THE QUILLEN DETERMINANT

1. Determinants in Finite Dimensions

The determinant of a linear transformation $A : V \longrightarrow W$ acting between finite-dimensional complex vector spaces is an element $\det A$ of a complex line L_A. The abstract element $\det A$ is called the *Quillen determinant* of A, and the complex line L_A is called the *determinant line* of A. A choice of (linear) isomorphism

$$\phi : L_A \longrightarrow \mathbb{C} \tag{1}$$

associates to $\det A$ the complex number

$$\det{}_{\phi} A := \phi(\det A) \in \mathbb{C} \ , \tag{2}$$

which can equivalently be written as the ratio

$$\det{}_{\phi} A = \frac{\det A}{\phi^{-1}(1)} \ , \tag{3}$$

taken in the 1-dimensional complex vector space L_A relative to the canonical generator $\phi^{-1}(1)$. It is not necessarily the case that $\det A$ determines a generator for L_A; specifically, if $\dim V = m$ and $\dim W = n$ then $\det A = 0$ if $m \neq n$ (by *fiat*), while if $m = n$ then $\det A = 0$ precisely when A is not invertible. For the moment, set $m = n$.

For $k \in \{0, 1, \ldots, n\}$ the k^{th} exterior power operator is defined by

$$\wedge^k A : \wedge^k V \longrightarrow \wedge^k W \ , \qquad \wedge^k A\,(v_1 \wedge v_2 \wedge \ldots \wedge v_k) := Av_1 \wedge Av_2 \wedge \ldots \wedge Av_k \ , \tag{4}$$

where $v_1, \ldots v_k \in V$ and $\wedge^0 V := \mathbb{C}$ and $\wedge^0 A := 1$. When $k = n$, $\mathrm{Det} V := \wedge^n V$ and $\mathrm{Det} W := \wedge^n W$ are complex lines and the determinant line of A is

$$L_A := \mathrm{Det} V^* \otimes \mathrm{Det} W \ , \tag{5}$$

while for any basis $\{e_1, \ldots, e_n\}$ for V, with dual basis $\{e_1^*, \ldots, e_n^*\}$ for V^*,

$$\det A := e_1^* \wedge \ldots \wedge e_n^* \otimes \ (\wedge^n A)(e_1 \wedge \ldots \wedge e_n) \in L_A \ . \tag{6}$$

There is a canonical isomorphism for $A \in \mathrm{Hom}(V, W)$, $B \in \mathrm{Hom}(U, V)$

$$L_{AB} \cong L_A \otimes L_B \tag{7}$$

coming from the isomorphism

$$\mathrm{Det} V^* \otimes \mathrm{Det} V \longrightarrow \mathbb{C} \tag{8}$$

defined by the canonical pairing $\mathrm{Det} V^* \times \mathrm{Det} V \rightarrow \mathbb{C}$, and this preserves the determinant elements

$$\det(AB) \longleftrightarrow \det A \otimes \det B \ . \tag{9}$$

1.1. **The Classical Determinant.** When $V = W$ these constructions take on a more familiar form. Then ϕ can be chosen to be the canonical isomorphism (8) and evaluation on $\det A \in L_A$ outputs the classical determinant

$$\det{}_{\mathbb{C}} A = \sum_{\sigma} (-1)^{\sigma} a_{1,\sigma(1)} \dots a_{n,\sigma(n)} \ , \tag{10}$$

where the sum is over permutations of $\{1, \dots, n\}$ and $(a_{i,j})$ is the matrix of A with respect to any basis of V — changing the basis may change the summands on the right-side of (10), but not their sum. It is fundamental that when $V = W$ the classical determinant is an intrinsic invariant of the operator A, independent of the choice of basis for V; when $V \neq W$ that is no longer so since there is then no *canonical* bilinear pairing $\mathrm{Det}V^* \times \mathrm{Det}W \longrightarrow \mathbb{C}$, the choice of a non-degenerate pairing is equivalent to a choice of ϕ in (1).

The identification of (10) from (6) and (8) amounts to the identity in $\mathrm{Det}V$

$$(\wedge^n A)(e_1 \wedge \dots \wedge e_n) = \det{}_{\mathbb{C}} A \, . \, e_1 \wedge \dots \wedge e_n \ . \tag{11}$$

Since $\wedge^n(AB) = \wedge^n A \circ \wedge^n B$, (11) in turn implies the characterizing multiplicativity property of the classical determinant

$$\det{}_{\mathbb{C}}(AB) = \det{}_{\mathbb{C}} A \, . \, \det{}_{\mathbb{C}} B \tag{12}$$

for $A, B \in \mathrm{End}(V)$, specializing the general fact in (7). Similarly, the group $\mathrm{Gl}(V, \mathbb{C})$ of invertible elements of $\mathrm{End}(V)$ is identified with those A with $\det_{\mathbb{C}} A \neq 0$.

The classical determinant can also be thought of in the following ways. First, the direct sum of the operators defined in (4) yields the total exterior power operator $\wedge A : \wedge V \longrightarrow \wedge V$ on the exterior algebra $\wedge V = \oplus_{k=0}^{n} \wedge^k V$ and this has trace

$$\mathrm{tr}\,(\wedge A) = \det{}_{\mathbb{C}}(I + A), \tag{13}$$

where I is the identity. Alternatively, one can do something a little more sophisticated and use the holomorphic functional calculus to define the logarithm $\log_\theta B$ of $B \in \mathrm{End}(V)$ by

$$\log_\theta B = \frac{i}{2\pi} \int_{\Gamma_\theta} \log_\theta \lambda \ (B - \lambda I)^{-1} \, d\lambda \ . \tag{14}$$

Here $\log_\theta \lambda$ is the branch of the complex logarithm defined by $\theta - 2\pi \leq \arg(\lambda) \leq \theta$ and Γ_θ is a positively oriented contour enclosing $\mathrm{spec}(B)$ but not any point of the spectral cut $R_\theta = \{re^{i\theta} \mid r \geq 0\}$. Then *if* B is invertible

$$\mathrm{tr}\,(\log_\theta B) = \log_\theta \det{}_{\mathbb{C}} B \ . \tag{15}$$

2. THE FREDHOLM DETERMINANT

The advantage of the constructions (13), (15) is that they extend to a restricted class of bounded linear operators on infinite dimensional Hilbert spaces. This is consequent on the fact that both of the formulas (13), (15) are computed as operator traces.

(Recall that a *trace* on a Banach algebra $\mathcal{B}$ is a linear functional $\tau : \mathcal{B} \to \mathbb{C}$ which has the property $\tau([a,b]) = 0$ for all a, b in $\mathcal{B}$, where $[a,b] := ab - ba$ is defined by the product structure on $\mathcal{B}$. Since one can define the logarithm $\log_\theta b$ of an element b of $\mathcal{B}$ with spectral cut R_θ by the formula (14), one in this case obtains a determinant $\det_{\tau,\theta}(b)$ on such elements by setting

$$\log_\theta \det{}_{\tau,\theta}(b) = \tau(\log_\theta b) \ . \tag{16}$$

If $a, b, ab \in \mathcal{B}$ have common spectral cuts θ, the traciality property (17) translates into the multiplicativity property $\det_{\tau,\theta}(ab) = \det_{\tau,\theta}(a) \det_{\tau,\theta}(b)$ via a version of the Campbell-Hausdorff formula.)

The *operator trace* arises as follows. Let H be a complex separable Hilbert space with inner-product $<,>$, let $\mathcal{C}(H)$ be the algebra of compact operators on H, and let

$$L_1 = \{A \in \mathcal{C}(H) \mid \ \|A\|_1^2 := \sum_{i=1}^{\infty} \mu_i(A^*A) < \infty\} \tag{17}$$

be the ideal of *trace class* operators, where the sum is over the real discrete eigenvalues $\mu_i(A^*A) \searrow +0$ of the compact self-adjoint operator A^*A. For any orthonormal basis $\{\eta_j\}$ of H the map

$$\mathrm{Tr} \, : L_1 \longrightarrow \mathbb{C} \ , \qquad A \longmapsto \mathrm{Tr}\,(A) := \sum_j < \eta_j, A\eta_j >$$

is a trace functional on $L_1(H)$, independent of the choice of basis. Lidskii's Theorem states that

$$\mathrm{Tr}\,(A) = \sum_{\lambda \in \mathrm{spec}(A)} \lambda \tag{18}$$

with the sum over the eigenvalues of A counted up to algebraic multiplicity; for general trace class operators this equality is highly non-trivial.

If A is trace class, then for each non-negative integer k so is each of the exterior power operators $\wedge^k A : \wedge^k H \longrightarrow \wedge^k H$, defined as in (4). Following (13), a determinant can therefore be defined on the semi-group $I + L_1 := \{I + A \mid A \in L_1\}$ of *determinant class* operators by the absolutely convergent sum

$$\mathrm{det_F(I + A) := Tr\,(\wedge A) = 1 + \sum_{k=1}^{\infty} Tr\,(\wedge^k A)} \ . \tag{19}$$

On the other hand, since Tr is tracial and $\log_\pi(I+A)$ defined by (14) is trace class, then according to (16) there is a determinant given on *invertible* determinant class operators by

$$\log_\pi \mathrm{det_F}(\mathrm{I}+\mathrm{A}) = \mathrm{Tr}\left(\log_\pi(\mathrm{I}+\mathrm{A})\right) , \tag{20}$$

which, as the left side already suggests, coincides with the Fredholm determinant.

The Fredholm determinant retains the characterizing properties of the classical determinant in finite dimensions, that $\mathrm{det_F} : \mathrm{I}+\mathrm{L}_1 \longrightarrow \mathbb{C}$ is multiplicative

$$\mathrm{det_F}\left((\mathrm{I}+\mathrm{A})(\mathrm{I}+\mathrm{B})\right) = \mathrm{det_F}(\mathrm{I}{+}\mathrm{A})\,\mathrm{det_F}(\mathrm{I}{+}\mathrm{B}) , \qquad \mathrm{A},\mathrm{B}\in\mathrm{L}_1 , \tag{21}$$

and $\mathrm{det_F}(\mathrm{I}+\mathrm{A}) \neq 0$ if and only if $I+A$ is invertible. It is, moreover, essentially unique; any other multiplicative functional on $I+L_1$ is equal to some power of the Fredholm determinant, or, equivalently, any trace on L_1 is a constant multiple of the operator trace. These properties (traciality of the operator trace, multiplicativity of the Fredholm determinant) do not, however, persist to any functional extension of the operator trace (resp. Fredholm determinant) on pseudodifferential operators acting on function spaces (fields over spacetime). In quantum physics this is a primary cause of *anomalies.* More precisely, determinants of differential operators arise in quantum field theories and string theory through the formal evaluation of their defining Feynman path integrals and the calculation of certain stable quantum numbers, which are in some sense ‘topological’.

From the latter perspective it instructive to be aware also of the following, third, construction of the Fredholm determinant, which equates the existence of a non-trivial determinant to the existence of non-trivial topology of the general linear group. First, in a surprising contrast to $\mathrm{Gl}(n,\mathbb{C})$, the general linear group $\mathrm{Gl}(H)$ of an infinite-dimensional Hilbert space H with the norm topology is contractible, and hence topologically trivial. By transgression properties in cohomology this implies any vector bundle with structure group $\mathrm{Gl}(H)$ is isomorphic to the trivial bundle. In order to recapture some topology (and hence, in applications, some physics) it is necessary to reduce to certain infinite-dimensional subgroups of $\mathrm{Gl}(H)$. The most obvious one is the group $\mathrm{Gl}(\infty)$ of of invertible operators differing from the identity by an operator of finite-rank. As the inductive limit of the $\mathrm{Gl}(n,\mathbb{C})$, the cohomology and homotopy groups of $\mathrm{Gl}(\infty)$ are a stable version of those of $\mathrm{Gl}(n,\mathbb{C})$. Precisely, $\mathrm{Gl}(\infty)$ is torsion free and its cohomology ring is an exterior algebra with odd degree generators, while Bott periodicity [B] identifies $\pi_k(\mathrm{Gl}(\infty))$ to be isomorphic to $\mathbb{Z}$ if k is odd and trivial if k is even. Topologically it is preferable to consider the closure of $\mathrm{Gl}(\infty)$ in $\mathrm{Gl}(H)$, which yields the group $\mathrm{Gl}_{\mathrm{cpt}}(H)$ of operators differing from the identity by a compact operator, but this is now a little ‘too large’ for analysis and differential geometry. Given our earlier comments, there is an intermediate natural choice of the Banach Lie group $\mathrm{Gl}_1(H)$ of operators differing from the identity by a trace-class operator (in fact, there is a tower of such Schatten class groups). Moreover, the inclusions $\mathrm{Gl}(\infty) \subset \mathrm{Gl}_1(H) \subset \mathrm{Gl}_{\mathrm{cpt}}(H)$ are homotopy equivalences, and so the cohomology of $\mathrm{Gl}_1(H)$ is just the exterior algebra mentioned above

$$H^*(\mathrm{Gl}_1(H)) = \wedge(\omega_1, \omega_3, \omega_5, \ldots) \ , \qquad \mathrm{deg}\omega_j = 2j-1 \ . \tag{22}$$

The advantage of considering $\mathrm{Gl}_1(H)$ is that precise analytical representatives for the classes ω_j can be written down

$$\omega_j = \left(\frac{i}{2\pi}\right)^j \frac{(j-1)!}{(2j-1)!} \, \mathrm{Tr}\,(\Theta^{2j-1}) \ ,$$

where Θ is the 1-form on $\mathrm{Gl}_1(H)$

$$\Theta = \mathrm{Tr}\,(Z^{-1}dZ) \ . \tag{23}$$

This equation makes sense because the derivative dZ is trace class, and hence so is $Z^{-1}dZ$. Now, locally $\Theta = d\log\mathrm{det}_{\mathrm{F}}(\mathrm{Z})$, so that the 1-form ω_1 pulled back by a path $\sigma : S^1 \to \mathrm{Gl}_1(H)$ is precisely the winding number of the curve traced out in $\mathbb{C}^*$ by the function $\mathrm{det}_{\mathrm{F}}(\sigma)$. In fact, this is a just a special case of the Bott periodicity theorem, which tells us that the stable homotopy group $\pi_{2j-1}(\mathrm{Gl}_1(H))$ is isomorphic to $\mathbb{Z}$ and an isomorphism is defined by assigning to a map $f : S^{2j-1} \to \mathrm{Gl}_1(H)$ the integer $\int_{S^{2j-1}} f^*\omega_j \in \mathbb{Z}$ (it is not obvious *a priori* that it is an integer).

Notice that it was not necessary to mentioned the Fredholm determinant of Z at this point. Indeed, the third definition of the Fredholm determinant is to see it as the integral of the 1-form Θ, define

$$\log_\pi \mathrm{det}_{\mathrm{F}}(\mathrm{I}+\mathrm{A}) := \int_\gamma \Theta \ , \tag{24}$$

where $\gamma : [0,1] \to \mathrm{Gl}_1(H)$ is any path with $\gamma(0) = I$ and $\gamma(1) = I + A$; this uses the connectedness of $\mathrm{Gl}_1(H)$ and independence of the choice of γ, as guaranteed by Bott periodicity.

Interestingly, this is closely tied in with the Atiyah-Singer index theorem for elliptic pseudodifferential operators (which in full generality uses the Bott periodicity theorem). Here, there is the following simple but quintessential version of that theorem which links it to the winding number of the determinant of the symbol of a differential operator

$$D = \sum_{\alpha| \leq m} a_\alpha(x) D_x^\alpha \tag{25}$$

on Euclidean space $\mathbb{R}^n$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multi-index of non-negative integers, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $D_x = i\partial/\partial x_i$. Here D acts on $C^\infty(\mathbb{R}^n, V)$ with V a finite-dimensional complex vector space and the coefficients of D are matrices varying smoothly with x which are required to decay suitably fast, $|D_x^\beta a_\alpha(x)| = O(|x|^{-|\beta|})$ as $|x| \to \infty$. If the symbol σ_D of D, defined by

$$\sigma_D(x,\xi) = \sum_{\alpha| \leq m} a_\alpha(x)\xi^\alpha \tag{26}$$

with $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, satisfies the ellipticity condition of being invertible on the $2n-1$ sphere S^{2n-1} in (x,ξ) space, then D is a Fredholm operator. The index theorem then states

$$\mathrm{index}\,(D) = \int_{S^{2j-1}} \sigma_D^*(\omega_j) \ ,$$

the higher dimensional analogue of the winding number of the determinant.

3. Fredholm operators and Determinant line bundles

The operators whose determinants are considered in this article are all Fredholm operators; this includes most operators of interest that arise in mathematics and physics. Recall that a linear operator $A : H_1 \to H_2$ between Hilbert spaces is *Fredholm* if it is invertible modulo compact operators; that is, there is a 'parametrix' $Q : H_2 \to H_1$ such that $QA - I$ and $AQ - I$ are compact operators on H_1 and H_2 respectively. Equivalently, the range $A(H_1)$ of A is closed in H_2, and the kernel $\mathrm{Ker}(A) = \{\eta \in H_1 \mid A\eta = 0\}$ and cokernel $\mathrm{Coker}(A) = H_2/A(H_1)$ of A are finite-dimensional. (This is equally true for Banach and Frechet spaces, we restrict attention to Hilbert spaces for brevity.) The space Fred of all such Fredholm operators with the norm topology has the homotopy type of the classifying space $\mathbb{Z} \times B\mathrm{Gl}(\infty)$. The first factor parametrizes the connected components of Fred, two Fredholm operators are in the same component if and only if they have the same index

$$\mathrm{index}\,(A) = \dim \mathrm{Ker}(A) - \dim \mathrm{Coker}(A) \ .$$

Mostly we restrict attention to the connected component Fred_0 of operators of index zero. The cohomology of $\mathsf{Fred}_0 \sim B\mathrm{Gl}(\infty)$ is a polynomial ring

$$H^*(\mathsf{Fred}_0) = \mathbb{R}[\mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3, \ldots]$$

whose generators may be formally realized as the even degree components of the Chern character of an infinite-dimensional bundle over Fred_0. In fact, the generators ω_{2j-1} of $H^*(\mathrm{Gl}_1(H))$ are related to the ch_j through transgression, see [CS]. We shall be interested here in the first generator ch_1, a transgression of the Fredholm determinant 'winding number 1-form' ω_1, which coincides with the real Chern class of a canonical complex line bundle $\mathrm{DET}_0 \to \mathsf{Fred}_0$. The fibre of DET_0 at $A \in \mathsf{Fred}_0$ is the determinant line $\mathrm{Det}(A)$ of the Fredholm operator A, which is defined as follows [S].

Just as for finite rank operators (Sect. 1.1), the determinant of a Fredholm operator $A : H^1 \rightarrow H^2$ exists abstractly not as a number but as an element $\det A$ of a complex line $\mathrm{Det}(A)$. For simplicity we suppose that $\mathrm{index}\,(A) = 0$. Elements of the *determinant line* $\mathrm{Det}(A)$ are equivalence classes $[E, \lambda]$ of pairs (E, λ), where $E : H^1 \rightarrow H^2$ such that $A - E$ is trace class and relative to the equivalence relation $(Eq, \lambda) \sim (E, \det{}_F(q)\lambda)$ for $q : H^1 \rightarrow H^1$ of determinant class and where $\det{}_F(q)$ is the Fredholm determinant of q. Complex multiplication on $\mathrm{Det}(A)$ is defined by $\mu\,[A, \lambda] = [A, \mu\lambda]$. The abstract, or Quillen, determinant of A is the preferred element $\det A := [A, 1]$ in $\mathrm{Det}(A)$.

Here are some essential properties of the determinant line. First, $\det A$ is non-zero if and only if A is invertible. Next, quotients of abstract determinants in $\mathrm{Det}(A)$ are given by Fredholm determinants; for if $A_1 : H^1 \rightarrow H^2, A_2 : H^1 \rightarrow H^2$ are Fredholm operators such that $A_i - A$ are trace class, then if A_2 is invertible we see that $A_2^{-1}A_1$ is determinant class and hence from the definition that

$$\frac{\det(A_1)}{\det(A_2)} = \det{}_F(A_2^{-1}A_1)\ , \tag{27}$$

where the quotient on the left side is taken in $\mathrm{Det}(A)$. The principal functorial property of the determinant line is that given a commutative diagram with exact rows and Fredholm columns

$$\begin{tikzcd}
0 \arrow[r] & H_1 \arrow[r] \arrow[d, "A"] & H_1^{'} \arrow[r] \arrow[d, "A^{'}"] & H_1^{''} \arrow[r] \arrow[d, "A^{''}"] & 0 \\
0 \arrow[r] & H_2 \arrow[r] & H_2^{'} \arrow[r] & H_2^{''} \arrow[r] & 0
\end{tikzcd} \tag{28}$$

then there is canonical isomorphism of complex lines

$$\mathrm{Det}(A^{'}) \cong \mathrm{Det}(A) \otimes \mathrm{Det}(A^{''}) \tag{29}$$

preserving the Quillen determinants $\det\,(A^{'}) \leftrightarrow \det\,(A) \otimes \det\,(A^{''})$. A consequence of this property is that given Fredholm operators $A : H_2 \rightarrow H_3$ and $B : H_1 \rightarrow H_2$ then

$$\mathrm{Det}(AB) \cong \mathrm{Det}(A) \otimes \mathrm{Det}(B)$$

with $\det\,(AB) \leftrightarrow \det\,(A) \otimes \det\,(B)$, generalizing the elementary property (9).

The principal context of interest for studying determinant lines is the case where one has a family $\mathcal{A} = \{A_x \mid x \in B\}$ of Fredholm operators parameterized by a manifold B, satisfying suitable continuity properties, and one aims to make sense of the determinant as a function $\mathcal{A} \rightarrow \mathbb{C}$. It is then of no difficulty to show that the corresponding family of determinant lines $\mathrm{DET}(\mathcal{A}) = \cup \mathrm{Det}(A_x)$ define a complex line bundle over B endowed with a canonical section $\det : B \longrightarrow \mathrm{DET}(\mathcal{A})$ assigning to $x \in B$ the Quillen determinant $\det\,(A_x) \in \mathrm{Det}(A_x)$ [Q, S].

To identify the Quillen determinant section with a function on $\mathcal{A}$ we need to identify a trivialization of the line bundle $\mathrm{DET}(\mathcal{A})$, giving a global basis for the fibres. This is the same thing as giving a non(or never)-vanishing section $\psi : B \to \mathrm{DET}(\mathcal{A})$, with respect to which we have the regularized determinant function (cf. (3))

$$x \longmapsto \det{}_{\psi}(A_x) := \frac{\det(A_x)}{\psi(x)} \ . \tag{30}$$

If $\mathcal{A}$ is trivializable, so a non-zero section exists, there will be many such sections and some extra data is needed to fix a natural choice of ψ.

Each of the properties mentioned above for determinant lines carries forward to determinant line bundles in a natural way. In particular, one easily deduces from (28), or from the exact sequence

$$0 \longrightarrow \mathrm{Ker}A_x \longrightarrow H_{1,x} \xrightarrow{A_x} H_{2,x} \longrightarrow \mathrm{Coker}A_x \longrightarrow 0$$

that if the kernels $\mathrm{Ker}A_x$ have constant dimension as x varies then there is a canonical isomorphism

$$\mathrm{Det}(\mathcal{A}) \cong \wedge^{\max}\mathrm{Ker}(\mathcal{A})^* \otimes \wedge^{\max}\mathrm{Coker}(\mathcal{A}) \ , \tag{31}$$

where $\mathrm{Ker}(\mathcal{A})$ is the finite-rank complex vector bundle over B with fibre $\mathrm{Ker}A_x$, and $\mathrm{Coker}(\mathcal{A})$ similarly. The interesting feature here is that it shows the determinant bundle to be the top exterior power of the index bundle $\mathrm{Ind}\,(\mathcal{A}) = [\mathrm{Ker}(\mathcal{A})] - [\mathrm{Coker}(\mathcal{A})] \in K(B)$ in the even K-theory of B, and in this sense determinant theory may be see as a particular aspect of index theory — understood is the very broadest sense; in fact, the computation of determinants is usually a considerably more complex and difficult task than computing an index.

4. Determinant Bundles for Differential Operators over Manifolds

The Quillen determinant has been of particular interest in the case of families of Dirac operators. Such a family is associated to a C^∞ fibration $\pi : M \longrightarrow B$ of closed boundaryless finite-dimensional Riemannian manifolds of even dimension. If there is a graded Hermitian vector bundle $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \to M$ of Clifford modules, then from the Riemannian structure one can construct a Levi-Civita connection on the vertical tangent bundle $T(M/B)$ which can be lifted to a Clifford connection on $\mathcal{E}$; for example, the spinor connection if we have a family of spin manifolds. This data yields a smooth family of first-order elliptic differential operators $\mathbf{D} = \{D_x : C^\infty(M_x; \mathcal{E}_x^+) \longrightarrow C^\infty(M_x; \mathcal{E}_x^-) \mid x \in B\}$ of chiral Dirac-type, with D_x a Dirac-type operator acting over the manifold $M_x = \pi^{-1}(x)$ parametrised by the fibration, along with a determinant line bundle $\mathrm{DET}(\mathbf{D}) \to B$ endowed with a canonical section $x \mapsto \det(D_x)$. There are various mathematical and physics contexts in which one would like to assign to the determinant section a naturally associated smooth function (a regularized determinant) $\det_{\mathrm{reg}} : B \longrightarrow \mathbb{C}$, which can, for example, then be integrated. As discussed in the previous section, this depends on identifying a trivializing (non-zero) section of $\mathrm{DET}(\mathbf{D})$. For such a section to exist the first Chern class $c_1(\mathrm{DET}(\mathbf{D}) \in H^2(B)$ must vanish, and this in turn can be computed as a term in the Atiyah-Singer Index Theorem for Families [AS]. Indeed, this is clear from the formal identification (31) which here takes on here a precise meaning.

The following simple example, which is the basic topological anomaly computation in string theory, may help to explain the type of computation. Let M_x be a copy of Σ a compact Riemann surface, so that M is a family of surfaces parametrized by B. Let $T = \cup T_x$ be the vertical complex tangent line bundle on M, where T_x is the complex tangent line bundle to M_x. Each fibre has an associated $\overline{\partial}$-operator $\overline{\partial}_x$ which we couple to the Hermitian bundle $\mathcal{E}_x := T_x^{\otimes m}$ for m a non-negative integer. In this way we get a family $\mathbf{D}_\Sigma$ of $\overline{\partial}$-operators coupled to $\mathcal{E} = T^{\otimes m}$ whose index bundle is the element $\mathrm{Ind}\,(\mathbf{D}_\Sigma) = f_!(T^{\otimes m}) \in K(B)$. The Atiyah-Singer Index Theorem for Families in this situation coincides with the Grothendieck-Riemann-Roch Theorem and this says that

$$\mathrm{ch}(f_!(T^{\otimes m})) = f_*(\mathrm{ch}(T^{\otimes m})\mathrm{Todd}(T))$$

where ch is the Chern character class and $\mathrm{Todd}(T)$ is the Todd class defined for a vector bundle F whose first few terms are

$$\mathrm{Todd}(F) = 1 + \frac{1}{2}c_1(F) + \frac{1}{12}c_1(F)^2 + \dots\ ,$$

and where $f_* : H^i(M) \to H^{i-1}(B)$ is integration over the fibres. That is, with $\xi = c_1(T)$

$$\begin{aligned} \mathrm{ch}(f_!(T^{\otimes m})) &= f_*((1 + m\xi + \frac{1}{2}m^2\xi^2 + \dots)(1 + \frac{1}{2}\xi + \frac{1}{12}\xi^2 + \dots)) \\ &= f_*(1 + (m + \frac{1}{2})\xi + \frac{1}{12}(m^2 + m + \frac{1}{6})\xi^2 + \dots)\ . \end{aligned}$$

So we have

$$c_1(f_!(T^{\otimes m})) = \frac{1}{12}(6m^2 + 6m + 1)f_*(\xi^2) \in H^2(B)\ . \tag{32}$$

But for any element of K-theory, $c_1(E) = c_1(\mathrm{DET}(E))$, and so the left side of (32) is the first Chern class of the determinant line bundle $\mathrm{DET}(\mathbf{D}_\Sigma)$. If we take, in particular, $B = Conf(\Sigma)$, the space of conformal classes of metrics on Σ (or compact subsets of this space), and couple the family $\mathbf{D}_\Sigma$ to a background trivial real bundle of rank $d/2$, or its negative in K-theory, then taking $m = 1$ (32) is easily seen to be modified to $c_1(\mathbf{D}_{\Sigma,-d/2}) = \frac{(d-26)}{24}f_*(\xi^2)$. It follows for this topological anomaly to vanish one must have background space time of dimension $d = 26$. The idea here is that $Conf(\Sigma)$ is a configuration space for Bosonic strings in $\mathbb{R}^d$ with the requirement that the determinant section of the determinant line bundle be conformally invariant, corresponding to the classical invariance of the string Lagrangian defining the string path integral from which the determinant arises. That is, in order to evaluate the path integral on the reduced configuration space one requires a trivialization of the determinant line bundle which defines a conformally invariant regularised determinant function. The above calculation says that there is a topological obstruction to this occurring when the background space dimension differs from 26.

This is the most basic example of determinant anomaly computations, which have acquired considerably more sophisticated constructions in modern versions of string theory and QFT. One immediate deficiency in the approach explained so far is that not all anomalies are topological and so even though the first Chern class of the determinant line bundle may vanish, there may still be local and global obstructions to the existence of a determinant function with the correct symmetry properties. To be more precise one needs to say not just that a trivialization of the determinant line bundle formally exists, but to actually be able to construct a specific preferred trivialization. For this more refined objective one needs to know more about the differential geometry of the determinant line. One approach is to fix a canonical choice of connection and, if the determinant bundle is topologically trivial, to construct a determinant section (up to phase) using the parallel transport of the connection.

The principal contribution to such a theory was made in a remarkable four page paper by Dan Quillen [Q] in which using zeta-function regularization he presented a beautiful construction of a metric and connection on the determinant line bundle for a family of $\overline{\partial}$-operators over a Riemann surface coupled to a holomorphic vector bundle. (This is the first paper one should read on determinant line bundles; Quillen's motivation, in fact, did not come from physics but from a problem in number theory.)

To outline this construction, which was extended to general families of Dirac-type operators in [BF], first we recall that if Δ is an invertible Laplacian type 2nd order elliptic differential operator acting on the space of sections of a vector bundle over a compact manifold of dimension n, then it has a spectrum consisting of real discrete eigenvalues $\{\lambda\}$ forming an unbounded subset of the positive real line. The zeta function of of Δ, which is discussed in detail elsewhere in the Encyclopedia, is defined in the complex half -plane $\mathrm{Re}(s) > \frac{n}{2}$ by

$$\zeta(\Delta, s) = \mathrm{Tr}\,(\Delta^{-s}) = \sum_{\lambda} \lambda^{-s}\ , \qquad \mathrm{Re}(s) > \frac{n}{2}\ ,$$

and extends to a meromorphic function of s on the whole complex plane. It turns out that the extension has no pole at $s = 0$ and this means that we may define the *zeta-function regularised determinant* of Δ by

$$\det{}_{\zeta}(\Delta) := \exp\left(-\frac{d}{ds}_{|s=0}\zeta(\Delta, s)\right) ,$$

since $\frac{d}{ds}|_{s=0}\lambda^s = \log\lambda$ this formally represents a *regularized* product of the eigenvalues of Δ. A metric is now defined on the determinant line bundle $\mathrm{DET}(\mathbf{D})$ by defining the norm square of the element $\det(D_x) \in \mathrm{Det}(D_x)$ by

$$\|\det(D_x)\|^2 := \det{}_{\zeta}(D_x^* D_x)$$

over the subset B_0 of $x \in B$ where D_x is invertible. Elsewhere in B one includes a factor defined by the induced L^2 metric in the kernel and cokernel. See [Q, BF] for full details.

A connection is defined by similarly constructing a regularized version of the connection we would define if we were working with finite rank bundles. First, one includes in the data associated to the fibration $\pi : M \longrightarrow B$ defining the family of operators $\mathbf{D}$ a splitting of the tangent bundle $TM = T(M/B) \oplus \pi^*(TB)$. This assumption and the Riemannian geometry of the fibration then yield a connection $\nabla^{(\pi)}$ defined along the fibres of the fibration. The connection form over B_0 is then defined by

$$\omega(x) = \mathrm{Tr}_{\zeta}(D_x^{-1}\nabla^{(\pi)} D_x)$$

where the zeta regularized trace Tr_{ζ} is defined on a vertical bundle endomorphism-valued one form $x \mapsto A_x$ on M by

$$\mathrm{Tr}_{\zeta}(A_x) := \mathrm{fp}_{s=0}\mathrm{Tr}\,(A_x(D_x^* D_x)^{-s})|^{\mathrm{mer}}$$

where the superscript indicates we are considering the meromorphically extended form, and $\mathrm{fp}_{s=0}(G(s))$ means the finite part of a meromorphic function G on $\mathbb{C}$; that is, the constant term in the Laurent expansion of $G(s)$ near $s = 0$.

A theorem of Bismut and Freed, generalizing Quillen's original computation, computes the curvature $\Omega^{(\mathrm{DET}(\mathbf{D}))}$ of this connection to be the 2-form component in the local Atiyah-Singer Families Index theorem; this is a refined version of the topological version of that theorem which we utilized earlier, it expresses the characteristic classes on B in terms of specific canonical differential forms constructed by integrating along the fibres of the fibration canonically defined vertical characteristic forms. More precisely, they prove the formula [BF, BGV]

$$\Omega^{(\mathrm{DET}(\mathbf{D}))} = (2\pi i)^{-n/2}\left(\int_{M/B}\widehat{A}(M/B)\,\mathrm{ch}(\mathcal{E})\right)_{[2]} , \qquad (33)$$

where $(\sigma)_{[2]} \in \Omega^2(B)$ means the 2-form component of a differential form σ on B. Here $\widehat{A}(M/B) = \det{}^{1/2}\left((R^{M/B}/2)/\sinh((R^{M/B}/2)\right)$ is the vertical $\widehat{A}$-genus differential form, while $\mathrm{ch}(\mathcal{E})$ is the vertical Chern character form associated to the curvature form of the bundle $\mathcal{E}$.

This theory seems a long way from the classical theory of stable characteristic classes and the Fredholm determinant discussed in earlier sections. There are, however, interesting parallels which may guide the search for an understanding of the geometry of families of elliptic operators, of which determinants form a component. The prototypical situation where determinants arise in the quantization of gauge theory is the following. Consider the infinite-dimensional affine space $\mathcal{A}$ of connections on a complex vector bundle E with structure group G sitting over S^n the n-sphere. The Lie group G is assumed to be compact. For each connection $A \in \mathcal{A}$ we consider a Dirac operator $D_A : C^\infty(S^n, S^+ \otimes E) \longrightarrow C^\infty(S^n, S^- \otimes E)$ where E is a Hermitian vector bundle coupled to the spinor bundles $S^\pm$. The group $\mathcal{G}$ of based gauge transformations acts on $\mathcal{A}$ and symmetry properties of conservation laws lead one to be interested in constructing a determinant function on the quotient space $\mathcal{A}/\mathcal{G}$. More precisely, $g \in \mathcal{G}$ transforms D_A to $D_{g.A}$ and by equivariance the Quillen determinant section pushes down to a section of a reduced determinant line bundle over $\mathcal{A}/\mathcal{G}$. As seen earlier, the topological obstruction to realizing this determinant section as a function on $\mathcal{A}/\mathcal{G}$ can be computed from the Atiyah-Singer index theorem for families applied to the corresponding index bundle $\mathrm{Ind}\,(\mathbf{D}_{\mathcal{A}/\mathcal{G}})$ in the $K(\mathcal{A}/\mathcal{G})$ by picking out the degree 2 component in $H^2(\mathcal{A}/\mathcal{G})$ of the Chern character $\mathrm{ch}(\mathrm{Ind}\,(\mathbf{D}_{\mathcal{A}/\mathcal{G}}))$. On the other hand, it turns out that this characteristic class is the transgression of the element of $H^1(\mathcal{G}, \mathbb{Z})$ defined by the zeta determinant trace

$$\begin{aligned} \Theta_\zeta &:= \mathrm{Tr}_\zeta \left((D_A^* D_{g.A})^{-1} d_{\mathcal{G}} (D_A^* D_{g.A}) \right) \\ &:= \mathrm{fp}_{s=0} \mathrm{Tr}\, (D_A^* D_{g.A})^{-1} d_{\mathcal{G}} (D_A^* D_{g.A}) \ (D_A^* D_{g.A})^{-s})|^{\mathrm{mer}} \end{aligned}$$

which counts the winding number of the zeta determinant $\mathcal{G} \longrightarrow \mathbb{C}^*$ defined by $\det_\zeta(D_A^* D_{g.A})$. This provides an interesting parallel of the classical theory described in Sect. 3. For more details of this and more advanced ideas take a look at [Si]. (A similar parallel holds between the topological derivation of the conformal anomaly outlined at the beginning of this section and what it called the Polyakov multiplicative anomaly formula for the zeta determinant of the Laplacian with respect to conformal changes in the metric on the surface.)

Aspects of more recent work in this direction have been the extension of the theory to manifolds with boundary, and how it encodes into the structures of topological and conformal field theories [S], [MS], and more generally into M-theory [FM].

References

[AlS] Alvarez O, Singer IM (1984), *Gravitational anomalies and the families index theorem*, Comm. Math. Phys. **96**, 409–417.

[AS] Atiyah MF, Singer IM (1984), *Dirac operators coupled to vector potentials*, Proc. Natl. Acad. Sci USA. **81**, 2597–2600.

[B] Bott R. (1959), *The stable homotopy of the classical groups*, Ann. Math. **70**, 313–337.

[BF] Bismut J-M, Freed DS (1986), *The analysis of elliptic families I*, Comm. Math. Phys. **106**, 159–176.

[BGV] Berline N, Getzler E, Vergne M (1992), *Heat Kernels and Dirac Operators.* Grundlehren der Mathematischen Wissenschaften **298**, Springer-Verlag, Berlin.

[CS] Chern SS, Simons J (1974), *Characteristic forms and geometric invariants*, Ann. Math. **99**, 48–69.

[FM] Freed DS, Moore G (2004), *Setting the quantum integrand of M-theory*, arXiv:hep-th/0409135.

[MS] Mickelsson J, Scott S (2001), *Functorial QFT, gauge anomalies and the Dirac determinant bundle*, Comm. Math. Phys. **219**, 567–605.

[Q] Quillen D (1985), *Determinants of Cauchy-Riemann opeators over a Riemann surface*, Funktsional Anal. i Prilozhen. **19**, 37–41.

[S] Segal G (2004), *The definition of conformal field theory* 'Topology, Geometry and Quantum Field Theory', Cambridge University Press, 421–577 ; *Topological field theory*, "Stanford lecture notes",http://www.cgtp.duke.edu/ITP99/segal..

[Si] Singer IM (1985), *Families of Dirac operators with applications to physics* Asterique, hors serie, 323–340.