ON CHERN-WEIL FORMS ASSOCIATED WITH SUPERCONNECTIONS

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Abstract

We define k-th Chern-Weil forms $c_k(\mathbb{A})$ associated to a superconnection $\mathbb{A}$ using ζ-regularisation methods extended to ΨDO valued forms. We show that they are cohomologous in the de Rham cohomology to $\mathrm{tr}(\mathbb{A}^{2k}\,\pi_P)$ involving the projection π_P onto the kernel of the elliptic operator P to which the superconnection $\mathbb{A}$ is associated. A transgression formula shows that the corresponding Chern-Weil cohomology classes are independent of the scaling of the superconnection. When P is a differential operator with scalar leading symbol, the k-th Chern-Weil form corresponds to the regularised k-th derivative at $t = 0$ of the Chern character $\mathrm{ch}(t\,\mathbb{A})$ and it has a local description

$$c_k(\mathbb{A}) = -\frac{1}{2p}\mathrm{res}\left(\mathbb{A}^{2k}\log(\mathbb{A}^2 + \pi_P)\right)$$

in terms of the Wodzicki residue extended to ΨDO-valued forms.

Introduction

Given a superconnection $\mathbb{A}$ adapted to a family of self-adjoint elliptic ΨDOs with odd parity parametrised by a manifold B, the ΨDO valued form $e^{-\mathbb{A}^2}$ is trace class so that one can define the associated Chern character [1],[2],[3]

$$\mathrm{ch}(\mathbb{A}) := \mathrm{tr}\left(e^{-\mathbb{A}^2}\right) \tag{1}$$

which defines a characteristic class independent of any scaling $t \mapsto \mathbb{A}_t$ of the super connection. Here, tr is the supertrace associated to the family of ΨDOs, which reduces to the usual trace when the grading is trivial. When the fibre of the manifold fibration over B reduces to a point, $\mathbb{A}$ can then be replaced by an ordinary connection ∇ on a vector bundle over B with Chern character $\mathrm{ch}(\nabla) := \mathrm{tr}\left(e^{-\nabla^2}\right)$. This can be expressed as a linear combination

$$\mathrm{ch}(\nabla) = \sum_{k=0}^{\mathrm{dimM}} (-1)^k \frac{c_k(\nabla)}{k!}$$

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of the associated Chern-Weil forms $c_k(\nabla) := \mathrm{tr}(\nabla^{2k})$ of degree $2k$. Conversely, the Chern-Weil forms can be interpreted as the coefficients of a Taylor expansion of the map $t \mapsto \mathrm{ch}_t(\nabla) := \mathrm{tr}\left(e^{-t\nabla^2}\right)$ at $t = 0$:

$$\partial_t^k \mathrm{ch}_t(\nabla)|_{t=0} = (-1)^k\, c_k(\nabla). \tag{2}$$

Just as ordinary Chern-Weil forms are traces of the k-th power of the curvature, we define k-th Chern-Weil forms associated with a superconnection $\mathbb{A}$ as weighted traces:

$$c_k(\mathbb{A}) := \mathrm{tr}^{\,\mathbb{A}^2}(\mathbb{A}^{2k})$$

of the k-th power of the curvature $\mathbb{A}^2$. To do this, we extend weighted traces [4], [10], [6] to families $\mathbb{A}$, $\mathbb{Q}$ of classical ΨDO valued forms by setting

$$\mathrm{tr}^{\,\mathbb{Q}}(\mathbb{A}) := \zeta(\mathbb{A}, \mathbb{Q} + \pi_{\mathbb{Q}_{[0]}}, z)|_{z=0}^{\mathrm{mer}} = \zeta(\mathbb{A}, \mathbb{Q}, 0)|^{\mathrm{mer}} + \mathrm{tr}(\mathbb{A}\, \pi_{\mathbb{Q}_{[0]}}), \tag{3}$$

including weights which are themselves ΨDO valued forms. Here, $\pi_{Q_{[0]}}$ is the projection onto the bundle of kernels $\cup_{x\in B}\mathrm{Ker}(Q_x)$ of the zero degree part $Q_{[0]}$ of Q (we assume that $\dim(\mathrm{Ker}(Q_x))$ is constant), while $\zeta(A, P, z)|^{\mathrm{mer}}$ is the mixed degree meromorphic differential form studied in [15] which extends the holomorphic form $\mathrm{Tr}\left(AP^{-z}\right)$ from a suitable half-plane $\mathrm{Re}(z) >> 0$. We show that the forms $c_k(\mathbb{A})$ are closed and that the associated characteristic classes are independent of the scaling of $\mathbb{A}$. This is equivalent to the fact proved in ([15]) that the zeta forms $\zeta(\mathbb{A}^2, -k) := \zeta(\mathbb{A}^{2k}, \mathbb{A}^2, 0)|^{\mathrm{mer}}$ are exact and independent of scaling; the (closed) Chern-Weil forms $c_k(\mathbb{A})$ differ from these forms by a term $\mathrm{tr}(\pi\, \mathbb{A}^{2k}\, \pi)$ involving the projection π on the kernel of the operator $\mathbb{A}_{[0]}$ to which the superconnection $\mathbb{A}$ is adapted. As a consequence of the exactness of $\zeta(\mathbb{A}^2, -k)$ we obtain that $c_k(\mathbb{A})$ is cohomologous to $\mathrm{tr}(\pi\, \mathbb{A}^{2k}\, \pi)$. Whenever $\mathbb{A}$ is a superconnection associated with a family of *differential operators* of order p, with scalar leading symbol, we relate them to the Chern character $\mathrm{ch}(\mathbb{A})$ via a formula which mimics equation (2): for $t > 0$,

$$\mathrm{fp}_{t=0}\, \partial_t^k \mathrm{ch}_t(\mathbb{A}) = (-1)^k\, c_k(\mathbb{A}), \tag{4}$$

where fp denotes the finite part (the constant term in the asymptotic expansion as $t \to 0+$). With the same assumptions, we show that the following local formula for these weighted Chern forms (see equation (19)) holds

$$c_k(\mathbb{A}) := -\frac{1}{2p}\, \mathrm{res}(\mathbb{A}^{2k}\, \log(\mathbb{A}^2 + \pi_P)),$$

where the right-side is the residue trace extended these families; this object would not be defined for general families of ΨDOs. This equation generalises formulae obtained in [13] which relate weighted traces of differential operators to Wodzicki residues extended to logarithms (see theorem 3) and provides a local expression on the grounds of the locality of the Wodzicki residue.

These results are summarised in Theorem 4.

The paper is organised as follows

1. ΨDO valued forms
2. Complex powers and logarithms of ΨDO valued forms
3. The Wodzicki residue and the canonical trace extended to ΨDO valued forms
4. Holomorphic families of ΨDO valued forms
5. Weighted traces of differential operator valued forms
6. Chern-Weil forms associated with a superconnection

1. ΨDO VALUED FORMS

In this section we recall the construction of form valued geometric families of ΨDOs from [15]. Consider a smooth fibration $\pi : M \to B$ with closed n-dimensional fibre $M_b := \pi^{-1}(b)$ equipped with a Riemannian metric $g_{M/B}$ on the tangent bundle $T(M/B)$. Let $|\Lambda_\pi| = |\Lambda(T^*(M/B))|$ be the line bundle of vertical densities, restricting on each fibre to the usual bundle of densities $|\Lambda_{M_b}|$ along M_b. Let $\mathcal{E} := \mathcal{E}^+ \oplus \mathcal{E}^-$ be a vertical Hermitian $\mathbb{Z}_2$-graded vector bundle over M and let $\pi_*(\mathcal{E}) := \pi_*(\mathcal{E}^+) \oplus \pi_*(\mathcal{E}^-)$ be the graded infinite dimensional Fréchet bundle with fibre $C^\infty\left(M_b, \mathcal{E}^b \otimes |\Lambda_{M_b}|^{\frac{1}{2}}\right)$ at $b \in B$, where $\mathcal{E}^b$ is the $\mathbb{Z}_2$-graded vector bundle over M_b obtained by restriction of $\mathcal{E}$. By definition, a smooth section ψ of $\pi_*(\mathcal{E})$ over B is a smooth section of $\mathcal{E} \otimes |\Lambda_\pi|^{\frac{1}{2}}$ over M, so that $\psi(b) \in C^\infty(M_b, \mathcal{E}^b \otimes |\Lambda_{M_b}|^{\frac{1}{2}})$ for each $b \in B$. More generally, the de Rham complex of smooth forms on B with values in $\pi_*(\mathcal{E})$ is defined by

$$\mathcal{A}(B, \pi_*(\mathcal{E})) = C^\infty\left(M, \pi^*(\wedge T^*B) \otimes \mathcal{E} \otimes |\Lambda_\pi|^{\frac{1}{2}}\right)$$

with $\otimes$ the $\mathbb{Z}_2$-graded tensor product. Let $C\ell(\mathcal{E})$ denote the infinite-dimensional bundle of algebras with fibre $C\ell(\mathcal{E}^b) = C\ell\left(M_b, \mathcal{E}^b \otimes |\Lambda_{M_b}|^{\frac{1}{2}}\right)$. A section $Q \in \mathcal{A}\left(B, C\ell(\mathcal{E})\right)$ defines a smooth family of classical ΨDOs with differential form coefficients parametrized by B.

Such an operator valued form Q is locally described by a *vertical* symbol

$$\mathbf{q}(x, y, \xi) \in C^\infty\left((U_M \times_\pi U_M) \times \mathbb{R}^n, \pi^*(\Lambda T^* U_B) \otimes \mathbb{R}^N \otimes \left(\mathbb{R}^N\right)^*\right) ,$$

where $\times_\pi$ is the fibre product, ξ may be identified with a vertical vector in $T_b(M/B)$, and U_M is a local coordinate neighbourhood of M over which $\mathcal{E}_{U_M} \simeq U_M \times \mathbb{R}^N$ is trivialized and $\mathbb{R}^N$ inherits the grading of

$\mathcal{E}$. With respect to the local trivialisation of $\pi_*(\mathcal{E})$ over $U_B = \pi(U_M)$ one has

$$\mathcal{A}\left(U, \pi_*(\mathcal{E})_{|U_B}\right) \simeq \mathcal{A}(U) \otimes C^\infty\left(M_{b_0}, \mathcal{E}^{b_0}\right)$$

with $M_{b_0} = \pi^{-1}(b_0)$ relative to a basepoint $b_0 \in U_B$, so that $\mathbf{q}$ can be written locally over U_B as a finite sum of terms of the form $\omega_k \otimes \mathbf{q}_{[k]}$, where $\omega_k \in \mathcal{A}^k(U_B)$ and $\mathbf{q}_{[k]} \in C^\infty\left(U_{b_0} \times \mathrm{I\!R}^n/\{0\}, \mathrm{I\!R}^N \times \left(\mathrm{I\!R}^N\right)^*\right)$ is a symbol (in the single manifold sense) of form degree zero so that for all multi-indices α, β and each compact subset $K \subset U_{b_0}$ the growth estimate holds

$$|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \mathbf{q}_{[k]}(x, y, \xi)| < C_{k,\alpha,\beta,K}(1 + |\xi|)^{q_k - |\gamma|}. \tag{5}$$

For clarity we will work only with local symbols which are *simple*, meaning they have they have the local form $\sum_{k=0}^{\mathrm{dimB}} \omega_k \otimes \mathbf{q}_{[k]}$, with just one term in each form degree, extending by linearity to general sums. The order of a simple symbol is defined to be the $(\dim B + 1)$-tuple $(q_0, \cdots, q_{\mathrm{dim}B})$ with q_k the order of the symbol $\mathbf{q}_k$; for simplicity we consider the case where $\mathbf{q}_k$ is constant on B.

In accordance with the splitting of the local symbol into form degree $\mathbf{q} = \mathbf{q}_{[0]} + \cdots + \mathbf{q}_{[\dim B]}$ the operator

$$(Q\psi)(x) = \frac{1}{(2\pi)^n} \int_{M/B} \int_{\mathrm{I\!R}^n} e^{i(x-y)\cdot\xi} \mathbf{q}(x, y, \xi) \psi(y) vol_{M/B} d\xi,$$

for ψ with compact support in U_M, splits as $Q = Q_{[0]} + Q_{[1]} + \cdots + Q_{[\dim B]}$, where Q is a simple family of ΨDOs in so far as locally there is one component $Q_{[k]} = \omega_k \otimes Q_k \in \mathcal{A}\left(U_B, \pi_*(\mathcal{E})_{|U_B}\right)$ in each form degree. $Q_{[k]}$ raises form degree by k and $Q_k(b) : C^\infty(M_b, \mathcal{E}_b^+) \to C^\infty(M_b, \mathcal{E}_b^-)$ is a pseudodifferential operator in the usual sense acting on sections of the bundle $\mathcal{E}_b := \mathcal{E}_{|M_b}$ over the fibre M_b.

The composition of ordinary symbols naturally extends to a composition of families of vertical symbols defined fibrewise by:

$$\mathbf{q} \circ \mathbf{q}' := \omega \wedge \omega' \otimes q \circ q'$$

where $q \circ q'$ is the ordinary composition of symbols corresponding to the ΨDO algebra multiplication

$$\mathcal{A}^i(B, \Psi^\nu(\mathcal{E})) \times \mathcal{A}^j(B, \Psi^\mu(\mathcal{E})) \longrightarrow \mathcal{A}^{i+j}(B, \Psi^{\nu+\mu}(\mathcal{E})). \tag{6}$$

By a standard method the vertical symbol $\mathbf{q}$ in (x, y)-form can be replaced an equivalent (modulo $S^{-\infty}$) symbol in x-form. A (simple) family of vertical symbols $\mathbf{q}$ of order $(q_0, \cdots, q_{\mathrm{dim}B})$ is then called *classical* if for each $k \in \{0, \cdots, \mathrm{dim}B\}$ one has $\mathbf{q}_{[k]}(x, \xi) \sim \sum_{j=0}^\infty \mathbf{q}_{[k],j}(x, \xi)$ with $\mathbf{q}_{[k],j}(x, t\xi) = t^{q_k - j} \mathbf{q}_{[k],j}(x, \xi)$ for $t \geq 1, |\xi| \geq 1$. A family of vertical ΨDOs Q is called *classical* if each of its local component simple symbols is classical.

Definition 1. *A smooth family of vertical* ΨDOs $Q \in \mathcal{A}\left(B, C\ell(\mathcal{E})\right)$ *is elliptic if its form degree zero component* $Q_{[0]}$ *is pointwise (with respect to the parameter manifold* B*) elliptic.*

In this case Q has spectral cut θ if $Q_{[0]}$ admits spectral cut θ. Likewise, it is invertible if $Q_{[0]}$ is invertible. Given that Q admits a spectral cut then it has a well-defined resolvent, which is a sum of simple families of ΨDOs . Setting $Q_{[>0]} := Q - Q_{[0]} \in \mathcal{A}^1\left(B, C\ell(\mathcal{E})\right)$ then there is an open sector $\Gamma_\theta \in \mathbb{C} - \{0\}$ containing the ray L_θ such that on any compact codimension zero submanifold B_c of B for large $\lambda \in \Gamma_\theta$ one has in $\mathcal{A}\left(B_c, C\ell(\mathcal{E})\right))$ using the indempotence of forms on B

$$\begin{aligned}(Q-\lambda)^{-1} &= \left(Q_{[0]}-\lambda\right)^{-1} \\ &\quad + \sum_{k=1}^{\dim B} (-1)^k \left(Q_{[0]}-\lambda\right)^{-1}\left(Q_{[>0]}(Q_{[0]}-\lambda)^{-1}\right)^k . \quad (7)\end{aligned}$$

In particular $\left((Q-\lambda)^{-1}\right)_{[0]} = \left(Q_{[0]}-\lambda\right)^{-1}$.

2. Complex powers and logarithms of ΨDO valued forms

Here we use the complex powers for ΨDO valued forms introduced in [15] to define and investigate the properties of the logarithm of a simple family of invertible admissible elliptic vertical ΨDOs .

Let Q be a smooth family of vertical admissible elliptic invertible ΨDOs , the orders $(q_0, \cdots, q_{\dim B+1})$ of which fulfill the assumption

$$q_0 = \mathrm{ord} Q_{[0]} > 0$$

and

$$q_k \le q_0 \quad \forall k \ge 1. \tag{8}$$

Under these assumptions one obtains an operator norm estimate in $\mathcal{A}(B)$ as $\lambda \to \infty$ in Γ_θ

$$\|(Q-\lambda I)^{-1}\|_{M/B}^{(l)} = O(|\lambda|^{-1})$$

where $\|\cdot\|_{M/B}^{(l)} : \mathcal{A}\left(B, C\ell(\mathcal{E})\right) \to \mathcal{A}(B)$ is the vertical Sobolev norm associated to the vertical metric.

Lemma 1. *Let Q be an admissible elliptic invertible* ΨDOs *valued form on B with spectral cut θ. Then*

$$Q_\theta^{-z} = \frac{i}{2\pi}\int_{C_{R,\theta}} \lambda_\theta^{-z}(Q-\lambda I)^{-1} d\lambda$$

defines a family of ΨDOs *in* $\mathcal{A}\left(B, C\ell(\mathcal{E})\right)$ *which is a finite sum of simple* ΨDO *families with of holomorphic orders*

$$\alpha(z) = -q_0 \cdot z + \alpha(0)$$

where $q_0 = \mathrm{ord} Q_{[0]}$ *and the constant term* $\alpha(0)$ *is determined by the* q_i *and the form degree. In particular,* $\left(Q_\theta^{-z}\right)_{[0]} = \left((Q_\theta)_{[0]}\right)^{-z}$. *Here* λ_θ^{-z} *is the branch of* λ^{-z} *defined by* $\lambda_\theta^{-z} = |\lambda|^{-z} e^{-iz \;\; Arg\lambda}, \quad \theta - 2\pi \leq Arg\lambda < \theta$ *and where* $C_{R,\theta}$ *is a contour as in section 1. When* $Q_{[0]}$ *has non negative leading symbol we can choose* $\theta = \frac{\pi}{2}$ *in which case this complex power is the Mellin transform of the corresponding heat-operator form:*

$$Q^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}\, e^{-tQ}\, dt.$$

Remark 1. *When* Q *is not invertible, we can apply the Lemma to* $Q + \pi_{Q_{[0]}}$ *which is invertible. Here* $\pi_{Q_{[0]}}$ *denote the projection onto the kernel of* $Q_{[0]}$.

Proof. Let us fist check the last formula, which is very straightforward. As for ordinary pseudodifferential operators, we write

$$\begin{aligned}
Q^{-z} &= \frac{i}{2\pi} \int_{C_{R,\theta}} \lambda_\theta^{-z} (Q - \lambda I)^{-1}\, d\lambda \\
&= \frac{i}{2\pi} \int_{C_{R,\theta}} \left(\frac{1}{\Gamma(z)} \int_0^\infty t^{z-1}\, e^{-t\,\lambda}\, dt \right) (Q - \lambda I)^{-1}\, d\lambda \\
&= \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1}\, \frac{i}{2\pi} \int_{C_{R,\theta}} e^{-t\,\lambda}\, (Q - \lambda I)^{-1}\, d\lambda \\
&= \frac{1}{\Gamma(z)} \int_0^\infty dt\, t^{z-1}\, e^{-t\,Q}.
\end{aligned}$$

To prove the first part of the lemma, we follow [15]. Let us first assume (8). Following Seeley's analysis, one can define the complex power $Q_\theta^{-z} = \frac{i}{2\pi} \int_{C_{R,\theta}} \lambda_\theta^{-z} (Q - \lambda I)^{-1} d\lambda$ for Re $(z) > 0$ in the usual way provided Q satisfies assumption (8). Since $((Q - \lambda I)^{-1})_{[0]} = \left(Q_{[0]} - \lambda I\right)^{-1}$ it follows that $\left(Q_\theta^{-z}\right)_{[0]} = \left((Q_\theta)_{[0]}\right)^{-z}$. The order of $\left(Q_\theta^{-z}\right)_{[d]}$ is derived from the expression of $((Q - \lambda I)^{-1})_{[d]}$, which from (7) can be rewritten as

$$\left((Q - \lambda)^{-1}\right)_{[d]} = \tag{9}$$

$$\sum_{k=0}^{\dim B} (-1)^k \sum_{p_1 + \cdots + p_k = d} \left(Q_{[0]} - \lambda\right)^{-1} W_{[p_1]} \left(Q_{[0]} - \lambda\right)^{-1} \ldots W_{[p_k]} \left(Q_{[0]} - \lambda\right)^{-1}$$

where $W_{[p_j]} = (Q - Q_{[>0]})_{[p_j]}$, a simple family of ΨDOs of pure form degree $p_j \in \{1, \ldots, \dim B\}$. Let $\alpha_j = \mathrm{ord}(W_{[p_j]})$. This contributes to $\left(Q_\theta^{-z}\right)_{[d]}$ a (simple) family of ΨDOs order $-q_0\, z - q_0\, k + \alpha_1 + \cdots + \alpha_k$. It follows that the the orders of each simple component of the complex power is holomorphic with derivative at 0 independent of k.

Complex powers can be extended to operators which do not satisfy assumption (8). By (9)

$$\partial_\lambda^m (Q-\lambda)^{-1}_{[d]} = \sum_{k=0}^{\dim B} \sum_{\substack{p_1+\ldots+p_k=d \\ m_0+\cdots+m_k=m}} \partial_\lambda^{m_0}\left(Q_{[0]}-\lambda\right)^{-1} W_{[p_1]}\,\partial_\lambda^{m_1}\left(Q_{[0]}-\lambda\right)^{-1}\ldots W_{[p_k]}\partial_\lambda^{m_k}(Q_{[0]}-\lambda)^{-1}. \tag{10}$$

Since $\partial_\lambda^{m_i}(Q_{[0]}-\lambda I)^{-1}$ is of order $-q(m_i+1)$, taking m sufficiently large, we can ensure that

$$\|\partial_\lambda^m (Q_{[0]}-\lambda I)^{-1}\|_{M/Z}^{(l)} = 0(|\lambda|^{-1})$$

without assumption (8). In this way, regularizing using integration by parts we may define

$$Q_\theta^{-z} = \frac{1}{(z-1)\cdots(z-m)}\frac{i}{2\pi}\int_{C_{R,\theta}} \lambda_\theta^{m-z}\partial_\lambda^m (Q-\lambda I)^{-1} d\lambda. \tag{11}$$

The order computation is essentially unchanged. □

Let Q be a smooth family of vertical admissible elliptic invertible ΨDOs. The logarithm is also built as in the single operator case by defining

$$\log_\theta Q := \frac{\partial}{\partial z}_{|_{z=0}}\left(Q_\theta^z\right).$$

As in the single operator case the logarithm is *not* quite a family of *classical* ΨDOs, but the non-logarithmic component is located only in the form degree zero component:

Lemma 2. *Let $Q \in \mathcal{A}(B, C\ell(\mathcal{E}))$ be a family of differential form valued vertical classical invertible elliptic* ΨDOs *with spectral cut θ. Then*

$$(\log_\theta Q)_{[0]} = \log_\theta Q_{[0]}.$$

If Q moreover satisfies assumption (8), then

$$\log_\theta Q - \log_\theta Q_{[0]} \in \mathcal{A}(B, C\ell(\mathcal{E})).$$

Remark 2.
- *Since the order of the components of $(Q^{-z})_{[d]}$ are of the form $-qz+\alpha(0)$, from Lemma 1, one expects $\log_\theta Q_{[d]}$ to contain $\log|\xi|$ terms, which would contradict the statement of Lemma 2. A closer look shows that the terms of the form $|\xi|^{-qz}$ that arise in $Q^{-z}_{[d]}$ when $d>0$ come with a factor of z and therefore do not yield any $\log|\xi|$ term when differentiated w.r.to z at $z=0$.*
- *A priori $|\xi|$ is $b \in B$-dependent, this reflecting the fact that the decomposition depends on the choice of metric on the fibre M_b.*

Proof. We drop the index θ to simplify notations. First, we show that $(\log Q)_{[0]} = \log Q_{[0]}$. For some integer m chosen large enough, we have for $\mathrm{Re}(z) > 0$:

$$\partial_z Q^{-z} = \frac{1}{(z-1)\cdots(z-m)}\frac{1}{2i\pi}\int_C \log\lambda\, \lambda^{m-z}\partial_\lambda^m (Q-\lambda I)^{-1} d\lambda$$

and hence

$$\log Q = (-1)^m \frac{1}{m!}\frac{1}{2i\pi}\int_C \log\lambda\, \lambda^m \partial_\lambda^m (Q-\lambda I)^{-1} d\lambda.$$

From (9), the degree d-form part of the resolvent reads:

$$\begin{aligned}&(Q-\lambda)^{-1}_{[d]}\\ = &\sum_{k=0}^{\dim B}(-1)^k \sum_{|[p,k]|=d}(Q_{[0]}-\lambda I)^{-1}W_{[p_1]}(Q_{[0]}-\lambda I)^{-1}\cdots W_{[p_k]}(Q_{[0]}-\lambda I)^{-1}\end{aligned}$$

where we have set $W := Q_{[>0]} = Q - Q_{[0]}$. It follows that

$$\begin{aligned}&\log Q_{[d]}\\ = &\frac{1}{2i\pi}\sum_{k=0}^{\dim B}(-1)^k \sum_{|[p,k]|=d}\ \sum_{m_1+\cdots+m_k=m}(-1)^k\frac{m_0!\cdots m_k!}{m!}R(\lambda,m,p,k)\log\lambda\, d\lambda\end{aligned}$$

with the same notations as above. In particular, setting $d = k = 0$ yields:

$$(\log Q)_{[0]} = \frac{1}{2i\pi}\int_C \log\lambda\, \partial_\lambda^m (Q_{[0]}-\lambda I)^{-m-1} d\lambda = \log\left(Q_{[0]}\right).$$

It follows that $\log Q = \log\left(Q_{[0]}\right) + (\log Q)_{[>0]}$.

To see that $\log Q - \log Q_{[0]} \in \mathcal{A}(B, C\ell(\mathcal{E}))$, recall that $\log Q = \partial_z Q^z|_{z=0}$ and that for each z the operator Q^z is represented with respect to local trivializations by a local polyhomogeneous symbol of mixed differential-form degree

$$\mathbf{q}^z(b,x,\xi) = \frac{i}{2\pi}\int_{C_0}\lambda_\theta^z\ \mathbf{b}[\lambda](b,x,\xi)\circ\left(\mathbf{w}(b,x,\xi)\circ\mathbf{b}[\lambda](b,x,\xi)\right)^k\ d\lambda \tag{12}$$

where $\circ$ is the vertical form-valued symbol product, C_0 is a finite closed key-hole contour enclosing the spectrum of the leading vertical symbol of $(Q-\lambda)^{-1}_{[0]} = (P-\lambda)^{-1}$, and where $\mathrm{Op}(\mathbf{b}[\lambda]) \sim (Q-\lambda)^{-1}$, $\mathrm{Op}(\mathbf{w}) \sim \mathsf{W} := Q - \mathsf{P}$.

The symbol $\mathbf{b}[\lambda]\circ(\mathbf{w}\circ\mathbf{b}[\lambda])^k$ in (12) is polyhomogeneous, with an asymptotic expansion into terms of decreasing homogeneity, each of mixed form degree, in the usual way. In general this is a complicated expression. Nevertheless, the log-type of $\log Q$ can be inferred just from the leading symbol (top homogeneity). This is a consequence of the following simple lemma.

Lemma 3. $\mathbf{q}^z|_{z=0}$ *is the identity vertical symbol* $\mathbf{I}$ *defined by*

$$\begin{aligned} \mathbf{I}_{[0]} &= (I,0,0,\ldots) \\ \mathbf{I}_{[p]} &= \mathbf{o} := (0,0,0,\ldots)\ , \qquad p>0\ , \end{aligned}$$

where $\mathbf{I}_{[p]}$ *indicates the component of form degree* p*, and the sequence on the right-side are the homogeneous terms.*

Proof. This is immediate from (12) since $\mathbf{b}[\lambda]\circ(\mathbf{w}\circ\mathbf{b}[\lambda])^k$ is $O(\lambda^{-2})$ for $k>0$ (general ΨDO s). When $k=0$ then the integrand is $O(\lambda^{-1})$, and we have the usual situation of form degree zero operators. ☐

Let $\mathbf{q}^z_w(x,\xi)$ denote the (mixed-form degree) term in the asymptotic expansion of Q^z of homogeneity w, and let $\mathbf{q}^z_{w_{\max}}(x,\xi)$ be the leading symbol (with maximum homogeneity). Then

$$\mathbf{q}^z_{w_{\max}}(x,\xi) = \frac{i}{2\pi}\int_{C_0}\lambda^z_\theta\mathbf{g}(x,\xi,\lambda)\,d\lambda\ , \tag{13}$$

where

$$\mathbf{g}(x,\xi,\lambda) = \mathbf{b}_{-m}[\lambda](x,\xi)\,\mathbf{w}_\nu(x,\xi)\,\mathbf{b}_{-m}[\lambda](x,\xi)\ldots\mathbf{w}_\nu(x,\xi)\,\mathbf{b}_{-m}[\lambda](x,\xi)$$

is the ordinary form-valued matrix product (i.e. not a symbol product) of leading order symbols $\mathbf{b}_{-m}[\lambda](x,\xi)$ of $\mathsf{P}-\lambda I$ (P of order $m>0$), and $\mathbf{w}_\nu$ the leading ΨDO -order symbol of W which has maximum homogeneity ν.

Each $\mathbf{b}_{-m}$ has the quasi-homogeneity property for $t>0$

$$\mathbf{b}_{-m}[t^m\lambda](x,t\xi) = t^{-m}\,\mathbf{b}_{-m}[\lambda](x,\xi)$$

and so

$$\mathbf{g}(x,t\xi,t^m\lambda) = t^{-m(k+1)+k\nu}\,\mathbf{g}(x,\xi,\lambda)\ .$$

Hence making the change of variable $\lambda = t^m\mu$ in (13) we have

$$\mathbf{q}^z_{w_{\max}}(x,t\xi) = t^{mz-mk+k\nu}\,\mathbf{f}^z_{w_{\max}}(x,\xi). \tag{14}$$

It follows that $\mathbf{q}^z$ has an expansion into terms of mixed form degree

$$\mathbf{q}^z(x,\xi)\sim\sum_{j\geq 0}\mathbf{q}^z_{mz-mk+k\nu-j}(x,\xi)\ .$$

¿From 3 we therefore have

$$\mathbf{q}^z_{mz-mk+k\nu-j}(x,\xi)|_{z=0} = \delta_{j,0}\,\mathbf{I} \tag{15}$$

(since from the lemma terms of positive form degree do not contribute).

The final conclusion for $\log\mathbf{q} = \partial_z\mathbf{q}^z_{z=0}$ now follows in the usual way by differentiating

$$\mathbf{q}^z_{mz-mk+k\nu-j}(x,\xi) = |\xi|^{mz-mk+k\nu-j}\,\mathbf{q}^z_{mz-mk+k\nu-j}\left(x,\frac{\xi}{|\xi|}\right)$$

with respect to z, then evaluating at $z=0$ and using (15) to get $\log\mathbf{q}\sim\sum_{j\geq 0}\log\mathbf{q}_j$ with

$$\log \mathbf{q}(x,\xi) = m \log |\xi| \, \delta_{j,0} \mathbf{I} + |\xi|^{k(\nu-m)-j} \partial_z|_{z=0} \mathbf{q}^z \left(x, \frac{\xi}{|\xi|}\right) \, .$$

Note, since $(\partial_z|_{z=0}\mathbf{q}^z(x,\xi/|\xi|)$ has order zero, that provided $\mathrm{ord}(\mathsf{P}) = m \geq \mathrm{ord}(\mathsf{W})$ (assumed) the second term is of order zero (or less). □

Remark 3. *A formal argument based on the Campbell-Hausdorff formula provides some intuition why the second part of the lemma holds. We show here how the Campbell-Hausdorff formula for ordinary* ΨDOs [11] *formally extended to families of vertical* ΨDOs *, yields that* $(\log Q)_{[0]}$ *is a classical vertical* ΨDO *. Indeed, the splitting* $Q = Q_{[0]} + Q_{[>0]}$ *yields*

$$\begin{aligned}
\log Q &= \log\left(Q_{[0]} + Q_{[>0]}\right) \\
&= \log\left[Q_{[0]}\left(I + Q_{[0]}^{-1} Q_{[>0]}\right)\right] \\
&\sim \log Q_{[0]} + \log(I + Q_0^{-1} Q_{[>0]}) + \sum_{k=2}^{\infty} C^{(k)}\left(\log Q_{[0]}, \log(I + Q_{[0]}^{-1} Q_{[>0]})\right)
\end{aligned}$$

where $C^{(k)}(M,N)$ *stands for a linear combination of Lie polynomials of degree* k *in* M *and* N *given by:*

$$\begin{aligned}
&C^{(k)}(M,N) \\
= &\sum_{j=1}^{\infty} c_j \sum_{\alpha_l+\beta_l>0,\, \sum_{j=1}^{l} \alpha_j+\beta_j+1=k} (AdM)^{\alpha_1}(AdN)^{\alpha_1} \cdots (AdM)^{\alpha_1}(AdN)^{\alpha_1} N
\end{aligned}$$

for some coefficients $c_j \in \mathbb{R}$ *and where* $(AdM)M' := [M,M']$*. Under assumption (8), the operator* $Q_{[0]}^{-1}Q_{[>0]}$ *has a vanishing form degree zero part and negative orders* $(\beta_{[1]}, \cdots, \beta_{[dimB+1]})$*. The logarithm therefore coincides with the logarithm on bounded operators and yields an asymptotic expansion:*

$$\log(I + Q_{[0]}^{-1} Q_{[>0]}) \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(Q_{[0]}^{-1} Q_{[>0]}\right)^k ,$$

which shows that $\log(I + Q_{[0]}^{-1} Q_{[>0]})$ *is a classical vertical* ΨDO *. It follows that each of the* $C^{(k)}(M,N)$*'s is also a classical vertical* ΨDO *. Indeed, these are built up from iterated brackets of an ordinary (corresponding to the* 0*-form degree part) logarithmic* ΨDO $\log Q_{[0]}$ *and the classical vertical* ΨDO $\log(I + Q_{[0]}^{-1} Q_{[>0]})$*. The ordinary symbol analysis shows that such brackets are classical and hence that* $\log Q - \log Q_{[0]} \in \mathcal{A}(B, C\ell(\mathcal{E}))$ *as claimed in the lemma.*

This argument therefore provides a heuristic but maybe more intuitive explanation of the lemma.

3. The Wodzicki residue and the canonical trace extended to geometric families

Both the Wodzicki residue and the cut-off integral defined on ordinary classical symbols extend to smooth families of vertical differential form valued classical symbols.

Let $Q \in \mathcal{A}(B, \Psi(\mathcal{E}))$ be a simple family of ΨDOs of non-integer order, so that in each form degree $\mathrm{ord}(Q_{[k]}) \in \mathbb{R}\backslash\mathbb{Z}$. Then working in local coordinates U_M on M where Q is represented by a smooth family of symbols $\sigma_Q(x\xi)$ find that the local matrix valued forms

$$\frac{1}{(2\pi)^n} \,⨍_{T_x^*M} \sigma_Q(x,\xi)\; d\xi$$

patch together to determine a global section of the bundle $\pi^*(\wedge T^*B) \otimes \mathrm{End}(\mathcal{E}) \otimes |\Lambda_\pi|$ over M; this is proved by an obvious fibrewise version of the usual existence proof of the Kontsevich Vishik canonical trace. Taking the fibrewise trace we consequently have an element

$$\mathrm{TR}_x(Q) := \frac{1}{(2\pi)^n} \,⨍_{T_x^*M} \mathrm{tr}_x(\sigma_Q(x,\xi))\; d\xi \in C^\infty(M, \pi^*(\wedge T^*B) \otimes |\Lambda_\pi|)$$

which can then be integrated over the fibres to define the canonical trace for families of non-integer order ΨDOs , a differential form on the parameter manifold B, by

$$\mathrm{TR}(Q) := \int_{M/B} \mathrm{TR}_x(Q) \;\in \mathcal{A}(B).$$

In the case when each component of Q has order less than $-n$, then $\mathrm{TR}(Q) = \mathrm{Tr}\,(Q)$, the usual fibrewise trace.

In a similar way, for a simple family $Q \in \mathcal{A}(B, \Psi(\mathcal{E}))$ of ΨDOs of any real (or complex) order one has a residue trace density

$$\mathrm{res}_x(Q) := \frac{1}{(2\pi)^n}\int_{S_x^*M} \mathrm{tr}_x(\sigma_Q(x,\xi)_{-n})\, dS(\xi) \in C^\infty(M, \pi^*(\wedge T^*B) \otimes |\Lambda_\pi|)$$

where $\sigma_Q(x,\xi)_{-n} = \sum_{k=0}^{\dim B} (\sigma_Q(x,\xi)_{-n})_{[k]}$ is the homogeneous part of the local symbol of homogeneity $-n$ and $dS(\xi)$ is the sphere measure. This can then be integrated over the fibres to define the residue trace for families of arbitrary order ΨDOs , once more defining a differential form on the parameter manifold B, by

$$\mathrm{res}(Q) := \int_{M/B} \mathrm{res}_x(Q) \;\in \mathcal{A}(B).$$

Notice that if all the components of Q have non-integer ΨDO order then $\mathrm{res}(Q)$ vanishes; as in the case of a single operator, the functionals TR and res are roughly complimentary.

As in the case of a single operator, $\mathrm{res} : \mathcal{A}(B, \Psi(\mathcal{E})) \longrightarrow \mathcal{A}(B)$ defines a trace, vanishing on (graded) brackets $[Q_1, Q_2]$ of families ΨDOs , while TR vanishes provided $[Q_1, Q_2]$ has non-integer order components.

Let us now extend the Wodzicki residue to forms on B with values in ΨDOs of logarithmic type.

Let $A \in \mathcal{A}(B, \Psi(\mathcal{E}))$ be a family of *differential operators*. Since by Lemma (2) the operator valued form $\log_\theta Q - \log_\theta Q_{[0]}$ is classical and since $\mathrm{res}_x(A \log Q_{[0]})dx$ defines a global top degree form on M, as A is a family of differential operators, by the results of [13], so does

$$\mathrm{res}_x(A \log Q) \ = \mathrm{res}_x(A(\log Q - \log Q_{[0]})) \ + \mathrm{res}_x(A \log Q_{[0]})\,.$$

Hence we may define the differential form, in this case, by

$$\mathrm{res}(A \log Q) = \int_{M/B} \mathrm{res}_x(A \log Q) \in \mathcal{A}(B).$$

The fact that we restrict to differential operators A ensures the independence of this extended residue on the choice of the metric on the fibre M_b since a change of metric brings in a vertical multiplication operator, which combined with the vertical differential operator A modifies the expression by another differential operator, for which the Wodzicki residue will vanish.

We comment that this is possibly taking place in a $\mathbb{Z}_2$-graded context, where the residue density is the super residue density and so forth.

4. Holomorphic families of ΨDO valued forms

We call a family $Q_z = \sum_{k=0}^{dimB} (Q_z)_{[k]} \in \mathcal{A}\left(B, C\ell(\mathcal{E})\right)$ parametrized by $z \in W \subset \mathbb{C}$ *holomorphic* if in each local trivialization of $\pi_*\mathcal{E}$ over a neighborhood U_B of b, $\left((Q_z)_{|_{U_B}}\right)_{[k]} = \omega_k \otimes Q_{k,z}$ for some $\omega_k \in \mathcal{A}(U_B)$ we have that $z \mapsto Q_{k,z} \in C\ell\left(M_b, \mathcal{E}_b\right)$ is holomorphic family of ΨDOs parametrised by W in the usual single operator sense [8, 9, 13]. In particular, the corresponding symbols $\mathbf{q}_z$ then define a holomorphic family of symbols in the usual sense.

Definition 2. *We call a holomorphic regularisation procedure a map* $\mathcal{R}$ *which to any* $A \in \mathcal{A}\left((B, C\ell(\mathcal{E})\right)$ *associates a holomorphic family* $A_z \in \mathcal{A}(B, C\ell(\mathcal{E}))$ *such that* $A_0 = A$ *and with order* $\alpha(z)$ *such that* $\alpha'_{[k]}(0) \neq 0$ *or any* $k \in \{0, \cdots, \mathrm{dim}B + 1\}$. *Similarly, one defines holomorphic regularisation procedures on the level of symbols in such a way that a regularisation procedure* $\mathcal{R} : A \mapsto A_z$ *induces one for the corresponding symbols.*

Let us illustrate these definitions with two examples.

Example 1. (1) *For any holomorphic map* H *such that* $H(0) = 1$, *the map* $\mathcal{R}^H : \mathbf{q} \mapsto H(z)\, |\xi|^{-z}\, \mathbf{q}$ *defines a holomorphic regularisation procedure on local classical vertical symbols. For a certain choice of* H *it gives back dimensional regularisation.*

(2) *Given a family* $Q \in \mathcal{A}\left(B, C\ell(\mathcal{E})\right)$ *of differential form valued vertical classical invertible elliptic* ΨDOs *with spectral cut* θ *such that* Q *moreover satisfies assumption (8), the map*

$$\mathcal{R}^Q : A \mapsto A\, Q_\theta^{-z}$$

is a holomorphic regularisation procedure on vertical classical ΨDOs *called* ζ*-regularisation.*

Theorem 1. (1) *For any family* $z \mapsto \mathbf{q}_z := \sum_{k=0}^{\mathrm{dimB}+1} \left(\mathbf{q}_z\right)_{[k]}(b, x, \xi)$ *of classical symbols locally parametrised by* $b \in B$ *and holomorphic on an open subset* $W \subset \mathbb{C}$ *with order* $z \mapsto \alpha(z) = (\alpha_{[0]}(z), \cdots, \alpha_{[\mathrm{dimB}+1]}(z))$ *such that* $z \mapsto \left(\alpha_{[k]}(b)\right)'(z)$ *does not vanish for any* k*, then the functions* $\fint_{T_x^* M_b} \left(\mathbf{q}_z\right)_{[k]}(b, x, \xi) d\xi$ *are meromorphic with simple poles in* $\left(\alpha_{[k]}(b)\right)^{-1}(\mathbb{Z}) \cap W$*. The pole of the map* $z \mapsto \fint_{T_x^* M_b} \left(\mathbf{q}_z\right)_{[k]}(b, x, \xi) d\xi$ *at a point* z_0 *in* $\left(\alpha_{[k]}(b)\right)(\mathbb{Z}) \cap W$ *is expressed in terms of a Wodzicki residue:*

$$\mathrm{Res}_{z=z_0} \fint_{T_x^* M_b} \left(\mathbf{q}_z\right)_{[k]}(b, x, \xi) d\xi = -\frac{1}{\left(\alpha_{[k]}(b)\right)'(z_0)} \mathrm{res}\left(\left(\mathbf{q}_{z_0}\right)_{[k]}(b) \right). \tag{16}$$

(2) *As a consequence, given a holomorphic family* $z \mapsto Q_z := \sum_{k=0}^{\mathrm{dimB}+1} \left(Q_z\right)_{[k]}$ *at point* b *on* $W \subset \mathbb{C}$ *of differential form valued vertical classical* ΨDOs *with holomorphic order* $z \mapsto \alpha(b) = (\alpha_{[0]}(b)(z), \cdots, \alpha_{[\mathrm{dimB}]}(b)(z))$ *such that* $z \mapsto \left(\alpha_{[k]}(b)\right)'(z)$ *does not vanish for any* $k \in \{0, \cdots, \mathrm{dim}B\}$*, the map* $z \mapsto \mathrm{TR}\left(\left(Q_z(b)\right)_{[k]} \right)$ *is meromorphic with simple poles in* $\left(\alpha_{[k]}(b)\right)^{-1}(\mathbb{Z}) \cap W$*. The pole of* $\mathrm{TR}\left(\left(Q_z(b)\right)_{[k]} \right)$ *at a point* z_0 *in* $\left(\alpha_{[k]}(b)\right)^{-1}(\mathbb{Z})$ *is expressed in terms of the Wodzicki residue of* $\left(\left(Q_{z_0}\right)_{[k]} \right)$ *at point* $b \in B$*:*

$$\mathrm{Res}_{z=z_0} \mathrm{TR}\left(\left(Q_z(b)\right)_{[k]} \right) = -\frac{1}{\left(\alpha_{[k]}(b)\right)'(z_0)} \mathrm{res}\left(\left(Q_{z_0}(b)\right)_{[k]} \right). \tag{17}$$

Proof. The similar result for ordinary classical ΨDOs [8] applied to each $\left(q_z(b)\right)_{[k]}$ and each $\left(Q_z(b)\right)_{[k]}$ yields the result. □

On the grounds of this theorem, the holomorphic regularisation $\mathcal{R}^Q : A \mapsto A\, Q_\theta^{-z}$ on differential form valued vertical classical ΨDOs gives rise to meromorphic maps

$$z \mapsto \zeta_\theta(A, Q, z) := \mathrm{TR}\left(A\, Q_\theta^{-z}\right)$$

with simple poles so that it makes sense to extract the finite part at $z = 0$. As a consequence of the above Theorem,we have as in the case of ordinary ΨDOs , the following formula relating the Wodzicki residue with the complex residue at $z = 0$

$$\mathrm{res}(A) = q_0\, \mathrm{Res}_{z=0}\mathrm{TR}\left(A\, Q^{-z}\right) = q_0\, \mathrm{Res}_{z=0}\mathrm{TR}\left(A\, Q_{[0]}^{-z}\right)$$

since $A\, Q_\theta^{-z}$ has order $\alpha_{[k]}(z) = -q_0\, z + \alpha_{[k]}(0)$ with q_0 the order of $Q_{[0]}$. Indeed, notice this formula is independent of the choice of Q apart from $q_0 = \mathrm{ord}(Q_{[0]}$.

Definition 3. *Let $Q \in \mathcal{A}\,(B, C\ell(\mathcal{E}))$ be a family of of differential form valued vertical classical invertible elliptic* ΨDOs *with spectral cut θ such that Q moreover satisfies assumption (8). Provided the dimension of the kernel* $\ker(Q(b)_{[0]})$ *is independent of b, the map $b \mapsto \left(\Pi_{Q(b)_{[0]}}\right)$ built from the orthogonal projection onto this kernel is smooth and for any $A \in \mathcal{A}\,(B, C\ell(\mathcal{E}))$,*

$$\begin{aligned} & \mathrm{tr}^Q(A)_{[k]} \\ := \;& \lim_{z\to 0}\left(\mathrm{TR}\left(A\,\left(Q + \pi_{Q_{[0]}}\right)_\theta^{-z}\right)_{[k]} - \frac{1}{z}\mathrm{Res}_{z=0}\mathrm{TR}\left(A\,\left(Q + \pi_{Q_{[0]}}\right)_\theta^{-z}\right)_{[k]}\right) \\ := \;& \zeta(A,\, Q, 0)|_{[k]}^{mer} + \mathrm{tr}\left(A_{[k]}\,\pi_{Q_{[0]}}\right), \end{aligned}$$

defines a differential form $\mathrm{tr}^Q(A)$ on B called the Q-weighted trace of A.

Let us compare these weighted traces to the finite part of heat-operator regularised traces.
When $Q_{[0]}$ has non negative leading symbol the operator $A\, e^{-\epsilon Q}$ is trace-class for positive ϵ and we can write (see e.g. [7]):

$$\begin{aligned} \mathrm{tr}\left(A\, e^{-\epsilon Q}\right) &= \sum_{n\ge 0}(-\epsilon)^n \int_{\Delta_n} du\, \mathrm{tr}\left(A\; e^{-u_0\epsilon Q_{[0]}}\, Q_{[>0]} \cdots e^{-u_{n-1}\epsilon Q_{[0]}} Q_{[>0]}\, e^{-u_n \epsilon Q_{[0]}}\right) \\ &= \sum_{n\ge 0}\sum_{|k|\ge 0} \frac{c(k)\, \epsilon^{|k|+2n-1}}{(|k|+n-1)!}\, \mathrm{tr}\left(A\, Q_{[>0]}^{(k_1)} \cdots Q_{[>0]}^{(k_n)}\, e^{-\epsilon Q_{[0]}}\right), \end{aligned}$$

where $c(k)$ is defined by induction for any mutiindex $k = (k_1, \cdots, k_n)$ by $c(k_1) = 1$ and

$$c(k_1, \cdots, k_n) = c(k_1, \cdots, k_{n-1})\frac{(k_1 + \cdots k_{n-1} + 1)\cdots(k_1 + \cdots k_{n-1} + n - 1)}{k_n!}.$$

For an operator B, the operator $B^{(i)}$ is also defined by induction; $B^{(0)} := B$ and for any non negative integer i, $B^{(i+1)} := [Q_{[0]}, B^{(i)}]$.
The sum over n is finite for each fixed form degree d whereas the sum over $k = (k_1, \cdots, k_n) \in \mathbb{N}^n$ is a priori infinite. However, if $Q_{[0]}$ is assumed to have *scalar leading symbol*, $B^{(k)}$ has order $b + k(q_0 - 1)$ where b is the order of B and q_0 the order of $Q_{[0]}$ and $\left(A\, Q_{[>0]}^{(k_1)} \cdots Q_{[>0]}^{(k_n)}\right)_{[d]}$

has order $a + nq_d + |k|(q_0 - 1)$ where q_d is the order of $Q_{[d]}$. It follows that for each fixed multiindex k, there are coefficients α_{j_k}, $j_k \geq 0$ and β_k such that

$$\begin{aligned}
&\epsilon^{|k|+2n-1} \operatorname{tr}\left(A\, Q_{[>0]}^{(k_1)} \cdots Q_{[>0]}^{(k_n)}\, e^{-\epsilon Q_{[0]}}\right) \\
\sim_{\epsilon \to 0} \quad & \sum_{j_k=0}^{\infty} \alpha_{j_k} \epsilon^{\frac{q_0|k|+q_0(2n-1)+j_k-(a+nq_d+|k|\,(q_0-1))-\dim M_b}{q_0}} + \beta_k \log \epsilon \\
\sim_{\epsilon \to 0} \quad & \sum_{j_k=0}^{\infty} \alpha_{j_k} \epsilon^{\frac{q_0(2n-1)+j_k-a-nq_d+|k|-\dim M_b}{q_0}} + \beta_k \log \epsilon
\end{aligned}$$

so that the fractional powers of ϵ increase with $|k|$; in the $\epsilon \to 0$ limit, they will not contribute for large enough $|k|$. Extracting a finite part when $\epsilon \to 0$, we can therefore define for any non negative integer d:

$$\begin{aligned}
&\mathrm{fp}_{\epsilon=0} \operatorname{tr}\left(A\, e^{-\epsilon Q}\right)_{[d]} \\
= \; & \sum_{n \geq 0} \mathrm{fp}_{\epsilon=0} \left[\sum_{|k| \geq 0} \frac{c(k)\, \epsilon^{|k|+2n-1}}{(|k|+n-1)!} \operatorname{tr}\left(A\, Q_{[>0]}^{(k_1)} \cdots Q_{[>0]}^{(k_n)}\, e^{-\epsilon Q_{[0]}}\right)_{[d]} \right].
\end{aligned}$$

Since for ordinary ΨDOs A, Q we have (see e.g. [12])

$$\mathrm{fp}_{\epsilon=0} \operatorname{tr}(A e^{-\epsilon Q}) = \operatorname{tr}^Q(A) + \gamma \operatorname{res}(A)$$

where γ is the Euler constant, weighted traces coincide with heat-kernel regularised traces for operators with vanishing residue, so this holds in particular for differential operators.

Applying this to each operator $\left(A\, Q_{[>0]}^{(k_1)} \cdots Q_{[>0]}^{(k_n)}\right)_{[d]}$ we get that provided $Q_{[d]}$ and $A_{[d]}$ are differential operators for any non negative integer d, then:

$$\mathrm{fp}_{\epsilon=0} \operatorname{tr}\left(A\, e^{-\epsilon Q}\right) = \operatorname{tr}^Q(A). \tag{18}$$

5. Weighted traces of differential operator valued forms; locality

A connection ∇ on $\mathcal{E} \otimes |\Lambda M_b|^{\frac{1}{2}}$ induces a connection ∇^{Hom} on $C\ell\,(\mathcal{E})$ which locally reads $\nabla^{\mathrm{Hom}} = d + [\Theta, \cdot]$ if ∇ reads $\nabla = d + \Theta$. Applying Theorem 1 to the holomorphic family $Q_z = A\,[\nabla, (Q_\theta + \pi_{Q_{[0]}})^{-z}]$ where $A \in \mathcal{A}(B, C\ell(\mathcal{E}))$ yields:

Theorem 2. *Let $Q \in \mathcal{A}\,(B, C\ell(\mathcal{E}))$ be a differential form on B with values in vertical classical elliptic* ΨDOs *with spectral cut θ and with kernel* $\operatorname{Ker} Q(b)_{[0]}$ *independent of b. Let $A \in \mathcal{A}(B, C\ell(\mathcal{E}))$. Given a connection ∇ on $\mathcal{E} \otimes |\Lambda M_b|^{\frac{1}{2}}$ then we have the equality of forms:*

$$d \operatorname{tr}^Q(A) = \operatorname{tr}^Q([\nabla, A]) + \frac{(-1)^{a+1}}{q_0} \operatorname{res}\left(A\,[\nabla, \log_\theta(Q + \pi_{Q_{[0]}})]\right)$$

where q_0 is the order of $Q_{[0]}$ and where a is the degree of A as a form.

Proof. For simplicity we assume Q is invertible, but the proof extends to the non invertible case replacing Q by $Q+\pi_{Q_{[0]}}$ in the complex powers. The proof goes as in [4] where Q was a ΨDO valued 0-form; indeed we have

$$\begin{aligned} d\,\mathrm{tr}^Q(A)-\mathrm{tr}^Q([\nabla,A]) &= \mathrm{fp}_{z=0}\left(d\mathrm{TR}(AQ_\theta^{-z})-\mathrm{TR}([\nabla,A]\,Q_\theta^{-z})\right)\\ &= (-1)^a\,\mathrm{fp}_{z=0}\mathrm{TR}\left(A\,[\nabla,Q_\theta^{-z}]\right)\\ &= (-1)^a\,\mathrm{Res}_{z=0}\left(\mathrm{TR}\left(\frac{A\,[\nabla,Q_\theta^{-z}]}{z}\right)\right)\\ &= \frac{(-1)^{a+1}}{q_0}\mathrm{res}\left(A\,[\nabla,\log_\theta Q]\right)\end{aligned}$$

where we have applied Theorem 1 to the holomorphic family $Q_z = A\,[\nabla,Q_\theta^{-z}]$ to get the last identity using the fact that that the degree k part of Q_θ^{-z} has order $-q_0\,z+cst$. □

Applying Theorem 1 to the holomorphic family $Q_z = A\,(Q_\theta+\pi_{Q_{[0]}})^{-z}$ where $A\in\mathcal{A}(B,C\ell(\mathcal{E}))$ is such that $A_{[i]}$ is a differential operator for any non negative integer i leads to the a description of the weighted trace of a differential operator valued differential form in terms of a Wodzicki residue.
In order to make these notes self-contained, we include the full proof for ΨDO valued forms although it mimicks the proof derived in [13] in the case of ordinary ΨDOs. As in [13] we use the following preliminary lemma.

Lemma 4. *Let $A\in\mathcal{A}(B,C\ell(\mathcal{E}))$ be a family of vertical* ΨDOs *such that $A_{[i]}(b)$ is a differential operator on M_b at any point $b\in B$ and for any non negative integer i. Then, for any x_inM_b, for any positive real number α*

$$z\mapsto -\!\!\!\!\!\!\int_{T_x^*M_b}|\xi|^{-\alpha\,z}\sigma_A(x,\xi)d\xi$$

is meromorphic with simple poles and if f.p.$_{z=0}$ *denotes its finite part at $z=0$ we have:*

$$0=-\!\!\!\!\!\!\int_{T_x^*M_b}\sigma_A(b,x,\xi)d\xi=\mathrm{f.p.}_{z=0}-\!\!\!\!\!\!\int_{T_x^*M_b}|\xi|^{-\alpha\,z}\sigma_A(b,x,\xi)d\xi.$$

Proof. The fact that $x\mapsto -\!\!\!\!\!\int_{T_x^*M_b}|\xi|^{-\alpha z}\sigma_A(b,x,\xi)d\xi$ defines a meromorphic function with simple poles follows from Theorem 1 applied to $\sigma_z(b,x,\xi)=|\xi|^{-\alpha\,z}\sigma_A(b,x,\xi)$ of order $\alpha(z)=-\alpha\,z+\alpha$ where α is the order of A. let us fix a non negative integer i. The symbol of the differential operator $A_{[i]}$ reads $\sigma_{A_{[i]}}(b,x,\xi)=\sum_{k=0}^{\mathrm{ord}A_{[i]}}\sigma_k(b,x,\xi)$ where for any multiindex $k=(k_1,\cdots,k_{\mathrm{dim}M_b})$, $\sigma_k(b,x,\xi)=a(b,x)\xi^k$ is positively homogeneous. Hence, its cut-off integral on the cotangent space at $x \in M_b$ reads (here $B^*_{b,x}(0,R)$ is the ball of radius R centered at 0 in $T^*_x M_b$):

$$\begin{aligned}
\fint_{T^*_x M_b} \sigma_{A_{[i]}}(b,x,\xi)d\xi &= \mathrm{f.p.}_{R\to\infty} \int_{B^*_{b,x}(0,R)} \sigma_{A_{[i]}}(b,x,\xi)d\xi \\
&= \sum_{k=0}^{\mathrm{ord}A_{[i]}} a_k(b,x)\mathrm{f.p.}_{R\to\infty} \int_{B^*_{b,x}(0,R)} \xi^k d\xi \\
&= \sum_{k=0}^{\mathrm{ord}A_{[i]}} a_k(b,x)\mathrm{f.p.}_{R\to\infty} \left(\int_0^R r^{k+n-1}dr\right) \int_{S^*_x M_b} \xi^k d\xi \\
&= \mathrm{f.p.}_{R\to\infty} \frac{R^{k+n}}{k+n} \int_{S^*_x M_b} \xi^k d\xi \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathrm{f.p.}_{z=0} \fint_{T^*_x M_b} \sigma_{A_{[i]}}(b,x,\xi)|\xi|^{-z}d\xi \\
= &\sum_{k=0}^{\mathrm{ord}A_{[i]}} a_k(b,x)\mathrm{f.p.}_{z=0} \fint_{T^*_x M_b} |\xi|^{-z}\xi^k d\xi \\
= &\sum_{k=0}^{\mathrm{ord}A_{[i]}} a_k(b,x)\mathrm{f.p.}_{z=0} \left(\mathrm{f.p.}_{R\to\infty} \int_0^R r^{k+n-z-1}dr\right) \int_{S^*_x M_b} \xi^k d\xi \\
= &\sum_{k=0}^{\mathrm{ord}A} a_k(b,x)\mathrm{f.p.}_{z=0} \left(\mathrm{f.p.}_{R\to\infty} R^{k+n-z}\right) \int_{S^*_x M_b} \xi^k d\xi \\
= &\ 0.
\end{aligned}$$

The fact that the finite part vanishes in the line before last follows from the fact that $\mathrm{f.p.}_{R\to\infty} R^{k+n-z}$ vanishes for $\mathrm{Re}(z)$ sufficiently small, as the finite part of a meromorphic extension of a function which vanishes on some half plane. □

We are now ready to prove the main result of this section:

Theorem 3. *Let* $Q \in \mathcal{A}(B, C\ell(\mathcal{E}))$ *be a differential form on* B *with values in vertical classical elliptic* ΨDOs *with spectral cut* θ *such that* Q *moreover satisfies assumption (8) and has kernel* $\mathrm{Ker}\, Q(b)_{[0]}$ *with constant dimension. Let* $A \in \mathcal{A}(B, C\ell(\mathcal{E}))$ *such that* $A_{[i]}$ *is a differential operator for any non negative integer* i *then we have the equality of forms:*

$$\mathrm{tr}^Q(A) = -\frac{1}{q_0}\mathrm{res}\left(A \log_\theta(Q + \pi_{Q_{[0]}})\right)$$

where q_0 *is the order of* $Q_{[0]}$ *and* $\pi_{Q_{[0]}}$ *the orthogonal projection onto the* $\mathrm{Ker}\, Q_{[0]}$.

Proof. Here again, we prove the result for invertible Q; the proof then extends to the non invertible case replacing Q by $Q + \pi_{Q_{[0]}}$ in the complex powers. Since $\left(Q_\theta^{-z}\right)_{[0]} = (Q_\theta)_{[0]}^{-z}$ has order $-q_0\, z$ with q_0 the order of $Q_{[0]}$, dropping the subscript θ to simplify notations, we write for any $b \in B$

$$\begin{aligned}
&\mathrm{tr}^Q(A)(b)\\
:=\ & \mathrm{f.p.}_{z=0} \int_{M_b} dx \fint_{T_x^* M_b} \mathrm{tr}_x\left(\sigma(A\, Q^{-z})(b,x,\xi)\right)\\
=\ & \mathrm{f.p.}_{z=0} \int_{M_b} dx \fint_{T_x^* M_b} d\xi\, \mathrm{tr}_x\left(\sigma(A\, Q^{-z})(b,x,\xi) - |\xi|^{-q_0 z}\sigma(A)(b,x,\xi)\right) +\\
+\ & \mathrm{f.p.}_{z=0} \int_{M_b} dx \fint_{T_x^* M_b} d\xi\, |\xi|^{-q_0\, z}\mathrm{tr}_x\sigma(A)(b,x,\xi)\\
=\ & \mathrm{f.p.}_{z=0} \int_{M_b} dx \fint_{T_x^* M_b} d\xi\, \mathrm{tr}_x\left(\sigma(A\, Q^{-z})(b,x,\xi) - |\xi|^{-q_0\, z}\sigma(A)(b,x,\xi)\right) \quad \text{by Lemma 4}\\
=\ & \mathrm{Res}_{z=0} \int_{M_b} dx \fint_{T_x^* M_b} d\xi\, \frac{\mathrm{tr}_x\left(\sigma(A\, Q^{-z})(b,x,\xi)\right) - |\xi|^{-q_0\, z}\mathrm{tr}_x\left(\sigma(A)(b,x,\xi)\right)}{z}.
\end{aligned}$$

Applying Theorem 1 to $\sigma_z(b,x,\xi) := \frac{\mathrm{tr}_x\sigma(A\, Q^{-z})(b,x,\xi) - |\xi|^{-q_0\, z}\mathrm{tr}_x\sigma(A)(b,x,\xi)}{z}$ then yields for any $d \in \{1, \cdots, \mathrm{dim}B\}$

$$\begin{aligned}
&\mathrm{tr}^Q(A)(b)_{[k]}\\
=\ & \mathrm{Res}_{z=0} \int_{M_b} dx \fint_{T_x^* M_b} d\xi\ \mathrm{tr}_x\sigma_z(b,x,\xi)_{[k]}\\
=\ & -\frac{1}{\left(\alpha_{[k]}(b)\right)'(0)} \left[\int_{M_b} dx \int_{S_x^* M_b} d\xi \left[\frac{\mathrm{tr}_x\sigma(A\, Q^{-z})(b,x,\xi) - |\xi|^{-q_0\, z}\mathrm{tr}_x\sigma(A)(b,x,\xi)}{z}\right]_{|z=0}\right]_{[k]}\\
=\ & \frac{1}{q_0}\left[\int_{M_b} dx \int_{S_x^* M_b} d\xi \frac{d}{dz}\left[\mathrm{tr}_x\sigma(A\, Q^{-z})(b,x,\xi) - |\xi|^{-q_0 z}\mathrm{tr}_x\sigma(A\, Q^{-z})(b,x,\xi)\right]_{|z=0}\right]_{[k]}\\
& \text{since} \quad \alpha_{[k]}(z) = \mathrm{ord}(\sigma_z)_{[k]} = -q_0\, z + \alpha_{[k]}(0)\\
& \text{and} \quad \text{since} \quad \left[\mathrm{tr}_x\sigma(A\, Q^{-z}) - |\xi|^{-q_0\, z}\mathrm{tr}_x\sigma(A)\right]_{|z=0} = 0\\
=\ & -\frac{1}{q_0}\int_{M_b}\int_{S_x^* M_b} d\xi \left[\mathrm{tr}_x\sigma(\log Q\, A)(b,x,\xi) - q_0 \log|\xi|\mathrm{tr}_x\sigma(A)(b,x,\xi)\right]_{[k]}\\
=\ & -\frac{1}{q_0}\left[\mathrm{res}\left(A\ \log Q\right)(b)\right]_{[k]}.
\end{aligned}$$

□

6. Chern-Weil forms associated with a superconnection

Definition 4. *A super connection* [18],[1], [3] *on* $\pi_*\mathcal{E}$ *adapted to a smooth family of formally self-adjoint elliptic* ΨDOs $P \in \mathcal{A}^0\left(B, C\ell^q(\mathcal{E})\right)$

with odd parity is a classical ΨDO $\mathbb{A}$ *on* $\mathcal{A}\left(B,\pi_*\mathcal{E}\right)$ *of odd-parity with respect to the* $\mathbb{Z}_2$*-grading such that:*

$$\mathbb{A}(\omega\cdot\sigma)=d\omega\wedge\sigma+(-1)^{|\omega|}\omega\wedge\mathbb{A}(\sigma)\quad\forall\omega\in\mathcal{A}(B),\sigma\in\mathcal{A}\left(B,\pi_*\mathcal{E}\right)$$

and

$$\mathbb{A}_{[0]}:=P$$

where as before, $\mathbb{A}=\sum_{i=0}^{\dim B}\mathbb{A}_{[i]}$ *and* $\mathbb{A}_{[i]}:\mathcal{A}^*\left(B,\pi_*\mathcal{E}\right)\mapsto\mathcal{A}^{*+i}\left(B,\pi_*\mathcal{E}\right)$.

The curvature of a super connection $\mathbb{A}$ is given by $\mathbb{A}^2\in\mathcal{A}\left(B,C\ell(\mathcal{E})\right)$. Notice that $\mathbb{A}_{[0]}^2=P^2$ so that $\mathbb{A}^2$ is elliptic with spectral cut π .We know from the previous paragraphs that provided Ker $\mathbb{A}^2(b)_{[0]}=$ Ker $P(b)$ is independent of b:

$$\zeta(\mathbb{A}^{2k},\,\mathbb{A}^2+\pi_P,z)\;\;:=\;\;\mathrm{TR}\left(\mathbb{A}^{2k}\left(\mathbb{A}^2+\pi_P\right)^{-z}\right)$$

$-\pi_P$ will denote the orthogonal projection onto the kernel of P–, is a ΨDO valued form in $\mathcal{A}\left(B,C\ell(\mathcal{E})\right)$ so that we can define its finite part:

$$\mathrm{tr}^{\,\mathbb{A}^2}(\mathbb{A}^{2k}):=\zeta(\mathbb{A}^{2k},\,\mathbb{A}^2+\pi_P,z)|_{z=0}^{mer}.$$

Theorem 4. *Let* $\mathbb{A}$ *be a super connection on* $\pi_*\mathcal{E}$ *adapted to a smooth family of formally self-adjoint elliptic* ΨDOs $P\in\mathcal{A}^0\left(B,C\ell^p\left(\mathcal{E}\right)\right)$ *of odd parity which satisfies assumption (8). Let us further assume that the kernel* Ker $\mathbb{A}^2(b)_{[0]}=$ Ker $P(b)$ *is independent of* b.
Then for any non negative integer k,

(1) *the associated Chern forms*

$$\mathrm{c}_k(\mathbb{A}):=\mathrm{tr}^{\,\mathbb{A}^2}(\mathbb{A}^{2k})$$

are closed forms on B *which are cohomologous in de Rham cohomology to* $\mathrm{tr}(\mathbb{A}^{2k}\pi_P)$.

(2) *The corresponding Chern-Weil classes are independent of the scaling of* $\mathbb{A}$ *with fixed kernel and we have the following transgression formula*

$$\partial_t\mathrm{c}_k(\mathbb{A}_t)=d\tau_k(\mathbb{A}_t)$$

where

$$\tau_k(\mathbb{A}_t)=k\,\mathrm{tr}^{\,\mathbb{A}_t^2}\left(\dot{\mathbb{A}}_t\,\mathbb{A}_t^{2(k-1)}\right)-\frac{1}{p}\,\mathrm{res}\left(\dot{\mathbb{A}}_t\left(\mathbb{A}_t^2+\pi_P\right)^{k-1}\right)$$

for any smooth one parameter family $\mathbb{A}_t$ *of superconnections associated with* P *of order* p.

(3) *If* P *has scalar leading symbol and if* $\mathbb{A}(b)$ *is a differential operator at each point* $b\in B$ *then the Chern-Weil classes relate to the Chern character by*

$$\mathrm{fp}_{t=0}\left(\partial_t^k\mathrm{tr}\left(e^{-t\,\mathbb{A}^2}\right)\right)=(-1)^k c_k(\mathbb{A}).$$

(4) *If* $\mathbb{A}(b)$ *is a differential operator at each point* $b \in B$ *then the associated Chern forms have a local description in terms of the Wodzicki residue:*

$$c_k(\mathbb{A}) = -\frac{1}{2p}\mathrm{res}\left(\mathbb{A}^{2k}\,\log(\mathbb{A}^2 + \pi_P)\right). \tag{19}$$

Moreover, τ_k *is also local and we have:*

$$\begin{aligned}&\tau_k(\mathbb{A}_t)\\ =\;& -\frac{k}{2p}\,\mathrm{res}\left(\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2 + \pi_P)^{k-1}\,\log(\mathbb{A}_t^2 + \pi_P)\right) - \frac{1}{p}\,\mathrm{res}\left(\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2 + \pi_P)^{k-1}\right).\end{aligned}$$

Proof. (1) Theorem 2 applied to $A = \mathbb{A}^{2k}$ and $Q = \mathbb{A}^2$ with $q_0 = 2p$ yields the closedness. Indeed, using the fact that $\nabla = \mathbb{A}$ commutes with the weight $\mathbb{A}^2$ in this case, we have:

$$\begin{aligned} d\,\mathrm{tr}^{\mathbb{A}^2}(\mathbb{A}^{2k}) &= \mathrm{tr}^{\mathbb{A}^2}([\mathbb{A}, \mathbb{A}^2]) - \frac{1}{2p}\mathrm{res}\left(\mathbb{A}^{2k}\,[\mathbb{A}^{2k}, \log(\mathbb{A}^2 + \pi_P)]\right)\\ &= -\frac{1}{2p}\mathrm{res}\left(\mathbb{A}^{2k}\,[\mathbb{A}^2, \log(\mathbb{A}^2 + \pi_P]\right)\\ &= -\frac{1}{2p}\mathrm{res}\left(\mathbb{A}^{2k}\,[\mathbb{A}^2 + \pi_P, \log(\mathbb{A}^2 + \pi_P)]\right)\\ &= 0\end{aligned}$$

where we have used the fact that $\mathrm{res}\left(P\,[\mathbb{A}^2, \log(\mathbb{A}^2 + \pi_P)]\right) = 0$ since P has finite rank combined with the fact that $\mathbb{A}^2 + \pi_P$ commutes with $\log(\mathbb{A}^2 + \pi_P)$.
Furthermore, fom [15] we know that $\zeta(\mathbb{A}^2, -k) := \zeta(\mathbb{A}^{2k}, \mathbb{A}^2, z)|_{z=0}^{mer}$ is exact, so that $\mathrm{tr}^{\mathbb{A}^2}(\mathbb{A}^{2k}) = \zeta(\mathbb{A}^{2k}, \mathbb{A}^2, z)|_{z=0}^{mer} + \mathrm{tr}(\mathbb{A}^{2k}\,\pi_P)$ is cohomologous to $\mathrm{tr}(\mathbb{A}^{2k}\,\pi_P)$.

(2) Applying Theorem 2 to $\nabla = \partial_t$, $A = \mathbb{A}_t^{2k}$, $Q := \mathbb{A}_t^2$ with $\mathbb{A}_t$ a smooth family of superconnections parametrised by $\mathbb{R}$ associated with a family P_t with constant kernel and corresponding

projection π_P we get

$$\begin{aligned}
\partial_t c_k(\mathbb{A}_t) &= \operatorname{tr}^{\mathbb{A}_t^2}\left(\partial_t\, \mathbb{A}_t^{2k}\right) \\
&- \frac{1}{2p}\operatorname{res}\left(\mathbb{A}_t^{2k}\,\partial_t\,\log(\mathbb{A}_t^2+\pi_P)\right) \\
&= \sum_{i=1}^{k}\operatorname{tr}^{\mathbb{A}_t^2}\left(\mathbb{A}_t^{2(i-1)}\,[\mathbb{A}_t,\,\dot{\mathbb{A}}_t]\,\mathbb{A}_t^{2(k-i)}\right) \\
&- \frac{1}{p}\int_0^1 \operatorname{res}\left(\mathbb{A}_t^{2k}\,(\mathbb{A}_t^2+\pi_P)^{-1-\lambda}\,[\mathbb{A}_t,\,\dot{\mathbb{A}}_t]\,(\mathbb{A}_t^2+\pi_P)^{\lambda}\right)d\lambda \\
&= \sum_{i=1}^{k}\operatorname{tr}^{\mathbb{A}_t^2}\left(\mathbb{A}_t^{2(i-1)}\,[\mathbb{A}_t,\,\dot{\mathbb{A}}_t]\,\mathbb{A}_t^{2(k-i)}\right) \\
&- \frac{1}{p}\int_0^1 \operatorname{res}\left((\mathbb{A}_t^2+\pi_P)^{k}\,(\mathbb{A}_t^2+\pi_P)^{-1-\lambda}\,[\mathbb{A}_t,\,\dot{\mathbb{A}}_t]\,(\mathbb{A}_t^2+\pi_P)^{\lambda}\right)d\lambda \\
&= k\operatorname{tr}^{\mathbb{A}_t^2}\left(\left[\mathbb{A}_t,\,\dot{\mathbb{A}}_t\;\mathbb{A}_t^{2(k-1)}\right]\right) \\
&- \frac{1}{p}\operatorname{res}\left(\left[\mathbb{A}_t,\,\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2+\pi_P)^{k}\,(\mathbb{A}_t^2+\pi_P)^{-1}\right]\right) \\
&= k\,d\operatorname{tr}^{\mathbb{A}_t^2}\left(\dot{\mathbb{A}}_t\;\mathbb{A}_t^{2(k-1)}\right) \\
&- \frac{1}{p}\,d\operatorname{res}\left(\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2+\pi_P)^{k-1}\right)
\end{aligned}$$

where we have used the fact that $\partial_t\,\mathbb{A}_t^2=[\mathbb{A}_t,\,\dot{\mathbb{A}}_t]$ as well as the cyclicity of the Wodzicki residue combined with the fact that it vanishes on finite rank operators (which we used to replace $\mathbb{A}_t^2$ by $\mathbb{A}_t^2+\pi_P$ in the third equality).

(3) First of all, a Volterra type formula for $\partial_t e^{-\mathbb{A}_t^2}$ with $\mathbb{A}_t := \sqrt{t}\,\mathbb{A}$ combined with the cyclicity of the trace yields:

$$\partial_t^k \operatorname{ch}(t\,\mathbb{A}) = (-1)^k \operatorname{tr}\left(\mathbb{A}^{2k}\,e^{-t\,\mathbb{A}^2}\right).$$

Since weighted traces coincide with the ordinary trace on trace-class operators and since the operator valued form $e^{-\mathbb{A}^2}$ is trace-class as a consequence of the ellipticity of the self-adjoint operator P, for any $t>0$ we have

$$\begin{aligned}
\operatorname{fp}_{t=0}\partial_t^k \operatorname{tr}\left(e^{-t\,\mathbb{A}^2}\right) &= (-1)^k\,\operatorname{fp}_{t=0}\operatorname{tr}\left(\mathbb{A}^{2k}\,e^{-t\,\mathbb{A}^2}\right) \\
&= (-1)^k \operatorname{tr}^{\mathbb{A}^2}\left(\mathbb{A}^{2k}\right) \\
&= (-1)^k c_k(\mathbb{A}).
\end{aligned}$$

Here we have used the fact that the leading symbol of P is scalar to make sense of the heat-kernel regularised trace $\operatorname{fp}_{t=0}\operatorname{tr}\left(\mathbb{A}^{2k}\,e^{-t\,\mathbb{A}^2}\right)$

and formula (18) to identify the weighted trace with the heat-kernel regularised trace since, by assumption, all the operators involved are differential operators.

(4) Applying Theorem 3 to $A = \mathbb{A}^{2k}$, $Q := \mathbb{A}^2$ then yields the local formula for Chern forms anounced in the last part of the theorem. In that case, τ_k is also local and we have

$$\begin{aligned}\tau_k(\mathbb{A}_t) &= -\frac{k}{2p}\,\mathrm{res}\left(\dot{\mathbb{A}}_t\,\mathbb{A}_t^{2(k-1)}\,\log(\mathbb{A}_t^2+\pi_P)\right)\\ &- \frac{1}{p}\,\mathrm{res}\left(\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2+\pi_P)^{k-1}\right)\\ &= -\frac{k}{2p}\,\mathrm{res}\left(\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2+\pi_p)^{k-1}\,\log(\mathbb{A}_t^2+\pi_P)\right)\\ &- \frac{1}{p}\,\mathrm{res}\left(\dot{\mathbb{A}}_t\,(\mathbb{A}_t^2+\pi_P)^{k-1}\right)\end{aligned}$$

using here again, the fact that the Wodzicki residue vanishes on finite rank operators in order to replace $\mathbb{A}_t^2$ by $\mathbb{A}_t^2+\pi_P$. □

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