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ζ-determinant and the Quillen determinant on the Grassmannian of elliptic self-adjoint boundary conditions

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Abstract. We announce a proof of the equality of the ζ-determinant of a Dirac operator over an odd-dimensional manifold with boundary with the Quillen determinant section computed in a canonical trivialization over the Grassmannian of elliptic self-adjoint boundary conditions. © Académie des Sciences/Elsevier, Paris

Le ζ-déterminant et le déterminant de Quillen sur l'espace grassmannien de conditions au bord elliptiques auto-adjointes

Résumé. *Dans cette Note, nous annonçons la preuve de l'égalité du ζ-déterminant d'un opérateur de Dirac sur une variété à bord, compacte, de dimension impaire, avec la section déterminant de Quillen calculée dans une trivialisation canonique sur l'espace du grassmannien aux conditions de bords elliptiques auto-adjointes.* © Académie des Sciences/Elsevier, Paris

Version française abrégée

Soit $\mathcal{D} : \mathrm{C}^{\infty}(M;\mathcal{S}) \to \mathrm{C}^{\infty}(M;\mathcal{S})$ un opérateur de Dirac compatible avec une variété M à bord, compacte, de dimension impaire, et de bord Y. On suppose que les structures sont des structures produit au voisinage de Y. On désigne par $\mathrm{Gr}_{\infty}^{*}(\mathcal{D})$ l'espace grassmannien lisse réduit de toutes les projections P de $\mathrm{L}^2(Y;\mathcal{S}|Y)$, de manière que $P - P(\mathcal{D})$ soit un opérateur de noyau lisse, où $P(\mathcal{D})$ est la projection de Calderón sur l'espace des données de Cauchy définies par $\mathcal{D}$ (*voir* [2]), et de plus $-\Gamma P \Gamma = I - P$. Dans la dernière égalité, Γ désigne la multiplication de Clifford par le vecteur cotangent unité et normal au bord.

Note présentée par Jean-Michel Bismut.

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Soit $\gamma_0 : \mathrm{H}^1(M;\mathcal{S}) \to \mathrm{H}^{1/2}(Y;\mathcal{S}_{|Y})$ l'opérateur de trace défini par la restriction des sections de Sobolev au bord. Pour tout $P \in \mathcal{G}\mathrm{r}^*_\infty(\mathcal{D})$ le problème à bord elliptique :

$$\begin{cases} \mathcal{D}_P = \mathcal{D} : \operatorname{dom}(\mathcal{D}_P) \to \mathrm{L}^2(M;\mathcal{S}), \\ \operatorname{dom}(\mathcal{D}_P) = \{\psi = \mathrm{H}^1(M;\mathcal{S}) : P\gamma_0\psi = 0\}, \end{cases}$$

définit un opérateur de Fredholm auto-adjoint de spectre réel et discret, et tel que $\operatorname{Ker}(\mathcal{D}_P) \subset \mathrm{C}^\infty(M;\mathcal{S})$ (*voir* [2]).

Le ζ-déterminant $\det_\zeta \mathcal{D}_P \in \mathbf{C}$ est bien défini sur $\mathrm{Gr}^*_\infty(\mathcal{D})$ (*voir* [10]). Sans faire de choix, la « fonction » déterminant sur $\mathrm{Gr}^*_\infty(\mathcal{D})$ apparaît comme une section det du fibré en doites déterminant de Quillen associé à la famille $\{\mathcal{D}_P | P \in \mathcal{G}\mathrm{r}^*_\infty(\mathcal{D})\}$. Toutefois, sur $\mathrm{Gr}^*_\infty(\mathcal{D})$ ce fibré a une trivialisation canonique qui permet de définir det comme une fonction complexe $\det_{\mathcal{C}} : \mathcal{G}\mathrm{r}^*_\infty(\mathcal{D}) \to \mathbf{C}$, appelée le « déterminant canonique » (*voir* [7], [8]).

Le résultat principal de cette Note est l'égalité projective de ces deux régularisations du déterminant de $\mathcal{D}_P$;

THÉORÈME 0.1. – *Sur* $\mathcal{G}\mathrm{r}^*_\infty(\mathcal{D})$ *on a*

$$\det_\zeta \mathcal{D}_P = \det_\zeta \mathcal{D}_{P(\mathcal{D})} \cdot \det_{\mathcal{C}} \mathcal{D}_P. \tag{1}$$

Remarque 0.2. – Dans la monographie [2], on a donné un exposé détaillé de la relation entre l'indice de l'opérateur non auto-adjoint $\mathcal{D}_P$, où P désigne un projecteur pseudo-différentiel quelconque de manière que $P - P(\mathcal{D})$ soit un opérateur lissant, et l'indice de l'opérateur correspondant sur le bord $PP(\mathcal{D}) : \operatorname{Ran} P(\mathcal{D}) \to \operatorname{Ran} P$. On a montré que

$$\operatorname{index} \mathcal{D}_P = \operatorname{index} PP(\mathcal{D}) \tag{2}$$

(ceci a été montré pour la première fois dans [1]), c'est-à-dire qur l'indice d'un opérateur agissant sur toute la variété est égal à l'indice d'un opérateur agissant seulement sur le bord. Le théorème 0.1 dit que le ζ-déterminant de l'opérateur $\mathcal{D}_P$ est, abstraction faite d'une constante naturelle multiplicative, égal au déterminant canonique de l'opérateur $\mathcal{D}_P$, qui est vraiment le déterminant de Fredholm de l'opérateur $PP(\mathcal{D})$. Dans ce sens, notre nouveau résultat étend (2) à une correspondance entre déterminants, même si le déterminant est un invariant beaucoup plus subtil.

Pour montrer le théorème 0.1, nous étudions la variation de $\det_\zeta \mathcal{D}_P$ et $\det_{\mathcal{C}} \mathcal{D}_P$ par changement de la condition au bord P. Nous fixons $P_1, P_2 \in \mathcal{G}\mathrm{r}^*_\infty(\mathcal{D})$ et nous considérons des chemins $P_{i,r} = U_r P_i U_r^{-1}$, où $\{U_r\}$ est une famille d'opérateurs elliptiques et unitaires. Nous étudions la différence

$$\log\det_\zeta \mathcal{D}_{P_{2,r}} - \log\det_\zeta \mathcal{D}_{P_{1,r}}.$$

Le résultat essentiel est :

PROPOSITION 0.3

$$\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0} \{\log\det_\zeta \mathcal{D}_{P_{2,r}} - \log\det_\zeta \mathcal{D}_{P_{1,r}}\} = \frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0} \{\log\det_{\mathcal{C}} \mathcal{D}_{P_{2,r}} - \log\det_{\mathcal{C}} \mathcal{D}_{P_{1,r}}\}. \tag{3}$$

En intégrant (3), nous obtenons

$$\frac{\det_\zeta \mathcal{D}_{UP_2U^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{UP_2U^{-1}}}{\det_\zeta \mathcal{D}_{P_2} / \det_{\mathcal{C}} \mathcal{D}_{P_2}} = \frac{\det_\zeta \mathcal{D}_{UP_1U^{-1}} / \det_{\mathcal{C}} \mathcal{D}_{UP_1U^{-1}}}{\det_\zeta \mathcal{D}_{P_1} / \det_{\mathcal{C}} \mathcal{D}_{P_1}} := \alpha(U).$$

Ici $\alpha := \alpha_{P_1,P_2} : \mathcal{U}_\infty \to \mathbf{C}^*$ est un caractère du groupe $\mathcal{U}_\infty$. En substituant $P_1 = P(\mathcal{D})$ et $P_2 = P$, nous avons $\det_\zeta \mathcal{D}_P = \det_\zeta \mathcal{D}_{P(\mathcal{D})} \cdot \det_{\mathcal{C}} \mathcal{D}_P \cdot \alpha(U)$.

Une étude plus approfondie de $\alpha(U)$ montre que $\alpha = 1$.

1. Introduction and statement of results

The ζ-determinant of a self-adjoint elliptic operator has been of central importance in spectral geometry, topology, and quantum field theory (QFT). In this paper we show that the ζ-determinant of an elliptic boundary problem on an odd-dimensional manifold is equal (up to a multiplicative constant) to the Fredholm determinant of a canonically associated operator over the boundary. This opens the door to further investigation of pasting formulae for the ζ-determinant, as suggested by recent studies in QFT.

Let M be a compact odd-dimensional Riemannian manifold with boundary $Y := \partial M$, and let $\mathcal{S}$ denote a vector bundle of Clifford modules over M. Assume that we have a compatible Dirac operator $\mathcal{D} : \mathrm{C}^\infty(M;\mathcal{S}) \to \mathrm{C}^\infty(M;\mathcal{S})$ acting on sections of $\mathcal{S}$. In this paper we consider only the case of product structures in a neighbourhood of the boundary Y. This means that there exists $\mathcal{N} = [0,1] \times Y$, a collar neighbourhood of the boundary $Y = \{0\} \times Y$, such that the Riemannian metric on M and Hermitian metric on $\mathcal{S}$ are pull-backs of the metric structures over the boundary. Over $\mathcal{N}$ the Dirac operator $\mathcal{D}$ then takes the form $\mathcal{D}|_{\mathcal{N}} = \Gamma(\partial_u + B)$, where $B : \mathrm{C}^\infty(Y;\mathcal{S}|Y) \to \mathrm{C}^\infty(Y;\mathcal{S}|Y)$ is the corresponding Dirac operator over Y, and $\Gamma : \mathcal{S}|Y \to \mathcal{S}|Y$ is Clifford multiplication by the normal vector du. To avoid technical trivialities we assume that $\mathrm{Ker}(B) = \{0\}$. Both B and Γ are independent of the normal coordinate u and they satisfy the identities

$$\Gamma^2 = -I, \qquad \Gamma^* = -\Gamma, \qquad \Gamma \cdot B + B \cdot \Gamma = 0.$$

The boundary Clifford bundle splits $\mathcal{S}|Y = \mathcal{S}^+ \oplus \mathcal{S}^-$ into the $\pm i$-eigenspaces of the anti-involution Γ, and $\mathcal{D}|\mathcal{N}$ has the representation

$$\mathcal{D}|\mathcal{N} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot \left(\frac{\partial}{\partial u} + \begin{pmatrix} 0 & B^+ \\ B^+ & 0 \end{pmatrix} \right),$$

where $B^\pm : F^\pm \to F^\mp$ are the chiral Dirac operators over the boundary acting on $F^\pm = \mathrm{C}^\infty(Y;\mathcal{S}^\pm)$.

Let $\Pi_>$ be the spectral projection on $\mathrm{L}^2(Y;\mathcal{S}|Y)$ with range equal to the subspace H^+ spanned by the eigensections of B with positive eigenvalues. We consider elliptic self-adjoint boundary conditions for $\mathcal{D}$ parameterized by the smooth restricted Grassmannian $\mathrm{Gr}^*_\infty(\mathcal{D})$ of all orthogonal projections P on $\mathrm{L}^2(Y;\mathcal{S}|Y)$ such that $P - \Pi_>$ is an operator with a smooth kernel and such that $-\Gamma P \Gamma = I - P$. For each $P \in \mathrm{Gr}^*_\infty(\mathcal{D})$ the associated elliptic boundary value problem

$$\mathcal{D}_P := \mathcal{D} : \mathrm{dom}\,(\mathcal{D}_P) \to \mathrm{L}^2(M;\mathcal{S})$$

with $\mathrm{dom}\,(\mathcal{D}_P) = \{\psi = \mathrm{H}^1(M;\mathcal{S}) : P(\psi|Y) = 0\}$ is a self-adjoint Fredholm operator with $\mathrm{Ker}(\mathcal{D}_P) \subset \mathrm{C}^\infty(M;\mathcal{S})$ and real discrete spectrum (*cf.* [2]).

The space $\mathrm{Gr}^*_\infty(\mathcal{D})$ is characterized by two important properties. First, it contains the Calderón projection $P(\mathcal{D})$ with range equal to $\mathcal{H}(\mathcal{D})$, the space of Cauchy data of the operator $\mathcal{D}$:

$$\mathcal{H}(\mathcal{D}) = \{f \in \mathrm{C}^\infty(Y;\mathcal{S}|Y) : \exists s \in \mathrm{C}^\infty(M;\mathcal{S}),\ \mathcal{D}s = 0,\ s|Y = f\}.$$

Second, $H^+ = \mathrm{range}(\Pi_>)$ is the graph of the elliptic unitary isomorphism $(B^+B^-)^{-1/2}B^+ : F^+ \to F^-$, the "phase" of B^+. Similarly, any other elliptic boundary condition $P \in \mathcal{G}r^*_\infty(\mathcal{D})$ is described precisely by the property that it is the graph of an elliptic unitary isomorphism $T : F^+ \to F^-$ such that $T - (B^+B^-)^{-1/2}B^+$ has a smooth kernel. In particular, there is a specific unitary isomorphism K such that $\mathcal{H}(\mathcal{D}) = \mathrm{graph}\,(K : F^+ \to F^-)$. The graph characterization induces a canonical trivialization σ_{graph} of the determinant line bundle $\mathcal{L}$ over $\mathrm{Gr}^*_\infty(\mathcal{D})$ associated to the family of elliptic boundary value problems $\{\mathcal{D}_P : P \in \mathcal{G}r^*_\infty(\mathcal{D})\}$ (*see* [7], [8]). Computing the ratio of the determinant section $P \to \det \mathcal{D}_P$ of $\mathcal{L}$ (the Quillen determinant) to σ_{graph} provides us with a canonical regularization of the determinant of $\mathcal{D}_P$ as a complex number $\det_{\mathcal{C}} \mathcal{D}_P$ called the canonical determinant. One finds (*see* [7], [8])

$$\det_C \mathcal{D}_P = \det_{F^-}\left[\frac{1}{2}(I + KT^{-1})\right], \tag{1}$$

where range $(P) = \mathrm{graph}\,(T : F^+ \to F^-)$. Here the right-hand side of (1) is the Fredholm determinant taken on F^- and since KT^{-1} has the form I+trace−class it is well defined.

On the other hand, for $P \in \mathcal{G}\mathrm{r}^*_\infty(\mathcal{D})$ the self-adjoint elliptic operator $\mathcal{D}_P$ has a well-defined ζ-function regularized determinant

$$\det_\zeta \mathcal{D}_P := e^{i\frac{\pi}{2}\left(\eta_{\mathcal{D}_P}(0) - \zeta_{\mathcal{D}_P^2}(0)\right)} \cdot e^{i\frac{1}{2}\zeta'_{\mathcal{D}_P^2}(0)}.$$

The existence of the meromorphic extensions of the functions $\zeta_{\mathcal{D}_P^2}(s)$ and $\eta_{\mathcal{D}_P}(s)$ to the whole complex plane, and their nice behaviour in a neighbourhood of $s = 0$ was established in the paper [10]. While the canonical determinant is a robust algebraic regularization which is relatively easy to compute, the ζ-determinant is, in contrast, a highly delicate analytic regularization which is notoriously difficult to compute and there is little a priori reason to suppose that the two regularizations might coincide. The main result if this announcement is the following remarkable fact:

THEOREM 1.1. – *The following equality holds over* $\mathcal{G}\mathrm{r}^*_\infty(\mathcal{D})$

$$\det_\zeta \mathcal{D}_P = \det_\zeta \mathcal{D}_{P(\mathcal{D})} \cdot \det_C \mathcal{D}_P. \tag{2}$$

That is,

$$\det_\zeta \mathcal{D}_P = c \cdot \det_{F^-}\left[\frac{1}{2}(I + KT^{-1})\right], \tag{3}$$

where the constant c is the value of the ζ-determinant evaluated at the Cauchy data space (Calderón) boundary condition.

Equivalently, (2) expresses the equality of the ratios $\det_\zeta \mathcal{D}_P / \det_\zeta \mathcal{D}_{P(\mathcal{D})}$ and $\det_C \mathcal{D}_P / \det_C \mathcal{D}_{P(\mathcal{D})}$.

Remark 1.2. – In the monograph [2] a detailed account was given of the relation between the index of the operator $\mathcal{D}_P$ for any P in the smooth Grassmannian $\mathcal{G}\mathrm{r}_\infty(\mathcal{D})$ of projections differing from $\Pi_>$ by a smoothing operator, and the index of the boundary operator $PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to \mathrm{Ran}\,P$. Outside of the real submanifold $\mathcal{G}\mathrm{r}^*_\infty(\mathcal{D})$ of $\mathcal{G}\mathrm{r}_\infty(\mathcal{D})$ the Fredholm operator $\mathcal{D}_P$ is not self-adjoint and it was shown that $\mathrm{index}\,\mathcal{D}_P = \mathrm{index}\,PP(\mathcal{D})$ (*see* [1] for the original presentation). Hence $\mathrm{index}\,\mathcal{D}_P$ is equal to the index of an operator which lives only on the boundary. Theorem 1.1 extends this somewhat unexpectedly to a correspondence between the determinants.

The proof of Theorem 1.1 is outlined in Section 2. Details will appear in [9]. Note that (3) generalizes well-known formulas for the ζ-determinant in dimension one (*see* in particular [3]).

2. Sketch of the proof

The proof of Theorem 1.1 adapts ideas suggested in [5] by Forman who obtained precise results in one and two dimensions. We show that this method generalizes to compute $\det_\zeta \mathcal{D}_P$ over $\mathcal{G}\mathrm{r}^*_\infty(\mathcal{D})$ in *any* odd dimension.

To establish Theorem 1.1 we study the variation of $\det_\zeta \mathcal{D}_P$ and $\det_C \mathcal{D}_P$ under a change of the boundary condition P. To compute the variation we exploit the transitive action $P \mapsto UPU^{-1}$ on $\mathcal{G}\mathrm{r}^*_\infty(\mathcal{D})$ of the group $\mathcal{U}_\infty$ of unitary isomorphisms of the form:

$$U = \begin{pmatrix} I_{F^+} & 0 \\ 0 & g \end{pmatrix}, \tag{4}$$

written relative to the grading $F^+ \oplus F^-$, where $g - I_{F^-}$ is a smoothing operator, with $I_{F^\pm}$ the identity operator on $F^\pm$. Using a "unitary twist" this action is extended to a unitary action on $C^\infty(M;\mathcal{S})$ which by spectral invariance allows us to compute the variation of $\det_\zeta \mathcal{D}_P$ with respect to P (*see* [4], [8], [10] for details). Fix P_1, $P_2 \in \mathcal{G}r^*_\infty(\mathcal{D})$ and consider the paths $P_{i,r} = U_r P_i U_r^{-1}$ determined by a smooth 1-parameter family $U_r \in \mathcal{U}_\infty$ with $-1 < r < 1$ and $U_0 = I$. We study the difference

$$\log\det_\zeta \mathcal{D}_{P_{2,r}} - \log\det_\zeta \mathcal{D}_{P_{1,r}}.$$

The crucial result is:

PROPOSITION 2.1. – *The following equality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0}\{\log\det_\zeta \mathcal{D}_{P_{2,r}} - \log\det_\zeta \mathcal{D}_{P_{1,r}}\} = \frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0}\{\log\det_C \mathcal{D}_{P_{2,r}} - \log\det_C \mathcal{D}_{P_{1,r}}\}. \tag{5}$$

Outline proof of Proposition 2.1. – We start with the discussion of the left side of equation (5). We need the following result proved in [10].

PROPOSITION 2.2. – (1). *The value of the ζ-function of $\mathcal{D}_P$ at $s = 0$ is constant on $\mathcal{G}r^*_\infty(\mathcal{D})$.*

(2). *The variation of the η-invariant of $\mathcal{D}_P$ is stable in the following sense: for any pair of projections P_1, $P_2 \in \mathcal{G}r^*_\infty(\mathcal{D})$*

$$\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0}(\eta_{\mathcal{D}_{P_{1,r}}}(0)) = \frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0}(\eta_{\mathcal{D}_{P_{2,r}}}(0)),$$

where the families $\{P_{i,r} = U_r P_i U_r^{-1}\}_{r\in(-1,1)}$ are constructed by using a family of unitary operators $\{U_r\}$ of the form (4), *such that g_r is a smooth family of unitary operators on F^- of the form $I+$ smoothing operator with $g_0 = I$.*

It follows from Proposition 2.2 that we have only to study the variation of the modulus $\det_\zeta |\mathcal{D}_{P_{i,r}}| = \mathrm{e}^{-\frac{1}{2}\zeta'_{\mathcal{D}^2_{i,r}}(0)}$ of the determinant $\det_\zeta \mathcal{D}_{P_{i,r}}$. We compute:

$$\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0}\left(\frac{1}{2}\zeta'_{\mathcal{D}^2_{P_{i,r}}}(0)\right) = -\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0}\mathcal{D}_{P_{i,r}}\right)\mathcal{D}^{-1}_{P_i}\mathrm{e}^{-\varepsilon\mathcal{D}^2_{P_i}}. \tag{6}$$

To evaluate the limit in (7) we have to study the inverse of the operator $\mathcal{D}_P$.

The operator $\mathcal{D}_P$ is invertible if and only if the operator $PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to \mathrm{range}(P)$ is invertible, and in this case we have:

PROPOSITION 2.3. – *The operator inverse $\mathcal{D}_P^{-1} : \mathrm{L}^2(M;\mathcal{S}) \to \mathrm{dom}(\mathcal{D}_P)$ of the operator $\mathcal{D}_P$ is given by the formula:*

$$\mathcal{D}_P^{-1} = \mathcal{D}^{-1} - \mathcal{D}^{-1}\gamma_0^*\Gamma(PP(\mathcal{D}))^{-1}P\gamma_0\mathcal{D}^{-1}, \tag{7}$$

where $\mathcal{D}^{-1}$ denotes the operator $\mathrm{r}_M \cdot \widetilde{\mathcal{D}}^{-1} \cdot \mathrm{e}_{\widetilde{M}}$. Here $\widetilde{\mathcal{D}}$ is the (invertible) double operator of $\mathcal{D}$ over the closed double $\widetilde{M}$ of M; $\mathrm{e}_{\widetilde{M}} : \mathrm{H}^s(M;\mathcal{S}) \to \mathrm{H}^s(\widetilde{M};\widetilde{\mathcal{S}})$ and $\mathrm{r}_M : \mathrm{H}^s(\widetilde{M};\widetilde{\mathcal{S}}) \to \mathrm{H}^s(M;\mathcal{S})$ are the corresponding extension and restriction operators; and γ_0^* is the dual to the operator $\gamma_0 : \mathrm{H}^s(M;\mathcal{S}) \to \mathrm{H}^{s-1/2}(Y;\mathcal{S}|Y)$ restincting sections to the boundary (see [2] *for details*).

Note that $(PP(\mathcal{D}))^{-1}$ is the restriction to range (P) of the elliptic pseudodifferential operator $((I-P)(I-P(\mathcal{D})) + PP(\mathcal{D}))^{-1}$. We observe that the difference

$$\mathcal{D}^{-1}_{P_1} - \mathcal{D}^{-1}_{P_2} = \mathcal{D}^{-1}\gamma^*\Gamma(P_1P(\mathcal{D}))^{-1}P_1\gamma_0\mathcal{D}^{-1} - \mathcal{D}^{-1}\gamma^*\Gamma(P_2P(\mathcal{D}))^{-1}P_2\gamma_0\mathcal{D}^{-1},$$

is a smoothing operator, which implies the next Proposition.

PROPOSITION 2.4. – *For any pair of projections* P_1, $P_2 \in \mathcal{G}\mathrm{r}^*_\infty(\mathcal{D})$ *such that* $\mathcal{D}_{P_1}$ *and* $\mathcal{D}_{P_2}$ *are invertible operators and any* 1*-parameter family of unitary operators* U_r *of the form* (4) *the following identity holds:*

$$\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0} \{\log\det{}_\zeta \mathcal{D}_{P_{2,r}} - \log\det{}_\zeta \mathcal{D}_{P_{1,r}}\} = \mathrm{Tr}\,\dot{D}(D^{-1}_{P_2} - D^{-1}_{P_1}), \tag{8}$$

where $\dot{\mathcal{D}}$ *denotes the operator* $\frac{\mathrm{d}}{\mathrm{d}r}(U_r^{-1}\mathcal{D}U_r)|_{r=0}$, *and the families* $P_{i,r}$ *are defined as in Proposition* 2.2.

The determinant lines of the operators $PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to \mathrm{Ran}\,(P)$ and $\mathcal{D}_P$ are canonically isomorphic and under the isomorphism their abstract Quillen determinant elements identified. Hence, $\det_C(PP(\mathcal{D}))$ is also defined by the right-side of (1), which gives us

$$\det{}_C\, \mathcal{D}_{P_{2,r}}(\det{}_C\, \mathcal{D}_{P_{1,r}})^{-1} = \det{}_C(P_{2,r}P(\mathcal{D}))(\det{}_C(P_{1,r}P(\mathcal{D})))^{-1}. \tag{9}$$

Equation (9) and the fact that $P(\mathcal{D}) = \gamma_0\mathcal{D}^{-1}\gamma^*\Gamma$ allow us to study the right-side of (5). Using formula (7) for the operator $\mathcal{D}_P^{-1}$ we compute the logarithm variation of the right-side of (9) as:

PROPOSITION 2.5.

$$\frac{\mathrm{d}}{\mathrm{d}r}\bigg|_{r=0} \log\det{}_C(P_{2,r}\gamma_0\mathcal{D}^{-1}\gamma_0^*\Gamma(P_{1,r}P(\mathcal{D}))^{-1}P_{1,r}) = \mathrm{Tr}\dot{\mathcal{D}}(\mathcal{D}^{-}_{P_{2,r}} - \mathcal{D}^{-}_{P_{1,r}}). \tag{10}$$

This completes the proof of Proposition 2.1.

The subspace of $\mathrm{Gr}^*_\infty(\mathcal{D})$ consisting of projections P such that $PP(\mathcal{D}) : \mathcal{H}(\mathcal{D}) \to \mathrm{Ran}\,P$ is an invertible operator is path connected (*see* [6], Proposition 3.12). This implies that two invertible operators $\mathcal{D}_{P_1}$ and $\mathcal{D}_{P_2}$ can always be connected through a path of invertible operators $\{\mathcal{D}_{P_t}\}$. Therefore we can integrate (5) and obtain:

$$\frac{\det_\zeta \mathcal{D}_{UP_2U^{-1}}/\det_C \mathcal{D}_{UP_2U^{-1}}}{\det_\zeta \mathcal{D}_{P_2}/\det_C \mathcal{D}_{P_2}} = \frac{\det_\zeta \mathcal{D}_{UP_1U^{-1}}/\det_C \mathcal{D}_{UP_1U^{-1}}}{\det_\zeta \mathcal{D}_{P_1}/\det_C \mathcal{D}_{P_1}} := \alpha(U).$$

For U, $V \in \mathcal{U}_\infty$ we have $\alpha(UV) = \alpha(U)\alpha(V)$ and hence $\alpha : \mathcal{U}_\infty \to \mathbf{C}^*$ is a character on the group $\mathcal{U}_\infty$. Substituting $P_1 = P(\mathcal{D})$, $P_2 = P = UP(\mathcal{D})U^{-1}$ we obtain $\det_\zeta \mathcal{D}_P = \det_\zeta \mathcal{D}_{P(\mathcal{D})} \cdot \det_C \mathcal{D}_P \cdot \alpha(U)$. Further inspection of $\alpha(U)$ as the metric in the collar neighbourhood $\mathcal{N}$ is blownup in the normal direction to the boundary shows that $\alpha = 1$. This completes the proof of Theorem 1.1.

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