CP^n INSTANTONS AND THE FAMILY'S INDEX THEOREM

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ABSTRACT. Using the Atiyah-Singer index theorem for families we present a calculation of the number of CP^n instantons of charge k = np + n on a Riemann surface of genus g = np + p.

I. Introduction

Let Σ be a compact Riemann surface of genus g and let CP^n denote complex projective n-space. By a CP^n-*instanton* on Σ we mean a holomorphic map $f : \Sigma \to CP^n$. The *charge* k of the instanton is the degree of the map f. Let $\mathcal{A}_{k,n,g}$ be the space of all holomorphic maps $\Sigma \to CP^n$ of degree k, and let $\mathcal{G}_n$ be the 'gauge' group of holomorphic automorphisms of CP^n, that is, $\mathcal{G}_n = PSL(n+1, C)$. The moduli space of CP^n-instantons over Σ of charge k is the quotient space $\mathcal{M}_{k,n,g} = \mathcal{A}_{k,n,g}/\mathcal{G}_n$. The space $\mathcal{M}_{k,n,g}$ is analogous to the moduli space of 4D Yang-Mills instantons, the dimension of which was calculated in ref.1 by an application of the Atiyah-Singer index theorem for a certain twisted Dirac operator. For each $p = 0, 1, \ldots$ we find that

$$\text{dim}\mathcal{M}_{k,n,g} = 0 \quad \text{when} \quad k = np + n \quad \text{and} \quad g = np + p, \tag{1}$$

and so if $\mathcal{M}_{k,n,g}$ is compact it then consists of only a finite number of points. We denote that number by $m(\Sigma, n, p)$. The purpose of this paper is to present a calculation of $m(\Sigma, n, p)$ using an application of the Atiyah-Singer index theorem for families. We do not address here the deeper question of for which choice of complex structure on CP^n the moduli space is compact, but refer rather to the recent work of McDuff and Salamon[2] for a discussion.

The interest in CP^n-instantons is precisely that they provide a 2D analogue of 4D instantons, in which the splitting of a complex function $\Sigma \to CP^n$ into its holomorphic and anti-holomorphic components corresponds to the splitting into the self-dual and anti-self-dual parts of the curvature 2-form of a connection on a $SU(2)$ bundle over a 4-manifold. A 4D instanton gives a classical energy solution of the Yang-Mills functional and similarly a CP^n instanton of charge k is a classical energy solution of the non-linear sigma model defined by the energy functional

$$\mathcal{E} : C_k^\infty(\Sigma; CP^n) \to R, \qquad \mathcal{E}(f) = \int_\Sigma \|df\|^2 \ ,$$

where $C_k^\infty(\Sigma; CP^n)$ is the space of maps of degree k and $\|.\|$ is the Hilbert-Schmidt norm. (Recall that the Hilbert-Schmidt norm for a linear map $T : V^0 \to V^1$ this norm is defined by $\|T\|^2 = \mathrm{Tr}(T^*T)$.) We note also that CP^n instantons, like Yang-Mills instantons, are conformally invariant. The relation between 2D and 4D instantons was made precise by Atiyah[3] who showed that 4D G-instantons correspond to holomorphic maps $CP^1 \to \Omega G$. (The based loop space ΩG of a compact Lie group G has most of the essential properties of CP^n.) For a 4D axially symmetric $SU(2)$-instanton on R^4 this reduces[3] to a CP^1-instanton on $S^2 \cong CP^1$, and taking $p = 0$ in our result we compute that $m(\Sigma, n, 0) = 1$, and hence that the moduli space of axially symmetric $SU(2)$-instantons on R^4 of charge 1 is a point. At the other extreme there is the result of Segal[4] which asserts that as the degree k tends to ∞ the space of based rational maps $S^2 \to CP^n$ has the same homotopy type as ΩCP^n. The case of higher genus CP^n-instantons $g > 0$ is currently is of particular interest in the context of the related theory of quantum cohomology[2,5].

We begin by explaining the assertion (1). To do that we identify the CP^n-instantons we are considering with a finite set of points in the Jacobian of holomorphic line bundles over Σ of Chern class k. We then calculate the number $m(\Sigma, n, p)$ in terms of the Chern character of the index bundle of a family of Dirac operators parametrized by the Jacobian.

II. The Moduli Space

The charge of a Yang-Mills instanton defined on an $SU(2)$ bundle $P \to X$ is the integer topological invariant $< [c_2(P)], [X] >$ pairing the 2nd Chern class of P with the fundamental homology class $[X]$, which determines P up to isomorphism. Similarly, in 2D the charge k of a CP^n instanton is an integer topological invariant of the map f, or, equivalently, of a holomorphic line bundle over Σ defined by f. More precisely, there is a one-to-one correspondence between CP^n-instantons $f : \Sigma \to CP^n$ of charge k and holomorphic line bundles $L \to \Sigma$ of degree $\deg(L) =< [c_1(L)], [\Sigma] >$ equal to k with $n+1$ holomorphic sections which do not all vanish at one point. For to f one assigns $(L = f^*(\mathcal{L}), f^*s_0, \dots, f^*s_n)$, where $\mathcal{L}$ is the hyperplane bundle over CP^n and $s_i \in \Gamma_{hol}(CP^n; L)$ is the ith 'homogeneous coordinate', and to $(L, \theta_0, \dots, \theta_n)$ with $\theta_i \in \Gamma_{hol}(\Sigma; L_f)$ one assigns the holomorphic map $f(x) = \pi(\psi(\theta_0(x)), \dots, \psi(\theta_n(x)))$, where ψ is any trivialization of L over $x \in \Sigma$ and $\pi : C^{n+1} - \{0\} \to CP^n$ is the quotient map. Let ω denote the standard Kahler form on CP^n so that $\omega_0 = \omega/2\pi$ is the generator of

$H^2(CP^n : Z) = Z$. By definition one has

$$\deg(f) =< f^*[\omega], [\Sigma] >,$$

where $f^* : H^2(CP^n; Z) \to H^2(\Sigma; Z)$ is the induced map on cohomology. Since $i\omega$ is the curvature form of the canonical holomorphic connection on $\mathcal{L}$ compatible with the Fubini-Study metric $\|.\|$, one has $< f^*[c_1(\mathcal{L})], [\Sigma] >=< [c_1(L)], [\Sigma] >= \deg(L)$. That is, the degree k of f as a map coincides with the degree of L as a holomorphic line bundle.

This correspondence establishes a canonical identification of the moduli space $\mathcal{M}_{k,n,g}$ with the space of isomorphism classes of holomorphic line bundles over Σ of degree k having an $(n+1)$-dimensional space of holomorphic sections[6]. In algebro-geometric terms $\mathcal{M}_{k,n,g}$ is the variety parametrizing complete linear series on Σ of degree $np+n$ and dimension at least $n+1$ [7]. Thus to calculate $m(\Sigma, n, p)$ we must calculate how many holomorphic line bundles (up to isomorphism) there are on Σ of degree $np+n$ with at least $n+1$ linearly independent sections.

To do that we begin by recalling some facts. The space of isomorphism classes of all holomorphic line bundles on Σ form an algebraic group, the Picard group denoted $\text{Pic}(\Sigma)$, and there is a natural identification $\text{Pic}(\Sigma) = H^1(\Sigma; \vartheta^*)$, where ϑ^* is the sheaf of non-vanishing local holomorphic functions on Σ. In a similar way, isomorphism classes of holomorphic line bundles over CP^n are parameterized by the cohomology group $H^1(CP^n; \vartheta^*)$, where ϑ^* now denotes the corresponding sheaf on CP^n. Since $H^1(CP^n; \vartheta^*)$ is generated by $[\mathcal{L}]$, the above remarks mean that we have the following commutative diagram of maps.

$$\begin{array}{ccc} H^1(CP^n; \vartheta^*) & \xrightarrow{c_1} & H^2(CP^n; Z) \\ \downarrow f^* & & \downarrow f^* \\ \text{Pic}(\Sigma) & \xrightarrow{c_1} & H^2(\Sigma; Z) \end{array}$$

The horizontal maps are given by the first Chern class of the bundle, and the top map is an isomorphism. The bottom map fits into the exact sequence

$$0 \longrightarrow H^1(\Sigma; \vartheta)/H^1(\Sigma; Z) \longrightarrow \text{Pic}(\Sigma) \longrightarrow H^2(\Sigma; Z) \longrightarrow 0,$$

and since $H^2(\Sigma; Z) = Z$ then $\text{Pic}(\Sigma)$ is disconnected with one component $J_k(\Sigma)$ for each $k \in Z$. The identity component $J_0(\Sigma)$ is identified with $H^1(\Sigma; \vartheta)/H^1(\Sigma; Z)$ and hence is canonically a torus Lie group of dimension $g = np + p$. Thus $\text{Pic}(\Sigma)$ is a central extension of Z by $J_0(\Sigma)$ and each component $J_k(\Sigma)$ is a compact torus of dimension $np + p$ (though not natuarally a Lie group for $k \neq 0$), called the Jacobian of degree k. As a set $J_k(\Sigma)$ parameterizes isomorphism classes of holomorphic line bundles over Σ of degree k, and the group structure on $\text{Pic}(\Sigma)$ is given by tensor product.

Let L be a holomorphic Hermitian line bundle over Σ. We assume from here on in that $\deg(L) = np + n$. The holomorphic structure on L is equivalent to giving a linear $\overline{\partial}$-operator

$$\overline{\partial}_L : \Omega^0(\Sigma; L) \longrightarrow \Omega^{0,1}(\Sigma; L) \tag{2}$$

which satisfies the integrability conditions $\overline{\partial}_L^2 = 0$ and $\overline{\partial}_L f.s = df.s + f.\overline{\partial}_L s$ for $f \in C^\infty(\Sigma), s \in \Omega^0(\Sigma; L)$. The operator $\overline{\partial}_L$ is Fredholm and hence has a well-defined index $\mathrm{ind}\overline{\partial}_L = \mathrm{dimKer}\overline{\partial}_L - \mathrm{dimKer}\overline{\partial}_L^*$. Because $\overline{\partial}_L$ is elliptic this depends only on the isomorphism class of L in $\mathrm{Pic}(\Sigma)$ and hence on $\deg(L)$. In fact, considering (2) as the Dolbeault complex for L we obtain the exact sequence

$$0 \longrightarrow H^0(\Sigma; L) \longrightarrow \Omega^0(\Sigma; L) \xrightarrow{\overline{\partial}_L} \Omega^{0,1}(\Sigma; L) \longrightarrow H^1(\Sigma; L) \longrightarrow 0,$$

and hence that

$$\mathrm{ind}\overline{\partial}_L = \mathrm{dim}H^0(\Sigma; L) - \mathrm{dim}H^1(\Sigma; L) = \frac{-1}{4\pi i}\int_\Sigma (R^\Sigma + 2R^L), \tag{3}$$

where the second equality is the Riemann-Roch-Hirzebruch theorem. R^L is the curvature form of the unique connection on L compatible with the Hermitian structure and the operator $\overline{\partial}_L$, and one has

$$-2\pi i \deg(L) = \int_\Sigma R^L. \tag{4}$$

R^Σ is the curvature of the canonical line bundle $T^{1,0}\Sigma$ and (taking L in (4) to be the trivial bundle and since $2g = \mathrm{dim}H^1(\Sigma)$),

$$1 - g = \frac{-1}{4\pi i}\int_\Sigma R^{(\Sigma)}. \tag{5}$$

Hence we obtain

$$\mathrm{ind}\overline{\partial}_L = 1 - g + \deg(L) = n - p + 1.$$

We can now identify $\mathcal{M}_{np+p,n,np+n}$ as follows. Because $\overline{\partial}_L$ is Fredholm it is an element of the space $\mathcal{F}$ of all Fredholm operators on a Sobolev space completion of $\Omega^0(\Sigma; L)$. In the metric topology defined by the operator norm $\mathcal{F}$ has one component $\mathcal{F}_r$ for each integer $r \in Z$, such that $T \in \mathcal{F}$ if and only if $\mathrm{ind}T = r$. Within $\mathcal{F}_r$, the space of operators with $\mathrm{dimKer}T \geq i$, and hence with $\mathrm{dimKer}T^* \geq j = r - i$, form an analytic subvariety $\mathcal{F}_{i,j}$ of codimension ij. Because of the homotopy equivalence $\mathcal{F} \simeq Z \times BU$, where $BU = \lim_{n\to\infty} BU(n)$ is the classifying space of the unitary group $U(n)$, the cohomology ring of $\mathcal{F}_r$ is identified as a polynomial ring

$$H^*(\mathcal{F}_r) = Z[c_1, c_2, \ldots],$$

where c_i is the ith universal Chern class. The cohomology class $[\mathcal{F}_{i,j}] \in H^{ij}(\mathcal{F})$ representing $\mathcal{F}_{i,j}$ is known[8] to be the determinant

$$\begin{vmatrix} c_i & c_{i+1} & \dots & c_{i+j-1} \\ c_{i-1} & c_i & & \\ \vdots & & \ddots & \vdots \\ c_{i-j+1} & & & c_i \end{vmatrix} = \pm \begin{vmatrix} h_j & h_{j+1} & \dots & h_{j+i-1} \\ h_{j-1} & h_j & & \\ \vdots & & \ddots & \vdots \\ h_{j-i+1} & & & h_j \end{vmatrix} \tag{6}$$

where

$$(1 + c_1 + c_2 + \dots)^{-1} = 1 + h_1 + h_2 + \dots . \tag{7}$$

($c_r = h_r = 0$ for $r < 0$.) By regarding the Jacobian $J = J_{np+p}$ as the parameter space of a family of elliptic operators we now get a map

$$J \xrightarrow{\phi} \mathcal{F}_{n-p+1}, \quad L \longmapsto \overline{\partial}_L.$$

$\mathcal{M}_{np+p,n,np+n}$ is thus identified with the subset of $L \in J$ with $\dim\Gamma_{hol}(\Sigma; L) \geq n+1$, but $\Gamma_{hol}(\Sigma; L) = \mathrm{Ker}\overline{\partial}_L$ and so there is a correspondence

$$\mathcal{M}_{np+p,n,np+n} \cong J \cap \phi^{-1}(\mathcal{F}_{n+1,p}), \tag{8}$$

which, provided that ϕ is transverse so that there are no degenerate points, is bijective. But the parameter space J is a torus of dimension $np + p$, while those L with $\dim\mathrm{Ker}\overline{\partial}_L \geq n + 1$ have codimension $(n + 1)p = np + p$, and hence

$$\dim\mathcal{M}_{np+p,n,np+n} = 0.$$

Thus (8) is (trivially) a diffeomorphism and so when $\mathcal{M}_{np+p,n,np+n}$ is compact it therefore consists of only a finite number of points. The questions of transversality and compactness are closely related and are discussed, for example, in ref.2.

III. Calculation of $m(\Sigma, n, p)$

With the assumption of compactness the number $m(\Sigma, n, p)$ is the number of points in the moduli space $\mathcal{M}_{np+p,n,np+n}$ and from (8) that is the same as

$$m(\Sigma, n, p) = < \phi^*[\mathcal{F}_{n+1,p}], [J] >, \tag{9}$$

where $[\mathcal{F}_{n+1,p}] \in H^{np+p}(\mathcal{F})$ and $<, >: H^{np+p}(J) \otimes H_{np+p}(J) \to R$. That is, $\phi^*[\mathcal{F}_{n+1,p}]$ intersects J in exactly $m(\Sigma, n, p)$ points.

To calculate (9) means identifying the cohomology class $\phi^*[\mathcal{F}_{n+1,p}]$, and from (6) that means identifying the classes $\phi^* c_i \in H^{2i}(J)$. We do that in the following way. By a theorem of Atiyah[9] there is a canonical isomorphism

$$\mathrm{Ind} : [J, \mathcal{F}] \longrightarrow K(J),$$

where the domain is the space of homotopy classes of maps from J into $\mathcal{F}$ and $K(J)$ is the K-theory of the compact manifold J. The map Ind assigns to a given class the index bundle of the corresponding family of Fredholm operators. Thus the map $\phi : J \to \mathcal{F}_{n+p-1}$ is the same thing as the index bundle of the family $\overline{\partial}_{\mathbf{L}} = \{\overline{\partial}_L : L \in J\}$ represented by the fibration

$$\pi : \Sigma \times J \longrightarrow J.$$

Although the dimensions of the vector spaces $\mathrm{Ker}\overline{\partial}_L$ vary with L, the virtual bundle with fibre $\mathrm{Ker}\overline{\partial}_L - \mathrm{Ker}\overline{\partial}_L^*$ makes sense as an element of $K(J)$, and this is the index bundle $\mathrm{Ind}\overline{\partial}_{\mathbf{L}} = \phi^*\mathrm{Ind}\mathcal{F}_{n+p-1}$ of the family. Here $\mathrm{Ind}\mathcal{F}_{n+p-1}$ is the index bundle over $\mathcal{F}_{n+p-1}$ and

$$\mathrm{ch}(\mathrm{Ind}\overline{\partial}_{\mathbf{L}}) = \phi^*((n+p-1) + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \ldots),$$

where $\mathrm{ch} : K(J) \to H^{even}(J;Q)$ is the Chern character map. Writing $\mathrm{ch}(\mathrm{Ind}\overline{\partial}_{\mathbf{L}}) = \sum_i \mathrm{ch}_i(\mathrm{Ind}\overline{\partial}_{\mathbf{L}})$, where $\mathrm{ch}_i(\mathrm{Ind}\overline{\partial}_{\mathbf{L}}) \in H^{2i}(J;Q)$ we have

$$\phi^* c_1 = \mathrm{ch}_1(\mathrm{Ind}\overline{\partial}_{\mathbf{L}}). \tag{10}$$

We can calculate the right-hand side using the families index theorem. More precisely, the line bundles L parameterized by J fit together to define a holomorphic line bundle $\mathbf{L} \to \Sigma \times J$ trivial in the fibre directions (the Poincaré bundle), and the theorem is that

$$\mathrm{ch}(\mathrm{Ind}\overline{\partial}_{\mathbf{L}}) = \int_\pi \mathrm{ch}(\mathbf{L})\mathrm{Todd}(\Sigma),$$

where $\int_\pi : H^{r+2}(\Sigma \times J) \to H^r(J)$ is the cohomology map induced by integration over the fibre. Here $\mathrm{ch}(\mathbf{L}) = e^{c_1(\mathbf{L})} \in H^{even}(\Sigma \times J;Q)$ is the Chern character of $\mathbf{L}$, and $\mathrm{Todd}(\Sigma) = [\det(R^\Sigma/(e^{R^\Sigma} - 1))]$. From (5) we have

$$\mathrm{Todd}(\Sigma) = [1 - R/2] = 1 + (1-g)\sigma,$$

where $1 \in H^0(\Sigma;Z) = Z$, and σ is the generator of $H^2(\Sigma;Z)$.

Since J is an $np+p$-dimensional torus then $H^*(J;Z) = \wedge H^1(J;Z)$ is an exterior algebra on $2np+2p$ generators $\alpha_1^*, \ldots, \alpha_{2np+2p}^* \in H^1(J;Z)$. Let $\alpha_1, \ldots, \alpha_{2np+2p} \in H^1(\Sigma;Z) \cong H^1(J;Z)^*$ be the dual basis. We choose the generators to form a symplectic basis, so that

$$\alpha_i.\alpha_{j+np+p} = \delta_{ij}\sigma, \quad \alpha_i.\alpha_j = \alpha_{i+np+p}.\alpha_{j+np+p} = 0, \quad i,j = 1, \ldots, np+p,$$

where the products are cup products. Then one has

$$c_1(\mathbf{L}) = (np+n)\sigma + \sum_i \alpha_i \otimes \alpha_{i+np+p}^*, \tag{11}$$

which is an element of $H^2(\Sigma;Z)\oplus H^1(\Sigma;Z)\otimes H^1(J;Z)\subset H^2(\Sigma\times J;Z)$. Equation (11) is a special case of a result proved in ref.6 (Chapter 2) for the Jacobian $J_0(\Sigma)$ of bundles of degree zero, and the same proof works for general $k\neq 0$. Thus

$$\mathrm{ch}(\mathbf{L}) = e^{(np+n)\sigma}.e^{\sum_i \alpha_i\otimes\alpha_i^*},$$

and since for dimensional reasons $\sigma^r=0$, $r\geq 2$, and $(\sum_i \alpha_i\otimes\alpha_i^*)^r=0$, $r\geq 3$, one has

$$\begin{aligned}\mathrm{ch}(\mathbf{L})\mathrm{Todd}(\Sigma) &= (1+(np+n)\sigma)(1+(1-np-p)\sigma)\left(1+\sum_i \alpha_i\otimes\alpha_i^* + \frac{1}{2}\left(\sum_i \alpha_i\otimes\alpha_i^*\right)^2\right)\\ &= (1+(n+p-1)\sigma)\left(1+\sum_i \alpha_i\otimes\alpha_i^* + \sigma\otimes\sum_i \alpha_i^*.\alpha_{i+np+p}^*\right).\end{aligned}$$

Integration over the fibre $\int_\pi : H^{r+2}(\Sigma\times J)\to H^r(J)$ just extracts the coefficient of σ, and so from (12)

$$\mathrm{ch}(\mathrm{Ind}\overline{\partial}_{\mathbf{L}}) = (n+p-1)+\sum_i \alpha_i^*.\alpha_{i+np+p}^*.$$

This is an element of $H^*(J)$, the first term $\mathrm{ch}_0(\mathrm{Ind}\overline{\partial}_{\mathbf{L}})=n+p-1$ is the pointwise index, the second term is $\mathrm{ch}_1(\mathrm{Ind}\overline{\partial}_{\mathbf{L}})\in H^2(J;Q)$ and hence from (10) one has

$$\phi^* c_1 = \sum_i \alpha_i^*.\alpha_{i+np+p}^*. \tag{12}$$

Moreover, it tells us that

$$\mathrm{ch}_i(\mathrm{Ind}\overline{\partial}_{\mathbf{L}}) = 0, \quad \text{for } i\geq 2,$$

so $c(\mathrm{Ind}\overline{\partial}_{\mathbf{L}})=e^{\phi^* c_1}$ and therefore, from (7), that $h(\mathrm{Ind}\overline{\partial}_{\mathbf{L}})=e^{-\phi^* c_1}$. Hence

$$h_i(\mathrm{Ind}\overline{\partial}_{\mathbf{L}}) = \frac{(-\phi^* c_1)^i}{i!},$$

and using (6) one calculates for $p\geq n$,

$$\phi^*\mathcal{F}_{n+1,p} = \begin{vmatrix} \frac{1}{p!} & \frac{1}{(p+1)!} & \cdots & \frac{1}{(p+n)!} \\ \frac{1}{(p-1)!} & \frac{1}{p!} & & \vdots \\ \vdots & & \ddots & \frac{1}{(p+1)!} \\ \frac{1}{(p-n)!} & & \frac{1}{(p-1)!} & \frac{1}{p!} \end{vmatrix} (\phi^* c_1)^{np+p}. \tag{13}$$

If $0\leq p<n$, then the diagonals beneath the $p+1$th row are zero. Because

$$(\phi^* c_1)^{np+p} = (np+p)!\alpha_1^*.\alpha_2^*\ldots\alpha_{2np+2p}^*,$$

and $\alpha_1.\alpha_2\ldots\alpha_{2np+2p}$ is, in the chosen basis, the canonical dual of the fundamental homology class $[J_{np+n}]$ we obtain from (9) and (13) our final result.