DETERMINANTS OF DIRAC BOUNDARY VALUE PROBLEMS OVER ODD-DIMENSIONAL MANIFOLDS

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Abstract. We present a canonical construction of the determinant of an elliptic self-adjoint boundary value problem for the Dirac operator D over an odd-dimensional manifold. For 1-dimensional manifolds we prove that this coincides with the ζ-function determinant. This is based on a result that elliptic self-adjoint boundary conditions for D are parameterized by a preferred class of unitary isomorphisms between the spaces of boundary chiral spinor fields. With respect to a decomposition $S^1 = X^0 \cup X^1$ we explain how the determinant of a Dirac-type operator over S^1 is related to the determinants of the corresponding boundary value problems over X^0 and X^1.

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1. Introduction

Let X be a compact odd-dimensional Riemannian spin manifold with boundary Y. We assume there is a collar neighbourhood $U = [0,1] \times Y$ of the boundary in which the Riemmanian metric is a product metric. Fix a choice of spin structure, and let S be the complex spinor bundle over X. The Dirac operator $D : C^\infty(X;S) \to C^\infty(X;S)$ is the first-order elliptic differential operator defined at $x \in X$ by $Ds = \sum_i e_i.\nabla_{e_i} s$, where ∇ is the canonical metric connection on S and $\{e_i\}$ is an orthonormal frame for T_xX. The e_i act on S by Clifford multiplication. The restriction of S to Y may be identified with the spinor bundle over Y with Z_2 grading $S_Y = S^+ \oplus S^-$. That induces a decomposition of the boundary spinor fields $F = F^+ \oplus F^-$ into positive and negative chirality with respect to which the Dirac operator D_Y over the boundary splits into the chiral operator $D_Y^+ : F^+ \to F^-$, whose index is calculated by evaluating the $\hat{A}$-cohomology class over Y, and its formal adjoint D_Y^-. We assume that D_Y is invertible.

By a *boundary value problem* D_W for D, we shall mean D with restricted domain $C^\infty_W(X;S) = \{\psi \in C^\infty(X;S) : P_W b\psi = 0\}$, where $P_W : C^\infty(Y;S) \to C^\infty(Y;S)$ is a pseudo-differential projection operator (of order 0) with range W, and $b : C^\infty(X;S) \to C^\infty(Y;S_Y)$ is the operator restricting sections to the boundary. We shall refer to W as a *boundary condition* for D.

The purpose of this paper is to present a construction of determinants of elliptic self-adjoint boundary value problems for D using the following theorem.

Theorem 1.1. *There is a canonical one-to-one correspondence between elliptic self-adjoint boundary conditions for the Dirac operator over an odd-dimensional spin manifold and L^2-unitary isomorphisms $g : F^+ \to F^-$ between the positive and negative boundary spinor fields which differ from $g_+ = i(D_Y^+ D_Y^-)^{-1/2} D_Y^+ : F^+ \to F^-$ by a smoothing operator.*

In [4] the basic unitary isomorphism g_+ is considered in the context of the index theorem for families for odd-dimensional manifolds with boundary.

A boundary condition W for D is referred to as *elliptic* if it lies in a certain infinite-dimensional Grassmannian Gr associated to the space of boundary spinor fields. Roughly this is the requirement that we only consider those boundary conditions which are commensurable with the Atiyah-Patodi-Singer boundary condition which, for even-dimensional X, was studied in detail in [3] and the index 'defect' identified as essentially the η-invariant of the boundary Dirac operator. To justify the use of the term elliptic we give in Appendix A of the paper a construction of a specific parametrix for D_W, from which it follows that D_W has the principal analytic properties of an elliptic operator over a closed manifold. We refer to [8] for detailed background on elliptic boundary value problems for Dirac operators. The elliptic boundary conditions considered here form a dense subset of those studied in [7],[8],[10].

We refer to the determinant of D_W constructed using Theorem (1.1) as the canonical determinant and we denote it by $\det_{\mathcal{C}} D_W$. Specifically, if K is the restriction to the boundary of the space of harmonic spinors Ker$D$, then K is a self-adjoint boundary condition for D and one has

Theorem 1.2. . *Let W be a self-adjoint boundary condition for D. If D_W is invertible, there is a canonical identification*

$$\det{}_{\mathcal{C}} D_W = \det \frac{1}{2}(1 - g_0 h), \tag{1.1}$$

where W and K are respectively the graphs of the unitary isomorphisms $g : F^+ \to F^-$ and $h : F^+ \to F^-$ from Theorem (1.1), and $g_0 = -g^{-1}$.

Here the right-hand side denotes the usual determinant as a number in C of an operator of the form $1 + t$ where $t : F^+ \to F^+$ is a trace-class for the L^2 norm [27], defined by $\det(1 + t) = \sum_{k=0}^{\infty} \mathrm{tr}(\wedge^k t)$.

A more enlightening way to view Theorem (1.1) and Theorem (1.2) is as follows. In the seminal paper of Quillen [18] it was explained that the determinant associated to a smooth family $\mathcal{A}$ of Dirac operators arises not as a function $\mathcal{A} \to C$ but rather as a section of a complex line bundle L over $\mathcal{A}$; the so-called determinant line bundle. Consequently the obstruction to writing the determinant as a globally defined function on $\mathcal{A}$ is precisely the obstruction to finding a global trivialization of L. If that obstruction vanishes one then looks for a canonical choice of trivialization that naturally extends the theory of finite-dimensional determinants. In [18] that is achieved for a family of $\overline{\partial}$-operators on a Hermitian vector bundle over a closed Riemann surface by defining a a flat connection on the determinant line bundle using a construction of ζ-function determinants of Laplacians. This procedure defines a natural trivialization of L and hence an identification of the determinant section as a function up to a phase ambiguity, and this has come to be accepted as essentially the canonical method for calculating such determinants. (In [5] the ζ-function metric was constructed for the determinant line bundle associated to a general family of Dirac operators over a closed manifold and the curvature, representing the first Chern class of the bundle, identified as the 2-form component of the local family's index theorem.)

In this general context, one may view the Grassmannian Gr as the parameter space $\mathcal{A}$ of a smooth family of Dirac boundary value problems. Over Gr one still has the Quillen determinant line bundle L with canonical section $\det : Gr \to L$ which takes W to $\det D_W$. In fact, the holomorphic line bundle L is isomorphic to the determinant line bundle of [17] which has non-zero first Chern class and hence is topologically non-trivial, and so no global trivialization of L exists. However, over the component Gr_{iso} (the isotropic Grassmannian) of the restricted Grassmannian parameterizing self-adjoint boundary conditions for D the determinant bundle L is canonically trivial. Indeed, that is the content of Theorem (1.1) which defines a trivialization $\sigma : Gr_{iso} \to L$, and from Theorem (1.2)

$$\det D_W = \det \frac{1}{2}(1 - g_0 h).\sigma(W)\,.$$

The determinant line bundle is discussed in Sect.4 of this paper.

The proof of Theorem (1.2) is straight-forward. The harder step comes in the identification of the canonical determinant with the ζ-function determinant $\det_\zeta D_W$. That determinant is defined via the the ζ-function norm $\|.\|_\zeta$ defined on the determinant line of D_W and *formally* $|\det_\zeta D_W| = \|\det D_W\|_\zeta$. In general though, it is not clear that $\det_\zeta D_W$ is defined for the same analytic reasons that compel one in the case of closed manifolds to consider the ζ-function determinant of the Laplacian rather than the operator itself; that is the origin of the phase ambiguity. When X is one dimensional, however, and D is a Dirac-type operator of the form $i\nabla_{d/dx}$ acting on the sections of a (trivial) $U(n)$-bundle $\mathcal{E}$ with unitary connection ∇, the ζ-function determinant can be defined directly. We take $X = [0,1]$ and denote the boundary fibres of $\mathcal{E}$ by $\mathcal{E}_0, \mathcal{E}_1$.

Theorem 1.3. *Let D_W be invertible. Then $\zeta_{D_W}(s) = \mathrm{tr}\, D_W^{-s}$ is well-defined for*

res$s > 1$ *and has an analytic continuation to all of $\mathbb{C}$. The ζ-function determinant exists and*

$$\det{}_{\zeta} D_W = \det(1 - g_0 h), \tag{1.2}$$

where W and K are respectively the graphs of the unitary isomorphisms $g : \mathcal{E}_0 \to \mathcal{E}_1$ and $h : \mathcal{E}_0 \to \mathcal{E}_1$ from Theorem (1.1) and $g_0 = -g^{-1}$. The isomorphism h is the parallel transport of the connection ∇.

Relative to a trivialization of $\mathcal{E}$ the isomorphisms g, h are identified as elements of the unitary group $U(n)$, changing the trivialization only changes $g_0 h$ by conjugation by an element of $U(n)$ and hence the right-hand side of (1.2) is unambiguously defined as a complex number. Theorem (1.3) is complementary to the work of [9] and [11] on determinants in 1 dimension.

The existence of the canonical trivialization σ of $L_{|Gr_{iso}}$, in addition to the usual ζ-function trivialization, is because of the extra degree of freedom introduced by the choice of boundary condition, and we exploit this fact repeatedly in our constructions. One effect of this extra degree of freedom is that there is something of a managerie of different but isomorphic determinant line bundles over Gr. In particular, in Sect.5c we use this to give a third distinct construction of the determinant for dim$X = 1$ which coincides with the ζ-function and canonical determinants up to a factor of i^n.

Having established these identifications it is natural to ask if there is a relation with determinants over closed manifolds. To answer that, consider the closed double $M = X \cup_Y X^-$, where X^- is a copy of X with reverse orientation, which by reflection has a Riemmanian metric equal to a product in a tubular neighbourhood $V = [-1,1] \times Y$ of $Y = \{0\} \times Y$. Over M is the double spinor bundle S_M formed by gluing together two copies of the spinor bundle S via the automorphism $\sigma : S_Y \to S_Y$. A section $\psi_M \in C^\infty(M; S_M)$ then consists of a pair of sections $(\psi, \psi^-) \in C^\infty(X; S) \oplus C^\infty(X^-; S)$ such that at $\{0\} \times Y$ the sections $\sigma\psi$ and ψ^- have the same values and normal derivatives of all orders. One then has the first-order elliptic 'Dirac' operator $D_M = D^0 \cup (-D^1) : C^\infty(M; S_M) \to C^\infty(M; S_M)$, defined by $D_M(\psi, \psi^-) = (D^0\psi, -D^1\psi^-)$. More generally, given Riemmanian spin manifolds X^0, X^1 with the same boundary Y, up to orientation, and with spinor bundles S^0, S^1 such that the restricted spinor bundles S^0_Y, S^1_Y coincide and all topological and geometric data agree at Y, one may form the corresponding elliptic operator $D_M = D^0 \cup -D^1$ over $M = X^0 \cup_Y X^1$. The canonical determinant of D_M is defined and we denote it by $\det_{\mathcal{C}} D_M$. We denote by K^0 and K^1 the respective restrictions to the boundary of the space of harmonic spinors of D^0 and D^1.

Theorem 1.4. . *Let M be odd-dimensional. If D_M is invertible, there is a canonical identification*

$$\det{}_{\mathcal{C}} D_M = \det \frac{1}{2}(1 - h_1 h_0), \tag{1.3}$$

where K^0 and K^1 are respectively the graphs of the unitary isomorphisms $h_0 : F^+ \to F^-$ and $h_1 : F^- \to F^+$ from Theorem (1.1).

In dimension 1, with $S^1 = X^0 \cup X^1$, the isomorphisms h_0 and h_1 represent the parallel transport along X^0 and X^1 with respect to a unitary connection $\nabla_{d/dx} : C^\infty(S^1; \mathcal{E}_{S^1}) \to C^\infty(S^1; \mathcal{E}_{S^1})$ on a Hermitian n-bundle $\mathcal{E}_{S^1}$ over S^1. Hence, since $h_1 h_0$ is the holonomy of ∇ around S^1 (and since the factor of $1/2$ on the right-hand side of (3) can be removed when $\dim X = 1$), then $\det_{\mathcal{C}} D_{S^1}$, where $D_{S^1} = i\nabla_{d/dx}$, coincides with the well-known value of the ζ-function determinant [1],[9].

For $\dim X = 1$ Theorem 1.3 and Theorem 1.4 are related as follows. Let $\mathcal{E}^i = \mathcal{E}_{S^1 | X^i}$ and let D^i be the restriction of D_{S^1} to $C^\infty(X^i; \mathcal{E}^i)$.

Theorem 1.5.

$$\det{}_{\mathcal{C}} D_{S^1} = \int_{U(n)} \det{}_{\mathcal{C}} D^0_W \det{}_{\mathcal{C}} D^1_{W^\perp} \, dW. \tag{1.4}$$

The integral in (1.4) is carried out over the unitary group under the isomorphism $U(n) \cong Gr_{iso}$ defined by Theorem (1.1). Thus Theorem (1.5) states that the determinant over the closed manifold is obtained by integrating away the choice of self-adjoint boundary condition in the determinants over the two halves. We could of course have written (1.4) in terms of ζ-function determinants, the difference is purely notational. An open question is whether (1.4) may indicate a relation between the ζ-function metric on the determinant line bundle for a general family of Dirac boundary value problems and the ζ-function metric for the corresponding family of Dirac operators over the closed double manifold.

Since the topic of determinants of boundary value problems has taken on a specific interest in mathematical physics with the development of topological quantum field theories [2],[26], we conclude this paper with some brief comments on the relation between Theorems (1.1)-(1.5) and $0+1$-dimensional TQFT. For a specific account of the relation of conformal field theory to Grassmannians and elliptic boundary value problems for $\overline{\partial}$-operators over a Riemann surface we refer to [25],[28].

2. A Grassmannian of Dirac Boundary Value Problems

In this section we describe the analytic constructions in more detail. Let X be a compact spin manifold with boundary Y. Then the Dirac operator D over X is formally self-adjoint with respect to the L^2-Hermitian inner-product

$$< \psi_1, \psi_2 >_S = \int_X (\psi_1, \psi_2)_S \, dx, \tag{2.1}$$

where dx denotes the Riemannian measure on X. This means that for all $\psi_1, \psi_2 \in \mathcal{S}$ with supports disjoint from the boundary of X one has

$$< D\psi_1, \psi_2 >_S = < \psi_1, D\psi_2 >_S .$$

In the collar neighbourhood $U = [0,1] \times Y$ of the boundary $Y \cong \{0\} \times Y$ we choose a Riemannian metric g on X which splits isometrically as $g_U = du^2 + g_Y$, where u is the normal coordinate to the boundary and g_Y the induced metric on Y. Over U the Dirac operator has the form

$$D_{|U} = \sigma\left(\frac{\partial}{\partial u} + A\right), \tag{2.2}$$

where the symbol map $\sigma = \sigma(D)(du) : S_Y \to S_Y$ is the bundle isomorphism given by Clifford multiplication by the inward unit normal du in T^*U. We note that $\sigma^2 = -1$ and that σ is an isometry with respect to the induced inner-product $<,>$ on S_Y. The boundary operator $A = D_Y\sigma : S_Y \to S_Y$ is a self-adjoint first-order elliptic differential operator independent of the normal coordinate u. Because Y is a closed manifold, A has a real and discrete spectrum λ with smooth eigenvectors ϕ_λ. Because D is formally self-adjoint the following equalities hold,

$$\sigma^* = -\sigma \qquad \sigma A + A\sigma = 0, \tag{2.3}$$

so that A is of degree 1 with respect to the mod 2 grading defined by σ.

The Grassmanian of elliptic boundary conditions is defined with respect to the energy polarization $F = H^+ \oplus H^-$ of the space of boundary spinor fields, where the subspaces H^+ and H^- are spanned, respectively, by those eigenvectors of A with non-negative and negative eigenvalues. The polarization is given by an involution $J : F \to F$ equal to $+1$ on H^+ and -1 on H^-, which defines canonical pseudo-differential projections

$$P^\pm = \frac{1}{2}(I \pm J) : F \to H^\pm.$$

One thus obtains the preferred boundary value problem $D_{H^+} : C^\infty_{H^+}(X;S) \to C^\infty(X;S)$ studied in [3].