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Logarithmic structures and TQFT

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Abstract. This is an expository article on logarithmic structures on semigroups and categories. Characters of logarithmic representations of semigroups coincide with a number of fundamental topological and spectral invariants. An area of current research is the extension of this construction to TQFT, which incorporates a further idea of sewing together invariants, requiring an extension of the notion of abstract logarithms to non-linear maps between categories. The motivating ideas and the basic definition for such a structure are given here along with some examples.

Contents

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1. The categorization of analysis

An interesting idea that has been impulsed in mathematics, at least in part, by the study of the structures that drive quantum field theory is the need to understand certain familiar mathematical concepts in rather more abstract contexts. For example, cyclic homology frees up de Rham cohomology from the commutative algebra of smooth functions on a manifold to general noncommutative algebras. On the other hand, topological quantum field theory suggests the essential structures of geometric analysis and index theory need to be understood in the way that they relate to the construction of representations of the cobordism category.

Such categorization may be seen, perhaps, as commencing with Grothendieck's idea of motives as a proposal for a universal cohomology theory (encompassing various competing cohomology theories in algebraic geometry). Likewise, the use of n-categories (or ∞-categories) as a multi-layer approach to representations of categories is a quite old idea. But in both cases these structures have in recent times become the focus of much work in homotopy theory and geometry, in topological quantum field theory and its spin-offs into geometric analysis.

The purpose of this paper is to outline how the analytical concept of a 'determinant' and, more fundamentally, a 'logarithm' might abstract itself naturally into this discussion. Logarithmic structures appear to lie behind many basic topological invariants and it is of interest to determine what the logarithmic structures are in topological quantum field theory, corresponding in principle to certain 'higher' sewing formulae.

2. Sewing formulae

The way in which topological and spectral geometric invariants add up relative to the division of a manifold into two (or more) pieces is of some importance in geometric analysis and functorial (possibly topological) QFT. Ideally one would aim to consider decompositions into simple submanifolds, roughly expressing the manifold as a CW-complex of lower dimensional simplices, and then be able to compute on those submanifolds before using sewing formulae to reconstitute the invariant on the complicated parent manifold. Such processes are inherently cohomological. However, even in the purely topological case where analytic considerations are left to one side, such a general objective is in dimensions greater than 2 currently perhaps still at the stage of a useful, if mathematically potent, myth needed to think about representations of cobordism n-categories. Nevertheless, deep progress has been made at a fundamental category-theoretic level (see Lurie and Hopkins [**13**]) along with the discovery in dimension 2 of exotic cohomology theories (such as elliptic cohomology, Chas-Sullivan products).

2.1. Analytic sewing formulae.

At the analytic level one might begin with a more modest objective. One might ask simpler questions, such as what the codimension zero splitting formula might be for the generalized zeta-function quasi-trace at zero $\zeta(A, Q, 0)$ for classical pseudodifferential operators A and Q, which, it is known, coincides with interesting spectral-geometric and topological invariants (such as the index of an elliptic operator). Here, Q is assumed to be elliptic operator of positive order $q > 0$ whose complex powers Q_θ^{-z} are defined. The operator AQ_θ^{-z} is trace class when the real part of z is large, and its trace (or supertrace given a $\mathbb{Z}_2$-grading) admits an extension $\zeta(A, Q, z) := \mathrm{Tr}\,(AQ_\theta^{-z})|^{\mathrm{mer}}$ to a meromorphic function defined on all of $\mathbb{C}$ which has a Laurent expansion around $z = 0$ of the form

$$\zeta(A, Q, z) = \frac{1}{q}\,\mathrm{res}\, A\,\frac{1}{z} + \zeta(A, Q, 0) + \zeta'(A, Q, 0)\, z + \cdots \qquad z \text{ near } 0. \tag{2.1.1}$$

The pole coefficient $\mathrm{res}\, A$ is the residue trace of A. In terms of sewing formulae relative to a splitting of the manifold rather little is known about these coefficients in general; the first few coefficients provide key invariants in geometric index theory while the higher coefficients are largely mysterious and highly non-local.

For the first term, the residue trace on manifolds with boundary with local Boutet de Monvel boundary conditions was looked into in [**10**] and [**11**], while less is known for global APS type boundary conditions. In either case pasting formulae for $\mathrm{res}\, A$ in terms of such boundary problems relative to a splitting of M have not yet, it appears, been studied. The next term in the expansion $\zeta(A, Q, 0)$ is in general rather complicated and non-local and no general pasting formulae are known. However, in certain cases progress has been made. Most outstandingly that is so for the eta-invariant

$$\eta(D, 0) := \zeta(D|D|^{-1}, |D|, 0)$$

of a self-adjoint Dirac type operator which arises as the correction term in the APS index theorem. In this case there is an exact pasting formula proved originally by Wojciechowski [**25, 26, 27**] and Bunke [**7**], which has been reproved and refined by other authors in numerous subsequent works, in particular in [**5**] and more from a TQFT view point in [**9**].

When $A = I$ write $\zeta(Q, z) := \zeta(I, Q, z)$. The expansion (2.1.1) in this case becomes

(2.1.2)

$$\zeta_\theta(Q, z) \;=\; \underbrace{-\left(\frac{1}{q}\,\mathrm{res}(\log_\theta Q) - \mathrm{tr}\,(\Pi_Q)\right)}_{=\zeta(Q,0)} z^0 \;+\; \underbrace{(\log \det{}_\zeta Q)}_{=-\zeta'(Q,0)}\, z \;+\; \cdots \qquad z \text{ near } 0.$$

where Π_Q is a projector onto the generalized kernel of Q. Then the first two terms of (2.1.2) are (logarithms of) determinants. The first is an exotic determinant, the 'residue determinant' equal to zero on ψdos with a well-defined classical determinant. Properties of the residue determinant and the extension of the residue trace to the log-classical ψdo $\log_\theta Q$ are detailed in [**18**] and [**20**]. The second is the log-zeta determinant, a quasi-determinant – an extension of the classical determinant but with a loss of the multiplicativity property. Pasting formulae for the first term $\zeta(Q, 0)$, relative to elliptic boundary problems for the restrictions of Q to M_1, M_2 given a partition $M = M_1 \cup_Y M_2$, have not yet been established; certainly this involves non-local terms associated to the dividing hypersurface Y, though the pasting formula does not appear to be hard to resolve using standard methods. For the next term a great deal more is known, at least for self-adjoint Dirac type operators. The original contribution in this direction for the case of APS boundary problems is the adiabatic pasting formula of Wojciechowski and Park [**16**], and has lead to a steady stream of refinements and extensions by other authors. It was suggested by Wojciechowski that a similar formula ought to hold for the curvature 2-form of the determinant line bundle for a family of Dirac operators, much as it does in the case of b-calculus [**15**], and recent work suggests that that is correct; see [**23**] for more on the curvature of the determinant line bundle in this case.

2.2. Sewing formulae in TQFT. A rather interesting way to think about sewing formulae has been thrown into the ring by mathematical topological quantum field theory (TQFT). Roughly speaking, recall that the idea of TQFT is that though the Feynman path integral (PI) formulation of QFT is currently eludes mathematical precision, one may nevertheless study the mathematical imprint that is left by its passing, the structures that would be left in its wake if the PI were rigorously defined.

2.3. Path integral formalism. Witten explained this motivation in the following concrete way. Consider a smooth compact n-dimensional manifold M with connected boundary $\partial M = Y$, then a PI has the general form

$$Z_M : \Gamma(Y) \to \mathbb{C}, \quad Z_M(f) = \int_{\Gamma_f(M)} e^{-S(\psi)} \mathcal{D}\psi, \tag{2.3.1}$$

where $\mathcal{D}\psi$ is a formal measure, $S : \Gamma(M) \to \mathbb{C}$ is the classical action functional on a space $\Gamma(M)$ of fields on M (the fields could, for example, be the space of smooth functions on M, or the space of sections of a vector bundle over M), while the subspace $\Gamma_f(M) = \{\psi \in \Gamma(M) \mid \psi_{|X} = f\}$ consists of smooth fields on M with boundary value $f \in \Gamma(Y)$. Thus we may view

$$Z_M \in Z(Y) = \text{a Hilbert space of distributions } \{u : \Gamma(Y) \to \mathbb{C}\}.$$

The precise class of distributions is to be specified, but $Z(Y)$ is supposed to be the Hilbert space of the theory so it might, for example, be a suitable Sobolev completion of $\Gamma(Y)$.

If M, on the other hand, has disconnected boundary $\partial M = \overline{Y_0} \sqcup Y_1$ then the PI defines the Schwartz kernel distribution

$$\mathbb{K}_M : \Gamma(Y_0) \times \Gamma(Y_1) \to \mathbb{C}, \qquad \mathbb{K}_M(f_0, f_1) = \int_{\Gamma_{(f_0,f_1)}(M)} e^{-S(\psi)} \mathcal{D}\psi,$$

or, what is formally the same thing, the linear operator

$$Z_M \in \operatorname{Hom}(Z(Y_0), Z(Y_1)), \qquad Z_M(u_0)(f_1) = \int_{\Gamma(Y_0)} \mathbb{K}_M(f_0, f_1) u_0(f_0)\, \mathcal{D}f_0.$$

If $Y_0 = Y_1 = Y$ this determines a bilinear form

$$\langle\ ,\ \rangle : Z(Y_0) \times Z(Y_1) \to \mathbb{C}, \qquad \langle u_0, u_1 \rangle = \int_{\Gamma(Y)} u_1(f) Z_M(u_0)(f)\, \mathcal{D}f.$$

Consider a compact boundaryless manifold $M = M_0 \cup_Y M_1$ partitioned into two halves by a connected hypersurface Y. Then the partition functions on M_0, M_1 have the form (2.3.1). Since the fields on M can be written as a fibre product

$\Gamma(M) = \Gamma(M_0) \times_{\Gamma(Y)} \Gamma(M_1)$, this suggests an equality

$$\begin{aligned}\int_{\Gamma(M)} e^{-S(\psi)}\mathcal{D}\psi &= \int_{\Gamma(Y)} \left(\int_{\Gamma_f(M_0)} e^{-S(\psi_0)}\mathcal{D}\psi_0 \int_{\Gamma_f(M_1)} e^{-S(\psi_1)}\mathcal{D}\psi_1 \right) \mathcal{D}f \\ &= \int_{\Gamma(Y)} Z_{M_0}(f) Z_{M_1}(f)\ \mathcal{D}f, \qquad (2.3.2)\end{aligned}$$

or, more schematically,

$$Z_M = \langle Z_{M_0}, Z_{M_1} \rangle. \qquad (2.3.3)$$

The Hamiltonian of the theory is defined by the Euclidean time evolution operator $e^{-tH} = Z_{Y\times[0,t]} \in \mathrm{End}\,(Z(Y))$; in a 'topological theory' $H := 0$ so one then expects

$$Z_{Y\times[0,t]} = I \in \mathrm{End}\,(Z(Y)). \qquad (2.3.4)$$

3. Categorical abstraction - homotopy theory

The above characterization suggests that the PI in dimension n may be viewed abstractly as a (particular) map Z which takes a boundaryless compact $(n-1)$-manifold Y to a vector space $Z(Y)$ and each n manifold M whose boundary is Y to a vector $Z_M \in Z(Y)$. If $Y = \emptyset$ is the empty $(n-1)$-manifold then from (2.3.1) Z_M is a number and $Z(\emptyset) = \mathbb{C}$. If $\partial M = \overline{X} \sqcup Y$ then Z_M defines a linear map between the vector spaces $Z(X)$ and $Z(Y)$.

Succinctly, the PI functor Z defines a representation of the cobordism category Cob_n. That is, Z is a functor $\mathsf{Cob}_n \to \mathsf{Vect}(k)$ to the category of vector spaces and linear maps over a field k.

Here, Cob_n is the category whose objects are smooth boundaryless compact $(n-1)$-manifolds and whose morphisms are smooth compact manifolds modulo orientation preserving diffeomorphisms. Thus a morphism $M \in \mathsf{mor}_n(X,Y)$ in Cob_n is a smooth compact n-manifold M equipped with a diffeomorphism $\partial M \cong \overline{X} \sqcup Y$; any two such morphisms are identified up to an orientation preserving diffeomorphism equal to the identity on the boundary. Composition of morphisms

$$\mathsf{mor}_n(X,Y) \times \mathsf{mor}_n(Y,Z) \to \mathsf{mor}_n(X,Z), \qquad (M,N) \mapsto M \cup_Y N,$$

is by pasting along the common boundary Y (up to specifying the smooth structure on $M \cup_Y N$). There is a symmetric monoidal product $\mathsf{Cob}_n \times \mathsf{Cob}_n \to \mathsf{Cob}_n$ defined by disjoint union of manifolds; with unit ('monoidal') element the empty manifold $\emptyset$. Likewise the usual tensor product defines a symmetric monoidal product on $\mathsf{Vect}(k)$ with identity the ground field k; again in $\mathsf{Vect}(k)$ identifications are made up to equivalence (isomorphism).

The formal definition, then, of a TQFT is a functor $Z : \mathsf{Cob}_n \to \mathsf{Vect}(k)$ preserving the symmetric monoidal structures.

Specifically, Z functorially maps $M \in \mathsf{mor}_n(X,Y)$ to a linear homomorphism $Z_M \in \mathrm{Hom}(Z(X), Z(Y))$, which can be equivalently written

$$Z_M \in Z(\overline{X}) \otimes Z(Y), \qquad Z(\emptyset) = k, \qquad Z(\overline{X}) \cong Z(X)^* \qquad (3.0.5)$$

with

$$Z_{M\cup_Y N} = Z_M \circ Z_N. \qquad (3.0.6)$$

In particular, since Cob_n is modulo diffeomorphisms then taking $M = N = X \times I$ gives that $Z_{X\times I}$ is an idempotent, which the functoriality says must, in fact, be the identity, corresponding to (2.3.4).

Since the boundary of a high-dimensional manifold M is really no topologically simpler than the manifold itself, it is important to allow for reasons of computability higher codimensional decompositions of M using manifolds with corners; in principle this may allow matters to be reduced to computing on k-simplices. Indeed, an implicit expectation is that a given TQFT ought to correspond to (a possibly exotic) generalized cohomology theory. In this respect the functorial sewing properties in a TQFT ought likewise to provide a computational tool along the lines of a generalized Mayer-Vietoris property. With this motivation, an *extended TQFT* with values in a symmetric monoidal n-category C is defined to be a symmetric monoidal functor $Z : \mathsf{Cob}_n^{\rm ext} \to \mathsf{C}$, where $\mathsf{Cob}_n^{\rm ext}$ is the extended cobordism n-category whose objects are points, whose (1-)morphisms are oriented 1-manifolds (cobordisms of points), whose 2-morphisms are cobordisms of 1-morphisms, and so on.

These definitions are due principally to Segal [**24**], Atiyah [**1**], Witten [**28**], and Hopkins and Lurie [**13**].

TQFT in dimensions 1 and 2 is relatively straightforward to characterize in a generic sort of way. A functor $Z : \mathsf{Cob}_1 \to \mathsf{Vect}(k)$ is determined by its possible evaluations on $c = [0,1]$. There are for c two possible boundary 0-manifolds p and $\overline{p}$ (with opposite orientations), to which Z assigns a finite dimensional vector space $V = Z(p)$ and its dual $V^* = Z(\overline{p})$. The meaning of $Z(c)$ changes according to whether it is regarded as a morphism in $\mathsf{mor}_n(p,p)$ in which case $Z(c) = I \in \mathrm{End}\, V$, or in $\mathsf{mor}_1(\emptyset, p \sqcup \overline{p})$ in which case $Z(c)$ is the map $\mathbb{C} \to \mathrm{End}\,(V), x \mapsto xI$ with I the identity, or in $\mathsf{mor}_1(p \sqcup \overline{p}, \emptyset)$ in which case $Z(c)$ is the trace map $\mathrm{tr} : \mathrm{End}\,(V) \to \mathbb{C}$. From this one can compute the value $Z(S^1)$ on the circle by writing $S^1 = S^1_+ \cup S^1_-$ as the union of its upper and lower semicircles, considering $S^1_+ \in \mathsf{mor}_1(\emptyset, p \sqcup \overline{p})$ and $S^1_- \in \mathsf{mor}_1(p \sqcup \overline{p}, \emptyset)$. Then according to (3.0.5), (3.0.6), $Z(S^1)$ is the composition of the maps $\mathbb{C} \to \mathrm{End}\,(V)$ and $\mathrm{End}\,(V) \to \mathbb{C}$ above, and hence $Z(S^1) = \dim V$. This is the description given in [**13**].

To give a 2-dimensional TQFT $Z : \mathsf{Cob}_2 \to \mathsf{Vect}(k)$, on the other hand, is the same thing as to specify a unital associative tracial algebra $(\mathcal{A}, \tau)$ (see below for more on traces). This is essentially because any real compact surface is a composition of copies of the disc D and copies of the 'pair of pants' surface P (a genus zero surface with two incoming and one outgoing boundary). Setting $\mathcal{A} = Z(S^1)$, then, Z_P defines according to the axioms of the TQFT an associative multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, while regarding D as an element of $\mathsf{mor}_2(\emptyset, S^1)$ defines the unit in $\mathcal{A}$ as the image of $1 \in k$, while as an element of $\mathsf{mor}_2(S^1, \emptyset)$ it defines a trace $\tau : \mathcal{A} \to k$. For more on this see [**24**], [**13**], and for a detailed account of the far richer open-closed theory with boundary conditions see Moore and Segal [**14**].

3.1. Introducing additional structure.

The basic assumptions of a TQFT can be modified or extended in many interesting ways. Simple modifications are, for example, to specify additional topological structure, such as spin structures, framings, characters and so forth. Alternatively, the target category $\mathsf{Vect}(k)$ could be refined to a category of chain complexes, as considered in [**14**] and [**8**]. On the other hand, quantizing classical field theories can require the use of fibred-TQFT in which Cob_n is replaced by the category $\mathbb{C}\mathsf{ob}_n$ of smooth fibrations, with objects of fibre dimension $n-1$ and morphisms which are fibrations whose total space is a manifold with boundary with fibre diffeomorphic to a compact manifold with boundary of dimension n, all modulo fibrewise orientation preserving diffeomorphism (thus Cob_n is the subcategory of $\mathbb{C}\mathsf{ob}_n$ for the case of a fibration over a point). Composition of morphisms in $\mathbb{C}\mathsf{ob}_n$ is by fibrewise sewing. Then $\mathsf{Vect}(k)$ is replaced by the category $\mathbb{V}\mathsf{ect}(k)$ of vector bundles and bundle homomorphisms, modulo isomorphism. The symmetric monoidal structures extend to the fibred versions and a fibred-TQFT is defined to be a symmetric monoidal functor $Z : \mathbb{C}\mathsf{ob}_n \to \mathbb{V}\mathsf{ect}(k)$, with obvious extensions to their n-category counterparts.

Additional structure introduced through 'boundary conditions' is incorporated via labels attached to the morphisms in the category, defining an open-closed TQFT. The meaning here of a boundary condition is possibly more abstract, or at least more geometric, than in the usual PDE sense. Specifically, D-branes refer to the set of closed, connected, oriented, smooth submanifolds of a given manifold M and one can consider the space of maps $\gamma : [0,1] \to M$ with $\gamma(0) \in X$, $\gamma(1) \in Y$ in given D-branes X and Y. The homology of the corresponding mapping space for a given set of labels is, roughly, what is meant by 'string topology' and this is known to define an open-closed conformal field theory (CFT). See [**4**] for more on this and references.

Similarly, but moving into more uncharted areas, there is no difficulty in formally contemplating marked and degenerate morphisms. A marked morphism

$$M \in \mathsf{mor}_n^{\Sigma}(X, Y)$$

is a cobordism endowed with an embedding $\Sigma \hookrightarrow \overset{o}{M}$ of a submanifold (of arbitrary codimension) into the interior of M; thus, in connecting X to Y the morphism M is constrained to pass through Σ. The motivation for this is more easily explained when discussing logarithmic structures (below). On the other hand, there is no particular reason to suppose a more complete theory of quantum fields can be restricted to manifolds of constant dimension; a degenerate morphism is one which may be of non-constant dimension, the associated TQFT requires the use of n-categories and such morphisms may be thought of as compositions of k-morphisms for varying k.

In a different direction, one may abandon purely topological considerations in favour of introducing more analytic structure, such as metrics, linear boundary conditions, connections, and so forth, meaning in practise dropping diffeomorphism invariance. The resulting invariants may then be geometric or spectral rather than topological and the Hilbert spaces of the theory will then in general be infinite dimensional (for a TQFT $Z(X)$ is necessarily finite dimensional and so the dual has a unique meaning, but for infinite dimensional $Z(X)$ then analytic choices must be made as to which spaces of distributions are being contemplated.)

This is appropriate for understanding the role of some spectral-geometric (which in some cases turn out to be topological) invariants such as those mentioned in § 2.1. An example of this is functorial QFT $Z : \mathsf{Cob}_1^B \to \mathsf{Vect}(k)$ with Cob_1^B whose objects are the same as Cob_1 (see the characterization of 1-dimensional TQFTs in § 3), and whose non-closed morphisms $(c_{a,b}, P) \in \mathsf{mor}_1^B$ consist of (unions of) $c_{a,b} = [a, b]$, up to an orientation preserving diffeomorphism equal to the identity at a and b, and a projection P in the Grassmannian $\mathrm{Gr}(\mathbb{C}^{2m}) = \{P \in \mathrm{End}\,\mathbb{C}^{2m} \mid P^2 = P, P^* = P\}$

parametrizing linear subspaces of $\mathbb{C}^{2m}$. Consider, then, the formal fermionic PI

$$Z(c_{a,b}, P) = \int_{C^{\infty}(c_{a,b},\mathbb{C}^m)} e^{-\langle\psi, D_P\psi\rangle}\, d\psi$$

defined by the constrained Yang-Mills action, where $D = i\nabla_{d/dx}$ with ∇ a Hermitian covariant derivative (a Dirac operator in dimension 1) and D_P denotes its restriction to those ψ satisfying the linear elliptic boundary condition $P(\psi(a) \oplus \psi(b)) = 0$. Then, formally, $Z(c_{a,b}, P) = \det D_P$. Since $\mathrm{Gr}(\mathbb{C}^{2m})$ is a Kähler manifold with canonical Kähler form equal to the i times the curvature of the canonical line bundle $L \to \mathrm{Gr}(\mathbb{C}^{2m})$, then, as instructed by geometric quantization, we define the Hilbert space $Z(a \sqcup b)$ to be dual of the space of the holomorphic sections of $L^* \to \mathrm{Gr}(\mathbb{C}^{2m})$. Then $Z(a \sqcup b)$ is canonically isomorphic to the exterior algebra $\bigwedge \mathbb{C}^{2m}$. The point is that $\det D_P$, as P varies defines a section of the determinant line bundle, and the determinant line bundle is canonically isomorphic to L. If we restrict to the real submanifold identified with the unitary group $U(m) \hookrightarrow \mathrm{Gr}_m(\mathbb{C}^{2m})$, $g \mapsto P_g$ the projection onto graph$(g : \mathbb{C}^m \to \mathbb{C}^m)$, over which L (and the determinant line bundle) are naturally trivial, then there is an essentially canonical identification

$$Z(c_{a,b}, P) = \det D_{P_g} = \det(I - g^{-1}h)$$

(for example by zeta function regularization) where $h \in U(n) := U(\mathbb{C}^n)$ is the parallel transport of ∇ along $c_{a,b}$. Alternatively, considered as a functorial QFT, we may, via the Pl ücker embedding, regard the determinant line as a ray in the Fock space $Z(a \sqcup b)$ and the determinant as defined 'absolutely' without boundary condition as the 'vacuum vector' $\det D \in Z(a \sqcup b)$. With regard to the trivialization over $U(n)$ one finds defining $Z(a) = Z(b) = \bigwedge \mathbb{C}^m$

$$\det D \leftrightarrow \bigwedge h \in Z(a \sqcup b) \cong \mathrm{End}\,(\bigwedge \mathbb{C}^m),$$

where $\bigwedge h := \sum_{k=0}^{m} \bigwedge^k h$ relative to $\bigwedge \mathbb{C}^m := \sum_{k=0}^{m} \bigwedge^k \mathbb{C}^m$.

On the other hand, $Z(S^1) = \det(I - h_{S^1})$ with h_{S^1} the holonomy around the circle. Write, as in the example in § 2.1, $S^1 = S^1_+ \cup S^1_-$. Then there is a canonical pairing (2.3.3), (3.0.6)

$$Z(-1 \sqcup 1) \otimes Z(1 \sqcup -1) \to Z(\emptyset) = \mathbb{C}, \qquad A \otimes B \mapsto \mathrm{tr}\,(AB).$$

Applied to our vacuum vectors this gives

$$Z(S^1) = \langle Z(S^1_+), Z(S^1_-)\rangle,$$

which relative to the trivializations is the identity

$$\det(I - h_{S^1}) = \mathrm{tr}\,(\bigwedge h_+ \circ \bigwedge h_-),$$

which really is just the identity $\det(I - h_{S^1}) = \mathrm{tr}\,(\bigwedge h_{S^1})$, with $h_\pm$ the parallel transports along $S^1_\pm$, consequent on $h_{S^1} = h_- \circ h_+$. On the other hand, there is a quite precise identification of this with the formal PI formula (2.3.2) via the isometry $\mathrm{End}\,(\bigwedge \mathbb{C}^m) \to L^2(U(m))$ given by $T \mapsto f_T$ with $f_T(g) = \mathrm{tr}\,(T \circ \bigwedge g)$. Since, via the Peter-Weyl theorem and Schur's lemma, this is an isometry it may be applied to the elements $\bigwedge h_+$, $\bigwedge h_-$, using Haar measure dg, to give the sewing formula for the circle for this symmetric monoidal functorial QFT as

$$\underbrace{\det(I - h_{S^1})}_{\text{pairing on End}\,(\bigwedge \mathbb{C}^m)} = \underbrace{\int_{U(n)} \det(I - g^{-1}h_+)\cdot \det(I + gh_-)\, dg,}_{\text{pairing on } L^2(U(m))} \tag{3.1.1}$$

which is a rigorous version of (2.3.2) with the boundary condition f in that formula replaced here by P_g.

Thus one may say that this 1-dimensional QFT yields the fundamental representation of the Lie group $U(n)$, while the sewing formula (2.3.2) is the orthogonality relation (3.1.1). The 2-dimensional version of this theory produces the fundamental loop group representations of $LU(n)$. This is far more subtle; in particular, the determinant line bundle over $LU(n)$ is non-trivial (unlike over $U(n)$ in dimension 1) leading to a projective representation. See [**19, 24**] for more on loop group representations and determinant lines. (Indeed, there is an analogue of this theory for gauge groups in all dimensions). An alternative, possibly analytically better formulation would be to consider b-morphisms, in the sense of Melrose's b-calculus.

Note, however, that this is not a topological QFT; $Z(S^1)$ is not, here, a topological invariant of S^1. There is though an obvious topological invariant associated to the pair (S^1, ∇); namely the winding number

$$w(\nabla) := \frac{1}{2\pi}\int_{S^1} \operatorname{tr}\left(h(t)^{-1}dh(t)\right) \in \mathbb{Z}.$$

of the map $S^1 \to \mathbb{C}^*$, $t \mapsto \det h(t)$. But considered in the sense of cobordism this is not multiplicative (as would be expected in TQFT) but rather is additive. Precisely, with

$$\log c_{a,b} := \frac{1}{2\pi}\int_{c_{a,b}} h(t)^{-1}dh(t)$$

one has for $a < b < b'$

$$\log c_{a,b'} = \log c_{a,b} + \log c_{b,b'} \tag{3.1.2}$$

and in particular

$$\log S^1 = \log S^1_+ + \log S^1_-. \tag{3.1.3}$$

Taking the trace of these finite matrices gives, setting $\log\det c := \operatorname{tr}(\log c)$,

$$\log\det S^1 = \log\det S^1_+ + \log\det S^1_-. \tag{3.1.4}$$

One could just exponentiate and use $\det h_{a,b}$, with $h_{a,b}$ the transport along $c_{a,b}$ to get a multiplicative TQFT, and this is what is generically done when additive invariants are encountered, but doing that loses sight of interesting structure: there is an operator-valued logarithm and a trace, which when combined on a closed manifold give a topological invariant $\log\det S^1$. Indeed, the winding number is a homotopy invariant identifying the Bott isomorphism $\pi_1(U(m)) \cong \mathbb{Z}$, which is naturally additive rather than multiplicative.

4. Categorization and logarithms

This brings us, then, to the idea of logarithmic structures and, in particular, their role in constructing invariants as logarithmic functors on the cobordism category.

Such structures lie behind a number of familiar topological and analytic invariants. Index invariants are a prime example of such a thing, both in odd and even K-theory. It is, however, the case that logarithm operators taking values in an algebra $\mathcal{A}$ tend to be rather special and quite hard to find, though there may be many traces on $\mathcal{A}$ – giving rise potentially to many determinants once a logarithm is identified.

With respect to logarithmic structures on the cobordism category, the identity (3.1.4) decomposing the topological winding number as the sum of two non-topological numbers is repeated for any local invariant; for example, the index of a Dirac operator $\eth$ on a compact boundaryless spin manifold M, viewed as the $\widehat{A}$-genus, can always be broken up additively as $\operatorname{ind}\eth = \int_{M_1}\widehat{A}(x) + \int_{M_2}\widehat{A}(x)$ with respect to a codimension zero partition $M = M_1 \cup_Y M_2$. The terms $\int_{M_i}\widehat{A}(x)$, however, are not of the form $\operatorname{tr}(\log T_i)$ with $\log T_i$ trace-class logarithmic operators on M_i. Though this is consequently not a logarithmic structure, there is one of the form $\operatorname{ind}\eth = \operatorname{ind}\eth_1 + \operatorname{ind}\eth_2$ where $\eth_1, \eth_2$ are APS-type elliptic boundary problems for $\eth$ on M_1 and M_2, providing a basic instance of a log-determinant functor on Cob_n. There is more on this below.

Log-determinant structures lie behind many ψdo spectral invariants. For example, if we look again at the expansion (2.1.2) then the first term is a log-determinant structure, the so-called residue determinant. The second term, the zeta determinant, comes from an honest logarithmic operator composed with what is only a quasi-trace defined by zeta function regularization; its log-determinant properties are therefore anomalous. Likewise, spectral flow, suspended eta-invariants, the odd and even Chern characters, analytic torsion are all characters of logarithmic representations.

In the remaining sections we look first at the appearance of logarithmic structures in a number of standard invariants, before briefly outlining the categorical formulation along the lines of functorial QFT.

4.1. Logarithms.

Determinants will be studied here as characters of logarithmic representations of semigroups taking values in a tracial algebra. Let $\mathcal{Z}$ be a topological semigroup and let $\mathcal{B}$ be a unital locally convex topological algebra. A *global logarithm operator* is a map

$$\log : \mathcal{Z} \to \mathcal{B}, \qquad a \mapsto \log a, \tag{4.1.1}$$

which for $a, b \in \mathcal{Z}$ satisfies

$$\log ab - \log a - \log b \ \in \ [\mathcal{B}, \mathcal{B}]. \tag{4.1.2}$$

That is, $\log ab = \log a + \log b + \sum_{j=1}^{N} [b_j, b'_j]$ some $b_j, b'_j \in \mathcal{B}$. Thus, the space of global logarithms with values in $\mathcal{B}$ is

$$\mathbb{L}\mathrm{og}(\mathcal{Z}, \mathcal{B}) \ := \ \mathrm{Hom}(\mathcal{Z}, \mathcal{B}/[\mathcal{B}, \mathcal{B}]) = \mathrm{Hom}(\mathcal{Z}, HC_0(\mathcal{B})). \tag{4.1.3}$$

relative to the linear structure of $\mathcal{B}/[\mathcal{B}, \mathcal{B}]$ ($HC_0(\mathcal{B})$ is the degree zero cyclic homology group). We may likewise consider the distributional subspace of continuous log functionals.

We may write (4.1.2) as $\log ab \approx \log a + \log b$. It follows that if $\mathcal{Z}$ is unital with identity element I (so that $\mathcal{Z}$ is a monoid) then $\log I \approx 0$ and hence for $b \in \mathrm{GL}(\mathcal{Z})$ that $\log b^{-1} \approx -\log b$ and $\log(bab^{-1}) \approx \log a$. If $\mathcal{Z}$ is unital with identity element I it can be advantageous to use linearity to refine $\log \in \mathbb{L}\mathrm{og}(\mathcal{Z},\mathcal{B})$ to the relative $\mathrm{Log} \in \mathbb{L}\mathrm{og}(\mathcal{Z},\mathcal{B})$ by $\mathrm{Log}\, a := \log a - \log I$ so that $\mathrm{Log}\, I = 0$.

Counting logs, as with counting traces, is done projectively insofar as 'uniqueness' is up to a scalar multiple. Precisely, since $\mathcal{B}$ is a linear space over $\mathbb{C}$ so therefore is the space (4.1.3) of logs

$$\log_1,\ \log_2 \in \mathbb{L}\mathrm{og}(\mathcal{Z},\mathcal{B}) \quad \Rightarrow \quad \lambda\log_1 + \mu\log_2 \in \mathbb{L}\mathrm{og}(\mathcal{Z},\mathcal{B}) \tag{4.1.4}$$

any $\lambda, \mu \in \mathbb{C}$. Thus, the number of log maps $\mathcal{Z} \to \mathcal{B}$ means the dimension over $\mathbb{C}$ of $\mathbb{L}\mathrm{og}(\mathcal{Z},\mathcal{B})$.

A logarithm whose construction depends on extraneous choices may only be defined in a neighbourhood of each element of the semigroup $\mathcal{Z}$, though each such local choice, or 'branch', is required to not be visible to any consequent determinant. A *local logarithm operator* on $\mathcal{Z}$ with values in $\mathcal{B}$ is an operator which for each $a \in \mathcal{Z}$ can be defined on some open neighbourhood $\mathcal{U}$ of a, called a branch of the log, as a map

$$\log_{\mathcal{U}} : \mathcal{U} \to \mathcal{B}, \qquad c \mapsto \log_{\mathcal{U}} c,$$

such that for any $a, b \in \mathcal{Z}$ there exist respective neighbourhoods $\mathcal{U}, \mathcal{V}, \mathcal{W}$ of a, b, ab in $\mathcal{Z}$ for which

$$\log_{\mathcal{W}} ab - \log_{\mathcal{U}} a - \log_{\mathcal{V}} b \in [\mathcal{B},\mathcal{B}]. \tag{4.1.5}$$

The space of local logarithms on $\mathcal{Z}$ with values in $\mathcal{B}$ will be denoted $\mathbb{L}\mathrm{og}_{\mathrm{loc}}(\mathcal{Z},\mathcal{B})$. An element of $\mathbb{L}\mathrm{og}_{\mathrm{loc}}(\mathcal{Z},\mathcal{B})$ may be denoted simply by log.

A trace on the algebra $\mathcal{B}$ taking values in a vector space V is a linear map $\tau : \mathcal{B} \to V$ such that $\tau([a,b]) = 0$ for $a, b \in \mathcal{B}$. Thus

$$\mathbb{T}\mathrm{races}(\mathcal{B}, V) := \mathrm{Hom}\,(\mathcal{B}/[\mathcal{B},\mathcal{B}], V)$$

and so

$$\text{scalar valued traces} = (\mathcal{B}/[\mathcal{B},\mathcal{B}])^*$$

A basic question in studying traces is whether

$$\tau(a) = 0 \quad \overset{??}{\Longrightarrow} \quad a = \sum_{j=1}^{m} [\, b_j,\, c_j\,],$$

or, equivalently, whether $\mathrm{Ker}\,(\tau) = [\mathcal{B},\mathcal{B}]$. It is readily verified that for scalar traces the following are equivalent:

- $\tau : \mathcal{B} \to \mathbb{C}$ the unique non-trivial trace on $\mathcal{B}$
- $\mathcal{B}/[\mathcal{B},\mathcal{B}]$ is 1-dimensional
- Any $a \in \mathcal{B}$ can be written w.r.t $q \in \mathcal{B}$ with $\tau(q) \neq 0$ as

$$a = \sum_{j=1}^{J} [\, b_j,\, c_j\,] + \frac{\tau(a)}{\tau(q)}\, q.$$

A global (resp. local) *log-determinant* is defined by an element $\log \in \mathbb{L}\mathrm{og}(\mathcal{Z},\mathcal{B})$ (resp. $\log \in \mathbb{L}\mathrm{og}_{\mathrm{loc}}(\mathcal{Z},\mathcal{B})$) along with a trace $\tau : \mathcal{B} \to R$ to a ring R via

$$\mathcal{Z} \xrightarrow{\log} \mathcal{B} \xrightarrow{\tau} R$$

which, in view of (4.1.2), has the log-multiplicativity property

$$\tau(\log ab) = \tau(\log a) + \tau(\log b) \qquad \text{for all } a, b \in \mathcal{Z}. \tag{4.1.6}$$

Thus the log-determinant $\tau(\log a)$ is the τ-character of the logarithmic representation log of the semigroup $\mathcal{Z}$ in $\mathcal{B}$, and the space of log-determinants $\mathcal{Z} \to R$ factored through $\mathcal{B}$ is the linear product

$$\log\det(Z, R) = \mathbb{L}\mathrm{og}(Z, \mathcal{B}) \times \mathbb{T}\mathrm{races}(\mathcal{B}, R).$$

A *global determinant structure* is a triple $(\log, \tau, e)$ where $e : R \to R'$ is a homomorphism of unital rings with the exponential property $e(x+y) = e(x) \cdot e(y)$. The multiplicative determinant functional associated to $(\log, \tau, e)$ is

$$\det_{\tau,\, e} := e \circ \tau \circ \log : \mathcal{Z} \to R', \qquad \det_{\tau,\, e}(a) := e(\tau(\log a)). \tag{4.1.7}$$

Given an exponential map, the log property (4.1.2) may be relaxed to

$$\log ab - \log a - \log b \;\in\; \mathrm{Ker}\,(e \circ \tau) \tag{4.1.8}$$

and still define a multiplicative determinant.

There is, then, the (rough and not necessarily finite) bound on the number of log-determinants $\mathcal{Z} \to V$

$$\#\ \text{log-dets}\ \mathcal{Z} \to V \;\leq\; \dim \mathbb{L}\mathrm{og}(\mathcal{Z}, \mathcal{B}) \times \dim \mathbb{T}\mathrm{races}(\mathcal{B}, V). \tag{4.1.9}$$

This need not be an equality in general since a trace may be identically zero on the range of some log-maps.

The linearity of the spaces $\mathbb{L}\mathrm{og}(\mathcal{Z}, \mathcal{B})$ and $\mathbb{T}\mathrm{races}(\mathcal{B}, \mathbb{C})$ define via composition with the exponential map $\exp : \mathbb{C} \to \mathbb{C}$ the canonical commutative semigroup structure on the space of determinants $\mathbb{D}\mathrm{ets}(\mathcal{Z}, \mathbb{C})$

$$\det{}_1, \det{}_2 \in \mathbb{D}\mathrm{ets}(\mathcal{Z}, \mathbb{C}) \quad \Rightarrow \quad \det{}_1 \cdot \det{}_2 \in \mathbb{D}\mathrm{ets}(\mathcal{Z}, \mathbb{C}), \tag{4.1.10}$$

$$(\det{}_1 \cdot \det{}_2)(a) := \det{}_1(a) \cdot \det{}_2(a).$$

4.2. Determinant structures on DGAs. Let $\mathcal{B} = (\Omega, d)$ be a differential graded algebra (DGA). A (graded) logarithm operator sensitive to d is a map $\log : \mathcal{Z} \to (\Omega, d)$, $a \mapsto \log a$, which satisfies

$$\log ab - \log a - \log b \;\in\; \underbrace{[\Omega, \Omega]}_{\text{graded commutator}} \;+\; d\Omega. \tag{4.2.1}$$

The space of such logarithms is

$$\mathbb{L}\mathrm{og}(\mathcal{Z}, (\Omega, d)) \;:=\; \mathrm{Hom}(\,\mathcal{Z},\, \Omega/([\Omega, \Omega] + d\Omega)\,).$$

A *closed graded determinant structure* means a triple $(\log : \mathcal{Z} \to (\Omega, d), \tau, e)$ with $\tau : \Omega \to R$ a closed graded trace. The closure of τ leads again to the log-multiplicativity property (4.1.6) and, further, that for $d^t\tau(\log a) := \tau(d \log a) = 0$ for all $a \in \mathcal{Z}$. The associated graded determinant is defined as before by $\det_{\tau,\, e}(a) := e(\tau(\log a))$.

5. Examples of logarithmic structures

We list here some examples of log-determinant structures; details may be found in [**22**].

5.1. Fredholm index — an exotic determinant. A Fredholm operator on a Hilbert space H is a bounded operator which is far from zero and close to the Banach Lie group $\mathrm{GL}(H) := \mathsf{B}(H)^{\times}$ of invertible elements in $\mathsf{B}(H)$. As such, the multiplicative semigroup $\mathrm{Fred}\, H$ of Fredholm operators on H lies at an opposite extreme in $\mathsf{B}(H)$ to any proper ideal, and in particular to the ideal $\mathsf{F}(H)$ of finite rank operators. Precisely, $A \in \mathsf{B}(H)$ is Fredholm if there is a parametrix $P \in B(H)$, so that $L_A := PA - I \ \in \mathsf{F}(H)$ and $R_A := AP - I \ \in \mathsf{F}(H)$. $\mathrm{Fred}\, H$ is a semigroup with respect to operator composition, but is highly non-linear, having homotopy type $\mathrm{Fred}\, H \simeq \mathbb{Z} \times \mathrm{BGl}(\infty)$.

The index defines a local determinant structure in which $\mathcal{Z} = \mathrm{Fred}\, H$ and $\mathcal{B} = \mathsf{F}(H)$ endowed with the classical trace tr, while $\log \in \mathbb{L}\mathrm{og}_{\mathrm{loc}}(\mathrm{Fred}\, H, (\mathsf{F}(H), \mathrm{Tr}\,))$ is defined pointwise by $\log_P A := [A, P]$ for P a parametrix for A. There is not a unique choice of parametrix and for this reason the logarithm so defined is local. To see that this is logarithmic (4.1.5), let $A, B \in \mathrm{Fred}$ and let $P, Q \in B(H)$ be respective parametrices, then for any parametrix R for AB one has

$$\log_R AB = \log_P A + \log_Q B + \underbrace{[L_A\, B,\, Q\,]}_{\in [\mathsf{F}(H), \mathsf{F}(H)]} + [A\, R_B,\, P\,] + [AB, R - QP]. \tag{5.1.1}$$

The character of the logarithm is the index of A

$$\mathrm{tr}\,(\log_P A) = \mathrm{tr}\,([A, P]) = \mathrm{ind}\, A, \tag{5.1.2}$$

and its log-multiplicativity, consequent to (5.1.1), is the additivity property of the index

$$(4.1.6) \quad \longleftrightarrow \quad \mathrm{ind}\, AB = \mathrm{ind}\, A + \mathrm{ind}\, B.$$

The associated determinant functional $\det_{\mathrm{ind}} A := e^{\mathrm{Tr}(\log_P A)} = e^{\mathrm{ind} A} \in \mathbb{Z}$ is constant on the connected components of $\mathrm{Fred}\, H$. The logarithm may be refined to the composite logarithm

$$\log\ : \mathrm{Fred}\, H \to \mathsf{F}(H) \to \mathsf{F}(H)/[\mathsf{F}(H), \mathsf{F}(H)] \tag{5.1.3}$$

which is independent of the choice of P and hence a global logarithm. The classical trace is the canonical generator of the complex line $(\mathsf{F}(H)/[\mathsf{F}(H), \mathsf{F}(H)])^*$ which evaluated on (5.1.3) maps $\log A$ to the index.

This determinant structure is *exotic* insofar as the log-determinant vanishes on the subdomain of the classical Fredholm determinant.

5.2. Restricted general linear group. Let $H = H^+ \oplus H^-$ be a $\mathbb{Z}_2$-graded Hilbert space with grading defined by an indempotent $F^2 = I$ with $F\xi = \pm\xi$ for $\xi \in H^{\pm}$. For $(\mathsf{J}, \|\cdot\|_{\mathsf{J}}, \tau)$ a (proper) normed trace ideal in $(B(H), \|\cdot\|)$ one has the restricted general linear group

$$\mathrm{GL}_{\mathrm{res},\mathsf{J}}\,(H) = \{A \in \mathrm{GL}(H) \mid [F, A] \in \mathsf{J}\}.$$

$\mathrm{GL}_{\mathrm{res},\mathsf{J}}\,(H)$ is a Banach lie group and with respect to the grading $H = H^+ \oplus H^-$, its elements have the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $b, c \in \mathsf{J}$. If $A^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ in this representation then $ax = I_+ - bz$, and $xa = I_+ - yc$ with I_+ the identity operator on H^+. The element x is thus a canonical J-parametrix for a and we have homotopy equivalences

$$\mathrm{GL}_{\mathrm{res},\mathsf{J}}\,(H) \overset{\simeq}{\to} \mathrm{Fred}_{\mathsf{J}}(H^+) \overset{\simeq}{\to} \mathrm{Fred}(H^+), \qquad A \mapsto a, \tag{5.2.1}$$

where $\mathrm{Fred}_{\mathsf{J}}(H^+)$ is the semigroup of bounded operators invertible modulo J. The element x provides a canonical choice of parametrix for a and so we may define the *global* logarithm

$$\log\,:\mathrm{GL}_{\mathrm{res},\mathsf{J}}\,(H)\to\mathrm{Fred}_{\mathsf{J}}(H^+),\qquad \log A:=[a,x]. \tag{5.2.2}$$

Higher logarithms $\log_k\in\mathbb{L}\mathrm{og}(\mathrm{GL}_{\mathrm{res},\mathsf{J}}\,(H),(\Omega,d_{\mathrm{F}}))$ can be constructed with respect to the DGA differential $d_{\mathrm{F}}(A)=[F,A]$ by $\log_k A=(A^{-1}[F,A])^{2k+1}$. The associated log-determinant is the odd Chern supertrace τ_s-character

$$\mathsf{ch}^-_{k,\tau}(A):=\tau_s((A^{-1}[F,A])^{2k+1}).$$

5.3. A universal logarithm. Let $\mathcal{A}$ be an associative algebra $\mathcal{A}$ with unit 1 over a ring R. The following universal DGA $\Omega^*(\mathcal{A})$ has a role in the construction of trace character invariants for $\mathcal{A}$. It is defined by setting $\Omega^0(\mathcal{A})=\mathcal{A}$ and

$$\Omega^1(\mathcal{A})=\mathcal{A}\otimes_R(\mathcal{A}/R).$$

Then $\Omega^1(\mathcal{A})$ has an $\mathcal{A}$-bimodule structure defined by

$$x\cdot(a\otimes_R b)\cdot y=xa\otimes_R by-xab\otimes_R y,\qquad a,b,x,y\in\mathcal{A}, \tag{5.3.1}$$

and there is the degree one differential $d:\mathcal{A}\to\Omega^1(\mathcal{A})$ define by $da:=1\otimes a$. By construction $d(1)=0$. It has the universality property that if Φ is an $\mathcal{A}$-bimodule with a derivation $\delta:\mathcal{A}\to\Phi$ with $\delta(1)=0$, then there is a bimodule homomorphism $\rho:\Omega^1(\mathcal{A})\to\Phi$ such that $\delta=\rho\circ d$.

From (5.3.1) we obtain that d is a bimodule derivation $d(ab)=da\cdot b+a\cdot db$. Define the linear spaces

$$\Omega^n(\mathcal{A})=\underbrace{\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\cdots\otimes_{\mathcal{A}}\Omega^1(\mathcal{A})}_{n\text{ factors}}. \tag{5.3.2}$$

Any element of $\Omega^n(\mathcal{A})$ can be written as a linear sum of elements of the form $a_0da_1\cdots da_n$ for some $a_0,\ldots,a_n\in\mathcal{A}$. The graded derivation $d:\Omega^n(\mathcal{A})\to\Omega^{n+1}(\mathcal{A})$, $a_0\,da_1\cdots da_n\mapsto da_0\,da_1\cdots da_n$ gives $\Omega^*(\mathcal{A})=\sum_{k\geq 0}\Omega^k(\mathcal{A})$ the structure of a DGA of degree 1 (However, $\Omega^*(\mathcal{A})$ is not an exterior algebra, it is called the algebra of non-commutative differential forms.) There is a natural algebra isomorphism

$$\Omega^n(\mathcal{A})\cong\mathcal{A}\otimes_R(\mathcal{A}/R)^{\otimes n},\qquad a_0da_1\cdots da_n\longleftrightarrow a_0\otimes_R(a_1\otimes_R\cdots\otimes_R a_n),$$

In particular, $da_1\cdots da_n\leftrightarrow 1\otimes_R(a_1\otimes_R\cdots\otimes_R a_n)$.

Set $\mathrm{GL}_\infty(\mathcal{A}):=\bigcup_n\mathrm{Gl}_n(\mathcal{A})$ with the direct limit topology, where $\mathrm{Gl}_n(\mathcal{A})$ consists of invertible elements of the set $M_\infty(\mathcal{A})$ of $n\times n$ matrices with entries in $\mathcal{A}$. Summing the diagonal elements gives the pre-trace (or Denis trace) $\mathtt{Tr}_{\mathbf{pre}}:M_\infty(\mathcal{A})\to\mathcal{A}$. It is then not hard to show the following.

Theorem 1. *Associated to a DGA* $(\Omega^*=\sum_{k\geq 0}\Omega^k,d)$ *and a homomorphism* $\rho:\mathcal{A}\to\Omega^0$ *there is for each* $k=1,2,\ldots$ *a global logarithm map*

$$\log_k\in\mathbb{L}og(\mathrm{GL}(\mathcal{A}),(\Omega^*,d))$$

$$\log_k\,:\mathrm{GL}(\mathcal{A})\to\Omega^{2k-1},\qquad\log_k a:=(a^{-1}da)^{2k-1}, \tag{5.3.3}$$

where we have written $(a^{-1}da)^{2k-1}:=(\rho(a)^{-1}d\rho(a))^{2k-1}$. *The even degree form* $a\mapsto\gamma_k(a)=(a^{-1}da)^{2k}$ *also is logarithmic but is trivial insofar as* $\gamma_k(a)\in d\,\Omega^*$.

It follows that there is a canonical odd logarithm associated to any unital algebra $\mathcal{A}$

$$\log_k : \mathrm{GL}_\infty(\mathcal{A}) \to \Omega^{2k-1}(\mathcal{A}), \qquad \log_k a := \mathtt{Tr}_{\mathbf{pre}}\big((a^{-1}da)^{2k-1}\big)\,. \tag{5.3.4}$$

The resulting canonical pairing with cyclic cohomology to the space of logarithmic characters on $\mathrm{GL}(\mathcal{A})$ defined by associating to $(\log, \tau)$ the log-determinant $\tau \circ \log$

$$\begin{tikzcd}
\mathbb{L}\mathrm{og}(\mathrm{GL}(\mathcal{A}), (\Omega^*(\mathcal{A}), d)) \otimes HC^*(\mathcal{A}) \arrow[r] & \mathbb{L}\mathrm{og}(\mathrm{GL}(\mathcal{A}), \mathbb{C}) \arrow[d, "\exp"] \\
 & \mathrm{Hom}(\mathrm{GL}(\mathcal{A}), \mathbb{C})
\end{tikzcd} \tag{5.3.5}$$

and then by exponentiation to the space of determinants on $\mathrm{GL}(\mathcal{A})$ may be viewed as an odd Chern character pairing.

5.4. Logs and $K_1(\mathcal{A})$.

If $\mathcal{Z} = \mathsf{G}$ is a topological group, then one has:

Lemma 1.

$$\mathbb{L}og(\mathsf{G}, \mathcal{B}) = \mathrm{Hom}(\mathsf{G}/\mathsf{G}^{(1)}, HC_0(\mathcal{B})) \tag{5.4.1}$$

with $\mathsf{G}^{(1)}$ *the commutator subgroup.*

In the case of graded logarithms on G *with values in* (Ω, d), (5.4.1) *becomes*

$$\mathbb{L}og(\mathsf{G}, (\Omega, d)) = \mathrm{Hom}\left(\mathsf{G}/\mathsf{G}^{(1)}, \Omega/\left([\Omega, \Omega] + d\,\Omega\right)\right). \tag{5.4.2}$$

For example, if $\mathcal{Z} = \mathsf{G} = \pi_1(X)$ with X a smooth path-connected topological space and $\mathcal{B} = \mathbb{Z}$ then $\mathsf{G}/\mathsf{G}^{(1)}$, the abelianization of π_1, is the integer coefficient singular homology group $H_1(X, \mathbb{Z})$. Hence $\mathbb{L}\mathrm{og}(\pi_1(X), \mathbb{Z}) \overset{(5.4.1)}{=} \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}) \cong H^1(X, \mathbb{Z})$, the final equality holding since $\mathrm{Ext}(H_0(X, \mathbb{Z}), \mathbb{Z})) = 0$, by the universal coefficient theorem. This implies $H^1(X, \mathbb{Z}) = \{f : \pi_1(X) \to \mathbb{Z} \mid f([\gamma] \circ [\gamma']) = f([\gamma]) + f([\gamma'])\}$ where a homotopy class $[\gamma]$ is defined by a continuous (or smooth) loop $\gamma : S^1 \to X$.

Algebraic K-functors $K_m(\mathcal{A})$ exist for each $m \in \mathbb{N}$ and these provide a source of higher invariants. The K_1 functor from the category of algebras to the category of abelian groups has the realization $K_1(\mathcal{A}) = \mathrm{GL}_\infty(\mathcal{A})/\mathrm{GL}_\infty(\mathcal{A})^{(1)}$. Equivalently, $K_1(\mathcal{A})$ is the first de Rham homology group $H_1(\mathrm{GL}_\infty(\mathcal{A}), \mathbb{Z})$. Taking $\mathsf{G} = \mathrm{GL}_\infty(\mathcal{A})$ in (5.4.1), a logarithm map on $\mathrm{GL}_\infty(\mathcal{A})$ is the same thing as a group homomorphism from $K_1(\mathcal{A})$ to $\mathcal{B}/[\mathcal{B}, \mathcal{B}]$ (with respect to the multiplicative structure on the former and the linear space structure on the latter), so that

$$\mathbb{L}\mathrm{og}(\mathrm{GL}_\infty(\mathcal{A}), \mathcal{B}) = \mathrm{Hom}(K_1(\mathcal{A}), HC_0(\mathcal{B})). \tag{5.4.3}$$

A canonical homological invariant of $K_1(\mathcal{A})$ is the odd Chern character map constructed from the logarithms

$$\mathsf{ch}_k^-([u]) = \mathtt{Tr}_{\mathbf{pre}}\left((u^{-1}du)^{2k-1}\right) \tag{5.4.4}$$

of (5.3.3). The target space for the odd Chern character ch^- is the negative cyclic homology group $HC_1^-(\mathcal{A})$, just as the target for the Chern character on $K_0(\mathcal{A})$ is $HC_0^-(\mathcal{A})$

Proposition 1. *The odd Chern character*

$$\mathsf{ch}^- : K_1(\mathcal{A}) \to HC_1^-(\mathcal{A})$$

is the logarithm map defined by

$$\mathsf{ch}^- = \sum_{k\geq 0} t^k\, k!\, \mathsf{ch}_k^-, \tag{5.4.5}$$

that is,

$$\mathsf{ch}^-([u]) = \sum_{k\geq 0} t^k\, k!\ \mathtt{Tr}_{\mathbf{pre}}\left((u^{-1}du)^{2k+1}\right). \tag{5.4.6}$$

It follows that

$$\mathsf{ch}^- \in \mathrm{Hom}\left(K_1(\mathcal{A}), \frac{\Omega^{\,\mathrm{tot}}}{([\Omega^{\,\mathrm{tot}}, \Omega^{\,\mathrm{tot}}] + d\,\Omega^{\,\mathrm{tot}})}\right),$$

where $\Omega^{\,\mathrm{tot}} := \Omega^{\,\mathrm{tot}}(\mathcal{A})^-_{[t]}$ is the graded algebra of negative cyclic chains. The odd Chern character is consequently a logarithm

$$\mathsf{ch}^- \in \mathrm{Hom}\left(K_1(\mathcal{A}), HC_1^-(\mathcal{A})\right).$$

In the case where $(\mathcal{A}, \|\cdot\|)$ is a unital Banach algebra, such as the C^*-algebra $C(M)$ of continuous functions on a compact manifold M, there is an alternative candidate for the odd K-theory of $\mathcal{A}$ defined by the topological quotient group

$$K_{-1}(\mathcal{A}) = \mathrm{GL}_\infty(\mathcal{A})/\,\mathrm{GL}_\infty(\mathcal{A})_0,$$

where $\mathrm{GL}_\infty(\mathcal{A})_0$ is the subgroup of elements of $\mathrm{GL}_\infty(\mathcal{A})$ homotopic to the identity. This is topological odd K-theory. For example, $K_{-1}(M) := K_{-1}(C(M)) = [M, Gl(\infty)]$ is the relevant odd K-theory for geometric index theory and the corresponding Chern character $\mathsf{ch}_{-1} : K_{-1}(M) \to H^{\mathrm{odd}}(M)$ equal to the trace of (5.4.5) is given by

$$\mathsf{ch}_{-1}(g) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(k-1)!}{(2k-1)!}\ \mathrm{Tr}\,(g^{-1}dg)^{2k-1}.$$

In particular each of the Bott isomorphisms $\pi_{2k+1}(Gl(\infty)) \cong \mathbb{Z}$ arise as log determinant structures. We refer to [**22**] for more details.

5.5. Trace ideals and de Rham. If H is an infinite dimensional Hilbert space then $\mathsf{B}(H) = [\mathsf{B}(H), \mathsf{B}(H)]$, every bounded operator is a commutator, and so there are no non-trivial traces on $\mathsf{B}(H)$, and so every determinant is trivial, even though there are non-trivial logarithm operators. Likewise, the ideal of compact operators $\mathsf{C}(H) = [\mathsf{C}(H), \mathsf{C}(H)]$ has no non-trivial trace. There is, however, an intricate system of proper trace ideals $\mathsf{F}(H) \subset (\mathsf{J}, \tau) \subset \mathsf{C}(H)$; for example, $(\mathsf{C}_1(H), \mathrm{Tr}\,)$ with $\mathsf{C}_1(H)$ the first Schatten ideal and Tr the classical trace, and $(\mathsf{C}^{1,\infty}(H), \tau_\infty)$ where $\mathsf{C}^{1,\infty}(H)$ is the Macaev ideal and τ_∞ is a Dixmier trace. It would be interesting to better understand the resulting Dixmier determinants and, more generally, the log-determinant structure of the $\mathsf{B}(H)$ trace ideals.

In particular, on a sub-semigroup $\mathcal{Z}_{\mathsf{J}}$ of (J, τ) of operators having an Agmon angle θ there is a candidate logarithm for $\mathbb{L}\mathrm{og}_{\,\mathrm{loc}}(\mathrm{GL}(\mathcal{Z}), \mathcal{B})$

$$\log_{\theta} a = \int_{\mathcal{C}} \log_{\theta}\lambda\ (a-\lambda)^{-1}\ d\!\!\!{}^{-}\!\lambda,$$

where ${\mathchar'26\mkern-12mu d}\lambda = (i/2\pi)d\lambda$, with associated log-determinant $a \mapsto \tau(\log_{\theta} a)$. The (classical) Fredholm determinant arises this way

$$\log_{\mathbb{C}}\mathrm{det}_{\mathsf{F}} : \mathrm{GL}(I + \mathsf{C}_1(H)) \to \mathbb{C}, \quad \log_{\mathbb{C}}\mathrm{det}_{\mathsf{F}}(I + A) := \mathrm{Tr}\,(\mathrm{Log}_{\theta}(I + A))).$$

There are, on the other hand, many Dixmier determinants,

$$\log_{\theta} : I + \mathsf{C}^{1,\infty}(H) \to \mathsf{C}^{1,\infty}(H), \qquad \log_{\mathbb{C}}\mathrm{det}_{\infty}(I + A) = \tau_{\infty}\log(I + A),$$

but these are currently largely mysterious.

The de Rham algebra $\Omega(M, \mathrm{End}\, E)$ on a compact manifold M of dimension n is a DGA with infinitely many traces. Specifically, for example, we have the trace

$$\mathrm{tr}_{Y,\sigma}(a) := \int_Y \sigma \wedge \mathrm{tr}\,(a_{[k]}) \qquad \text{for } \sigma \in \Omega(M),\ Y \subset M.$$

For $Y = \{x_0\}$ a point and $\sigma = 1$ this is the delta distribution $\mathrm{tr}_{x_0}(a) := \mathrm{tr}\,(a_{[0]}(x_0))$, or, for $Y = M, \sigma = 1$, it is the standard trace $\mathrm{tr}_M(a) = \int_M \mathrm{tr}\,(a_{[n]})$, while on a spin manifold the coupled Atiyah-Singer density is the trace $\mathrm{tr}_{M,\widehat{A}}(\exp F^2)$.

On the de Rham algebra there are relatively few logarithmic structures, though there are many determinants since there are many traces. We may, for example, consider the logarithm on admissible endomorphism-valued forms $\mathsf{P} \in \Omega(M, \mathrm{End}\, E)$ defined as above by

$$\log_{\theta}\mathsf{P} = \int_{\mathcal{C}} \log_{\theta}\lambda\,(\mathsf{P} - \lambda)^{-1}\ {\mathchar'26\mkern-12mu d}\lambda \ \in \Omega(M, \mathrm{End}\, E).$$

The resolvent has a relatively simple structure; writing $\mathsf{P} = A + \mathsf{Q}$ with $A := \mathsf{P}_{[0]}$ the 0-form component

$$(\mathsf{P} - \lambda I)^{-1} = (A - \lambda)^{-1} + \sum_{k=1}^{\dim M} (-1)^k (A - \lambda)^{-1} \left(\mathsf{Q}(A - \lambda)^{-1}\right)^k.$$

Evaluating this using the trace tr_M, the resulting log-determinant structure is the Chern class on a (super)connection $\mathsf{c}(\mathbb{A}^2) = \mathrm{sdet}(I + \mathbb{A}^2) := \exp\left(\mathrm{tr_s}\log(I + \mathbb{A}^2)\right)$, giving the Chern class as a determinant structure

$$c : K^0(M) \to H^*(M).$$

On the other hand, for $g \in C^{\infty}(M, \mathrm{GL}(I + \mathsf{C}_1(H))$ we have a graded logarithm operator $\log_m g := (g^{-1}dg)^{2m-1}$; that is,

$$\log_m gh - \log_m g - \log_m h \ \in [\Omega^{\mathrm{odd}}, \Omega^{\mathrm{odd}}] + d\,\Omega^{\mathrm{odd}}.$$

From this one may build as in § 5.4 the odd topological Chern character logarithm

$$\mathsf{ch}^-(g) \ \in \ \mathbb{L}\mathrm{og}\left(C^{\infty}(M, \mathrm{GL}(I + \mathsf{C}_1(H)),\ (\Omega^{\mathrm{odd}}(M, \mathsf{C}_1(H)), d)\right).$$

Combined with any trace on $\Omega(M, \mathrm{End}\, E)$ defines a log-determinant. For example, on an odd-dimensional spin manifold the log-determinant

$$\mathrm{tr}_{\widehat{A}}(\mathsf{ch}^-(g)) = \int_M \widehat{A}(M)\mathsf{ch}^-(g)$$

computes spectral flow of a Dirac operator twisted by g.

5.6. Pseudodifferential log structures. A pseudodifferential operator (ψdo) of order m acting on the sections of a vector bundle $E \to M$ is a continuous operator $A : C^\infty(M,E) \to C^\infty(M,E)$ whose Schwartz kernel $k_A \in \mathcal{D}'(M\times M, E^*\boxtimes E)$ is an oscillatory integral of the form $k_A(x,y) = \int e^{i\langle x-y,\xi\rangle}\, s(x,y,\xi)\, d\xi$ with amplitude s an (x,y)-symbol of order m. Equivalently,

$$k_A(x,y) = \int e^{i\langle x-y,\xi\rangle} a(x,\xi)\, d\xi \qquad \mathrm{mod}\ \Psi^{-\infty}(M,E).$$

for a reduced order m symbol $a(x,\xi)$ (independent of y). We restrict to the algebra $\Psi^{\mathbb{Z}}(M,E)$ of integer order classical ψdos; 'classical' means that in each localization on M there is an asymptotic expansion $a(x,\xi) \sim \sum_j \mathsf{a}_{m-j}(x,\xi)$ with $\mathsf{a}_{m-j}(x,t\xi) = t^{m-j}\mathsf{a}_{m-j}(x,\xi)$ for $t, |\xi| \geq 1$. Then there is a unique (and exotic) scalar trace

$$\mathrm{res} : \Psi^{\mathbb{Z}}(M,E) \to \mathbb{C}, \qquad \mathrm{res}\ A = \int_M \mathrm{res}_x(A),$$

the *residue trace*, where the residue density on M is

$$\mathrm{res}_x(A) = \int_M \int_{|\xi|=1} \mathrm{tr}\,(a_{-n}(x,\eta))\, d_S\eta\ |dx|.$$

(It is 'exotic' insofar as it vanishes on trace-class ψdos, such as smoothing operators.) This means that these locally defined expressions, for local coordinates x in a neighbourhood on M with $|dx|$ meaning local Lebesgue measure, patch together to define a globally defined density on M.

But there are numerous other traces on subalgebras of $\Psi^{\mathbb{Z}}(M,E)$. For example, the classical trace

$$\mathrm{Tr} : \Psi^{-\infty}(M,E) \to \mathbb{C},\ \mathrm{Tr}\, A = \int_M \mathrm{tr}\,(k_A(y,y)),$$

is the unique trace on the subideal of smoothing operators.

There are various ways of constructing logarithms on ψdos and hence defining determinant structures, and these may provide significant spectral, and possibly topological, invariants. We mention only the 'classical' logarithm here. Specifically, for $A \in \Psi^m(M,E)$ with Agmon angle we have the logarithm operator

$$\log_\theta A := -\left.\frac{d}{ds}\right|_{s=0} A_\theta^{-s} \in \Psi^{0,1}_{\log}(M,E).$$

Here, $\Psi^{k,l}(M,E)$ is the space of log-classical ψdos of order k and log-degree 1, meaning that P in local coordinates has symbol of the form

$$\mathsf{p}(x,\xi) \sim \sum_{j\geq 0}\sum_{l=0}^{k} \mathsf{a}_{k-j,\,l}(x,\xi)\log^l|\xi|,$$

while $\Psi^{0,1}_{\log}(M,E)$ is the subspace of $\Psi^{0,1}(M,E)$ of order zero log-classical ψdos of log-degree 1 with local symbol of the form $c\log|\xi| + \mathsf{a}_0(x,\xi)$ with $\mathsf{a}_0(x,\xi) \sim \sum_{j\geq 0}\mathsf{a}_{-j}(x,\xi)$ classical of order zero; so $\log_\theta A$ is almost, but not quite, classical of order 0 (but has log-degree 1). It is known that

$$\log_\theta AB - \log_\phi A - \log_\psi B \in [\Psi^*, \Psi^*]$$

is a commutator of classical ψdos and hence is a logarithmic structure on admissible ψdos. The residue trace is known to extend to a linear functional on $\Psi^{0,1}_{\log}(M,E)$ (but in general no further into $\Psi^{k,l}(M,E)$) with the tracial property that although

$\Psi^{0,1}_{\log}(M,E)$ is not an algebra it vanishes on $[\Psi^{0,1}_{\log}(M,E),\Psi^{0,1}_{\log}(M,E)]$. Hence one has a log-determinant structure

$$\log\det_{\rm res}A := \mathrm{res}\,(\log A)$$

for A,B classical and A,B,AB with principal angles, called the *residue determinant.* Thus

$$\det_{\rm res}(AB) = \det_{\rm res}(A)\cdot\det_{\rm res}(B).$$

The residue determinant has the interesting property that

$$\log\det_{\rm res}A = -\alpha\,(\zeta(A,0)\,+\dim\mathrm{Ker}\,(A)). \tag{5.6.1}$$

For admissible operators A,B,AB, the log-determinant property therefore implies the identity

$$(\alpha+\beta)\,\zeta(AB,0) = \alpha\,\zeta(A,0)+\beta\,\zeta(B,0)$$

with $\alpha,\beta>0$ the orders of A,B, respectively.

This log-determinant structure is a local invariant and hence reasonably computable. For example, it is not hard to compute on a closed surface Σ with Laplacian $\Delta_g = -\sum_{i,j} g^{ij}(x)\nabla_i\nabla_j + \varepsilon_x(\Delta_g)$ that

$$\log\det_{\rm res}(\Delta_g+tI) = \frac{\mathrm{Area}(\Sigma)\,\mathrm{rk}(E)}{2\pi}\,t - \frac{1}{2\pi}\int_\Sigma \mathrm{tr}\,(\varepsilon_x(\Delta_g))\,dx - \frac{\chi(\Sigma)\,\mathrm{rk}(E)}{3}.$$

That computability may be used to give an elementary (insofar as it uses only symbol computations), if not short, proof of the 'local Atiyah-Singer Index formula' for a coupled Dirac operator $\not\partial : C^\infty(M,S^+\otimes E)\to C^\infty(M,S^-\otimes E)$. Precisely, from (5.6.1) we obtain:

Theorem 2. *There is an equality of densities, or n-forms,*

$$-\frac{1}{2}\left(\mathit{res}_x(\log\not\partial^-\not\partial^+) - \mathit{res}_x(\log\not\partial^+\not\partial^-)\right) = \frac{1}{(2\pi i)^{\frac{n}{2}}}\left(\widehat{A}(M,R)\,\mathsf{ch}(E,F)\right)_{[n]}$$

Here, F is the curvature form of E while $\widehat{A}(M,R)=\det^{1/2}\left(\frac{R/2}{\sinh(R/2)}\right)$ is the $\widehat{A}$-genus form with respect to the Riemannian curvature R of M.

A proof of this (using joint work with Don Zagier) may be found in [**22**].

The quasi log-determinant structure obtained by evaluating the zeta function extension of the classical trace on the logarithm $\log_\theta A$ has just a simple pole at $z=0$ (the second order pole vanishes for this particular log-classical operator). Precisely, one computes

$$\begin{aligned}\mathsf{TR}(\log_\theta A\cdot Q_\theta^z)|^{\rm mer} &= \frac{1}{z}\left(\frac{1}{\alpha}\mathrm{res}\,(\log A) - \frac{\alpha}{q^2}\,\mathrm{res}\log Q\right)\\ +\Bigg(\int_M \mathrm{Tr}_x(\log_\theta A) &- \frac{1}{q}\mathrm{res}_x(\log_\theta A\log_\theta Q) + \frac{\alpha}{2q^2}\mathrm{res}_x(\log_\theta^2 Q)\Bigg)\,z^0 + \cdots\end{aligned}$$

So the residue of $\mathsf{TR}(\log_\theta A\cdot Q_\theta^{-z})$ is 'the' residue determinant

$$\log\det_{Q,\rm res}A = \frac{1}{\alpha}\mathrm{res}\,(\log A) - \frac{\alpha}{q^2}\,\mathrm{res}\log Q \tag{5.6.2}$$

or, formally, '$\det_{Q,\rm res}A = \det_{\rm res}(A^{1/\alpha})/\det_{\rm res}(Q^{\alpha/q^2})$'. Thus the residue determinant arises as a pole in $\zeta(\log_\theta A,Q,z)$ just as the residue trace arises as a pole in (2.1.1). Of course, $\log\det_{Q,\rm res}A$ is slightly different from the residue determinant defined above; in fact, it is a log-determinant structure of the form consisting of the residue trace evaluated on $\log_\theta A$ plus the residue trace evaluated on a second logarithm, but we will omit details here.

There are other interesting log-determinant structures on $\Psi^*(M,E)$ which we will not pursue here; for example, leading symbol traces on $\Psi^{0,1}(M,E)$ evaluated on $\log_\theta A$ define the 'leading symbol determinant' considered in [**12**], while other pseudodifferential logarithms combined with these traces define a raft of other invariants. Important pseudodifferential invariants which turn out to be log-determinant structures are Melrose's suspended eta invariant, spectral flow, and analytic torsion.

6. Log structures on the cobordism category

Here, we briefly outline the categorical formulation of a log-determinant structure along the lines of functorial QFT. Unlike TQFT this cannot be a linear representation of Cob_n. Logarithms are inherently non-linear, unlike (path) integrals, and so logarithms are not going to be functors in the usual sense. Nevertheless, with the notion of logarithmic representation at hand the way to proceed is essentially clear.

We will give only a brief indication of the structures here; a more detailed account will appear shortly.

6.1. Definition of a log on Cob_n. A logarithmic structure on the category Cob_n over a field k means that:

[1] For every object $X \in \mathsf{obj}_n := \mathsf{obj}(\mathsf{Cob}_n)$ (closed compact $(n-1)$-manifold) there is an algebra $\mathcal{A}_X$. In particular, $\mathcal{A}_\emptyset = k$.

[2] For each pair $X, Y \in \mathsf{obj}(\mathsf{Cob}_n)$ there is an inclusion $i_X : \mathcal{A}_X \to \mathcal{A}_{X \sqcup Y}$ splitting a projection map $p_{X\sqcup Y, X} : \mathcal{A}_{X\sqcup Y} \to \mathcal{A}_X$ (that is, $p_{X\sqcup Y,X} \circ i_X$ is the identity map on $\mathcal{A}_X$).

[3] For each marked morphism $M \in \mathsf{mor}_n^W(X,Y)$ (thus $\partial M \cong \overline{X} \sqcup Z$ and there is an embedding $W \hookrightarrow \overset{o}{M}$) there is an element

$$\log_W M \in \mathcal{A}_{X\sqcup W \sqcup Z}.$$

[4] With respect to the composition by sewing

$$\mathsf{mor}_n^W(X,Y) \times \mathsf{mor}_n^\Sigma(Y,Z) \to \mathsf{mor}_n^{W\sqcup Y \sqcup \Sigma}(X,Z), \qquad (M, M') \mapsto M \cup_Y M',$$

one has

$$\log_{W\sqcup Y\sqcup\Sigma}(M \cup_Y M') - \log_W M - \log_\Sigma M' \ \in [\mathcal{A}_\sqcup, \mathcal{A}_\sqcup] \tag{6.1.1}$$

where $\mathcal{A}_\sqcup := \mathcal{A}_{X\sqcup W\sqcup Y\sqcup\Sigma\sqcup Z}$.

[5] If Cob_n is considered modulo diffeomorphisms, if [1] is replaced by the requirement that for every object $X \in \mathsf{obj}_n := \mathsf{obj}(\mathsf{Cob}_n)$ there is an *abelian group* $\mathcal{A}_X$, and if (6.1.1) is tightened to

$$\log_{W\sqcup Y\sqcup\Sigma}(M \cup_Y M') = \log_W M + \log_\Sigma M' \tag{6.1.2}$$

then the logarithmic structure is said to be *topological.*

If the $\mathcal{A}_X$ are algebras, then they are assumed to be algebras over the field k.

Marked morphisms were mentioned in § 3.1. The equalities (6.1.1), (6.1.2) are with respect to the inclusions of $\log_W M$ and $\log_\Sigma M'$ into $\mathcal{A}_\sqcup$.

In the first four axioms, for brevity, we are implicitly supposing that cobordisms may be regarded as different unless they actually coincide – that is, normally one requires everything modulo diffeomorphism (as in [5]).

A difference here, then, from TQFT, is that one expects a generalized relative-cohomology theory; that is, relative to the marking on M. It is essential to use marked morphisms; because logarithms have memory, they know where they were sewn together. This is in contrast to (3.0.6) which forgets the partition of the manifold. The idea here is that it is necessary to distinguish between a closed n-manifold M and, on the other hand, M with an embedded submanifold W. There are therefore two possibilities (given W)

$$M \in \mathsf{mor}_n(\emptyset,\emptyset) \qquad \Rightarrow \qquad \text{scalar valued } \log_k M \in \mathcal{A}_\emptyset = k$$

or

$$M \in \mathsf{mor}_n^W(\emptyset,\emptyset) \qquad \Rightarrow \qquad \text{operator valued } \log_W M \in \mathcal{A}_W.$$

In view of [2] there is a 'trace' $\tau_W : \mathcal{A}_W \to \mathcal{A}_\emptyset = k$ and we expect

$$\log_k M = \tau_W(\log_W M).$$

There are a number of variations one can contemplate including in the definition, and various consequences that must be explained, but for this short review we must be content to leave matters for a future exposition.

6.2. Example: Dirac index. The most elementary example of a scalar valued logarithmic structure is to take the relative Euler number $\chi(M,\partial M)$, which is additive with respect to sewing together manifolds. The trace-log structure is seen at the chain complex level.

On the other hand, for a general compatible Dirac operator $\eth$ acting on a bundle of Clifford modules $E \to M$ over a closed manifold M define

$$\log_k M = \operatorname{ind} \eth,$$

while to $M = M_0 \cup_Y M_1 \in \mathsf{mor}_n^Y(\emptyset,\emptyset)$ one assigns as follows. Let $\eth^i$ be the restriction of $\eth$ to M_i (which we assume has a collar neighbourhood near the boundary). Let P_{K_i} be the corresponding Calderón projector in the space $\mathsf{B}(L^2(Y, E_{|Y}))$ of boundary sections. Then set

$$\log_Y M := [P_{K_1}^{\perp} + P_{K_0}, L_{0,1}] : L^2(Y, E_Y) \to L^2(Y, E_Y)$$

where $L_{0,1}$ is a parametrix for the Fredholm operator $P_{K_1}^{\perp} + P_{K_0}$. The character of this operator is the index $\operatorname{ind} \eth$, but this particular logarithm operator depends on the choice of splitting codimension 1 manifold Y. To the restriction $\eth^i$ of $\eth$ to M^i one sets

$$\log_Y M_i := [P_{K_i}^{\perp} + \Pi_i, L_i] : L^2(Y, E_Y) \to L^2(Y, E_Y),$$

with Π_i is the APS projection for M_i and L_i a parametrix for $P_{K_i}^{\perp} + \Pi_i$. This logarithm operator has character equal to the index of the elliptic boundary problem $\eth^i_{\Pi_i}$.

It is then not hard to see using well known identifications that this defines a logarithmic structure on Cob_n with log-multiplicativity the well known sewing formula for the index

$$\operatorname{ind} \eth = \operatorname{ind}(\eth^0_{\Pi_0}) + \operatorname{ind}(\eth^1_{\Pi_1}).$$

This extends to the geometric fibre cobordism category $\mathbb{C}\mathsf{ob}_n$, defined in § 3.1 but here also endowed with a vertical metric on each fibration and spin structure, with compatibility with boundary structures. In this case an object is a geometric fibration $(N,B) := \pi : N = \bigcup_{b\in B} N_b \to B \in \mathsf{obj}(\mathbb{C}\mathsf{ob}_n)$ of compact closed manifolds and we define

$$\log(N,B) = K_0(B)$$

the even K-theory on the base; this depends therefore only on the parameter manifold B. To a morphism $(M,B) \in \mathsf{mor}_n((N,B),(N',B))$, a geometric fibration $M \to B$ of manifolds with boundary with $\partial M \cong \overline{N} \sqcup N'$, one assigns the index bundle

$$\log(M,B) = \mathrm{Ind}\,(\eth_{\Pi}) \in K_0(B)$$

where $\eth_{\Pi}$ is now the family of Dirac operators with APS boundary conditions defined by the geometric fibration. Take $\partial M = \emptyset$ with a fibrewise partition $M = M_0 \cup_Y M_1 \to B$. Let $\eth^i$ be the restriction of $\eth$ to M_i. Then it is not hard to see that

$$\mathrm{Ind}\,(\eth^0_{\Pi}) + \mathrm{Ind}\,(\eth^1_{\Pi}) = \mathrm{Ind}\,(\eth_{\Pi}) \quad \text{ in } K_0(B).$$

A similar logarithmic sewing formula holds when $\partial M \neq \emptyset$.

A natural trace map in this case defining the log-determinant is the Chern class $c : K_0(B) \to H^{\mathrm{even}}(B)$.

The logarithmic structure here uses the following quite general log-determinant property.An object in the category $\mathsf{C}_{\mathsf{Fred}}$ is a vector bundle $\mathcal{H} = \bigcup_{x\in X} H_x \to X$ of Hilbert spaces H_x over a compact manifold X. A morphism $\mathsf{B} \in \mathtt{mor}(\mathcal{H}^0, \mathcal{H}^1)$ is a bundle homomorphism $\mathcal{H}^0 \to \mathcal{H}^1$ defined by a continuous family of Fredholm operators, assigning to each $x \in X$ a Fredholm operator $B_x := \mathsf{B}(x) \in \mathrm{Fred}(H^0_x, H^1_x)$. A log map is defined on $\mathsf{C}_{\mathsf{Fred}}$ with values in the ring $A = K_0(X)$ by associated the index bundle Ind B. The log-property is then topological (exact)

$$\mathrm{Ind}\,(\mathsf{A}\circ\mathsf{B}) = \mathrm{Ind}\,\mathsf{A} + \mathrm{Ind}\,\mathsf{B} \qquad \text{in} \;\; K_0(X). \tag{6.2.1}$$

An 'exponential map' e in this case is to take the top exterior power giving the determinant line bundle

$$\mathrm{det}_{0,1} : \mathrm{mor}(\mathcal{H}^0, \mathcal{H}^1) \to \mathrm{Vect}(X), \qquad \mathsf{B} \mapsto \mathrm{Det\,Ind}\ \mathsf{B} = \Lambda^{\max}(\mathrm{Ker}\ \ \mathsf{B})^* \otimes \Lambda^{\max}\mathrm{Cok}\ \mathsf{B}$$

which inherits the determinant property

$$\mathrm{Det\,Ind}\,(\mathsf{A}\circ\mathsf{B}) \cong \mathrm{Det\,Ind}\,\mathsf{A} \otimes \mathrm{Det\,Ind}\,\mathsf{B}.$$

The logarithmic and trace structure may be refined to the smooth categories by defining the logarithm to be the differential form valued family of vertical ψdos

$$\log M = \log_{\pi}(I + \mathbb{A}^2) \in \mathcal{A}(M, \mathrm{End}\,(E))$$

and the trace to be integration over the fibre $\int_{M/B} \circ\mathrm{tr}\ : \mathcal{A}(M, \mathrm{End}\,(E)) \to \mathcal{A}(B)$; this is straightforward and well understood for the case of closed manifolds, and appears to hold for the non-empty boundary case without difficulty but has not been written down in detail in the literature. For more on this aspect see [**2**], [**3**], [**21, 22**].

6.3. Example: various. We mention without further detail that the *spectral flow* and *analytic torsion* define natural logarithmic representations of Cob_n.

In a different way one may construct logarithmic representations of Cob_2 using the logarithmic structures in § 5 applied to the *2D topological quantum field theory* with tracial algebra $\mathcal{A} = (Z(S^1), \tau)$ – this case is simplified by there being only one compact connected non-empty boundary manifold of dimension one.

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