde{\nabla}^{\partial M} : \Gamma(\partial M, E_{\partial M}) \to \Gamma(\partial M, E_{\partial M} \otimes T^*M)$ is the induced connection on $E_{\partial M}$, defining $\nabla^{\partial M} : \mathcal{A}^0(Y, \mathcal{H}(E_{\partial M})) \to \mathcal{A}^1(Y, \mathcal{H}(E_{\partial M}))$ by

$$\nabla^{\partial M}_\xi s = \widetilde{\nabla}^{\partial M}_{\xi_H} s, \qquad s \in \Gamma(\partial M, E_{\partial M}). \tag{2.2.4}$$

Secondly, that with respect to the boundary splitting

$$T(\partial M) \cong T(\partial M/Y) \oplus T_H \partial M$$

induced by

$$T\mathcal{U} \cong \mathbb{R} \oplus T\partial M \tag{2.2.5}$$

and the splitting (2.2.1), one has for $\xi \in C^\infty(Y, TY)$ that

$$(\xi_H)_{|\mathcal{U}} \in C^\infty(\mathcal{U}, T_H(\partial M)),$$

that is,

$$du(\xi_H) = 0,$$

where du is extended from $\mathcal{U}$ to M by zero. One then has from (2.2)

Lemma 2.1

$$\gamma \circ \widetilde{\nabla}_{\xi_H} = \widetilde{\nabla}^{\partial M}_{\xi_H} \circ \gamma, \qquad \xi \in C^\infty(Y, TY), \tag{2.2.6}$$

as maps $\Gamma(M, E) \to \Gamma(\partial M, E_{\partial M})$.

The curvature of the connection (2.2.2) evaluated on $\xi, \eta \in C^\infty(Y, TY)$

$$R(\xi,\eta) \in \Gamma(Y, \, \mathrm{End}(\mathcal{H}(E))\,)$$

is the smooth family of first-order differential operators (as in [2] Prop(1.11))

$$R(\xi,\eta) := \widetilde{\nabla}_{\xi_H}\widetilde{\nabla}_{\eta_H} - \widetilde{\nabla}_{\eta_H}\widetilde{\nabla}_{\xi_H} - \widetilde{\nabla}_{[\xi,\eta]_H} = \widetilde{R}(\xi,\eta) + \widetilde{\nabla}_{[\xi,\eta]_H - [\xi_H,\eta_H]}$$

where $\widetilde{R}(\xi,\eta) \in \Gamma(M, \mathrm{End}\, E)$ is the curvature of $\widetilde{\nabla}$. The above compatibility assumptions state that $\gamma_*(\xi_H), \gamma_*(\eta_H) \in C^\infty(\partial M, T_H(\partial M))$ and

$$R(\xi_H,\eta_H)\circ\gamma = R^{\partial M}(\gamma_*(\xi_H),\gamma_*(\eta_H)) \;\in \Gamma(Y, \mathrm{End}\,(\mathcal{H}(E_{\partial M}))),$$

where $R^{\partial M}(\alpha,\beta)$ is the curvature of (2.2.4).

2.2.1 Example: spin connection

For our purposes here, it is not necessary to specify which particular connection on E is being used. However, to compute the local index form curvature for a fibration of compact Riemannian spin manifolds (with or without boundary) then $\widetilde{\nabla}$ must be the Bismut connection [2], [1] and E a twisted vertical spinor bundle. Then $T(M/Y)$ is oriented and spin, while a metric g on TM in $\mathcal{U}$ is assumed to be the pull back of a metric $g^{\partial M}$ on $T\partial M$, so that $g^M_{|\mathcal{U}} = du^2 + g^{\partial M}$. If the connection on any twisting bundle is also of product type in the collar, then the situation of §2.2 holds, and the Bismut connection follows [2], [1].

3 Families of Pseudodifferential Operators

A smooth family of ψdos of constant order μ associated to a fibration $\pi : N \stackrel{X}{\to} Y$ of compact boundaryless manifolds, with $\dim(X) = n$, with vector bundle $E \to N$ means a classical ψdo

$$\mathsf{A} : \Gamma(N, E^+) \to \Gamma(N, E^-)$$

with Schwartz kernel $k_{\mathsf{A}} \in \mathcal{D}'(N \times_\pi N, E \boxtimes E)$ a vertical distribution, where the fibre product $N \times_\pi N$ consists of pairs $(x, x') \in N \times N$ which lie in the same fibre, i.e. $\pi(x) = \pi(x')$, such that in any local trivialization k_{A} is an oscillatory integral with *vertical symbol* $\mathsf{a} \in S^\nu_{\mathrm{vert}}(N/Y)$ of order ν for which ξ is *restricted* to the vertical momentum space, along the fibre. We refer to A as a vertical ψdo associated to the fibration and denote this subalgebra of ψdos on N by

$$\Gamma(Y, \Psi^\nu(E^+, E^-)) = \Psi^\nu_{\mathrm{vert}}(N, E^+, E^-).$$

In a similar way, for a fibration $\pi : M \to Y$ of compact manifolds with boundary the pseudodifferential boundary operator (ψdbo) calculus as developed by Grubb

[9], generalizing the Boutet de Monvel algebra, may be applied to define a vertical calculus of operators with oscillatory integral kernels along the fibres comprising trace operators from interior to boundary sections, vertical Poisson operators taking sections over the boundary ∂M into the interior, and restricted ψdo and singular Green's operators over the interior of M. This vertical ψdbo algebra is denoted

$$\Gamma(Y, \Psi_{\flat}(E^+, E^-)) = \Psi_{\mathrm{vert},\flat}(M, E^+, E^-).$$

The algebras $\mathsf{A} \in \Gamma(Y, \Psi^{\nu}(E^+, E^-))$ (see [21]) and $\Psi_{\mathrm{vert},\flat}(M, E^+, E^-)$ of generalized ψdos are described in more detail in the Appendix.

For a local trivialization of the fibration and of E one may locally identify a vertical ψdo A with a single ψdo (or ψdbo) A_y acting on a fixed space and depending on a local parameter y in Y.

3.1 Families of Dirac-type operators

Let D be a family of Dirac-type operators associated to the fibration $\pi : M \to Y$ of compact manifolds with boundary with vector bundles $E^{\pm} \to M$, such that in $\mathcal{U}$

$$\mathsf{D}_{|\mathcal{U}} = \Upsilon \left(\frac{\partial}{\partial x_n} + \mathsf{D}_{\partial M} \right),$$

where $\mathsf{D}_{\partial M} \in \Psi_{\mathrm{vert}}(\partial M, E_{\partial M})$ a family of Dirac-type operators associated to the boundary fibration of closed manifolds, and $\Upsilon \in \Gamma(\partial M, \mathrm{End}\,(E_{\partial M}))$ is a bundle isomorphism.

3.1.1 Vertical Poisson and Calderón operators

Let $\widehat{M} = M \cup_{\partial M} (-M) \to Y$ be the fibration of compact boundaryless manifolds with fibre the double manifold $\widehat{X}_y = X_y \cup_{\partial X_y} (-X_y)$. With the product structure (2.0.5), D extends by the proof for a single operator, as in [7] Chap.9, to an *invertible* vertical first-order differential operator $\widehat{\mathsf{D}} \in \Psi^1(\widehat{M}, \widehat{E}^+, \widehat{E}^-)$, where $\widehat{E}^{\pm}_{|M} = E^{\pm}$ and $r^+\widehat{\mathsf{D}}e^+ = \mathsf{D}$. As indicated in Appendix (A.1) and accounted for in detail in [21], there is therefore a smooth family of resolvent ψdos of order -1 $\widehat{\mathsf{D}}^{-1} \in \Psi^{-1}_{\mathrm{vert}}(\widehat{M}, \widehat{E}^-, \widehat{E}^+)$. Define

$$\mathsf{D}_+^{-1} := r^+\widehat{\mathsf{D}}^{-1}e^+ \ \in \ \Psi^{-1}_{\mathrm{vert},\,\flat}(M, E^-, E^+).$$

Since $\widehat{\mathsf{D}}\widehat{\mathsf{D}}^{-1} = \mathsf{I}$ on $\Gamma(\widehat{M}, \widehat{\mathcal{E}})$, with I the vertical identity operator, and since D is local

$$\mathsf{D}\mathsf{D}_+^{-1} = \mathsf{I} \quad \text{on } \ \Gamma(M, E^-).$$

Thus there is a short exact sequence $0 \to \mathrm{Ker}\,(\mathsf{D}) \to \Gamma(M,E^+) \xrightarrow{\mathsf{D}} \Gamma(M,E^-) \to 0$, where

$$\mathrm{Ker}\,(\mathsf{D}) = \{s \in \Gamma(M,E^+) \mid \mathsf{D}s = 0 \ \text{ in } \ M\backslash\partial M\}. \tag{3.1.1}$$

On the other hand, D_+^{-1} is not a left-inverse but (by an obvious modification of [24], [25], [7] §12)

$$\mathsf{D}_+^{-1}\mathsf{D} = \mathsf{I} - \mathsf{K}\gamma \qquad \text{on } \ \Gamma(M,E^+), \tag{3.1.2}$$

where γ is the restriction operator (2.1.3) and the *vertical Poisson operator associated to D* is

$$\mathsf{K} = \mathsf{D}_+^{-1}\gamma^*\Upsilon, \tag{3.1.3}$$

with γ as in (A.2.3). Composing with boundary restriction defines the *vertical Calderón projection* ([8], [24], [25], [7])

$$P(\mathsf{D}) := \gamma \circ \mathsf{K} \ \ \in \ \Gamma(Y, \Psi^0_{\mathrm{vert}}(E_{\partial M})) := \ \Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M}) \tag{3.1.4}$$

with range the space of vertical Cauchy data

$$\mathrm{ran}(P(\mathsf{D})) = \gamma\mathrm{Ker}\,(\mathsf{D}) = \{f \in \Gamma(\partial M, E_{\partial M}) \mid f = \gamma s, \ s \in \mathrm{Ker}\,(\mathsf{D})\}.$$

This may be formally characterized as the space of sections of the infinite-dimensional subbundle $\mathcal{K}(\mathsf{D}) \subset \mathcal{H}(E_{\partial M})$ with fibre $K(D_y) = \gamma\mathrm{Ker}\,(D_y)$ at $y \in Y$ (and, likewise, $\mathrm{Ker}\,\mathsf{D}$ as the space of sections of the formal subbundle of $\mathcal{H}(E^+)$ with fibre $\mathrm{Ker}\,D_y$). However, as with $\mathcal{H}(E_{\partial M})$ in §2.1, concretely one only works with the space of sections of $\mathcal{K}(\mathsf{D})$

$$\Gamma(Y,\mathcal{K}(\mathsf{D})) := \{f \in \Gamma(\partial M, E_{\partial M}) \mid f = \gamma s, \ s \in \mathrm{Ker}\,(\mathsf{D})\} = \mathrm{ran}(P(\mathsf{D})).$$

(Note, on the other hand, $\mathcal{K}(\mathsf{D})$ is not the space of sections of a subbundle of $E_{\partial M}$.)

By the fibrewise Unique Continuation property, the restriction $\gamma : \mathrm{Ker}\,\mathsf{D} \to \Gamma(Y,\mathcal{K}(\mathsf{D}))$ defines a canonical isomorphism with right-inverse

$$\mathsf{K} : \Gamma(Y,\mathcal{K}(\mathsf{D})) \xrightarrow{\cong} \mathrm{Ker}\,(\mathsf{D}). \tag{3.1.5}$$

3.2 Well-posed boundary problems for D

The vertical Calderón projection (3.1.4) provides the reference ψdo on boundary sections with respect to which is defined any vertical well-posed boundary condition for D.

3.2.1 Smooth families of boundary ψdo projections

We consider smooth families of ψdos on $\Gamma(Y,\mathcal{H}(E_{\partial M}))$ which are perturbations of the Calderón projection of the form

$$\mathsf{P} = P(\mathsf{D}) + \mathsf{S} \quad \in \Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M}), \tag{3.2.1}$$

where

$$\mathsf{S} \in \Psi^{-\infty}_{\rm vert}(\partial M, E_{\partial M})$$

is a vertical smoothing operator (smooth family of smoothing operators), cf. Appendix. From Birman-Solomyak [6], Seeley [25] (see also [7]) (3.2.1) may be replaced by the projection onto ran(P) to define an equivalent boundary problem. So we may assume $\mathsf{P}^2 = \mathsf{P}$ and $\mathsf{P}^* = \mathsf{P}$, where the adjoint is with respect to the Sobolev completions and vertical inner-product defined by metric on $E_{\partial M}$ and the choice of vertical density $d_{\partial\, \rm M/Y}x' \in \Gamma(\partial M, |\wedge^{n-1} T^*(\partial M/Y)|)$.

The family APS projection $\Pi_> = \{\Pi^y_> \mid y \in Y\}$ is only smooth in y when $\dim {\rm Ker}\,(\mathsf{D}_{\partial M})_y$ is constant [3]. Nevertheless, we refer to (3.2.1) as a vertical ψdo of APS-type.

The choice of P in (3.2.1) distinguishes the subspace of the space of boundary sections

$$\Gamma(Y, \mathcal{W}) := {\rm ran}(\mathsf{P}) = \{\mathsf{P}f \mid f \in \Gamma(\partial M, E_{\partial M})\} \;\subset\; \Gamma(\partial M, E_{\partial M}) := \Gamma(Y, \mathcal{H}(E_{\partial M})). \tag{3.2.2}$$

Here, $\mathcal{W}$ is the formal infinite-rank subbundle of $\mathcal{H}(E_{\partial M})$ with fibre $W_y = {\rm ran}(\mathsf{P}_y) \subset \Gamma(\partial X_y, (E_{\partial M})_y)$, whose local bundle structure follows from the invertibility of the operators $P_{y'}P_y : W_y \to W_{y'}$ for y' near y. Analytically, though, just as with $\mathcal{K}(\mathsf{D})$, one works in practise with (3.2.2).

Given any two choices P, P' of the form (3.2.1) one has the smooth family of Fredholm operators

$$\mathsf{P}' \circ \mathsf{P} : \Gamma(Y, \mathcal{W}) \to \Gamma(Y, \mathcal{W}') \tag{3.2.3}$$

where $\Gamma(\partial M, \mathcal{W}) := {\rm ran}(\mathsf{P})$. We may write this as a section of the formal bundle ${\rm Hom}(\mathcal{W}, \mathcal{W}')$ in so far as we declare the sections of the latter to precisely be the subspace of $\Psi_{\rm vert}(\partial M, E_{\partial M})$

$$\Gamma(Y, {\rm Hom}(\mathcal{W}, \mathcal{W}')) := \{\mathsf{P}' \circ \mathsf{A} \circ \mathsf{P} \mid \mathsf{A} \in \Psi_{\rm vert}(\partial M, E_{\partial M})\}. \tag{3.2.4}$$

Note here that

$$\mathsf{P}' \circ \mathsf{P} \in \Psi^0_{\rm vert}(\partial M, E_{\partial M})$$

is a smooth vertical ψdo on boundary sections. The reference to it as a 'smooth family of Fredholm operators' means additionally that there is smooth vertical ψdo on boundary sections

$$\mathsf{Q}_{\mathsf{P},\mathsf{P}'} \in \Psi^0_{\rm vert}(\partial M, E_{\partial M})$$

such that

$$\mathsf{Q}_{\mathsf{P},\mathsf{P}'} \circ (\mathsf{P}' \circ \mathsf{P}) = \mathsf{P} + \mathsf{P}\mathsf{S}'\mathsf{P}, \qquad \mathsf{S}' \in \Psi^{-\infty}_{\rm vert}(\partial M, E_{\partial M}), \tag{3.2.5}$$

and hence that $\mathsf{Q}_{\mathsf{P},\mathsf{P}'}$ is a parametrix for (3.2.3); that is, restricted to $\Gamma(Y, \mathcal{W})$ (3.2.5) is

$$(\mathsf{Q}_{\mathsf{P},\mathsf{P}'} \circ (\mathsf{P}' \circ \mathsf{P}))_{|\mathcal{W}} = \mathsf{I}_{\mathcal{W}} + \mathsf{P}\mathsf{S}'\mathsf{P}, \tag{3.2.6}$$

where $\mathsf{I}_{\mathcal{W}}$ denotes the identity on $\Gamma(Y, \mathcal{W})$. Indeed, we may take, for example, $\mathsf{Q}_{\mathsf{P},\mathsf{P}'} = \mathsf{P} \circ \mathsf{P}'$.

3.2.2 Vertical APS-type boundary problems

The choice of P in (3.2.1) additionally distinguishes the subspace of interior sections on the total space of the fibration (which is not itself the space of sections of some subbundle of E^+)

$$\Gamma(Y,\mathcal{H}_{\mathsf{P}}(E^+)) := \mathrm{Ker}\,(\mathsf{P}\circ\gamma) = \{s\in\Gamma(M,E^+)\mid \mathsf{P}\gamma s = 0\} \ \subset\ \Gamma(M,E^+) := \Gamma(Y,\mathcal{H}(E^+)). \tag{3.2.7}$$

We may consider the infinite-dimensional bundle $\mathcal{H}_{\mathsf{P}}(E)\to Y$ with fibre at $y\in Y$ the space of C^∞ sections of E^+ over X_y which lie in $\mathrm{Ker}\,(P_y\circ\gamma)$, related to $\mathcal{W}$ via the exact sequence $0\to\mathcal{H}_{\mathsf{P}}(E^+)\to\mathcal{H}(E^+)\overset{\mathsf{P}\circ\gamma}{\to}\mathcal{W}\to 0$. Concretely, however, one works in practise with (3.2.7).

A smooth family of APS-type boundary problems is the restriction of D to the subspace (3.2.7)

$$\mathsf{D}_{\mathsf{P}} := \mathsf{D} :\ \mathrm{Ker}\,(\mathsf{P}\circ\gamma) = \Gamma(Y,\mathcal{H}_{\mathsf{P}}(E^+)) \to \Gamma(M,E^-).$$

D_{P} restricts over X_y to $D_{P_y} := (D_y)_{P_y} : \mathrm{dom}(D_{P_y}) \to \Gamma(X_y, E_y^-)$ in a local trivialization of the fibration of manifolds, an APS boundary problem in the usual single operator sense.

The existence of the Poisson operator (3.1.3) reduces the construction of a vertical parametrix for D_{P} to the construction of a parametrix for the operator (3.2.3) on boundary sections

$$\mathsf{S}(\mathsf{P}) := \mathsf{P}\circ P(\mathsf{D}) :\ \Gamma(Y,\mathcal{K}(\mathsf{D}))\to\Gamma(Y,\mathcal{W}),$$

since the Poisson operator restricts to a bundle isomorphism between $\mathrm{Ker}\,(\mathsf{S}(\mathsf{P}))$ and $\mathrm{Ker}\,(\mathsf{D}_{\mathsf{P}})$, and likewise between the bundles of cokernels. Explicitly, let $U\subset Y$ be the open subset of points in Y where $\mathsf{S}(\mathsf{P})$ is invertible. That is, relative to any local trivialization of the geometric fibration $M\to Y$ and bundles at $y\in Y$ the Fredholm family $S(P)$ parametrises an operator $S_y(P_y) = P_y\circ P(D_y) : K(D_y)\to \mathrm{ran}(P_y)$ in the usual single operator sense; $y\in U$ if $S_y(P_y)$ is invertible. Over U we define

$$\mathsf{K}(\mathsf{P})_{|U} := \mathsf{K}\circ P(\mathsf{D})\mathsf{S}(\mathsf{P})_{|U}^{-1}\mathsf{P} :\ \Gamma(\pi_\partial^{-1}(U),E_{\partial M})\to\Gamma(\pi_\partial^{-1}(U),E^+), \tag{3.2.8}$$

where $\pi_\partial : \partial M\to Y$ is the boundary fibration. Then Green's theorem for the vertical densities along the fibres locally refines (3.1.2) to

$$(\mathsf{D}_{\mathsf{P}})_{|U}^{-1}\mathsf{D} = \mathsf{I}_{|U} - \mathsf{K}(\mathsf{P})_{|U}\gamma :\ \Gamma(\pi_\partial^{-1}(U),E^+)\to\Gamma(\pi_\partial^{-1}(U),E^+). \tag{3.2.9}$$

Moreover, if $\mathsf{D}_{\mathsf{P}'}$ is also invertible over U

$$(\mathsf{D}_{\mathsf{P}})_{|U}^{-1} = (\mathsf{D}_{\mathsf{P}})_{|U}^{-1}\mathsf{D}(\mathsf{D}_{\mathsf{P}'})_{|U}^{-1} = \mathsf{D}_{\mathsf{P}'}^{-1} - \mathsf{K}(\mathsf{P})_{|U}\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1} :\ \Gamma(\pi_\partial^{-1}(U),E^-)\to\Gamma(\pi_\partial^{-1}(U),E^+), \tag{3.2.10}$$

We note, globally on M, that:

Proposition 3.1 *With the above assumptions the relative inverse is a vertical smoothing operator*

$$(D_P)^{-1}_{|U} - (D_{P'})^{-1}_{|U} \quad \in \; \Gamma(U, \Psi^{-\infty}_{\mathrm{vert},\flat}(E_{|\pi_\partial^{-1}(U)})).$$

More generally, for a general APS-type vertical ψdo *projection* $P \in \Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M})$ *a global parametrix for the smooth family of APS-type boundary problems* D_P : $\mathrm{Ker}\,(P \circ \gamma) \to \Gamma(M, E^-)$ *is given by*

$$D_+^{-1} - K\,Q_{P,\,P(D)}\,\gamma D_+^{-1} \quad \in \; \Gamma(Y, \Psi^{-\infty}_{\mathrm{vert},\flat}(E_{|M})), \tag{3.2.11}$$

where $Q_{P,\,P(D)}$ *is any parametrix as in* (3.2.6) *for* $S(P)$, *for example* $Q_{P,\,P(D)} = P(D) \circ P$.

Proof. We have $\mathsf{P} = P(\mathsf{D}) + \mathsf{S}$, $\mathsf{P}' = P(\mathsf{D}) + \mathsf{S}'$ for vertical smoothing operators

$$\mathsf{S},\; \mathsf{S}' \in \Psi^{-\infty}_{\mathrm{vert}}(\partial M, E_{\partial M}).$$

Hence

$$\mathsf{P} - \mathsf{P}' \in \Psi^{-\infty}_{\mathrm{vert}}(\partial M, E_{\partial M}) \tag{3.2.12}$$

and

$$\mathsf{P}(\mathsf{I} - \mathsf{P}') = -\mathsf{P}\,\mathsf{S}' \in \Psi^{-\infty}_{\mathrm{vert}}(\partial M, E_{\partial M}) \tag{3.2.13}$$

are vertical smoothing operator operators. By (3.2.10)

$$(\mathsf{D}_\mathsf{P})^{-1}_{|U} - (\mathsf{D}_{\mathsf{P}'})^{-1}_{|U} = -\mathsf{K}(\mathsf{P})\mathsf{P}\gamma(\mathsf{D}_{\mathsf{P}'})^{-1}_{|U} = -\mathsf{K}(\mathsf{P})_{|U}\mathsf{P}(\mathsf{I}-\mathsf{P}')\gamma(\mathsf{D}_{\mathsf{P}'})^{-1}_{|U} \quad \text{over } M_{|U} = \pi^{-1}(U)$$

which by (3.2.13) and the composition rules of the ψdbo calculus (cf §A.2) is smoothing.

The assertion that (3.2.11) is a parametrix is an obvious slight modification of the argument leading to (3.2.10).

□

4 The Determinant Line Bundle

From Proposition 3.1 the choice of P restricts D to a family D_P of Fredholm operators. It also has the consequence that the kernels of the restricted operators no longer define a vector bundle (formally (3.1.1) does), rather they define a virtual bundle $\mathrm{Ind}\,\mathsf{D}_\mathsf{P} \in K(Y)$. Likewise, from §3.2, $\mathsf{S}(\mathsf{P}) : \Gamma(Y, \mathcal{K}((\mathsf{D})) \to \Gamma(Y, \mathcal{W})$ is a smooth Fredholm family defining an element $\mathrm{Ind}\,\mathsf{S}(\mathsf{P}) \in K(Y)$. The determinant line bundles $\mathrm{Det}\,\mathsf{D}_\mathsf{P}$ and $\mathrm{Det}\,\mathsf{S}(\mathsf{P})$ are the top exterior powers of these elements, at least in K-theory. To make sense of them as smooth complex line bundles we use the following trivializations, with respect to which the zeta connection will be constructed.

4.1 Determinant lines

The determinant of a Fredholm operator $T : H \to H'$ exists abstractly not as a number but as an element $\det T$ of a complex line $\mathrm{Det}\, T$. A point of $\mathrm{Det}\, T$ is an equivalence class $[S, \lambda]$ of pairs (S, λ), where $S : H \to H'$ differs from T by a trace-class operator and relative to the equivalence relation $(Sq, \lambda) \sim (S, \lambda \det_F q)$ for $q : H \to H$ of Fredholm-determinant class. (A compact operator $A : H \to H'$ is trace-class if $\sum_{\mu \in \mathrm{spec}(\sqrt{A^*A})} |\mu| < \infty$.) Scalar multiplication on $\mathrm{Det}\, T$ is $\mu.[S, \lambda] = [S, \mu\lambda]$. The determinant $\det T := [T, 1]$ is non-zero if and only if T is invertible, and there is a canonical isomorphism

$$\mathrm{Det}\, T \cong \wedge^{\max} \mathrm{Ker}\, T^* \otimes \wedge^{\max} \mathrm{Cok}\, T.$$

For Fredholm operators $T_1, T_2 : H \to H'$ with $T_i - T$ trace class and T_2 invertible

$$\frac{\det T_1}{\det T_2} = \det{}_F (T_1 T_2^{-1}), \tag{4.1.1}$$

where the quotient on the left-hand side is taken in $\mathrm{Det}\, T$ and $\det_F$ on the right-hand side in H'.

4.2 The line bundle $\mathrm{Det}\, \mathsf{S}(\mathsf{P})$

For each smooth family of smoothing operators $\sigma = \{\sigma_y\} \in \Gamma(Y, \Psi^{-\infty}_{\mathrm{vert}}(E_{\partial M})) = \Psi^{-\infty}_{\mathrm{vert}}(\partial M, E_{\partial M})$ define

$$\mathsf{P}_\sigma = \mathsf{P} + \mathsf{P}\sigma\mathsf{P} \in \Gamma(Y, \Psi^0_{\mathrm{vert}}(E_{\partial M})) = \Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M}) \tag{4.2.1}$$

and the open subset of Y

$$U_\sigma := \{ y \in Y \mid \mathsf{S}(\mathsf{P}_\sigma)_y := (P_y + P_y\, \sigma_y\, P_y) \circ P(D_y) : K(D_y) \to \mathrm{ran}(P_y) \ \text{ invertible} \}. \tag{4.2.2}$$

Over U_σ one has the canonical trivialization

$$U_\sigma \ \to \mathrm{Det}\, \mathsf{S}(\mathsf{P})_{|U_\sigma} = \bigcup_{y \in U_\sigma} \mathrm{Det}\, \mathsf{S}(\mathsf{P})_y, \qquad y \mapsto \det \mathsf{S}(\mathsf{P}_\sigma)_y := [(\mathsf{P}_\sigma \circ P(\mathsf{D}))_y, 1],$$

where $\mathsf{S}(\mathsf{P})_y := P_y \circ P(D_y) : K(D_y) \to \mathrm{ran}(P_y)$. Note that $\det \mathsf{S}(\mathsf{P}_\sigma)_y \neq 0$, and that $\mathsf{S}(\mathsf{P}_\sigma)_y - \mathsf{S}(\mathsf{P})_y$ is the restriction of a smoothing operator so that

$$\det \mathsf{S}(\mathsf{P}_\sigma)_y \ \in \ \mathrm{Det}\, \mathsf{S}(\mathsf{P})_y \backslash \{0\}.$$

Over the intersection $U_\sigma \cap U_{\sigma'} \neq \emptyset$ the transition function by (4.1.1) is the function

$$U_\sigma \cap U_{\sigma'} \ \to \ \mathbb{C}^*, \quad y \mapsto \det{}_F \left(\mathsf{S}(\mathsf{P}_\sigma)_y \circ \mathsf{S}(\mathsf{P}_{\sigma'})_y^{-1} \right), \tag{4.2.3}$$

where the Fredholm determinant is taken on $\mathrm{ran}(P_y)$ and varies holomorphically with y.

4.3 The line bundle Det D_{P}

The bundle structure of Det D_{P} is defined by perturbing D_{P_y} to an invertible operator. It is crucial for the construction of the ζ-connection to do so by perturbing the ψdo P_y, not D_y.

To do this we mediate the local trivializations of Det D_{P} through those of Det $\mathsf{S}(\mathsf{P})$ in §4.2.

Precisely, the family of ψdos P_σ in (4.2.1) is of APS-type

$$\mathsf{P}_\sigma - P(\mathsf{D}) \quad \in \Psi^{-\infty}_{\mathrm{vert}}(\partial M, E_{\partial M}), \tag{4.3.1}$$

defining the vertical boundary problem $\mathsf{D}_{\mathsf{P}_\sigma} : \Gamma(Y, \mathcal{H}_{\mathsf{P}_\sigma}(E^+)) \to \Gamma(Y, \mathcal{H}(E^-))$. From §3.2.2

$$U_\sigma := \left\{ y \in Y \mid (\mathsf{D}_{\mathsf{P}_\sigma})_y : \mathrm{dom}((D_{\mathsf{P}_\sigma})_y \to \Gamma(X_y, E^-_y) \ \text{ invertible} \right\}, \tag{4.3.2}$$

over which there is the local trivialization

$$U_\sigma \ \to \mathrm{Det}\, \mathsf{D}_{\mathsf{P}_\sigma|U_s}, \qquad y \mapsto \det((\mathsf{D}_{\mathsf{P}_\sigma})_y) = [(\mathsf{D}_{\mathsf{P}_\sigma})_y, 1] \in \mathrm{Det}\, (\mathsf{D}_{\mathsf{P}_\sigma})_y. \tag{4.3.3}$$

The equivalence of (4.2.2) and (4.3.2) is the identification for any APS-type $\widetilde{P} \in \Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M})$ of the kernel (resp. cokernel) of $(D_{\widetilde{P}})_y$ with that of $S(\widetilde{P})_y$ defined by the Poisson operator K_y. It follows that there is a *canonical* isomorphism

$$\mathrm{Det}\, (D_{\widetilde{P}})_y \ \cong \ \mathrm{Det}\, S(\widetilde{P})_y \quad \text{with} \quad \det(D_{\widetilde{P}})_y \longleftrightarrow \det S(\widetilde{P})_y. \tag{4.3.4}$$

The local trivialization of Det D_{P} is then defined through the canonical isomorphisms of complex lines applied to (4.3.3)

$$\mathrm{Det}\, (D_{\mathsf{P}_\sigma})_y \overset{(4.3.4)}{\cong} \mathrm{Det}\, \mathsf{S}(\mathsf{P}_\sigma)_y = \mathrm{Det}\, \mathsf{S}(\mathsf{P})_y \overset{(4.3.4)}{\cong} \mathrm{Det}\, (\mathsf{D}_{\mathsf{P}})_y,$$

where the central equality is from §4.2. By construction the transition functions for Det D_{P} are precisely (4.2.3); that is, as functions of $y \in U_\sigma \cap U_{\sigma'}$

$$\det(\mathsf{D}_{\mathsf{P}_\sigma})_y = \det{}_F \left(\mathsf{S}(\mathsf{P}_\sigma)_y \circ \mathsf{S}(\mathsf{P}_{\sigma'})^{-1}_y\right) \det(\mathsf{D}_{\mathsf{P}_{\sigma'}})_y \qquad \text{in } \ \mathrm{Det}\, (\mathsf{D}_{\mathsf{P}})_y. \tag{4.3.5}$$

Thus the bundle structure of Det D_{P} is constructed using that of Det $\mathsf{S}(\mathsf{P})$, as with all other spectral invariants of D_{P} owing to the facts in §3.2.2.

With respect to smooth families of boundary conditions $\mathsf{P}, \mathsf{P}' \in \Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M})$

$$\mathrm{Det}\, \mathsf{D}_{\mathsf{P}} \cong \mathrm{Det}\, \mathsf{D}_{\mathsf{P}'} \otimes \mathrm{Det}\, (\mathsf{P} \circ \mathsf{P}'),$$

which may be viewed as a smooth version of the K-theory identity

$$\mathrm{Ind}\, \mathsf{D}_{\mathsf{P}} = \mathrm{Ind}\, \mathsf{D}_{\mathsf{P}'} + \mathrm{Ind}\, (\mathsf{P} \circ \mathsf{P}'). \tag{4.3.6}$$

These are a consequence of the following general (useful) identifications.

Theorem 4.1 *Let $A_1 : \mathcal{H}' \to \mathcal{H}''$, $A_2 : \mathcal{H} \to \mathcal{H}'$ be smooth (resp. continuous) families of Fredholm operators acting between Fréchet bundles over a compact manifold Y. Then there is a canonical isomorphism of C^∞ (resp C^0) line bundles*

$$\mathrm{Det}\, A_1 A_2 \cong \mathrm{Det}\, A_1 \otimes \mathrm{Det}\, A_2$$

with $\det A_1 A_2 \longleftrightarrow \det A_1 \otimes \det A_2$. *In $K(Y)$ one has*

$$\mathrm{Ind}\, A_1 A_2 = \mathrm{Ind}\, A_1 \ + \ \mathrm{Ind}\, A_2$$

For a proof of Theorem 4.1 see [22].

5 Hermitian Structure

The (Quillen) ζ-metric on $\mathrm{Det}\, \mathsf{D}_{\mathsf{P}}$ is defined over U_σ by evaluating it on the non-vanishing section $\det \mathsf{D}_{\mathsf{P}_\sigma}$

$$\| \det(\mathsf{D}_{\mathsf{P}_\sigma})_y \|_\zeta^2 = \det{}_\zeta(\Delta_{\mathsf{P}_\sigma})_y, \tag{5.0.7}$$

where the right-side is the ζ-determinant of the vertical Laplacian boundary problem for an APS-type ψdo P

$$\Delta_{\mathsf{P}} = \Delta := D^* D \ : \ \mathrm{dom}(\Delta_{\mathsf{P}}) \to \Gamma(M, E^-) \tag{5.0.8}$$

with $\mathrm{dom}(\Delta_{\mathsf{P}}) = \{s \in \Gamma(M, E^+) \mid \mathsf{P}\gamma s = 0, \ \mathsf{P}^\star \gamma D s = 0\}$ and $\mathsf{P}^\star := \Upsilon(I - P_y)\Upsilon^*$ the adjoint vertical boundary condition.

From [20] Thm(4.2) we know that

$$\| \det(\mathsf{D}_{\mathsf{P}_\sigma})_y \|_\zeta^2 = \frac{\det_F \big(\mathsf{S}(\mathsf{P}_\sigma)_y^* \, \mathsf{S}(\mathsf{P}_\sigma)_y\big)}{\det_F \big(\mathsf{S}(\mathsf{P}_{\sigma'})_y^* \, \mathsf{S}(\mathsf{P}_{\sigma'})_y\big)} \, \| \det(\mathsf{D}_{\mathsf{P}_{\sigma'}})_y \|_\zeta^2,$$

which is the patching condition with respect to the transition functions (4.3.5) for (5.0.7) to define a global metric on the determinant line bundle $\mathrm{Det}\, \mathsf{D}_{\mathsf{P}}$.

6 Connections on $\mathrm{Det}(\mathsf{D}_{\mathsf{P}})$

There are two natural ways to put a connection on the determinant bundle $\mathrm{Det}\, \mathsf{D}_{\mathsf{P}}$. The first of these is associated to the boundary fibration and its curvature may be viewed as a relative η-form. The second is the ζ-function connection and the object of primary interest here.

6.1 A connection on $\mathrm{Det}\,\mathsf{S}(\mathsf{P})$

The first connection is defined on $\mathrm{Det}\,\mathsf{S}(\mathsf{P})$, which defines a connection on $\mathrm{Det}\,\mathsf{D}_{\mathsf{P}}$ via the isomorphism (by construction) between these line bundles.

The endomorphism bundle $\mathrm{End}\,(\mathcal{H}(E_{\partial M}))$ whose sections are the boundary vertical ψdos

$$\Gamma(Y, \mathrm{End}\,(\mathcal{H}(E_{\partial M}))) := \Psi^*_{\mathrm{vert}}(\partial M, E_{\partial M})$$

has an induced connection (also denoted $\nabla^{\partial M}$) from $\nabla^{\partial M}$ on $\Gamma(Y, \mathcal{H}(E_{\partial M}))$ in (2.2.4) by

$$\nabla^{\partial M}_{\xi}\mathsf{A} := [\,\nabla^{\partial M}_{\xi},\, \mathsf{A}] \;\in \Psi^*_{\mathrm{vert}}(\partial M, E_{\partial M}),$$

where $\xi \in C^{\infty}(Y, TY)$. That is,

$$(\nabla^{\partial M}_{\xi}\mathsf{A})f = \widetilde{\nabla}^{\partial M}_{\xi_H}(\mathsf{A}f) - \mathsf{A}(\widetilde{\nabla}^{\partial M}_{\xi_H} f), \qquad f \in \Gamma(\partial M, E_{\partial M}). \tag{6.1.1}$$

Let $P(\mathsf{D}) \in \Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M})$ be the Calderón vertical ψdo projection, and let $\mathsf{P} \in \Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M})$ be any other vertical APS-type boundary condition (3.2.1). Then there are induced connections

$$\nabla^{\mathcal{W}} = \mathsf{P}\cdot\nabla^{\partial M}\cdot\mathsf{P}, \qquad \nabla^{\mathcal{K}} = P(\mathsf{D})\cdot\nabla^{\partial M}\cdot P(\mathsf{D})$$

defined on the Frechét bundles $\mathcal{W}$ and $\mathcal{K}(\mathsf{D})$, in the sense that

$$\nabla^{\mathcal{W}}_{\xi} : \; \Gamma(Y, \mathcal{W}) := \{\mathsf{P}s \mid s \in \Gamma(\partial M, E_{\partial M})\} \;\to\; \Gamma(Y, \mathcal{W})$$

with

$$\nabla^{\mathcal{W}}_{\xi} s = \mathsf{P}\,\widetilde{\nabla}^{\partial M}_{\xi_H}(\mathsf{P}s), \qquad s \in \Gamma(Y, \mathcal{W}),$$

satisfies the Leibnitz rule, and likewise for $\nabla^{\mathcal{K}}$. We therefore have the induced connection $\nabla^{\mathcal{K},\mathcal{W}}$ on the restricted hom-bundle $\mathrm{Hom}(\mathcal{K}(\mathsf{D}), \mathcal{W})$, where, as in (3.2.4),

$$\Gamma(Y, \mathrm{Hom}(\mathcal{K}(\mathsf{D}), \mathcal{W})) \;:=\; \{\mathsf{P}\circ\mathsf{C}\circ P(\mathsf{D}) \mid \mathsf{C} \in \Psi^*_{\mathrm{vert}}(\partial M, E_{\partial M})\},$$

defined by

$$(\nabla^{\mathcal{K},\mathcal{W}}_{\xi}\mathsf{A})s = \nabla^{\mathcal{W}}_{\xi}(\mathsf{A}s) - \mathsf{A}(\nabla^{\mathcal{K}}_{\xi}s), \qquad s \in \Gamma(\partial M, E_{\partial M}),\; \mathsf{A} \in \Gamma(Y, \mathrm{Hom}(\mathcal{K}, \mathcal{W})). \tag{6.1.2}$$

One then has a connection on $\mathrm{Det}\,\mathsf{S}(\mathsf{P})$ by setting over U_{σ}

$$\nabla^{\mathsf{S}(\mathsf{P})}_{|U_{\sigma}} \det \mathsf{S}(\mathsf{P}_{\sigma}) = \omega^{\mathsf{S}(\mathsf{P}_{\sigma})} \det \mathsf{S}(\mathsf{P}_{\sigma})$$

where the locally defined 1-form in $\Omega^1(U_{\sigma})$ is

$$\omega^{\mathsf{S}(\mathsf{P}_{\sigma})} \;=\; \mathrm{Tr}\,(\mathsf{S}(\mathsf{P}_{\sigma})^{-1}\nabla^{\mathcal{K},\mathcal{W}_{\sigma}}\,\mathsf{S}(\mathsf{P}_{\sigma})), \tag{6.1.3}$$

with $\Gamma(Y, \mathcal{W}_{\sigma}) = \mathrm{ran}(\mathsf{P}_{\sigma})$. The trace on the right-side of (6.1.3) is the usual vertical trace (along the fibres, as recalled in the Appendix), by construction taken over $\Gamma(Y, \mathcal{K}(\mathsf{D})) \subset \Gamma(\partial M, E_{\partial M})$.

Notice, here, that $\mathsf{A}^{-1}\nabla_\xi^{\mathcal{K},\mathcal{W}}\mathsf{A}$ will not be a *trace class* family of ψdos for a general invertible vertical ψdo $\mathsf{A} \in \Gamma(Y, \mathrm{Hom}(\mathcal{K},\mathcal{W})) \subset \Psi^*_{\mathrm{vert}}(\partial M, E_{\partial M})$. That this is nevertheless the case when $\mathsf{A} = \mathcal{S}(\mathsf{P})$, so that the right-side of (6.1.3) is well defined, is immediate from (4.3.1) and (6.1.2).

The local 1-forms define a global connection with respect to (4.2.3) by the identity in $\Omega^1(U_\sigma \cap U_{\sigma'})$

$$d_\xi \det{}_F \left(\mathsf{S}(\mathsf{P}_\sigma)\circ \mathsf{S}(\mathsf{P}_{\sigma'})^{-1}\right) \;=\; \mathrm{Tr}\,(\mathsf{S}(\mathsf{P}_\sigma)^{-1}\nabla_\xi^{\mathcal{K},\mathcal{W}_\sigma}\mathsf{S}(\mathsf{P}_\sigma)) - \mathrm{Tr}\,(\mathsf{S}(\mathsf{P}_{\sigma'})^{-1}\nabla_\xi^{\mathcal{K},\mathcal{W}_{\sigma'}}\mathsf{S}(\mathsf{P}_{\sigma'})).$$

which is a standard Fredholm determinant identity $d_\xi \det_F \mathsf{C} = \mathrm{Tr}\,(\mathsf{C}^{-1}\nabla_\xi \mathsf{C})$ for a smooth family of Fredholm-determinant class operators $y \mapsto \mathsf{C}(y)$.

6.1.1 Curvature of $\nabla^{\mathsf{S}(\mathsf{P})}$

The curvature of the connection $\nabla^{\mathsf{S}(\mathsf{P})}$ on the complex line bundle $\mathrm{Det}\,\mathsf{S}(\mathsf{P}) \to Y$ is the globally defined 2-form

$$R^{\mathcal{K},\mathcal{W}} = (\nabla^{\mathsf{S}(\mathsf{P})})^2 \;\in\; \Omega^2(Y)$$

determined by

$$R^{\mathcal{K},\mathcal{W}}_{|U_\sigma} \;=\; d\,\omega^{\mathsf{S}(\mathsf{P}_\sigma)} \;\in\; \Omega^2(U_\sigma). \tag{6.1.4}$$

Remark. *No use is made of the interpretation of $\mathcal{W}$ as a 'Fréchet bundle'. The 2-form $R^{\mathcal{K},\mathcal{W}}$ is constructed concretely as the vertical trace of a vertical ψdo-valued form on M (cf. Appendix).*

6.1.2 Why $R^{\mathcal{K},\mathcal{W}}$ is a relative eta form

The APS η-invariant of a single invertible Dirac-type operator ∂ over a closed manifold N is

$$\eta(\partial) \;=\; \frac{1}{2\sqrt{\pi}}\int_0^\infty t^{-1/2}\mathrm{Tr}\,(\partial e^{-t\partial^2})\;dt \;=\; \mathrm{Tr}\,(\partial|\partial|^{-s-1})|^{\mathrm{mer}}_{s=0}.$$

(Here, the notation $\mathrm{Tr}\,(Q(s))|^{\mathrm{mer}}_{s=0}$ for a family of operators $Q(s)$ depending holomorphically on s and of trace-class for $\mathrm{Re}(s) \gg 0$, means the constant term around $s = 0$ (the 'finite part') in the Laurent expansion of the meromorphic extension $\mathrm{Tr}\,(Q(s))|^{\mathrm{mer}}$ of the trace of $Q(s)$ from $\mathrm{Re}(s) \gg 0$ to all of $\mathbb{C}$, assuming this is defined.)

Equivalently,

$$\eta(\partial) = \mathrm{Tr}\,((\Pi^\partial_> - \Pi^\partial_<)|\partial|^{-s})|^{\mathrm{mer}}_{s=0} \tag{6.1.5}$$

is the zeta function quasi-trace of the involution $\Pi^\partial_> - \Pi^\partial_<$ defined by the order zero ψdo projections $\Pi^\partial_> = \frac{1}{2}(I + \partial|\partial|^{-1})$ and $\Pi^\partial_< = \frac{1}{2}(I - \partial|\partial|^{-1}) = (\Pi^\partial_>)^\perp$ onto the positive and negative spectral subspaces of ∂.

Consider ψdo projections P, P' with $P - \Pi_>$ a and $P' - \Pi_>$ smoothing operators. Since $P - P'$ is smoothing the relative variant of (6.1.5) exists without regularization

$$\eta(P, P') = \mathrm{Tr}\,\left((P - P^{\perp}) - (P' - (P')^{\perp})\right).$$

One then has $\eta(\Pi_>^{\partial}, \Pi_>^{\partial'}) = \eta(\partial) - \eta(\partial')$ for $\partial - \partial'$ a finite-rank ψdo, and the relative index formula

$$\frac{\eta(P, P')}{2} = \mathrm{ind}\,(\partial_P) - \mathrm{ind}\,(\partial_{P'}),$$

which is the pointwise content of (4.3.6). This is the form degree zero in the boundary Chern character form $\eta(\mathsf{P}', \mathsf{P})$ whose component in $\Omega^{2k}(Y)$ is up to a constant the vertical trace

$$\eta(\mathsf{P}, \mathsf{P}')_{[2k]} = \mathrm{Tr}\,\left((\nabla^{\mathcal{W}})^{2k} - (\nabla^{\mathcal{W}'})^{2k}\right).$$

In particular, $R^{\mathcal{K},\mathcal{W}} = \eta(P(\mathsf{D}), \mathsf{P})_{[2]}$.

6.2 The zeta function connection on $\mathrm{Det}\,\mathsf{D}_{\mathsf{P}}$

The ζ-connection on $\mathrm{Det}\,\mathsf{D}_{\mathsf{P}}$ is defined locally on U_σ by

$$\nabla^{\zeta,\mathsf{P}_\sigma} \det \mathsf{D}_{\mathsf{P}_\sigma} = \omega^{\zeta,\mathsf{P}_\sigma} \det \mathsf{D}_{\mathsf{P}_\sigma} \tag{6.2.1}$$

with

$$\omega^{\zeta,\mathsf{P}_\sigma} \;=\; -\,\mathrm{Tr}\,(\Delta_{P(\mathsf{D})}^{-s}\mathsf{D}_{\mathsf{P}_\sigma}\nabla^{\mathsf{P}}\mathsf{D}_{\mathsf{P}_\sigma}^{-1})\Big|_{s=0}^{\mathrm{mer}} \;\in\; \Omega^1(U_\sigma), \tag{6.2.2}$$

where $\mathrm{Tr}\,:\Gamma(Y,\,\Psi_{\flat}^{-\infty}(E)) \;=\; \Psi_{\mathrm{vert},\flat}^{-\infty}(M,E) \to C^\infty(Y)$ is the vertical trace (integral over the fibres, see Appendix).

The definition of $\omega^{\zeta,\mathsf{P}_\sigma}$ has particular features which make it work (and be the essentially canonical choice). These are as follows.

The operator $\mathsf{D}_{\mathsf{P}_\sigma}$ on the right-side of (6.2.2) means that

$$(\nabla_\xi^{\mathsf{P}}\mathsf{D}_{\mathsf{P}_\sigma}^{-1})s \in \mathrm{dom}(\mathsf{D}_{\mathsf{P}_\sigma}), \qquad s \in \Gamma(M, E^-). \tag{6.2.3}$$

Ensuring that (6.2.3) holds is the job of the connection ∇^{P}, which is constructed in §6.2.1 (this issue is not present in the case of boundaryless manifolds). That is,

$$\omega_\sigma^{\zeta,\mathsf{P}_\sigma} \;=\; -\;\mathrm{Tr}\,(\Delta_{P(\mathsf{D})}^{-s}\mathsf{D}\nabla^{\mathsf{P}}\mathsf{D}_{\mathsf{P}_\sigma}^{-1})\Big|_{s=0}^{\mathrm{mer}} \;\in \Omega^1(U_\sigma)$$

while the additional subscript in (6.2.2) indicates (6.2.3). ∇^{P} has also to be such that the local 1-forms (6.2.2) patch together to define a global ζ-connection on $\mathrm{Det}\,\mathsf{D}_{\mathsf{P}}$.

The regularized trace of $\mathsf{D}_{\mathsf{P}_\sigma}\nabla^{\mathsf{P}}\mathsf{D}_{\mathsf{P}_\sigma}^{-1}$ in (6.2.2) is defined for any vertical APS ψdo projection P using the complex power $\Delta_{P(\mathsf{D})}^{-s}$ of the Calderón Laplacian $\Delta_{P(\mathsf{D})} = \mathsf{D}\mathsf{D}^*$ (cf.(5.0.8)).

This differs from the case of boundaryless manifolds which, recall, works as follows. Suppose $\mathcal{D}$ is a smooth family of Dirac-type operators associated to a fibration $\pi : N \to Y$ of compact boundaryless manifolds. Then the determinant line bundle $\mathrm{Det}\,\mathcal{D}$ may be constructed with respect to local charts $U_{\mathsf{s}} = \{y \in Y \mid \mathcal{D} + \mathsf{s} \text{ invertible}\}$ with $\mathsf{s} \in \Psi^{-\infty}_{\mathrm{vert}}(N, E^+, E^-)$ a vertical smoothing operator. Over U_{s} one has the trivialization $y \mapsto \det(\mathcal{D}_y + \mathsf{s}_y) \in \mathrm{Det}\,(\mathcal{D}_y + \mathsf{s}_y)$ and the ζ-connection 1-form is $-\,\mathrm{Tr}\,(\Delta_{\mathsf{s}}^{-s}(\mathcal{D}+\mathsf{s})\nabla(\mathcal{D}+\mathsf{s})^{-1})|^{\mathrm{mer}}_{s=0}$, where Δ_{s} is the Laplacian of $\mathcal{D} + \mathsf{s}$. What makes the patching work in this case is

$$\mathrm{Tr}\,(\,(\Delta_{\mathsf{s}}^{-s} - \Delta_{\mathsf{s}'}^{-s})(\mathsf{D}+\mathsf{s})\nabla(\mathcal{D}+\mathsf{s})^{-1})\big|^{\mathrm{mer}}_{s=0} \;=\; 0 \quad \text{for} \;\; \mathsf{s}, \mathsf{s}' \in \Psi^{-\infty}_{\mathrm{vert}}(N, E^+, E^-). \tag{6.2.4}$$

This is easily seen from the precise formulae of [15]. This might suggest that the local ζ-connection form on $\mathrm{Det}\,\mathsf{D}_{\mathsf{P}}$ be defined as $\mathrm{Tr}\,\left(\Delta_{\mathsf{P}_\sigma}^{-s}\mathsf{D}\nabla^{\mathsf{P}}\mathsf{D}_{\mathsf{P}_\sigma}^{-1}\right)\big|^{\mathrm{mer}}_{s=0}$. But these forms do not patch together, because the analogue of the left-hand side of (6.2.4) does not vanish. This one knows from the pole structure of the meromorphic continuation of the trace to all of $\mathbb{C}$, from [10, **?**, **?**, 9, **?**] the constant term in the Laurent expansion at zero depends on $\mathsf{P}_\sigma, \mathsf{P}_{\sigma'}$.

In contrast, the connection forms $\omega^{\zeta,\mathsf{P}_\sigma}$ do patch together (Theorem 6.2).

This carries a certain naturality; the family of vertical APS boundary problems $\mathsf{D}_{P(\mathsf{D})}$ is distinguished by the fact that it is invertible (at all points $y \in Y$), and thus so is $\Delta_{P(\mathsf{D})}$, providing a global regularizing operator not available in the case of general family $\mathcal{D}$ over boundaryless manifolds. In general, changing the regularizing family $Q(s)$ of elliptic ψdos used to define the connection form $\mathrm{Tr}\,(Q(s)(\mathcal{D}+\mathsf{s})\nabla(\mathcal{D}+\mathsf{s})^{-1})|^{\mathrm{mer}}_{s=0}$ results in additional residue trace terms.

6.2.1 A connection on $\mathrm{Hom}(\mathcal{H}(E^-), \mathcal{H}_{\mathsf{P}}(E^+))$

To define a connection on $\mathrm{Det}\,\mathsf{D}_{\mathsf{P}}$ requires a connection on the bundle $\mathrm{Hom}(\,\mathcal{H}(E^-), \mathcal{H}_{\mathsf{P}}(E^+))$ whose sections are the subspace of vertical ψdbos with range in $\mathrm{Ker}\,(\mathsf{P}\circ\gamma) = \mathrm{dom}(\mathsf{D}_{\mathsf{P}})$

$$\Gamma\left(M, \mathrm{Hom}(\,\mathcal{H}(E^-), \mathcal{H}_{\mathsf{P}}(E^+))\right) \;:=\; \{\mathsf{A} \in \Psi_{\mathrm{vert},\beta}(M, E^-, E^+) \mid \mathsf{P}\gamma\mathsf{A}s = 0,\; s \in \Gamma(M, E^-)\}.$$

All that this requires is a natural connection ∇^{P} on $\mathcal{H}_{\mathsf{P}}(E^+)$, meaning a connection $\widetilde{\nabla}^{\mathsf{P}}$ on $\Gamma(Y, \mathcal{H}(E^+)) = \Gamma(M, E^+)$ which preserves $\mathrm{Ker}\,(\mathsf{P}\circ\gamma) = \mathrm{dom}(\mathsf{D}_{\mathsf{P}})$. That is, such that

$$\mathsf{P}\gamma\nabla^{\mathsf{P}}_{\xi}s := \mathsf{P}\gamma\widetilde{\nabla}^{\mathsf{P}}_{\xi_H}s = 0 \quad \text{for } s \in \Gamma(M, E^+) \;\; \text{with} \;\; \mathsf{P}\gamma s = 0. \tag{6.2.5}$$

For, then, there is the induced connection (also denoted ∇^{P}) on $\mathrm{Hom}(\,\mathcal{H}(E^-), \mathcal{H}_{\mathsf{P}}(E^+))$

$$\nabla^{\mathsf{P}}_{\xi} : \Gamma(M, \mathrm{Hom}(\,\mathcal{H}(E^-), \mathcal{H}_{\mathsf{P}}(E^+))) \to \Gamma(M, \mathrm{Hom}(\,\mathcal{H}(E^-), \mathcal{H}_{\mathsf{P}}(E^+))),$$

$$(\nabla^{\mathsf{P}}_{\xi}\mathsf{A})s := \nabla^{\mathsf{P}}_{\xi}(\mathsf{A}s) - \mathsf{A}(\nabla^{M}_{\xi}s) := \widetilde{\nabla}^{\mathsf{P}}_{\xi_H}(\mathsf{A}s) - \mathsf{A}(\widetilde{\nabla}_{\xi_H}s) \tag{6.2.6}$$

where $\widetilde{\nabla}$ is the connection (2.2.4) and $\xi \in \Gamma(Y, TY)$. We then evidently have

$$\mathsf{P}\gamma(\nabla^{\mathsf{P}}_{\xi}\mathsf{A})s = 0 \quad \text{for } s \in \Gamma(M, E^-).$$

This is how $\nabla^{\mathsf{P}}\mathsf{D}^{-1}_{\mathsf{P}_\sigma}$ in (6.2.2) is defined, and why $\mathsf{D}\nabla^{\mathsf{P}}\mathsf{D}^{-1}_{\mathsf{P}_\sigma}{=}\mathsf{D}_{\mathsf{P}}\nabla^{\mathsf{P}}\mathsf{D}^{-1}_{\mathsf{P}_\sigma}$.

The task, then, is to define the connection ∇^{P} in (6.2.5). The connection (2.2.3) ∇^M on $\mathcal{H}(E^+)$ does not restrict to a connection on $\mathcal{H}_{\mathsf{P}}(E^+)$ (except when P is constant in $y \in Y$ as in the example of §1.0.8), i.e. (6.2.5) does not hold for ∇. We define ∇^{P} by adding a correction term to ∇^M in an essentially canonical way, as follows.

First, for an APS-type vertical boundary ψdo $\mathsf{P} \in \Gamma(Y, \Psi^0_{\mathrm{vert},\flat}(E_{\partial M})) := \Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M})$ we have its covariant derivative

$$\nabla^{\partial M}_{\xi}\mathsf{P} \;\in\; \Gamma(Y, \Psi^0_{\mathrm{vert},\flat}(E_{\partial M})),$$

where $\nabla^{\partial M}$ is the connection (6.1.1). Let $\phi : [0, \infty) \to \mathbb{R}$ be a smooth function with $\phi(u) = 1$ for $0 \le u < 1/4$ and $\phi(u) = 0$ for $u > 3/4$. Define

$$m_\phi : M \to \mathbb{R}$$

with support in the collar neighbourhood $\mathcal{U}$ of ∂M by

$$m_\phi(x) \;=\; \begin{cases} 0, & x \in M \backslash \mathcal{U}, \\ \phi(u), & x = (u, z) \in \mathcal{U} = [0, 1) \times \partial M. \end{cases}$$

Then we define

$$\nabla^{\mathsf{P}} \;:=\; \nabla^M \;+\; m_\phi\, \mathsf{P}(\nabla^{\partial M}\mathsf{P})\gamma.$$

Thus for $\xi \in \Gamma(Y, TY)$ and $s \in \Gamma(M, E^+)$

$$\nabla^{\mathsf{P}}_{\xi} s \;:=\; \widetilde{\nabla}^{\mathsf{P}}_{\xi_H} s \;:=\; \widetilde{\nabla}_{\xi_H} s \;+\; m_\phi\, \mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H} P)\gamma s \tag{6.2.7}$$

and the second (endomorphism) term acts by

$$(m_\phi\, \mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H} P)\gamma s)(x) \;=\; \begin{cases} 0, & x \in M \backslash \mathcal{U}, \\ \phi(u)\, \mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H} P)(s(0, z)) & x = (u, z) \in \mathcal{U}. \end{cases} \tag{6.2.8}$$

Because of the restriction map γ the Leibnitz property does not hold for ∇^{P} on $\Gamma(M, E^+)$ as a $C^\infty(M)$ module. It does hold, however, for $\Gamma(M, E^+)$ as a $C^\infty(Y)$ module (2.1.2), which is exactly what we need; that is, for the $C^\infty(Y)$ multiplication (2.1.2)

$$\nabla^{\mathsf{P}} f \cdot s \;=\; df \cdot s + f \cdot \nabla^{\mathsf{P}} s \quad \text{for } f \in C^\infty(Y), \;\; s \in \Gamma(Y, \mathcal{H}_{\mathsf{P}}(E^+)). \tag{6.2.9}$$

Proposition 6.1 *∇^P defines a connection on $\mathcal{H}_P(E^+)$. That is,* (6.2.5) *holds so that*

$$\nabla^P_{\xi} : \Gamma(Y, \mathcal{H}_P(E^+)) \to \Gamma(Y, \mathcal{H}_P(E^+)) \tag{6.2.10}$$

and satisfies the Leibnitz property (6.2.9).

Proof The Leibnitz property of the first term (2.2.3) of ∇^{P} is standard:

$$\nabla^M_\xi(f\cdot s) = \widetilde{\nabla}_{\xi_H}\left((f\circ\pi)s\right) = \xi_H\,(f\circ\pi)\,.\,s + (f\circ\pi)\widetilde{\nabla}_{\xi_H}s = df(\xi)\,\cdot\,s + f\cdot\nabla^M_\xi s$$

using the Leibnitz property of $\widetilde{\nabla}$ for the second equality and the chain rule for the third. Thus (6.2.9) is equivalent to the linearity for $f\in C^\infty(Y)$ and $s\in\Gamma(M,E^+)$

$$m_\phi\,\mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H}P)\gamma(f\cdot s) = f\cdot m_\phi\,\mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H}P)\gamma s, \tag{6.2.11}$$

and this holds because f acts as a constant on each fibre X_y of M, by definition (2.1.2). Precisely, we may assume $x=(u,z)\in\mathcal{U}$, the expressions being zero otherwise, and then from (6.2.8)

$$\begin{aligned}
m_\phi\,\mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H}P)\gamma(f\cdot s)(u,z) &= \phi(u)\,\mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H}P)\,(f(\pi(0,z))s(0,z))\\
&= f(\pi(0,z))\phi(u)\,\mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H}P)(s(0,z))\\
&= f(\pi(u,z))\phi(u)\,\mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H}P)(s(0,z))\\
&= (f\cdot m_\phi\,\mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H}P)\gamma s)(u,z)
\end{aligned}$$

which is (6.2.11). To see (6.2.5), we have, applying $\mathsf{P}\circ\gamma$ to (6.2.7),

$$\begin{aligned}
\mathsf{P}\gamma\nabla^{\mathsf{P}}_\xi s &= \mathsf{P}\gamma\widetilde{\nabla}_{\xi_H}s + \mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H}P)(\gamma s)\\
&= \mathsf{P}\,\widetilde{\nabla}^{\partial M}_{\xi_H}(\gamma s) + \mathsf{P}(\widetilde{\nabla}^{\partial M}_{\xi_H}P)(\gamma s),
\end{aligned} \tag{6.2.12}$$

using (2.2.6) for the second equality. From (6.1.1)

$$\mathsf{P}(\nabla^{\partial M}_\xi\mathsf{P})(h) = \mathsf{P}\,\widetilde{\nabla}^{\partial M}_{\xi_H}(\mathsf{P}h) - \mathsf{P}\widetilde{\nabla}^{\partial M}_{\xi_H}h, \qquad h\in\Gamma(\partial M, E_{\partial M}).$$

So with $h=\gamma s$ and the assumption of (6.2.5)

$$\mathsf{P}(\nabla^{\partial M}_\xi\mathsf{P})(\gamma s) = -\,\mathsf{P}\,\widetilde{\nabla}^{\partial M}_{\xi_H}(\gamma s), \qquad h\in\Gamma(\partial M, E_{\partial M}),$$

and hence (6.2.12) vanishes.

□

6.2.2 Curvature of $\nabla^{\zeta,\mathsf{P}}$

The curvature of the connection $\nabla^{\zeta,\mathsf{P}}$ on the complex line bundle $\mathrm{Det}\,\mathsf{D}_{\mathsf{P}}\to Y$ is the globally defined two form

$$F_\zeta(\mathsf{D}_{\mathsf{P}}) \;=\; (\nabla^{\zeta,\mathsf{P}})^2 \;\in\; \Omega^2(Y)$$

determined locally by

$$F_\zeta(\mathsf{D}_{\mathsf{P}})_{|U_\sigma} \;=\; d\,\omega^{\zeta,\,\mathsf{P}_\sigma} \;\in\; \Omega^2(U_\sigma). \tag{6.2.13}$$

Theorem 6.2 *The locally defined ζ 1-forms* (6.2.2) *define a connection on the determinant line bundle* $\mathrm{Det}\, D_{\mathsf{P}}$ *with curvature*

$$F_\zeta(D_{\mathsf{P}}) \;=\; F_\zeta(D_{P(D)}) \;+\; R^{\mathcal{K},\mathcal{W}}. \tag{6.2.14}$$

$F_\zeta(D_{P(D)})$ *is canonically exact; precisely*

$$\beta_\zeta \;:=\; \mathrm{Tr}\,(\Delta^z_{P(D)} D\,\nabla^{P(D)} D^{-1}_{P(D)})|^{\mathrm{mer}}_{z=0} \quad \in \Omega^1(Y)$$

is a globally defined 1-form and

$$F_\zeta(D_{P(D)}) \;=\; d\,\beta_\zeta.$$

6.3 Proof of Theorem 6.2

For the patching of the connection forms, the issue is that there are two candidates for the local connection over $U_\sigma \cap U_{\sigma'}$ defined by (6.2.1). Let l be a smooth section of $\mathrm{Det}\,\mathsf{D}_{\mathsf{P}}$ over $U_\sigma \cap U_{\sigma'}$. Then

$$l = f_\sigma \cdot \det \mathsf{D}_{\mathsf{P}_\sigma} = f_{\sigma'} \cdot \det \mathsf{D}_{\mathsf{P}_{\sigma'}}$$

for smooth functions $f_\sigma, f_{\sigma'} : U_\sigma \cap U_{\sigma'} \to \mathbb{C}$. The covariant derivative of l is therefore

$$\nabla^{\zeta,\mathsf{P}_\sigma}(f_\sigma \cdot \det \mathsf{D}_{\mathsf{P}_\sigma}) = df_\sigma \cdot \det \mathsf{D}_{\mathsf{P}_\sigma} + f_\sigma \cdot \omega^{\zeta,\mathsf{P}_\sigma}\ \det \mathsf{D}_{\mathsf{P}_\sigma}$$

and also

$$\nabla^{\zeta,\mathsf{P}_{\sigma'}}(f_{\sigma'} \cdot \det \mathsf{D}_{\mathsf{P}_{\sigma'}}) = df_{\sigma'} \cdot \det \mathsf{D}_{\mathsf{P}_{\sigma'}} + f_{\sigma'} \cdot \omega^{\zeta,\mathsf{P}_{\sigma'}}\ \det \mathsf{D}_{\mathsf{P}_{\sigma'}}$$

and these must coincide. From (4.3.5)

$$f_{\sigma'} = \det{}_F\left(\mathsf{S}(\mathsf{P}_\sigma) \circ \mathsf{S}(\mathsf{P}_{\sigma'})^{-1}\right) f_\sigma \qquad \text{on} \quad U_\sigma \cap U_{\sigma'}.$$

Hence, using §6.1, the patching condition for the locally defined connection forms is

$$\omega^{\zeta,\mathsf{P}_\sigma} \;-\; \omega^{\zeta,\mathsf{P}_{\sigma'}} \;=\; \omega^{\mathsf{S}(\mathsf{P}_\sigma)} \;-\; \omega^{\mathsf{S}(\mathsf{P}_{\sigma'})} \qquad \text{on} \quad U_\sigma \cap U_{\sigma'}.$$

We will prove a slightly more general statement, which also captures (6.2.14). Let $\mathsf{P}, \mathsf{P}' \in \Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M})$ be any two vertical ψdo APS projections and let U be the open subset of Y where both $(\mathsf{D}_{\mathsf{P}})_y$ and $(\mathsf{D}_{\mathsf{P}'})_y$ are invertible. Then

$$\omega^{\zeta,\mathsf{P}} \;-\; \omega^{\zeta,\mathsf{P}'} \;=\; \omega^{\mathsf{S}(\mathsf{P})} \;-\; \omega^{\mathsf{S}(\mathsf{P}')} \qquad \text{on} \quad U. \tag{6.3.1}$$

Here,

$$\omega^{\zeta,\mathsf{P}} \;=\; -\,\mathrm{Tr}\,(\Delta^{-s}_{P(\mathsf{D})} \mathsf{D}_{\mathsf{P}} \nabla^{\mathsf{P}} \mathsf{D}_{\mathsf{P}}^{-1})\Big|^{\mathrm{mer}}_{s=0} \qquad \text{and} \qquad \omega^{\mathsf{S}(\mathsf{P})} = \mathrm{Tr}\,(\mathsf{S}(\mathsf{P})^{-1} \nabla^{\mathcal{K},\mathcal{W}}_\xi\, \mathsf{S}(\mathsf{P})).$$

From (6.1.2), $\omega^{\mathsf{S}(P(\mathsf{D}))} = 0$ and hence, using (6.1.4) and (6.2.13), (6.3.1) also proves (6.2.14) globally in $\Omega^2(Y)$; note that the right-side of (6.1.4) and (6.2.13) are independent of the choice of σ, i.e. $d\omega^{\zeta,\mathsf{P}_\sigma} = d\omega^{\zeta,\mathsf{P}_{\sigma'}} = F_\zeta(\mathsf{D}_\mathsf{P})_{|U_\sigma\cap U_{\sigma'}}$, and likewise for $\omega^{\mathsf{S}(\mathsf{P}_\sigma)}$. Clearly, establishing (6.3.1) de facto proves the identity for the perturbations of P and P' on each chart U_σ, and hence shows (6.2.14) globally.

To see (6.3.1), since the vertical trace defining the zeta form is taken on $\Gamma(M, E^-)$ (or, rather, $L^2(M, E^-)$) we have

$$-\left(\omega^{\zeta,\mathsf{P}} - \omega^{\zeta,\mathsf{P}'}\right) = \operatorname{Tr}\left(\Delta_{P(\mathsf{D})}^{-s}\left(\mathsf{D}_\mathsf{P}\nabla^\mathsf{P}\mathsf{D}_\mathsf{P}^{-1} - \mathsf{D}_{\mathsf{P}'}\nabla^{\mathsf{P}'}\mathsf{D}_{\mathsf{P}'}^{-1}\right)\right)\Big|_{s=0}^{\mathrm{mer}}. \tag{6.3.2}$$

(Note that

$$\operatorname{Tr}(\Delta_{P(\mathsf{D})}^{-s}\mathsf{D}_\mathsf{P}\nabla^\mathsf{P}\mathsf{D}_\mathsf{P}^{-1}) - \operatorname{Tr}(\Delta_{P(\mathsf{D})}^{-s}\mathsf{D}_{\mathsf{P}'}\nabla^{\mathsf{P}'}\mathsf{D}_{\mathsf{P}'}^{-1}) = \operatorname{Tr}\left(\Delta_{P(\mathsf{D})}^{-s}\left(\mathsf{D}_\mathsf{P}\nabla^\mathsf{P}\mathsf{D}_\mathsf{P}^{-1} - \mathsf{D}_{\mathsf{P}'}\nabla^{\mathsf{P}'}\mathsf{D}_{\mathsf{P}'}^{-1}\right)\right)$$

for large $\mathrm{Re}(s)$, and by the uniqueness of continuation this extends to all of $\mathbb{C}$.) From (3.2.12) and (6.1.1)

$$\nabla^\mathsf{P} - \nabla^{\mathsf{P}'} = m_\phi\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P}) - \mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\right)\gamma \quad \in\ \Gamma(Y, \Psi_{\mathrm{vert},\flat}^{-\infty}(E)), \tag{6.3.3}$$

and hence using Proposition 3.1

$$\nabla^\mathsf{P}\mathsf{D}_\mathsf{P}^{-1} - \nabla^{\mathsf{P}'}\mathsf{D}_{\mathsf{P}'}^{-1} = \left(\nabla^\mathsf{P} - \nabla^{\mathsf{P}'}\right)\mathsf{D}_{\mathsf{P}'}^{-1} + \nabla^\mathsf{P}\left(\mathsf{D}_\mathsf{P}^{-1} - \mathsf{D}_{\mathsf{P}'}^{-1}\right) \quad \in\ \Gamma(Y, \Psi_{\mathrm{vert},\flat}^{-\infty}(E)). \tag{6.3.4}$$

Hence

$$\mathsf{D}_\mathsf{P}\nabla^\mathsf{P}\mathsf{D}_\mathsf{P}^{-1} - \mathsf{D}_{\mathsf{P}'}\nabla^{\mathsf{P}'}\mathsf{D}_{\mathsf{P}'}^{-1} = \mathsf{D}\left(\nabla^\mathsf{P}\mathsf{D}_\mathsf{P}^{-1} - \nabla^{\mathsf{P}'}\mathsf{D}_{\mathsf{P}'}^{-1}\right) \ \in\ \Gamma(Y, \Psi_{\mathrm{vert},\flat}^{-\infty}(E))$$

is also a smooth family of smoothing operators (with C^∞ kernel). It follows that we may swap the order of the operators inside the trace on the right-side of (6.3.2) to obtain

$$\begin{aligned}
-(\omega^{\zeta,\mathsf{P}} - \omega^{\zeta,\mathsf{P}'}) &= \operatorname{Tr}\left(\left(\mathsf{D}_\mathsf{P}\nabla^\mathsf{P}\mathsf{D}_\mathsf{P}^{-1} - \mathsf{D}_{\mathsf{P}'}\nabla^{\mathsf{P}'}\mathsf{D}_{\mathsf{P}'}^{-1}\right)\Delta_{P(\mathsf{D})}^{-s}\right)\Big|_{s=0}^{\mathrm{mer}} \\
&= \operatorname{Tr}\left(\mathsf{D}\left(\nabla^\mathsf{P}\mathsf{D}_\mathsf{P}^{-1} - \nabla^{\mathsf{P}'}\mathsf{D}_{\mathsf{P}'}^{-1}\right)\Delta\Delta_{P(\mathsf{D})}^{-s-1}\right)\Big|_{s=0}^{\mathrm{mer}} \\
&= \operatorname{Tr}\left(\mathsf{D}\left(\nabla^\mathsf{P}\mathsf{D}_\mathsf{P}^{-1} - \nabla^{\mathsf{P}'}\mathsf{D}_{\mathsf{P}'}^{-1}\right)\Delta\Delta_{P(\mathsf{D})}^{-1}\right) \\
&= \operatorname{Tr}\left(\mathsf{D}\left(\nabla^\mathsf{P}\mathsf{D}_\mathsf{P}^{-1} - \nabla^{\mathsf{P}'}\mathsf{D}_{\mathsf{P}'}^{-1}\right)\right)
\end{aligned}$$

using that $\Delta_{P(\mathsf{D})}^{-s-1}$ is vertically norm continuous for $\mathrm{Re}(s) > -1$ and, in particular, at $s = 0$, and hence that we may take s down to zero without continuation of the vertical trace.

Using (6.3.3), (6.3.4) and (3.2.10) we therefore have

$$\begin{aligned}\omega^{\zeta,\mathsf{P}} - \omega^{\zeta,\mathsf{P}'} = \quad & \mathrm{Tr}\left(\mathsf{D}\nabla^M\left(\mathsf{K}(\mathsf{P})\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\right)\right) && \text{(I)}\\ & - \mathrm{Tr}\left(\mathsf{D}\, m_\phi\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P}) - \mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\right)\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\right) && \text{(II)}\\ & + \mathrm{Tr}\left(\mathsf{D}\, m_\phi\,\mathsf{P}(\nabla^{\partial M}\mathsf{P})P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\right) && \text{(III)}\end{aligned}$$

using the fact that each term is a vertical smoothing operator, as in the proof of Proposition 3.1 for terms (I) and (III). We will deal with these terms in reverse order.

Term (III):

Again, in view of (6.3.3) we may permute the order of operators in the trace to obtain

$$\begin{aligned}\text{Term (III)} \quad & = \quad \mathrm{Tr}\left(\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\mathsf{D}\, m_\phi\,\mathsf{P}(\nabla^{\partial M}\mathsf{P})P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\right)\\ & \overset{(3.2.9)}{=} \quad \mathrm{Tr}\left(\mathsf{P}\gamma(\mathsf{I} - \mathsf{K}(\mathsf{P}')\gamma)\, m_\phi\,\mathsf{P}(\nabla^{\partial M}\mathsf{P})P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\right)\\ & \overset{(3.1.4),\ (3.2.8)}{=} \quad \mathrm{Tr}\left(\mathsf{P}(\mathsf{I} - P(\mathsf{D})\mathsf{S}(\mathsf{P}')^{-1}\mathsf{P}')\,\mathsf{P}(\nabla^{\partial M}\mathsf{P})P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\right)\\ & = \quad \mathrm{Tr}\left(P(\mathsf{D})\left(\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\ -\ \mathsf{S}(\mathsf{P}')^{-1}\mathsf{P}'\right)\,\mathsf{P}(\nabla^{\partial M}\mathsf{P})P(\mathsf{D})\right),\end{aligned}$$

cycling the operator $P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P} = P(\mathsf{D}) \circ P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}$ around for the final equality.

Term (II):

Since $\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1} = \mathsf{P} \circ \mathsf{P}\gamma(\mathsf{I} - \mathsf{P}')\mathsf{D}_{\mathsf{P}'}^{-1}$ is a composition of vertically smoothing and L^2-bounded operators we may cycle the operators in the trace to obtain

$$\begin{aligned}\text{Term (II)} \quad & = \quad -\,\mathrm{Tr}\left(\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\mathsf{D}\, m_\phi\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P}) - \mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\right)\right)\\ & \overset{(3.2.9)}{=} \quad -\,\mathrm{Tr}\left(\gamma(\mathsf{I} - \mathsf{K}(\mathsf{P}')\mathsf{P}')\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P}) - \mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\right)\right)\\ & = \quad -\,\mathrm{Tr}\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P}) - \mathsf{P}'(\nabla^{\partial M}\mathsf{P}') - P(\mathsf{D})\mathsf{S}(\mathsf{P}')^{-1}\mathsf{P}'\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P}) - \mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\right)\right).\end{aligned}$$

Term (I):

From the functoriality of connections on the hom-bundles (6.1.1), (6.2.6),

$$\begin{aligned}\nabla^M\left(\mathsf{K}(\mathsf{P})\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\right) \quad & := \quad \nabla^M\left(\mathsf{K} \circ P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\right)\\ & := \quad \nabla^M(\mathsf{K}) \circ P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\ +\ \mathsf{K}\,\nabla^{\partial M}\left(P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\right).\end{aligned}$$

Hence from (3.1.5)

$$\mathsf{D}\nabla^M\left(\mathsf{K}(\mathsf{P})\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\right) = \mathsf{D}\nabla^M(\mathsf{K}) \circ P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}$$

and therefore

$$\begin{aligned}
\text{Term (I)} &= \operatorname{Tr}\left(\mathsf{D}\nabla^M(\mathsf{K})\circ P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\right)\\
&= \operatorname{Tr}\left(P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\gamma\mathsf{D}_{\mathsf{P}'}^{-1}\mathsf{D}\nabla^M(\mathsf{K})\,P(\mathsf{D})\right)\\
&= \operatorname{Tr}\left(P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}\gamma(\mathsf{I}-\mathsf{K}(\mathsf{P}')\mathsf{P}'\gamma)\nabla^M(\mathsf{K})\,P(\mathsf{D})\right)\\
&\overset{(2.2.6),\ (3.1.4)}{=} \operatorname{Tr}\left(P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}(\mathsf{I}-\mathsf{K}(\mathsf{P}')\mathsf{P}')\nabla^{\partial M}(P(\mathsf{D}))\,P(\mathsf{D})\right)\\
&= \operatorname{Tr}\left(P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}(\mathsf{I}-\mathsf{K}(\mathsf{P}')\mathsf{P}')\nabla^{\partial M}(P(\mathsf{D}))\,P(\mathsf{D})\right)\\
&= \operatorname{Tr}\left(\left(P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}-P(\mathsf{D})\mathsf{S}(\mathsf{P}')^{-1}\mathsf{P}'\right)\nabla^{\partial M}(P(\mathsf{D}))\,P(\mathsf{D})\right).
\end{aligned}$$

Summing the expression for terms (I), (II) and (III),

$$\begin{aligned}
\omega^{\zeta,\mathsf{P}}-\omega^{\zeta,\mathsf{P}'} = \operatorname{Tr}\Big(&P(\mathsf{D})\,\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}(\nabla^{\partial M}\mathsf{P})\,P(\mathsf{D}) - P(\mathsf{D})\mathsf{S}(\mathsf{P}')^{-1}\mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\,P(\mathsf{D})\\
&+ P(\mathsf{D})\,\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}(\nabla^{\partial M}P(\mathsf{D}))\,P(\mathsf{D}) - P(\mathsf{D})\mathsf{S}(\mathsf{P}')^{-1}\mathsf{P}'(\nabla^{\partial M}P(\mathsf{D}))\,P(\mathsf{D})\Big)\\
&+ \operatorname{Tr}\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P})-\mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\right).
\end{aligned}$$

From

$$\mathsf{S}(\mathsf{P})^{-1}\nabla^{\mathcal{K},\mathcal{W}_\sigma}\,\mathsf{S}(\mathsf{P}) = P(\mathsf{D})\,\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}(\nabla^{\partial M}\mathsf{P})\,P(\mathsf{D}) + P(\mathsf{D})\mathsf{S}(\mathsf{P})^{-1}\mathsf{P}(\nabla^{\partial M}P(\mathsf{D}))\,P(\mathsf{D})$$

we are therefore left with

$$\omega^{\zeta,\mathsf{P}} - \omega^{\zeta,\mathsf{P}'} = \omega^{\mathsf{S}(\mathsf{P})} - \omega^{\mathsf{S}(\mathsf{P}')} + \operatorname{Tr}\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P})-\mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\right).$$

Since $\mathsf{P}\in\Psi^0_{\mathrm{vert}}(\partial M, E_{\partial M})$ is an indempotent we have

$$\nabla^{\partial M}\mathsf{P} = \nabla^{\partial M}(\mathsf{P}^2) = \mathsf{P}\nabla^{\partial M}(\mathsf{P}) + (\nabla^{\partial M}\mathsf{P})\circ\mathsf{P}$$

and hence composing with P on the left that

$$\mathsf{P}(\nabla^{\partial M}\mathsf{P})\circ\mathsf{P} = 0.$$

Hence

$$\operatorname{Tr}\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P})-\mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\right) = \operatorname{Tr}\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P})\circ\mathsf{P}^\perp - \mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\circ(\mathsf{P}')^\perp\right)$$

with $\mathsf{P}^\perp = \mathsf{I}-\mathsf{P}$. Writing the operator inside the trace as

$$\mathsf{P}\nabla^{\partial M}\mathsf{P}\circ(\mathsf{P}^\perp-(\mathsf{P}')^\perp) + (\mathsf{P}\nabla^{\partial M}\mathsf{P}-\mathsf{P}'\nabla^{\partial M}\mathsf{P}')\circ(\mathsf{P}')^\perp,$$

the bracketed vertical ψdos are smoothing and we may cycle the operators through the trace leaving

$$\operatorname{Tr}\left(\mathsf{P}(\nabla^{\partial M}\mathsf{P})-\mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\right) = \operatorname{Tr}\left(\mathsf{P}^\perp\mathsf{P}\nabla^{\partial M}\mathsf{P}\circ\mathsf{P}^\perp - (\mathsf{P}')^\perp\mathsf{P}'(\nabla^{\partial M}\mathsf{P}')\circ(\mathsf{P}')^\perp\right) = 0.$$

□

Appendix

A Vertical Pseudodifferential Operators

A.1 Families of ψdos on a closed manifold

A smooth family of ψdos of constant order μ associated to a fibration $\pi : N \xrightarrow{X} Y$ of compact boundaryless manifolds, with $\dim(X) = n$, with vector bundle $E \to N$ means a classical ψdo

$$\mathsf{A} : \Gamma(N, E^+) \to \Gamma(N, E^-)$$

with Schwartz kernel $k_{\mathsf{A}} \in \mathcal{D}'(N \times_\pi N, E \boxtimes E)$ a vertical distribution, where the fibre product $N \times_\pi N$ consists of pairs $(x, x') \in N \times N$ which lie in the same fibre, i.e. $\pi(x) = \pi(x')$, such that in any local trivialization k_{A} is an oscillatory integral with *vertical symbol* $\mathsf{a} \in S^\nu_{\mathrm{vert}}(N/Y)$ of order ν. Here, ξ is *restricted* to the vertical momentum space, along the fibre. We refer to A as a vertical ψdo associated to the fibration and denote this subalgebra of ψdos on N by

$$\Gamma(Y, \Psi^\nu(E^+, E^-)) = \Psi^\nu_{\mathrm{vert}}(N, E^+, E^-).$$

Thus in any local trivialization $N_{|U_Y} \cong U_Y \times X_y$ over an open subset $U_Y \subset Y$ with $y \in U_Y$, and a trivialization $E \cong U_Y \times V_y \times \mathbb{R}^N$ of E with V_y an open subset of X_y, a vertical amplitude of constant (in y) order ν is an element

$$\mathsf{a} = \mathsf{a}(y, x, x', \xi) \in \Gamma\left(U_Y \times (V_y \times V_y) \times \mathbb{R}^n \backslash \{0\},\ \mathrm{End}\,(\mathbb{R}^N)\right)$$

satisfying the estimate on compact subsets $K \subset N$

$$|\partial^\alpha_x \partial^\gamma_{x'} \partial^\delta_y \partial^\beta_\xi \mathsf{a}| < C_{\alpha,\gamma,\delta,\beta,K}(1 + |\xi|)^{\nu - |\beta|}.$$

We denote this as $\mathsf{a} \in S^\nu_{\mathrm{vert}}(N/Y)$. Here, ξ may be identified with an element of the vertical (or fibre) cotangent space $T^*_x(N/Y)$. The kernel of A is then *locally* written on $U_Y \times V_y$ as the distribution with singular support along the diagonal

$$k_{\mathsf{A}}(y, x, x') = \int_{\mathbb{R}^n} e^{i(x - x').\,\xi}\ \mathsf{a}(y, x, x', \xi)\ đ\xi.$$

If A has order $\nu < -n$ this integral is convergent and with respect to a vertical volume form $d_{N/Y}x$ the trace $\mathrm{Tr}\,\mathsf{A}$ is the smooth function on Y

$$(\mathrm{Tr}\,\mathsf{A})(y) = \int_{N/Y} \mathrm{tr}\,(k_{\mathsf{A}}(y, x, x))\ d_{N/Y}x \quad \in C^\infty(Y), \qquad \nu < -n.$$

If $\mathsf{w} \in S^{-\infty}_{\mathrm{vert}}(N/Y) = \bigcap_\nu S^\nu_{\mathrm{vert}}(N/Y)$ then the kernel is an element

$$k_{\mathsf{W}} \in \Gamma(N \times_\pi N, E^* \boxtimes E)$$

and defines a vertical smoothing operator (smooth family of smoothing operators)

$$\mathsf{W} \in \Gamma(Y, \Psi^{-\infty}(E)).$$

Any vertical ψdo of order ν may be written in the form $\mathsf{A} = \mathrm{OP}(\mathsf{a}) + \mathsf{W}$ with $\mathsf{a} = \mathsf{a}(y, x, \xi)$ a vertical symbol and W a vertical smoothing operator. Assuming this representation, we will consider here only classical vertical ψdos, meaning that the symbol has an asymptotic expansion $\mathsf{a} \sim \sum_{j\geq 0} \mathsf{a}_j$ with a_j positively homogeneous in ξ of degree $\nu - j$. The leading symbol a_0 has an invariant realization as a smooth section

$$\mathsf{a}_0 \in \Gamma(T^*(N/Y), \varphi^*(\mathrm{End}\,(E))) \ ,$$

where $\varphi : T^*(N/Y) \to N$. If a_0 is an invertible bundle map then $\mathsf{A} \in \Gamma(Y, \Psi(E))$ is said to be an *elliptic family*. If there exists θ such that $\mathsf{a}_0 - \lambda\mathcal{I}$ is invertible for each $\lambda \in R_\theta = \{re^{i\theta} \mid r > 0\}$, where $\mathcal{I}$ is the identity bundle operator, then A is elliptic with principal angle θ. In the latter case one has the resolvent family

$$(\mathsf{A} - \lambda\mathsf{I})^{-1} \ \in \Gamma(Y, \Psi^{-\nu}(E))$$

and the complex powers

$$\mathsf{A}_\theta^z := \frac{i}{2\pi} \int_{\mathcal{C}} \lambda_\theta^z \, (\mathsf{A} - \lambda\mathsf{I})^{-1} \, d\lambda \ \in \Gamma(Y, \Psi^{z\nu}(E)),$$

where $\mathcal{C}$ is a contour running in along R_θ from infinity to a small circle around the origin, clockwise around the circle, then back out to infinity along R_θ, as accounted for in detail in [21]. A principal angle, and hence the complex powers, can only exist if the pointwise index is zero.

For example, if $\mathsf{p} \in S^m_{\mathrm{vert}}(N/Y)$ is a polynomial of order $m \in \mathbb{N}$ in ξ and elliptic, then the corresponding vertical ψdo $\mathsf{D} \in \Gamma(Y, \Psi^m(E^+, E^-))$ is a smooth family of elliptic differential operators of order m. Specifically, this is the case for a geometric fibration of Riemannian spin manifolds §2.2.1 with associated smooth family of twisted Dirac operators [2, 1]. (The space $\Gamma(Y, \Psi(E^+, E^-))$ of vertical ψdos between different bundles $E^\pm$ is defined by a trivial elaboration of the above.)

A.2 Families of pseudodifferential boundary operators

Let $\pi : M \to Y$ be a smooth fibration of compact manifolds with boundary with vector bundle $E \to M$ and let

$$\widetilde{\pi} : \widetilde{M} \to Y$$

be a smooth fibration of compact boundaryless manifolds with vector bundle $\widetilde{E} \to M$ such that

$$M \subset \widetilde{M}, \quad \widetilde{E}_{|M} = E, \quad \text{and} \quad \widetilde{\pi}_{|M} = \pi.$$

We consider the following vertical families of pseudodifferential boundary operators (vertical ψdbos) as defined in the single operator case by Grubb [9], elaborating the algebra of Boutet de Monvel. First, one has the truncated, or restricted, ψdos

$$\mathsf{A}_+ : \Gamma(M,E) \to \Gamma(M,E), \qquad \mathsf{A}_+ = r^+ \mathsf{A} \varepsilon^+, \tag{A.2.1}$$

where

$$\mathsf{A} \in \Psi_{\rm vert}(\widetilde{M}, \widetilde{E})$$

is a vertical ψdo associated to the fibration of closed manifolds, and

$$r^+ : \Gamma(\widetilde{M}, \widetilde{E}) \to \Gamma(M,E), \quad e^+ : \Gamma(M \backslash \partial M, E) \to \Gamma(\widetilde{M}, \widetilde{E})$$

are the 'brutal' restriction and extension-by-zero operators. To avoid e^+ or r^+ introducing any new singularities A is assumed to be of integer order and to satisfy the *transmission condition* at ∂M, which in local coordinates $(x_n, w) \in \mathcal{U} = (0,1] \times \partial M$ in the collar neighbourhood of the boundary of M is the requirement

$$\partial_x^\beta \partial_\xi^\alpha \mathsf{a}_j(0, w, -\xi_n, 0) = (-1)^{\nu - j - |\alpha|} \partial_x^\beta \partial_\xi^\alpha \mathsf{a}_j(0, w, \xi_n, 0) \ \text{ for } |\xi_n| \geq 1,$$

where $\xi = (\xi_n, \xi_w)$ relative to (2.2.5).

More generally, one considers

$$\mathsf{A}_+ + \mathsf{G} : \Gamma(M,E) \to \Gamma(M,E) \tag{A.2.2}$$

with G a vertical singular Green's operator (vertical sgo), which we return to in a moment.

A *vertical trace operator* of order $\mu \in \mathbb{R}$ and class $r \in \mathbb{N}$ is an operator from interior to boundary sections of the form

$$\mathsf{T} : \Gamma(M,E) \to \Gamma(\partial M, E_{\partial M}), \qquad \mathsf{T} = \sum_{0 \leq j < r} \mathsf{S}_j \gamma_j + \mathsf{T}',$$

where the $\mathsf{S}_j \in \Psi_{\rm vert}(\partial M, E_{\partial M})$ are vertical ψdos on the boundary fibration of closed manifolds, as in §A, while $\gamma_j s(x_n, w) = \partial_{x_n}^j s(0, w)$ are the restriction maps to the boundary. The additional term is an operator of the form $\mathsf{T}' = \gamma \mathsf{A}_+$ for some restricted ψdo (A.2.1).

A *vertical Poisson operator* of order $\mu \in \mathbb{R}$ here will be an operator from boundary to interior sections of the form

$$\mathsf{K} : \Gamma(\partial M, E_{\partial M}) \to \Gamma(M,E), \qquad \mathsf{K} = r^+ \mathsf{B} \gamma^* \mathcal{C}, \qquad \gamma : \Gamma(\widetilde{M}, \widetilde{E}) \to \Gamma(\partial M, E_{\partial M}), \tag{A.2.3}$$

where $\mathsf{B} \in \Psi_{\rm vert}^{\mu-1}(\widetilde{M}, \widetilde{E})$ is a family of ψdos in the sense of §A, while $\mathcal{C} \in \Psi_{\rm vert}^m(\partial M, E_{\partial M})$ is a vertical differential operator on the boundary fibration of order m; however, $m = 0$ in the following. Note, the restriction map γ is here coming from $\widetilde{M}$, rather than M (same notation).

To make the composition rules work one includes vertical sgo operators in (A.2.2) of order ν and class $r \in \mathbb{N}$, these have the form

$$\mathsf{G} = \sum_{0 \le j < r} \mathsf{K}_j \gamma_j + \mathsf{G}',$$

where K_j is a vertical Poisson operator of order $\nu - j$, and G' is defined in local coordinates near ∂M by an oscillatory integral on a sgo symbol g satisfying standard estimates in ξ [9].

As with closed manifolds, if $\mathsf{A} + \mathsf{G}$ in (A.2.2) has order $\nu < -n$ and we assume the order of $\mathcal{C}$ in (A.2.3) is $m = 0$ then the 'distribution kernel' is continuous and the trace $\mathrm{Tr}\,\mathsf{A}$ is the smooth function, or differential form for de Rham valued symbols, on Y

$$\mathrm{Tr}\,(\mathsf{A} + \mathsf{G})(y) = \int_{M/Y} k_{\mathsf{A}+\mathsf{G}}(y, x, x)\, d_{\mathrm{M/Y}} x \quad \in \mathcal{A}(Y), \qquad (\mathrm{order}(\mathsf{A} + \mathsf{G}) << 0).$$

For each of the above classes of ψdbos one considers the subclass of operators defined by polyhomogeneous symbols, appropriately formulated [9]. We denote the resulting algebra by

$$\Gamma(Y,\, \Psi_\flat(E)) \;=\; \Psi_{\mathrm{vert},\,\flat}(M, E).$$

When the kernel is an element

$$k_{\mathsf{A}+\mathsf{G}} \in \Gamma(M \times_\pi M, E^* \boxtimes E)$$

then the operator defines a vertical smoothing operator (smooth family of smoothing operators)

$$\mathsf{A} + \mathsf{G} \in \Gamma(Y,\, \Psi_\flat^{-\infty}(E)) \;=\; \Psi^{-\infty}_{\mathrm{vert},\,\flat}(M, E).$$

We refer to [9] and references therein for a precise account of the pseudodifferential boundary operator calculus, which extends to the case of vertical operators in a way similar to the case for compact boundaryless manifolds [21].

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