Introduction

A trace provides a means of computing invariants of an algebra. If the algebra is a module for a semigroup representation associated to a geometric or topological structure, such as a manifold or a foliation, or a fractal or a quantum field theory, then the trace characters of the representation may provide a source of geometric, topological, spectral, or physical invariants. Building up enough such invariants may allow such structures to be characterized and distinguished between. It is of interest therefore to know how many traces there are on a given geometric algebra and, given a semigroup representation, to compute its characters.

The algebra of integer order classical pseudodifferential operators (ψdos) acting over a compact boundaryless manifold, for instance, is endowed with a unique trace, the residue trace. As one restricts to sufficiently small subalgebras, however, so new traces appear, such as the classical trace on the sub-ideal of smoothing operators. Any functional extension of the classical trace to the full algebra of ψdos is not tracial, but defines a quasi-trace; the residue trace arises in any such process as an obstruction to the traciality of the extension. The resulting residue trace characters and quasi-trace characters provide an array of spectral geometric and topological invariants of the manifold.

Such invariants acquire their full impact by the comparison of two trace formulae, or the equality obtained by computing a given trace in two different ways. Generically, in terms of the classical trace of an operator A acting on functions over a compact manifold M, the objective is a broad equality of the type

$$\underbrace{\sum \text{eigenvalues of A}}_{\text{spectral side}} = \underbrace{\sum \text{terms depending on M}}_{\text{geometric side}}$$

Such spectral geometric trace formulae are what are studied in geometric analysis in trace theory of ψdos and, on the other hand, in the theory of Selberg trace formulae as modular and automorphic forms (the Langlands programme). In the latter one has, typically, a Selberg-Arthur trace formula

$$\underbrace{\mathrm{Tr}\,(R(\phi))}_{\text{spectral side}} = \underbrace{\sum_{\gamma\in\{\Gamma\}} \mathrm{vol}(H_\gamma/\Gamma_\gamma) \int_{H/H_\gamma} \phi(x\gamma x^{-1})\; dx}_{\text{geometric side}} \qquad (0.0.0.1)$$

where H is a locally compact unimodular topological group, Γ a discrete subgroup for which H/Γ is a compact manifold, while for $\phi \in C_0^\infty(H)$ the unitary representation $R(\phi)$ of H on $\psi \in L^2(H/\Gamma)$ is $(R(\phi)\psi)(x) = \int_H \phi(y)\psi(x\cdot y)\; dy$. Here, H_γ is the centralizer of γ in H and $\{\Gamma\}$ the set of conjugacy classes. $R(\phi)$ decomposes into irreducible representations π of finite multiplicity $m(\pi, R)$ with character $\mathrm{Tr}\,(R(\phi)) = \sum_\pi m(\pi, R)\,\mathrm{tr}\,(\pi(\phi))$. (See [Arthur 2005] for more precision.)

This may be usefully viewed in the following way. A smoothing operator acting on C^∞ functions on M is a continuous linear operator $A : C^\infty(M) \to C^\infty(M)$ acting by $(A\psi)(x) = \int_M \mathrm{k}(x,y)\psi(y)$ with Schwartz kernel $\mathrm{k}(x,y)$ which is everywhere smooth on $M \times M$. There is a unique trace on the algebra of such operators given (up to a scalar multiple) by

$$\mathrm{Tr}\,(A) = \int_M \mathrm{k}(x,x). \tag{0.0.0.2}$$

This is the classical trace, equal to the convergent sum over the discrete eigenvalues of the (compact) operator A

$$\mathrm{Tr}\,(A) = \sum \mu.$$

For $R(\phi)$, by a change of variable, using the fact that the measure is H-invariant, we see $R(\phi) : C^\infty(H/\Gamma) \to C^\infty(H/\Gamma)$ is smoothing and, summing over Γ-translates, is defined by the Schwartz kernel

$$\mathrm{k}_{R(\phi)}(x,y) = \sum_{\gamma\in\Gamma} \phi(y\gamma x^{-1})\ dx \otimes dy. \tag{0.0.0.3}$$

Substituting (0.0.0.3) in (0.0.0.2) gives the Selberg-Arthur trace formulae (0.0.0.1).

More familiar instances of (0.0.0.1) are seen in low dimension. When M is a single point, then $C^\infty(M,\mathbb{C}^n) = \mathbb{C}^n$ and A is an $n \times n$ matrix (a_{ij}), while the trace formula is the elementary identity

$$\underbrace{\sum_{i=1}^n \mu_i}_{\text{spectral side}} \quad = \quad \underbrace{\sum_{i=1}^n a_{ii}}_{\text{geometric side}}$$

where μ_i are the eigenvalues of A. When $M = S^1$ one obtains the classical Poisson summation formula. This follows by viewing S^1 as the quotient of its universal cover $H = \mathbb{R}$ by $\Gamma := \pi_1(S^1) = \mathbb{Z}$. Indeed, quite generally, when H is the universal covering space of a compact Riemannian manifold (M,g) and $\Gamma = \pi_1(M)$ is the fundamental group the trace formula becomes a sum over primitive geodesics with $\mathbb{D} \subset H$ a fundamental domain

$$\mathrm{Tr}\,(A) = \sum_{l\geq 1}\ \sum_{\textbf{prim geodesics}\ \gamma} \int_{\mathbb{D}} \mathrm{tr}\ \left(\mathbb{K}(x, x\gamma^l)\right).$$

A particular case of this is the classical Selberg trace formula on a Riemann surface Σ of genus greater than one (the genus one case is Poisson summation). The universal covering of Σ is the upper-half plane $\mathbb{H}_+$, viewed as the unimodular group $H = \mathrm{SO}(2,\mathbb{R})\backslash\mathrm{SL}(2,\mathbb{R})$, and coupling the metric Laplacian Δ_g on Σ to a flat unitary bundle with connection via a representation $\chi : \pi_1(\Sigma) \to U(N)$ the Arthur-Selberg trace formula in this case is the formula

$$\begin{aligned}\mathrm{Tr}\,\left(e^{-t\Delta_g}\right) \quad &= \quad \frac{\mathrm{Area}\ \Sigma}{4\pi}\int_{-\infty}^{\infty} r\tanh\left(\pi r\right)\,\exp\left(-\frac{t}{4} - r^2 t\right)\,dr \\ &\quad + \sum_{k\geq 1}\sum_{\{\gamma\}} \chi^k(\gamma^{-1}) \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{t}{4} - \frac{k^2\,\mathsf{l}_\gamma^2}{t}\right) \frac{\mathsf{l}_\gamma}{\sinh(k\,\mathsf{l}_\gamma)}\end{aligned}$$

with l_γ the length of the unique primitive geodesic in the conjugacy class $\{\gamma\}$ in Γ This is the basis for the idea that the spectrum (of Δ_g) and the length spectrum (set of lengths of geodesics) are in duality.

So here the operator $e^{-t\Delta_g}$ is the heat operator, a smoothing operator which solves for $t > 0$ the heat equation $\partial_t f(x,t) + \Delta_g f(x,t) = 0$ on M with delta function initial data. This is an operator of deep interest in geometric analysis because it connects local data as $t \to 0+$ (when the heat distribution is like a delta function concentrated over the initial point on M) which is asymptotically computable, with global data as $t \to \infty$ (the heat flow controlled by the geometry of M). The idea is that the local and global data communicate through this flow to give one classical theorems, when Δ_g is replaced by a suitable Dirac-Laplacian Δ, such as the Gauss-Bonnet-Chern theorem and, more generally, the analytic Atiyah-Singer index theorem (discussed and proved in here in Chapter 3), in which certain topological invariants are computed by integrating densities over M constructed from the Riemannian metric and finitely many of its derivatives (curvature tensors, geodesic length, and so on).

This type of phenomenon, in contrast to the Selberg-Arthur trace formulae above, is a truly pseudodifferential affair, involving an asymptotic analysis of the heat kernel (the kernel of the smoothing operator $e^{-t\Delta}$). On the other hand, that spectral-geometric data can be extracted in a quite equivalent way from the zeta function

$$\zeta(\Delta, z) = \mathrm{Tr}\,(\Delta^{-z})\big|^{\mathrm{mer}} \qquad (0.0.0.4)$$

associated to the complex power Δ^{-z}. This, like $e^{-t\Delta}$, is a continuous operator on $C^\infty(M)$ and also defined by a Schwartz kernel $\mathrm{k}_z(x,y)$ — the crucial difference, however, is that $\mathrm{k}_z(x,y)$ is distributional on $M \times M$ with singularities (points of M where it fails to be C^∞) exactly located along the diagonal $M \hookrightarrow M \times M$. Those singularities are severe if $\mathrm{Re}(z) \leq n/2$, where $n = \dim M$, and the integral (0.0.0.3) defining the classical trace is then not defined. Regularization of traces is about how the true Schwartz kernel $\mathrm{k}_z(x,y)$ can be 'corrected' for such z to a function which is integrable along the diagonal. Choosing a way to do that is what is called a *regularized trace* or a *quasi-trace* of the operator. In the case of complex powers of simpler classes of operator there is a way to do this via meromorphic continuation. Specifically, for Δ that means meromorphically continuing the holomorphic function $\zeta(\Delta, z) = \sum \mu^{-z}$ from the half-plane $\mathrm{Re}(z) > n/2$ where Δ^{-z} is trace class, and this is what is indicated in (0.0.0.4). If we took $M = S^1$ then the classical Riemann zeta function $\sum_{n\geq 1} n^{-z}$ occurs in this way, and, indeed, for other special choices of M so do many of the classical zeta or L-functions of analytic number theory. The special choices of M generally require a particularly simple geometry, such as for tori or perhaps some higher genus Riemann surfaces. This brings us back into the territory of Arthur-Selberg trace formulae, which makes precise some of the special geometries that are needed to realize L-functions and modular forms in this way.

Let us stay with the complex powers for a moment, though. The most frequently studied method for constructing quasi-traces of a ψdo A is to consider slightly generalized spectral zeta functions such as

$$\zeta(A,\Delta,z) = \mathrm{Tr}\,(A\Delta^{-z})\Big|^{\mathrm{mer}}. \qquad (0.0.0.5)$$

Again, this fails to have a classical trace in the half plane $\mathrm{Re}(z) \leq (m+n)/2$ where m is the order of A. On the other hand, for $\mathrm{Re}(z) > (m+n)/2$ the Schwartz kernel of $A\Delta^{-z}$ is continuous at the diagonal and $\mathrm{Tr}\,(A\Delta^{-z})$ is defined. If A belongs to the class of *classical* ψdos, which is essentially the smallest semigroup containing elliptic differential operators and their parametrices, then $\mathrm{Tr}\,(A\Delta^{-z})$ can be meromorphically continued to the whole complex plane, the result is the zeta function (0.0.0.5). A quasi-trace of A is then defined by

$$\mathsf{TR}_{\mathsf{quasi}}(A) := \mathrm{fp}_{z=0}\left(\mathrm{Tr}\,(A\Delta^{-z})\Big|^{\mathrm{mer}}\right) \qquad (0.0.0.6)$$

where the finite-part $\mathrm{fp}_{z=0}(f)$ picks out the constant term, the coefficient a_0 of z^0, in the Laurent expansion of a complex function $f(z) = \frac{a_{-1}}{z} + a_0\, z^0 + O(z)$ around $0 \in \mathbb{C}$. It is not hard to see that $\zeta(A,\Delta,z)$ can have at most a simple pole at $z = 0$. What is more interesting is that that pole is exactly the residue trace of A, the unique trace on classical ψdos referred to at the beginning of the introduction.

Whilst the residue trace is a trace (it vanishes on commutators), quasi traces are not. Any extension of the classical trace such as (0.0.0.6) will not in general vanish on commutators. More emphatically, the whole process depends on the choices made, specifically the left-side of (0.0.0.6) depends on the choice of regularizing operator Δ^{-z} if $m \in [-n,\infty) \cap \mathbb{Z}$.

So one objective of this text is to give an overview of such methods for classical ψdos on closed boundaryless manifolds. The presentation aims to keep in focus the two somewhat different perspectives on traces described above: on the one hand, the enigmatic exact spectral geometric formulae that occur through the Selberg-Arthur trace formula, at least from a ψdo operator trace perspective, and on the other hand, the microlocal pseudodifferential analysis of traces and quasi traces via the symbol calculus of classical ψdos.

To some extent the symbol calculus of ψdo theory in there to avoid having to think about the distributional Schwartz kernel view point, through which exact spectral geometric trace formulae are computed, since the singularity structure of the Schwartz kernel is accessible entirely via the symbol calculus. However, the full force of trace theory is to be found in the two view points on ψdos being brought to bear simultaneously; indeed, constructing regularized traces is, as commented above, precisely the business of how to subtract-off the singular part of the kernel, and this is done using just the symbol calculus.

But there are also deeper resonances. One of the interesting ideas impulsed in mathematics by functorial quantum field theory and non-commutative geometry is the categorization of geometry and analysis. The relation of the quantum nature of matter to the electrostatic Riemannian u1niverse continues to be something elusive to mathematical precision and one of the consequent conceptual exigencies is that mathematical thinking be free-ed up from the C^∞ category to consider the structural meaning of some mathematical concepts in their own right. This comes from the need to adapt classical geometric structures to more radically singular spaces, such as foliated spaces, and in this way to better handle the methods thrown into the ring by quantum field theory (whatever that may be). Indeed, many invariants do not need to be restricted to smooth manifolds but are readily made sense of in more abstract categories. This applies, for example, to the extension of the Chern character from topological to general algebraic K theory ranging in negative cyclic homology (commented in Chapter 1).

Whilst the idea of a trace is well established abstractly through the decisive role of traces in constructing characters of semigroup, or category, representations, it is less known but equally well the case that determinants adapt well to more abstract environments, and in Chapter 2 we develop that theme more fully. The concept of determinant encodes a considerably richer structure than the classical notion alone. As background to this, in one small area of the pond, at least, determinants have been something of a zeitgeist for the developments between mathematics and theoretical physics, originating in the role of the determinant as a putative output of certain Feynman path integrals in quantum field theory and string theory.

The content has been arranged as follows. There are five chapters. The first two chapters are something of an overview of the theory. Chapter 1 deals with some general ideas of traces on algebras, while in the final section the case of tracial structures on ψdos is looked into in more detail. Chapter 2 is concerned with developing the abstract notion of determinant structures, that many crucial invariants involve not only traces but also an ambient logarithmic structure. Most proofs and detailed constructions concerning ψdo traces and determinants are to be found in Chapter 4. Chapter 3 contains explicit computations of ψdo zeta traces and determinants, including a discussion of Selberg-Arthur trace formulae. The examples included here are for the most part quite elementary instances, but directions to more complex computations are given in the chapter notes. Part II of Chapter 3 deals with the transition formulae between resolvent trace asymptotics, heat trace asymptotics, and the singularity structure of spectral zeta functions, and is applied to relative determinant formulae. The final part of Chapter 3 is concerned with residue trace computations and an application of those methods to a proof of the analytic (or 'local') Atiyah-Singer index formula. Chapter 5 takes matters off in a more specialized direction, with the beginnings of how the trace and determinant constructions for single operators described in the earlier chapters can be adapted to families of ψdos associated to geometric fibrations.

We need also to indicate what is not included. First, this is not a text book on ψdos, but rather a book on trace and determinant structures on ψdos. Indeed, with the numerous fine texts and articles already written on basic ψdo theory, as indicated in the Notes to each chapter, there is little need to repeat those expositions here. We have only discussed here smooth compact boundaryless manifolds. Much significant work has been taking place on singular spaces, such as manifolds with boundary, complete hyperbolic manifolds, manifolds with fractal boundaries, and so forth, but that is not entered into here. (For more on these aspects see, for example, [Grubb 2007], [Melrose 1995b], [Schulze 2007], [Guillopé and Zworski 1999].) Indeed, many (or most) topics in the quite vast subject area of traces on ψdos are not dealt with here. We do not, for example, discuss eta invariants or wave trace invariants, but it is to be hoped that other authors may contemplate further texts in this field focused on such topics.

Each chapter is augmented by a final section of Notes. This is where references to source material and some additional comments may be found. There are no bibliographic references in the text itself. The references given are undoubtedly incomplete and we do not pretend to have given anything near a full bibliography or proper historical trail, such an objective is probably anyway not achievable as many concepts and ideas are well known to specialists in the field long before they may have appeared in print. Proofs of most of the basic constructions of traces and determinants on ψdos are given, but, where not, we have tried to give reasonable bibliographic directions to detailed sources.

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