On perpetual American put valuation and first passage in a regime-switching model with jumps*

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Abstract

In this paper we consider the problem of pricing a perpetual American put option in an exponential regime-switching Lévy model. For the (dense) class of phase-type jumps and finitely many regimes we derive an explicit expression for the value function. The solution of the corresponding first passage problem under a state-dependent level rests on an embedding and a matrix Wiener-Hopf factorisation for this class of processes.

Key words: American put options, matrix Wiener-Hopf factorisation, phase type, regime switching, first passage

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1 Introduction

Consider a riskless bond and a stock whose price processes are given by $\{B_t, t \geq 0\}$ and $\{S_t, t \geq 0\}$ respectively, where

$$B_t = \exp\left(\int_0^t r(Z_s)\mathrm{d}s\right), \quad S_t = \exp(X_t), \qquad X_0 = x, \tag{1}$$

with $r(\cdot) \geq 0$ the instantaneous interest rate, Z a finite state Markov process and $X = \{X_t, t \geq 0\}$ a regime-switching phase-type Lévy process (that will be specified below in Section 2). When X_t is a Brownian motion with drift and r is constant, the model (1) reduces to the classical Black-Scholes model (BS)

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for the price of a risky asset. Although widely used as a benchmark model it is well documented that the BS model is not flexible enough to accurately model stock prices and replicate market call prices simultaneously across strikes and maturities. A substantial literature is devoted to the study of a regime-switching geometric Brownian motion model, where the parameters μ, σ, r of the BS model depend on the current state of Z, a finite state Markov process. The process Z is included to model (perceived) changes in economic factors and their influence on the stock price. See Guo [14, 15] for further references and background on the regime-switching model. In the context of option pricing, Guo [13, 14] and Guo and Zhang [16] obtained closed form solutions of European, perpetual American put and lookback options for a two-state regime switching Brownian motion; For the case of N states, Jobert and Rogers [18] considered the perpetual American put and numerically solved the finite time American put problem.

In a parallel line of research, the geometric Brownian motion is replaced by an exponential Lévy process, modelling sudden stock price movements by jumps. Lévy models have been widely studied and models that have been proposed include the infinite jump activity models, such as the NIG [6], CGMY [10], KoBoL [8] and hyperbolic processes [11], and the finite activity, jump-diffusion models. In the latter category, for instance, Kou [21] investigated the pricing of European and barrier options in the case of double-exponential jumps; Asmussen et al. [4] considered perpetual American and Russian options under phase-type jumps.

In the present study we consider the model (1) that combines both the important features of regime-switching and phase-type jumps. This model allows, at least in principle, for a flexible specification of the jump-distribution, since the phase-type distributions are dense in the class of all distributions on a half-line (see [4, Prop. 1]). Under this model, we obtain explicit, analytically tractable results for the value-function of a perpetual American put and corresponding optimal exercise strategy under this model. Rather than by directly solving the corresponding variational inequality, we achieve this by solving the first-passage time problem of X_t under the level $k(Z_t)$, a level k that depends on the current regime Z_t. We note that in Asmussen et al. [4] first passage across a constant level of a (reflected) two-state regime-switching phase-type Lévy processes was considered, leaving open the case of a state-dependent level. Here we follow a different approach from [4] that rests amongst others on new matrix Wiener-Hopf factorisation results. These results extend to the setting of regime-switching phase-type Lévy processes the matrix Wiener-Hopf factorisation approach developed by London et al. [22] and Rogers [25]. On a final note we mention that the knowledge of the first-passage time distribution allows for the analytical valuation of many quantities of interest in financial applications, such as barrier-type constracts, credit risky contracts in a structural model and ruin probabilities of insurance companies (see also the Example in Section 4), where dependence of the barrier on the regime may naturally occur.

The rest of the paper is organised as follows. Section 2 sets notations and describes the problem setting. The solution to the first passage problem under a state-dependent level is developed in Sections 3, 4 and 6. The perpetual American put problem is solved in 2.2, with an explicit example worked out in Section 7. Proofs that are not give in the text are deferred to the Appendix.

2 Problem formulation

2.1 Setting

Denote by $Z = \{Z_t, t \geq 0\}$ a continuous time irreducible Markov process with finite state space $E^0 = \{1, \dots, N\}$ and intensity-matrix G, and let $X = \{X_t, t \geq 0\}$ be a regime-switching jump-diffusion given by

$$X_t = x + \int_0^t \mu(Z_s)\mathrm{d}s + \int_0^t \sigma(Z_s)\mathrm{d}W_s + \sum_{s\leq t}\sum_{i\in E^0} \Delta J_i(s)\mathbf{1}_{\{\Delta J_i(s)\neq 0, Z_s=i\}}, \tag{2}$$

where $x \in \mathbb{R}$, μ, σ are real-valued functions that map E^0 with $\sigma(\cdot) > 0$, $W = \{W_t, t \geq 0\}$ is a Wiener process and J_i are independent compound Poisson processes with jumps arriving at rate λ. The processes X and Z are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is the completed filtration generated by (X, Z).

The jumps of the compound Poisson processes J_i are assumed to be distributed according to the probability density function $f_i = f_{p_i,\alpha_i^+,T_i^+,\alpha_i^-,T_i^-}$ that is a composition of phase-type densities. More precisely, with 1_n denoting a n-column vector of ones, $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^m$ being probability column vectors and $T \in \mathbb{R}^{n\times n}$ and $U \in \mathbb{R}^{m\times m}$ being generator matrices such that $t = (-T)1_n$ and $u = (-U)1_m$ are not equal to the zero vector, $f = f_{p,\alpha,T,\beta,U}$ reads as

$$f_{p,\alpha,T,\beta,U}(x) = \begin{cases} p\ \alpha' \mathrm{e}^{Tx} t & \text{if } x > 0 \\ q\ \beta' \mathrm{e}^{-Ux} u & \text{if } x < 0 \end{cases}$$

with $q = 1 - p$ and $'$ denoting transposition. Notice that the positive jumps of J_i arrive at rate $\lambda^+ = p_i\lambda$ and are distributed according to a phase-type distribution $\mathrm{PH}(\alpha_i^+, T_i^+)$. A similar observation can be made for the negative jumps. See Neuts [23] and Asmussen [1] for further background on phase-type distributions.

Given a map $k : E^0 \to \mathbb{R}$, the first-passage time problem for X_t under $k(Z_t)$ consists in finding the function v given by

$$v(x, i) = v_{b,k}(x, i) = \mathbb{E}_{x,i}[\mathrm{e}^{-R_T + bX_T} h(Z_T)], \tag{3}$$

where $\mathbb{E}_{x,i}[\cdot] = \mathbb{E}[\cdot | X_0 = x, Z_0 = i]$, $R_T = \int_0^T a(Z_s)\mathrm{d}s$ and T is defined as follows:

$$T = T(k) = \inf\{t \geq 0 : X_t \leq k(Z_t)\}. \tag{4}$$

Here a and h map E^0 to $\mathbb{R}_+$ and a and b are assumed to be such that $v_{b,k}$ is finite.

2.2 Perpetual American put

Let the bond and risky asset price processes be given as in (1) with X_t as in (2) such that $E[S_1] < \infty$. Without loss of generality we can assume that this specification is already under a pricing measure $\mathbb{P}^*$ and we will write $\mathbb{P} = \mathbb{P}^*$.[1] In particular, the following restriction holds for the parameters of X (see also [4, Sec. 2]):

$$\frac{\sigma^2(i)}{2} + \mu(i) + \lambda_i(\widehat{F}_i(1) - 1) = r(i) \qquad i \in E^0, \tag{5}$$

where

$$\widehat{F}_i(s) = p_i\alpha_i'(-sI - T_i)^{-1}t_i + q_i\beta_i'(sI - U_i)^{-1}u_i$$

denotes the moment-generating function of f_i, the pdf of the jump-sizes of X in state i.

In this setting we consider a perpetual American put with strike $K > 0$, a contract that gives its holder the right to exercise it at any moment t and receive the payment $K - S_t$. By standard theory of pricing American style options [7, 19] it follows that, if $S_0 = s = e^x$ and $Z_0 = i$, an arbitrage free-price for this contract is given by

$$W(s,i) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_{x,i}\left[\mathrm{e}^{-\int_0^\tau r(Z_s)\mathrm{d}s}(K - \mathrm{e}^{X_\tau})^+\right], \tag{6}$$

where $u^+ = \max\{u, 0\}$, $\mathcal{T}_{0,\infty}$ denotes the set of $\{\mathcal{F}_t\}_{t\geq 0}$-measurable finite stopping times, and X_t is given by (2) with $X_0 = x$. Guo & Zhang [16] and Jobert & Rogers [18] have shown that, if X is a regime switching Brownian motion, then an optimal stopping time in (6) is equal to

$$T(k^*) = \inf\{t \geq 0 : X_t \leq k^*(Z_t)\}, \tag{7}$$

for some map (or vector) $k^* : E^0 \to \mathbb{R}$. If X is a regime-switching Lévy process then the optimal stopping time will still be of the form (7). Indeed, since (X, Z) is a Markov process the general theory of optimal stopping [26] implies that an optimal stopping time τ^* in (6) is given by

$$\tau^* = \inf\{t \geq 0 : W^*(S_t, Z_t) \leq (K - S_t)^+\}.$$

This stopping time reduces to the first passage time of the form (7) since $s \mapsto W(s,i)$ is a positive convex decreasing function that dominates $(K - s)^+$ (by definition of $W(s,i)$) and that satisfies $W(0,i) = K$. The former follows observing that $s \mapsto (K - s\mathrm{e}^{X_\tau - X_0})^+$ is convex and decreasing and that taking the expectation and the supremum over all stopping times preserves these properties.

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[1] The market consisting of a risk free bond and a risky asset with price processes as specified in (1) is arbitrage-free if $E[S_1] < \infty$. Indeed, in the Appendix a probability measure $\mathbb{P}^*$ of Cramér-Esscher type is constructed that is equivalent to $\mathbb{P}$ and under which the process $B_t^{-1}S_t$ is a martingale. It is shown that under this measure, X is still of the form (2) but with changed parameters. For background on Markov additive processes see also Asmussen [3].

Theorem 1 *The value function in* (6) *is given by* $W(s,i) = V_{k^*}(s,i)$ *where*

$$V_k(s,i) = Kv_{0,k}(x,i) - v_{1,k}(x,i) \quad i \in E^0, s = \mathrm{e}^x, \tag{8}$$

with the optimal stopping time given by (7) *where* $k^* = (k_1^*, \ldots, k_N^*)$ *satisfies*

$$\lim_{x \downarrow k_j} V_k'(\mathrm{e}^x, j) = -\mathrm{e}^{k_j} \qquad j \in E^0. \tag{9}$$

2.3 First passage

To circumvent overshoot calculations, the first step in the solution of (3) is the observation, earlier made in [3, 4], that the phase-type distribution of the jumps enables one to give an explicit pathwise construction of a related *continuous* Markov additive process A, called the *fluid embedding* of X – see Figure 1. Section 3 is devoted to the specification of form of the fluid embedding for processes X of the form (2).

The first passage problem (3) is thus reduced to a first hitting time problem for a Markov additive process which can be solved by characterising the laws of corresponding up- and down-crossing ladder processes – see Figure 2. London et al. [22] developed matrix Wiener-Hopf factorisation results for fluctuating additive processes. Rogers [25] gave elegant martingale proofs for these factorisation results and considered Brownian perturbations. Extending [22, 25] to our setting, we solve the matrix Wiener-Hopf factorisation for the embedding A, in Section 4.

Figure 3 illustrates the different ways in which first passage in (3) can occur. The next step, in Section 5, is to use the Wiener-Hopf factorisation to calculate the distribution at the first moment of leaving a finite interval or a regime-switch, whichever comes earlier. We combine the foregoing results and present the solution to (3) in Section 6.

3 Fluid Embedding

In this section we will give a precise description of the embedding of X, as illustrated in Figure 1. To that end, let Y be an irreducible continuous time Markov process with finite state space $E \cup \partial$, where ∂ is an absorbing (graveyard) state, and denote by $A = \{A_t, t \geq 0\}$ the Markov additive process given by

$$A_t = A_0 + \int_0^t s(Y_s)\mathrm{d}W_s + \int_0^t m(Y_s)\mathrm{d}s, \tag{10}$$

where s and m are maps from $E \cup \partial$ to $\mathbb{R}$ with $s(\partial) = m(\partial) = 0$. The process A is the fluid-embedding of X if the generator of Y restricted to E equal to Q_0 where, in block notation,

$$Q_a = \begin{pmatrix} G - D_a & A^+ & A^- \\ t^+ & T^+ & O \\ t^- & O & T^- \end{pmatrix}. \tag{11}$$

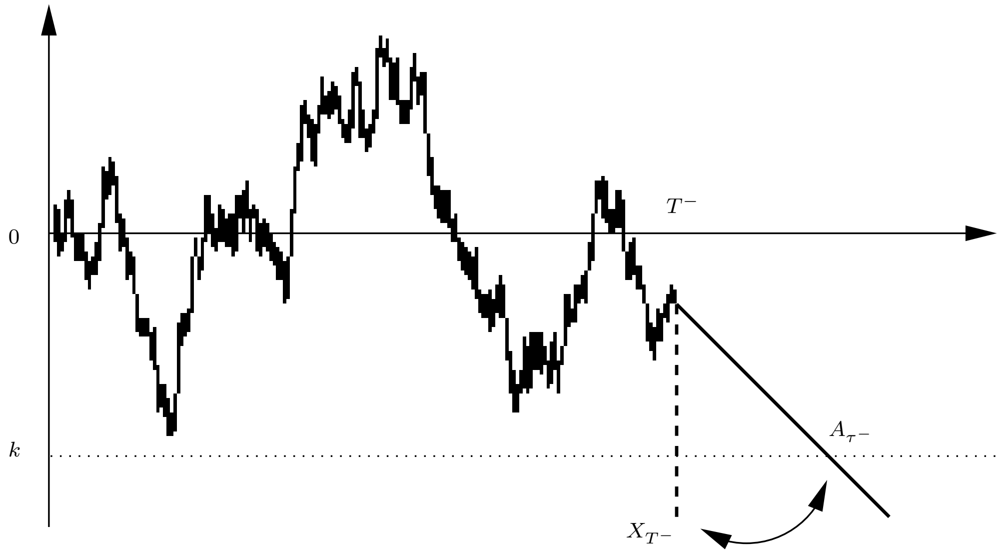


Figure 1: Shown is a sample path of X until the first time T^- that X enters $(-\infty, k)$. The process A has no positive jumps and always hits a level at first passage.

Here D_a is an $N \times N$ diagonal matrix with $(D_a)_{ii} = \lambda_i + a_i$, O are zero matrices of appropriate sizes and, again in block-notation,

$$A^{\pm} = \begin{pmatrix} \lambda_1^{\pm}\alpha_1^{\pm\prime} & & \\ & \ddots & \\ & & \lambda_N^{\pm}\alpha_N^{\pm\prime} \end{pmatrix}, T^{\pm} = \begin{pmatrix} T_1^{\pm} & & \\ & \ddots & \\ & & T_N^{\pm} \end{pmatrix}, t^{\pm} = \begin{pmatrix} t_1^{\pm} & & \\ & \ddots & \\ & & t_N^{\pm} \end{pmatrix}.$$

From the form of Q_a we see that E can be partitioned as $E = E^0 \cup E^+ \cup E^-$ where E^0 is the state-space inherited from Z and E^+/E^- are the states in which A is a linear drift of slope $+1$ or -1 that originate from the positive and negative jumps respectively of X. Correspondingly, m and s are specified by $s(j) = \sigma_j$ and $m(j) = \mu_j$, for $j \in E^0$, and $s(j) = 0$ for all other states and $m(j)$ is $+1$ or -1 according to whether $j \in E^+$ or $j \in E^-$.

Write $T_0(t) = \int_0^t \mathbf{1}_{(Y_s \in E^0)} \mathrm{d}s$ for the time before t spent by Y in E^0 and set

$$\tilde{T} = \tilde{T}(k) = \inf\{t \geq 0 : A_t \leq k(Y_t), Y_t \in E^0\}.$$

Since A can be constructed by the path-transformation of X illustrated in Figure 1, the processes (A, Y) and (X, Z) can be related to each other via the time-change T_0. More precisely, it holds that $(A \circ T_0^{-1}, Z \circ T_0^{-1})$ and (X, Z) are equal in law. This implies in particular that the triples $(T_0(\tilde{T}), A_{\tilde{T}}, Y_{\tilde{T}})$ and (T, X_T, Z_T) are equal in distribution. From the construction it also follows that replacing Q_0 by the generator Q_a for some E^0-vector a with non-negative coordinates corresponds

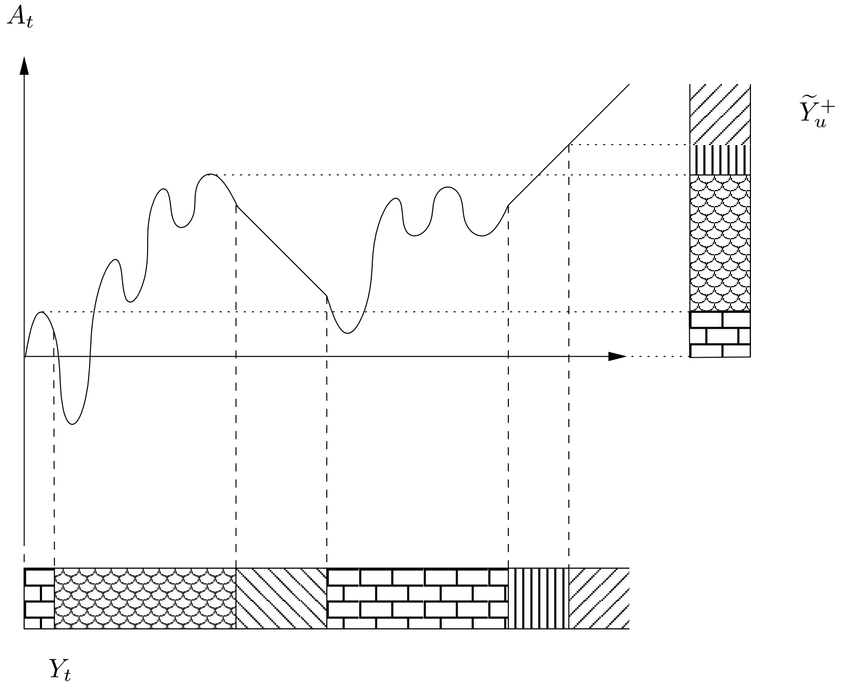


Figure 2: Pictured is a stylised sample path of the process A. The dashed vertical lines denote the jump times of Y. The horizontal dotted lines indicate the jump times of the associated Markov process $\widetilde{Y}^+$.

to discounting (or 'killing') at rate $a(i)$ when $Y_t = i$:

$$v_{b,k}(x,i) = \mathbb{E}_{x,i}\left[\mathrm{e}^{bA_{\tilde{T}}}\tilde{h}\left(Y_{\tilde{T}}\right)I_{(\tilde{T}<\zeta)}\right], \tag{12}$$

where Y now evolves according to the generator Q_a and $\mathbb{E}_{i,x}[\cdot] = \mathbb{E}[\cdot|Y_0 = i, A_0 = x]$. Further, I_A is the indicator of the set A,

$$\zeta = \inf\{t \geq 0 : Y_t \notin E\} \tag{13}$$

(with $\inf \emptyset = \infty$) and $\tilde{h} : \tilde{E} \to \mathbb{R}$ is given by $\tilde{h}(j) = h(i)$ for $j \in \widetilde{E}_i := \{i\} \cup E_i^+ \cup E_i^-$.

4 Matrix Wiener-Hopf factorisation

A classical approach [20, 22] to solve the first passage problem of the Markov process (A, Y) across a constant level rests on a characterisation of the up-crossing and downcrossing ladder processes $\widetilde{Y}^+$, $\widetilde{Y}^-$ of (A, Y). These are defined as the processes that observe Y only when A is at its maximum and at its minimum respectively, that is,

$$\widetilde{Y}_t^+ = Y(\tau_t^+) \qquad \text{and} \qquad \widetilde{Y}_t^- = Y(\tau_t^-), \tag{14}$$

where

$$\tau_t^+ = \inf\{s \geq 0 : A_s > t\} \qquad \text{and} \qquad \tau_t^- = \inf\{s \geq 0 : A_s < t\}.$$

It is easily verified that the ladder processes $\widetilde{Y}^+$ and $\widetilde{Y}^-$ are again Markov processes with state spaces $E^0 \cup E^+$ and $E^0 \cup E^-$ respectively. We will characterise in this section the generators Q_a^+ and Q_a^- of $\widetilde{Y}^+$ and $\widetilde{Y}^-$ along with the initial distributions η^+, defined by

$$\eta^+(i,j) = \tilde{\mathbb{P}}_{i,0}\left[\widetilde{Y}_0^+ = j, \tau_0^+ < \zeta\right] \tag{15}$$

for $i \in E^-$, $j \in E^+ \cup E^0$, and η^- defined as η^+ with $+$ and $-$ interchanged. Note that, if $Y_0 = i \in E^-$, initially the paths of A are decreasing and $\eta^+(i,\cdot)$ equals the distribution of Y at the end of the negative excursion of A away from A_0.

Denote by $\mathcal{Q}(n)$ the set of irreducible $n \times n$ generator matrices (generator matrices have non-negative off-diagonal elements, and non-positive row-sums) and write $\mathcal{P}(n,m)$ for the set of $n \times m$ matrices whose rows are sub-probability vectors. Let V and Σ denote the $|E| \times |E|$ diagonal matrices given by $\text{diag}(m(i))$ and $\text{diag}(s(i))$, respectively. The matrix Q_a is called *recurrent* if its rows sum up to zero, $Q_a\mathbf{1} = 0$; otherwise it is called *transient*. By considering the process A at the subsequent times it visits a certain state, $r \in E$ say, and noting that this defines a random walk, we note that in the recurrent case either $\sup_{t\geq 0} A_t = \infty$ or $\lim_{t\to\infty} A_t = -\infty$, $\mathbb{P}_{0,i}$-a.s.

Write N, N^+ and N^- for the number of elements of E^0, E^+ and E^-, respectively and let $N_0^+ = N + N^+$ and $N_0^- = N + N^-$.

Definition. Let G^+ [resp. G^-] and C^+ [resp C^-] be elements of $\mathcal{Q}(N_0^+)$ [resp. $\mathcal{Q}(N_0^-)$] and of $\mathcal{P}(N^-, N_0^+)$ [resp. $\mathcal{P}(N^+, N_0^-)$]. A quadruple (C^+, G^+, C^-, G^-) is called a *Wiener-Hopf factorization* of (Y, A) if

$$\Xi(-G^+, W^+) = O \qquad \text{and} \qquad \Xi(G^-, W^-) = O \tag{16}$$

where, for matrices W with $|E|$ rows,

$$\Xi(S, W) = \frac{1}{2}\Sigma^2 W S^2 + VWS + Q_a W \tag{17}$$

and W^+ and W^- are given in obvious block notation by

$$W^+ = \begin{pmatrix} I^+ & O \\ O & I_0 \\ \multicolumn{2}{c}{C^+} \end{pmatrix} \qquad W^- = \begin{pmatrix} \multicolumn{2}{c}{C^-} \\ I_0 & O \\ O & I^- \end{pmatrix}, \tag{18}$$

where I_0, I^+ and I^- are identity matrices of sizes $N \times N$, $N^+ \times N^+$ and $N^- \times N^-$ respectively and O denotes a zero matrix of the appropriate size.

Theorem 2 *(i) The quadruple* $(\eta^+, Q^+, \eta^-, Q^-)$ *is a Wiener-Hopf factorization of* (Y, A)*.*

(ii) The Wiener-Hopf factorisation $(\eta^+, Q^+, \eta^-, Q^-)$ *is unique if* Q *is transient or if* Q *is recurrent* A *oscillates (that is,* $\sup_t A_t = -\inf_t A_t = \infty$*).*

(iii) If Q *is recurrent and* $\lim_{t\to\infty} A_t = -\infty$*, there are precisely two Wiener-Hopf factorisations of* (Y, A) *given by* $(\eta^+, Q^+, \eta^-, Q^-)$ *and*

$$\left(\eta^+(I - \mathbf{1}\mu) + \mu, Q^+(I - \mathbf{1}\mu), \eta^-, Q^-\right),$$

where μ *is the left eigenvector of* Q^+ *corresponding to its largest eigenvalue, normalised so that* $\mu\mathbf{1} = 1$*.*

Proof of Theorem 2 For $\ell \in \mathbb{R}$ let $\Phi^\pm = \Phi^\pm_\ell$ be given by the matrices

$$\Phi^+_\ell(x) = W^+ \exp(Q^+(\ell - x)) \qquad \Phi^-_\ell(x) = W^- \exp(Q^-(x - \ell)). \tag{19}$$

where W^+ is given by (18) with $C^+ = \eta^+$ [resp. W^- is given by (18) with $C^- = \eta^-$]. The proof rests on the martingale property of $M^+_t = f_+(Y_{t\wedge\tau^+_\ell}, A_{t\wedge\tau^+_\ell})$ and $M^-_t = f_-(Y_{t\wedge\tau^-_\ell}, A_{t\wedge\tau^-_\ell})$ where

$$f_+(i, x) = e_i'\Phi^+(x)h \qquad f_-(i, x) = e_i'\Phi^-(x)h', \tag{20}$$

where h, h' are N^+_0 and N^-_0 column vectors, respectively. Here, and in the sequel, e_i denotes a (column) vector of appropriate size with $e_i(j) = 1$ if $j = i$ and zero else. To verify that M_+ is a martingale, observe first that, since $\widetilde{Y}^+$ is a Markov process with generator $Q^+ = Q^+_a$ and initial distribution η^+ defined by (15), Markov chain theory implies that for $x \leq \ell$

$$\mathbb{E}_{i,x}\left[h\left(\widetilde{Y}^+_\ell\right) I_{(\tau^+_\ell < \zeta)}\right] = e_i'\Phi^+(x)h. \tag{21}$$

The martingale property of M_+ then follows from (20) – (21) as a consequence of the Markov property of (Y, A). An application of Itô's lemma shows that $f = (f_+(i, u), i \in E)$ satisfies for $u \leq \ell$

$$\tfrac{1}{2}s(i)^2 f''(i, u) + m(i)f'(i, u) + \sum_j q_{ij}(f(j, u) - f(i, u)) = 0, \tag{22}$$

where f' and f'' denote the first and second derivatives of f with respect to u respectively. Substituting the expression (19) – (20) into equation (22) we find (since h was arbitrary) that Q^+ and η^+ satisfy the first set of equations of the system (16). The proof for Q^- and η^- is analogous and omitted. The proofs of Theorem 2 (ii)–(iii) are deferred to the Appendix. □

Example (Ruin probability) Let X_t in (2) with $x > 0$ be the surplus of an insurance company. Note that, for $\sigma = 0$, $N = 1$, and in the absence of positive jumps, (2) reduces to the classical Cramér-Lundberg model. Denote by

$\rho = \inf\{t \geq 0 : X_t < 0\}$ the ruin time of X. Above results imply that, under the model (2) the probability that the company will get ruined in regime $\ell \in E^0$ is given by

$$\mathbb{P}_{x,i}(\rho < \infty, Z_\rho = \ell) = e_i' \Phi_0^-(x) f_\ell,$$

where $f_\ell(j) = 1$ if $j \in E_\ell := \{\ell\} \cup E_\ell^+ \cup E_\ell^-$ and zero else and Φ_0^- is given by (19) with $Q^- = Q_0^-$. More generally, for a penalty function $\pi = \pi(x, z)$, depending on the size of the undershoot x at ruin and the regime z in which ruin takes place, the Gerber-Shiu [12] expected discounted penalty function reads as

$$\mathbb{E}_{x,i}[\mathrm{e}^{-R_\rho} \pi(X_\rho, Z_\rho)] = e_i' \Phi_0^-(x) \bar{g},$$

where, $R_\rho = \int_0^\rho a(Z_s)\mathrm{d}s$, $\bar{g}$ is the N^--vector with $\bar{g}(\ell) = \mathbb{E}_{0,\ell}[\pi(A_{T_0^{-1}(\rho)}, Y_{T_0^{-1}(\rho)})]$ and Φ_0^- is given by (19) with $Q^- = Q_a^-$. Taking note of the fact that, under $\mathbb{P}_{0,\ell}$, the time till Y enters E^0 follows a phase-type distribution with pdf $p(s) = e_\ell' \mathrm{e}^{sT_m^-} t_m^-$, $\bar{g}$ reads as

$$\bar{g}(\ell) = \begin{cases} \pi(0, \ell) & \text{for } \ell \in E^0 \\ \int_0^\infty \pi(-s, m) e_\ell' \mathrm{e}^{sT_m^-} t_m^- \mathrm{d}s & \text{for } \ell \in E_m^- \end{cases}.$$

5 First exit from a finite interval

The two-sided exit problem for A from the interval $[k, \ell]$ for $-\infty < k < \ell < +\infty$ is to find the distribution of the position of (A_τ, Y_τ) at the first exit time $\tau = \tau_{k,\ell} = \inf\{t \geq 0 : A_t \notin [k, \ell]\}$. By considering appropriate linear combinations of the martingales M^+, M^- the two-sided exit problem can be solved explicitly in terms of $(\eta^+, Q^+, \eta^-, Q^-)$. To that end, set Z^+, Z^-, in block notation, equal to

$$Z^+ = \begin{pmatrix} O \quad I_0 \\ \eta^+ \end{pmatrix} \mathrm{e}^{Q^+(\ell - k)}, \quad Z^- = \begin{pmatrix} \eta^- \\ I_0 \quad O \end{pmatrix} \mathrm{e}^{Q^-(\ell - k)}.$$

and define the matrices

$$\Psi^+(x) = \left(W^+ \mathrm{e}^{Q^+(\ell - x)} - W^- \mathrm{e}^{Q^-(x-k)} Z^+\right) \left(I - Z^- Z^+\right)^{-1}, \tag{23}$$

$$\Psi^-(x) = \left(W^- \mathrm{e}^{Q^-(x-k)} - W^+ \mathrm{e}^{Q^+(\ell - x)} Z^-\right) \left(I - Z^+ Z^-\right)^{-1}, \tag{24}$$

$$\Psi^\circ(s, x) = \left(\mathrm{e}^{sx} I - \mathrm{e}^{s\ell} \Psi^+(x) J^+ - \mathrm{e}^{sk} \Psi^-(x) J^-\right) [-K(s)]^{-1}, \tag{25}$$

where I is an identity matrix, $W^\pm$ are defined as in (18) with $C^+ = \eta^+$ and $C^- = \eta^-$ [resp. $J^\pm$ as the transpose of (18) with $C^\pm$ replaced by zero matrices] and

$$K(s) = \frac{1}{2} \Sigma^2 s^2 + Vs + Q_a. \tag{26}$$

We will write $\Psi_{k,\ell}^+ / \Psi_{k,\ell}^- / \Psi_{k,\ell}^\circ$ if we wish to clarify the dependence on k and ℓ.

The complete solution of the two-sided exit problem reads as follows:

Proposition 1 *Assume that* $s(i)^2b^2/2+m(i)b < -q_{ii}$ *for all* $i \in E$. *Let* $h^+, h^-, h^\dagger$ *be functions that map* $E^0 \cup E^+$, $E^0 \cup E^-$ *and* E *to* $\mathbb{R}$, *respectively. Then, for* $x \in [k, \ell]$ *and* $i \in E$, *it holds that*

$$\begin{aligned}
\mathbb{E}_{i,x}[h^+(Y_\tau)I_{(A_\tau=\ell,\tau<\zeta)}] &= e_i'\Psi^+(x)h^+, & (27)\\
\mathbb{E}_{i,x}[h^-(Y_\tau)I_{(A_\tau=k,\tau<\zeta)}] &= e_i'\Psi^-(x)h^-, & (28)\\
\mathbb{E}_{i,x}[\mathrm{e}^{bA_{\zeta-}}h^\dagger(Y_{\zeta-})I_{(\zeta<\tau)}] &= e_i'\Psi^\circ(b,x)\Delta_{h^\dagger}q_a, & (29)
\end{aligned}$$

where $q_a = (-Q_a)\mathbf{1}$, ζ *is defined by* (13) *and* $\Delta_{h^\dagger}$ *is the diagonal matrix with elements* $h^\dagger(j)$.

Proof Define g_+ and g_- by the right-hand side of (27) and (28) respectively. It is straightforward to verify from the (23) – (24) that for $x = k$ and $x = \ell$ it holds that

$$\begin{aligned}
g_+(i,k) &= 0 \text{ for } i \in E^- \cup E^0, \qquad g_+(i,\ell) = h^+(i) \text{ for } i \in E^+ \cup E^0\\
g_-(i,k) &= h^-(i) \text{ for } i \in E^- \cup E^0, \qquad g_-(i,\ell) = 0 \text{ for } i \in E^+ \cup E^0.
\end{aligned}$$

In view of these boundary conditions and the fact that any linear combination of M^+ and M^-, defined in the line just above (20), is a bounded martingale, Doob's optional stopping theorem yields that

$$g_+(i,x) = \mathbb{E}_{i,x}[g_+(Y_\tau, A_\tau)I_{(\tau<\zeta)}] = \mathbb{E}_{i,x}[h^+(Y_\tau)I_{(A_\tau=\ell,\tau<\zeta)}]$$

where $\tau = \tau_{k,\ell}$. Similarly, it follows that $g_-(i,x) = \mathbb{E}_{i,x}[h^-(Y_\tau)I_{(A_\tau=k,\tau<\zeta)}]$. To prove the third identity consider the map $h^* : E \to \mathbb{R}$ given by

$$h^*(i) = \mathbb{E}_{i,0}[\mathrm{e}^{sA_{\zeta-}}h^\dagger(Y_{\zeta-})].$$

By conditioning on the first jump epoch ξ of Y it follows that

$$\begin{aligned}
h^*(i) &= \mathbb{E}_{0,i}\left[\mathrm{e}^{sA_\xi}\right]\left(\frac{q_{i\partial}}{-q_{ii}}h^\dagger(i) + \sum_{j\in E\setminus\{i\}}\frac{q_{ij}}{-q_{ii}}h^*(j)\right)\\
&= -\frac{h^\dagger(i)q_{i\partial} + \sum_{j\neq i}q_{ij}h^*(j)}{s(i)^2s^2/2 + m(i)s + q_{ii}},
\end{aligned}$$

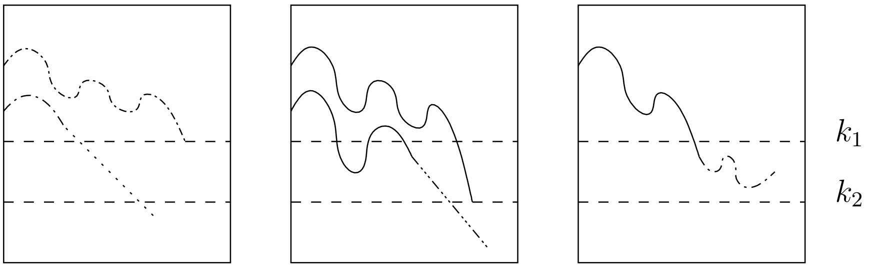
where $q_{ij}\,[q_{i\partial}]$ denotes the intensity of a transition $i \to j$ $[i \to \partial]$. After reordering and writing the above expression in matrix form it follows that $K(s)h^* = \Delta_{h^\dagger}Q_a\mathbf{1}$ so that

$$h^* = [-K(s)]^{-1}\Delta_{h^\dagger}(-Q_a)\mathbf{1}. \tag{30}$$

In view of the strong Markov property and (27) – (28), it follows that

$$\begin{aligned}
\mathbb{E}_{i,x}\left[\mathrm{e}^{sA_{\zeta-}}I_{(\zeta<\tau)}\right] &= \mathbb{E}_{i,x}[\mathrm{e}^{sA_{\zeta-}}] - \mathbb{E}_{i,x}\left[\mathrm{e}^{sA_{\zeta-}}I_{(\zeta>\tau)}\right]\\
&= \mathrm{e}^{sx}h^*(i) - \mathbb{E}_{i,x}\left[\mathrm{e}^{sA_\tau}I_{(\tau<\zeta)}h^*(Y_\tau)\right]\\
&= e_i'\left(\mathrm{e}^{sx}h^* - \mathrm{e}^{sk}\Psi^-(x)J^-h^* - \mathrm{e}^{s\ell}\Psi^+(x)J^+h^*\right).
\end{aligned}$$

By inserting (30) into the above equation, the result then follows in view of the definition (25). □



(a) (b) (c)

Figure 3: First passage under the level (k_2, k_1) takes place while $Y \in E^0$ and can take place in two ways: if A hits the level k_i while $Y = i$, $i = 1, 2$ (illustrated in (a) and (b)) or by a jump of Y (illustrated in (a), (b) and (c)). The path of A has a different line style according to the state of Y.

6 First passage under state-dependent levels

By combining the ingredients from the previous sections the first-passage function in (12) of A_t under $k(Y_t)$ can be explicitly expresesed in terms of the matrix Wiener-Hopf factorisation. For simplicity we assume that the levels are ordered as $k_1 > k_2 > \ldots > k_N$ (the general case of possibly equal levels follows by a similar reasoning). In Figure 3 the different ways are illustrated in which first passage across a state-dependent level can occur. Recall that in the problem (12) first-passage can only occur when Y is in E^0 (and not when Y is in the 'embedded jump states') As a consequence, if A_t lies inside the interval $[k_j, k_{j-1}]$, we observe that first passage of k before A exits the interval may occur in either of two ways, whichever happens earlier: Y jumps into a state $\{1, \ldots, j-1\}$ or A hits the level k_j while Y is in state j. For that reason in the derivation of the first passage function for starting values of A between k_j and k_{j-1}, we shall make use of the Markov process $Y^{(j)}$ that is the restriction of Y to $\widetilde{E}_j = E \backslash \cup_{i=1}^{j-1} E_i$ (with $Y^{(1)} = Y$). Here $E_i = \{i\} \cup E_i^+ \cup E_i^-$ denotes the state-space corresponding to the ith regime of X with embedded jumps. Let $Q_{(j)}$ be the corresponding restriction of Q_a and define $q^{(j)}$ as the matrix of exit rates from $\widetilde{E}_j$, so that Q_a can be partitioned in block notation as

$$Q_a = \begin{pmatrix} \multicolumn{2}{c}{R^{(j)}} \\ q^{(j)} & Q_{(j)} \end{pmatrix} \qquad j = 2, \ldots, N,$$

for some matrix $R^{(j)}$. By using the strong Markov property and results from Proposition 1 and Theorem 2, the value of $v_{b,k}(x, i)$ can be expressed in terms of the unknowns $v_{b,k}(k_j, i)$. For these unknowns a system of equations can be derived by invoking smoothness and continuity properties of $v_{b,k}$ above the barrier k.

Denote, for some constants $C_j(\ell), D_j(\ell)$, the vectors $h_j^-, h_j^+ : \widetilde{E}_j \to \mathbb{R}$ by

$$h_j^+(\ell) \;=\; \begin{cases} C_{j-1}(\ell) & \text{if } \ell \in E^0 \backslash \{1, \dots, j-1\} \\ D_{j-1}(\ell) & \text{else} \end{cases} \qquad (j = 2, \dots, N) \tag{31}$$

$$h_j^-(\ell) \;=\; \begin{cases} \mathrm{e}^{bk_j} & \text{if } \ell = j \\ C_j(\ell) & \text{if } \ell \in E^0 \backslash \{1, \dots, j\} \\ D_j(\ell) & \text{else} \end{cases} \qquad (j = 1, \dots, N), \tag{32}$$

and set $h_j^\dagger = (h(\ell), \ell \in \widetilde{E}_j)$. We shall write $\Psi_j^+/\Psi_j^-/\Psi_j^\circ$ as shorthand for $\Psi^+_{(k_j,k_{j-1})}/\Psi^-_{(k_j,k_{j-1})}/\Psi^\circ_{(k_j,k_{j-1})}$, respectively and denote $f(x+)$ and $f(x-)$ the left- and right-limit of the function f at x. The following characterististation of v holds true:

Theorem 3 *Assume that* $s(i)^2 b^2/2 + m(i)b < -q_{ii}$ *for all* $i \in E$. *The function* $v = v_{b;k}$ *in* (12) *is given by*

$$v(x,i) = \begin{cases} e_i' \Phi^-(x) h_1^- & \text{if } x > k_1 \\ \\ e_i' \left[\Psi_j^+(x) h_j^+ + \Psi_j^-(x) h_j^- + \Psi_j^\circ(b,x) \Delta_{h_j^\dagger} q^{(j)} \right] & \text{if } j = 2, \dots, N \\ & k_j < x \le k_{j-1} \end{cases} \tag{33}$$

where $\Phi^- = \Phi^-_{k_1}$ *and* h_j^+, h_j^- *are specified by* (31) – (32) *where* $C_j(\ell)$ *and* $D_j(\ell)$ *satisfy the system of equations,*

$$\begin{aligned} v'(k_j+,\ell) &= v'(k_j-,\ell) \qquad \ell \in E^0 \backslash \{1, \dots, j\}, & (34) \\ v(k_j+,\ell) &= v(k_j-,\ell) \qquad \ell \in \widetilde{E}_j \backslash [E^0 \cup E_j^-], & (35) \\ (D_j(\ell), \ell \in E_j^-) &= \mathrm{e}^{bk_j} (bI - T_j^-)^{-1} t_j^-, & (36) \end{aligned}$$

where $j = 1, \dots, N-1$ *and* $'$ *denotes differentation with respect to* x.

7 A worked Example

To illustrate the application of Theorem 1 we will below explicitly calculate the value function of the American put in a particular model. Suppose that Z is a Markov chain with state space $E^0 = \{1, 2\}$ and transition matrix

$$G = \begin{pmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{pmatrix}$$

and that X evolves as a Brownian motion with drift when Z is in state 1, $\mu_1 t + \sigma_1 W_t$, and as the jump-diffusion $\mu_2 t + \sigma_2 W_t - J_t$ when Z is in state 2, with J a compound Poisson process with intensity rate λ and $\exp(\alpha)$ jumps. Then the embedding of (X, Z) has state space $E = \{1, 2, 2^\star\}$, say, with corresponding transition matrix

$$Q = Q_r = \begin{pmatrix} -q_1 - r_1 & q_1 & 0 \\ q_2 & -q_2 - r_2 - \lambda & \lambda \\ 0 & \alpha & -\alpha \end{pmatrix}. \tag{37}$$

We consider the stopping time $T(k_1^*, k_2^*)$ for three different configurations of the optimal levels: $k_1^* < k_2^*$, $k_1^* = k_2^*$ and $k_1^* > k_2^*$. For $x > \max\{k_1^*, k_2^*\}$ the value function of the put is determined by the generator matrix Q_r^- of the corresponding down-crossing ladder process, which we determine by invoking the matrix Wiener-Hopf factorisation results from Section 4. Noting that $E^+ = \emptyset$ and $E^- = \{2^\star\}$ it follows that $Q^- = Q_r^-$ satisfies

$$\frac{1}{2}\Sigma^2(Q^-)^2 + VQ^- + Q_r = O \tag{38}$$

where

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

If $\beta(\theta)$ denotes a right-eigenvector of Q_r^- corresponding to eigenvalue θ it follows by right-multiplying (38) with $\beta(\theta)$ that the matrix $K(\theta) := \frac{1}{2}\Sigma^2\theta^2 + V\theta + Q_r$ is singular and $K(\theta)\beta(\theta) = 0$. It is a matter of algebra to verify that θ satisfies $g(\theta) = 0$ where

$$g(\theta) = F_1(\theta)((\alpha + \theta)F_2(\theta) - \lambda\theta) - q_1 q_2(\alpha + \theta) \tag{39}$$

with

$$F_j(\theta) = \frac{1}{2}\sigma_j^2\theta^2 + \mu_j\theta - q_j - r_j, \quad j = 1, 2. \tag{40}$$

The following result lists the properties of the roots of $g(\theta) = 0$:

Lemma 1 *Suppose that $r_1, r_2 > 0$. Then $g(\theta)$ has five different real roots which satisfy the distribution characteristics $\theta_1 < \theta_2 < \theta_3 < 0 < \theta_4 < \theta_5$. As a consequence, $Q^- = Q_r^-$ has three distinct eigenvalues θ_1, θ_2 and θ_3.*

Since eigenvectors $\beta[\theta_i]$ corresponding to the different eigenvalues θ_i, $i = 1, 2, 3$, are linearly independent, the matrix Q^- explicitly reads as

$$Q^- = \begin{pmatrix} \theta_1\beta[\theta_1] & \theta_2\beta[\theta_2] & \theta_3\beta[\theta_3] \end{pmatrix} \begin{pmatrix} \beta[\theta_1] & \beta[\theta_2] & \beta[\theta_3] \end{pmatrix}^{-1}. \tag{41}$$

1. Case $k = k_1^* = k_2^*$. For $x > k$, the value function of the American put reads as

$$W(\mathrm{e}^x, i) = e_i' \mathrm{e}^{Q^-(x-k)} H(k) \quad \text{with} \qquad H(k) = \begin{pmatrix} K - \mathrm{e}^k \\ K - \mathrm{e}^k \\ K - \mathrm{e}^k \frac{\alpha}{\alpha+1} \end{pmatrix},$$

where Q^- is given in (41) and k solves

$$-\mathrm{e}^k = e_i' Q^- H(k) \qquad i = 1, 2.$$

2. Case $k_2^* > k_1^*$. To deal with the case that $k_1^* < x < k_2^*$ and $Z = 1$, we note that, if the process Z is restricted to state 1, X is equal to a Brownian motion with drift, $\mu_1 t + \sigma_1 B_t$, killed at rate $q_1 + r_1$. The generator matrices of this restriction of Z and the corresponding ladder processes, denoted by $\bar{Q}, -\bar{Q}^+, \bar{Q}^-$, reduce in this case to scalars given by $\bar{Q} = (-q_1 - r_1)$ and the positive resp. negative root of the equation

$$\frac{1}{2}\sigma_1^2 x^2 + \mu_1 x - q_1 - r_1 = 0.$$

The associated two-sided exit probabilities from the interval $[k_1^*, k_2^*]$ read as

$$\begin{aligned} \Psi_1^-(x) &= \frac{\mathrm{e}^{\bar{Q}^-(x-k)} - \mathrm{e}^{\bar{Q}^+(\ell-x)}\mathrm{e}^{\bar{Q}^-(\ell-k)}}{1 - \mathrm{e}^{\bar{Q}^+(\ell-k)}\mathrm{e}^{\bar{Q}^-(\ell-k)}} \\ \Psi_1^+(x) &= \frac{\mathrm{e}^{\bar{Q}^+(\ell-x)} - \mathrm{e}^{\bar{Q}^-(x-k)}\mathrm{e}^{\bar{Q}^+(\ell-k)}}{1 - \mathrm{e}^{\bar{Q}^-(\ell-k)}\mathrm{e}^{\bar{Q}^+(\ell-k)}}, \end{aligned}$$

with $k = k_1^*$ and $\ell = k_2^*$. Putting everything together shows that the value function of the American put in this case is given by

$$W(\mathrm{e}^x, i) = \begin{cases} e_i' \mathrm{e}^{Q^-(x-k_2^*)}\overline{H}(k_2^*) & x \geq k_2^*, \ i = 1, 2, \\ \\ H^+(x) + \Psi_1^+(x)[C - H^+(k_2^*)] + \Psi_1^-(x) H^-(k_1^*) & k_1^* < x \leq k_2^*, \ i = 1, \end{cases}$$

where

$$\overline{H}(k) = \begin{pmatrix} C \\ K - \mathrm{e}^k \\ K - \mathrm{e}^k \frac{\alpha}{\alpha+1} \end{pmatrix}, \qquad H^-(k) = \mathrm{e}^k \frac{r_1}{q_1}, \qquad H^+(x) = K - \mathrm{e}^x \frac{q_1 + r_1}{q_1},$$

using equation (5). Here C is determined by

$$W'(\mathrm{e}^{k_2}-, 1) = e_1' Q^- \overline{H}(k_2)$$

and the levels k_1^*, k_2^* satisfy the smooth fit equations

$$\Psi_1^{-\prime}(k_1) H^-(k_1) + \Psi_1^{+\prime}(k_1)[C - H^+(k_2)] = \mathrm{e}^{k_1}\frac{r_1}{q_1} \quad \text{and} \qquad -\mathrm{e}^{k_2} = e_2' Q^- \overline{H}(k_2),$$

where prime denotes differentiation w.r.t. x.

3. Case $k_1^* > k_2^*$ For $k_2 < x < k_1$ and $Z = 2$, we are led to consider the Markov process $Y^{(2)}$ with state space $\{2, 2^\star\}$ and generator matrix

$$Q_{(2)} = \begin{pmatrix} -q_2 - r_2 - \lambda & \lambda \\ \alpha & -\alpha \end{pmatrix}.$$

In this case it can be checked from (16), (17), and (18) with Q_a replaced by $Q_{(2)}$ that $-Q^+_{(2)}$ is a scalar given by the positive root of

$$\frac{\sigma_2^2}{2}x^2 + \mu_2 x + \lambda\alpha(x+\alpha)^{-1} = q_2 + r_2 + \lambda$$

and $\eta^+ = \alpha/[-Q^+_{(2)} + \alpha]$. We can calculate $Q^-_{(2)}$ in a similar way as we calculated Q^- above. Writing $\tilde{Q}^+ = Q^+_{(2)}$ and $\tilde{Q}^- = Q^-_{(2)}$, the corresponding matrices of two-sided exit probabilities from $[k_2^*, k_1^*]$ read as

$$\begin{aligned}
\Psi_2^+(x) &= \frac{1}{c}\left[\begin{pmatrix} 1 \\ \eta^+ \end{pmatrix} \mathrm{e}^{\tilde{Q}^+(\ell - x)} - \mathrm{e}^{\tilde{Q}^-(x-k)} \begin{pmatrix} 1 \\ \eta^+ \end{pmatrix} \mathrm{e}^{\tilde{Q}^+(\ell-k)}\right] \\
\Psi_2^-(x) &= \left[\mathrm{e}^{\tilde{Q}^-(x-k)} - \mathrm{e}^{\tilde{Q}^+(\ell-x)} M \mathrm{e}^{\tilde{Q}^-(\ell-k)}\right]\left[I - \mathrm{e}^{\tilde{Q}^+(\ell-k)} M \mathrm{e}^{\tilde{Q}^-(\ell-k)}\right]^{-1}
\end{aligned}$$

with $k = k_2^*$ and $\ell = k_1^*$, where

$$c = 1 - (\ 1 \quad 0\ ) \mathrm{e}^{\tilde{Q}^-(\ell-k)} \begin{pmatrix} 1 \\ \eta^+ \end{pmatrix} \mathrm{e}^{\tilde{Q}^+(\ell-k)} \qquad \text{and} \qquad M = \begin{pmatrix} 1 & 0 \\ \eta^+ & 0 \end{pmatrix}.$$

The value function reads as

$$W(\mathrm{e}^x, i) = \begin{cases} e_i' \mathrm{e}^{Q^-(x-k_1^*)} \widetilde{H}(k_1^*) & x \geq k_1^*,\ i = 1, 2, \\ \\ \widetilde{H}^+(x) + f' \left[\Psi_2^+(x)[C - \widetilde{H}^+(k_1^*)] + \Psi_2^-(x)\widetilde{H}^-(k_2^*)\right] & k_2^* < x \leq k_1^*,\ i = 2, \end{cases}$$

where, $f = (\ 1 \quad 0\ )'$ and

$$\widetilde{H}(k) = \begin{pmatrix} K - \mathrm{e}^k \\ C \\ D \end{pmatrix}, \qquad \widetilde{H}^+(x) = K - \mathrm{e}^x \frac{q_2 + r_2}{q_2}, \qquad \widetilde{H}^-(k) = \mathrm{e}^k \frac{r_2}{q_2} \begin{pmatrix} 1 \\ \frac{\alpha}{\alpha+1} \end{pmatrix},$$

where we used again equation (5). Here C and D are determined by the two linear equations

$$W(\mathrm{e}^{k_1}-, 2^\star) = W(\mathrm{e}^{k_1}+, 2^\star) \qquad W'(\mathrm{e}^{k_1}-, 2) = W'(\mathrm{e}^{k_1}+, 2)$$

and the levels k_1^*, k_2^* satisfy the two equations

$$\begin{aligned}
f'\left[\Psi_2^{-\prime}(k_2)\widetilde{H}^-(k_2) + \Psi_2^{+\prime}(k_2)\{C - \widetilde{H}^+(k_1)\}\right] &= \mathrm{e}^{k_2}\frac{r_2}{q_2} \\
e_1' Q^- \widetilde{H}(k_1) &= -\mathrm{e}^{k_1}.
\end{aligned}$$

8 Conclusion

In this paper we explicitly solved the valuation problem of a perpetual American put under an exponential regime-switching Lévy model with phase-type jumps. The presence of regime-switches and jumps in a stock price model are supported by the economics literature and empirical literature. In order to solve the American put problem we derived an explicit solution to the first passage problem of a regime-switching Lévy process with phase-type jumps across a state-dependent level. To achieve this we first embedded the process into a continuous Markov additive process and subsequently derived the matrix Wiener-Hopf factorisation and solved explicitly the two-sided exit problem. We applied the first passage results to yield an analytical solution for the value-function of a perpetual American option driven by such a process. As illustration we worked out an explicit example in a two-regime setting.

The derived first-passage results may also find their application in the setting of structural credit risk models and valuation of (state-dependent) barriers options. We further note that the class of processes considered in this paper is flexible as it allows for any finite number of regimes and phase-type distributions are dense in the class of all distributions on a half-line. This flexibility together with the analytical tractability of first passage probabilities suggest that this class could potentially also serve to approximate barrier option prices in stochastic volatility or stochastic interest rate models. Another possible extension concerns the valuation of finite time American put options. In the setting of the Black-Scholes model Carr [9] proposed to approximate a finite time American put price by iteratively evaluating a series of perpetual American options. The solutions of the first-passage and perpetual American put problems could be employed to extend Carr's ideas to the setting of regime-switching Lévy processes.

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Appendix

A Proofs

A.1 Cramér Martingale Measure

In this section and the next we present a contruction of an equivalent martingale measure for the process (X, Z) and show how the parameters change under this change of measure. For background of Markov additive processes refer to Asmussen [3]. An important role in the construction of the change of measure is played by the process $X_a = \{X_a(t), t \geq 0\}$ defined by $X_a(t) = \int_0^t a(Z_s)\mathrm{d}X_s$, a stochastic integral against X of a function a on E^0, that is to be specified shortly. It is straightforward to verify that the process X_a is still of the form (2), but with changed parameters, and its characteristic matrix $K_a[s] = G + \Delta_a[s]$, where $\Delta_a[s]$ is the diagonal matrix with elements $\kappa_i(a_i s)$. Writing $h = h_a$ and $\lambda = \lambda_a$ for the Perron-Frobenius right-eigenvector and eigenvalue of $K_a[1]$, respectively, we define the candidate change of measure $L = \{L_t, t \geq 0\}$ by

$$L(t) = \mathrm{e}^{X_{\boldsymbol{a}}(t)-\lambda t} h(Z(t))/h(Z(0)), \tag{42}$$

with $h(i) = h_i$ the ith coordinate of h. Denoting by a_i, $i \in E^0$, the root of

$$\kappa_i(a_i + 1) = r(i) + \kappa_i(a_i),$$

where $\kappa_i(x) = \log E[\mathrm{e}^{xX_1^i}]$ is the Laplace exponent of $X_t^i = \mu_i t + \sigma_i W_t + J_i(t)$ and the function a is defined by $a(i) = a_i$. It is shown in the following result that the measure $\mathbb{P}^*$ with Radon-Nikodym derivative $\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}}\big|_{\mathcal{F}_t} = L(t)$, is indeed an equivalent martingale measure:

Proposition 2 *Suppose that* $\mathbb{E}[S_1]$ *is finite.*

(i) The process $L = \{L_t, \mathcal{F}_t, t \geq 0\}$ *is a positive mean one martingale and* $\mathbb{P}^*$ *is a probability measure;*

(ii) Under $\mathbb{P}^*$, $\exp\left(X_t - \int_0^t r(Z_s)\mathrm{d}s\right), t \geq 0$ *is a martingale.*

Proof (i) Let (X, Z) be of the form (2), with corresponding characteristic matrix K and suppose that g maps E^0 to $\mathbb{R}$. Asmussen and Kella [5] have shown that M_t is a martingale where

$$\mathrm{e}^{bX_t - ct} g(Z_t) - \mathrm{e}^{bX_0} g(Z_0) - \int_0^t \mathrm{e}^{bX_u - cu} g(Z_u)\mathrm{d}u(K[b] - cI), \tag{43}$$

is a row vector of martingales, where $c \geq 0$ and b such that the diagonal elements of $K[b]$ are finite, which holds true for $b = 1$ due to our supposition Choosing in (43) the process X to be equal to X_a, b to be one and c to be λ_a and setting $g = h_a$, it follows that $\mathrm{e}^{X_a(t)-\lambda t} h_a(Z_t) - \mathrm{e}^x h_a(Z_0)$ is a zero mean martingale. As h_a is positive, the process L in (42) is thus a positive mean one martingale. The proof of (ii) can be found in the next section. □

A.2 Change of measure

Proposition 3 *(i) Under* $\mathbb{P}^*$*,* Z *has intensity matrix* G^* *with elements*

$$g^*_{ij} = g_{ij}h(j)/h(i),\ i \neq j,\ \text{and } g^*_{ii} = -\sum_{j\neq i} g_{ij}h(j)/h(i),$$

where g_{ij} *is the* ij*th element of* G*;*

(ii) Under $\mathbb{P}^*$*, the process* X *is still of the form* (2) *with* $\sigma^*(i) = \sigma(i)$*,*

$$\mu^*(i) = \mu(i) + \alpha_i\sigma_i^2 - \int_0^1 y(1-\mathrm{e}^{\alpha_i y})\lambda_i^{(+)}F_i^{(+)}(\mathrm{d}y)$$

and with J^*_i *compound Poisson processes with changed jump rates*

$$\lambda_i^{(+)*} = \lambda^{(+)}\hat{F}^{(+)}[-\gamma] \text{ and } \lambda_i^{(-)*} = \lambda^{(-)}\hat{F}^{(-)}[\gamma]$$

and distributions of the positive and negative jumps of phase-type with representations

$$(\alpha_i^{(+)*}, T_i^{(+)*}) = (\alpha_i^{(+,a_i)}, T_i^{(+,a_i)}) \text{ and } (\alpha_i^{(-)*}, T_i^{(-)*}) = (\alpha_i^{(-,a_i)}, T_i^{(-,a_i)}),$$

respectively, where the parameters are transformed according to

$$\begin{aligned}
(\alpha^{(+,\gamma)}, T^{(+,\gamma)}) &= (\alpha^{(+)}\Delta_+/\hat{F}^{(+)}[-\gamma], \Delta_+^{-1}T^{(+)}\Delta_+ + \gamma I) \qquad (44)\\
(\alpha^{(-,\gamma)}, T^{(-,\gamma)}) &= (\alpha^{(-)}\Delta_-/\hat{F}^{(-)}[\gamma], \Delta_-^{-1}T^{(-)}\Delta_- - \gamma I) \qquad (45)
\end{aligned}$$

where Δ_+ *and* Δ_- *are the diagonal matrices with respectively* $(k_+)_i$ *and* $(k_-)_i$ *on the diagonal with* $k_+ = (-\gamma I - T^{(+)})^{-1}t^{(+)}$*,* $k_- = (\gamma I - T^{(-)})^{-1}t^{(-)}$ *and* I *is an identity matrix of appropriate size.*

Proof (i)-(ii) We first show how to find the characteristic matrix of X under $\mathbb{P}^*$. Denote by $\widetilde{X}_a$ the process $X_a + bX$, where b is such that the elements of the characteristic matrix of $\widetilde{X}_a$ are finite and let f be a function that maps E^0 to $\mathbb{R}$. Applying Itô's lemma to $\mathrm{e}^{\widetilde{X}_a(t)-\lambda t}h_a(Z_t)f(Z_t)$ shows that

$$\begin{aligned}
\mathrm{e}^{\widetilde{X}_a(t)-\lambda t}h(Z_t)f(Z_t) - \mathrm{e}^{\widetilde{X}_a(0)}h(Z_0)f(Z_0) - \int_0^t \lambda\mathrm{e}^{\widetilde{X}_a(u)-\lambda u}h(Z_u)f(Z_u)\mathrm{d}u&\\
- \int_0^t \mathrm{d}u\ \mathrm{e}^{\widetilde{X}_a(u)-\lambda u}\times&\\
\times\left[\sum_{i\in E^0}(\mathbf{1}_{\{Z_u=i\}}h(i)f(i)\kappa_i(a_i+b) + \sum_{j\neq i}g_{ij}(h(j)f(j) - h(i)f(i)))\right]&,
\end{aligned}$$

is a $\mathbb{P}$-martingale, where we wrote $\lambda = \lambda_a$ and $h = h_a$. Since

$$\lambda_a(h_a)_i = (K_a[1]h_a)_i = \sum_{j\neq i} g_{ij}(h_a(j) - h_a(i)) + \kappa_i(a_i)h_a(i),$$

it follows from taking expectations and rearranging terms that, in vector notation, it holds that

$$\mathbb{E}_{0,i}[L_t \mathrm{e}^{bX(t)} \mathbf{1}_{Z_t}] = \mathbf{1}_i + \int_0^t \mathbb{E}_{0,i}[L_u \mathrm{e}^{bX(u)} \mathbf{1}_{Z_u}](G^* + \Delta^*[b])\mathrm{d}u \qquad (46)$$

where G^* is as in the statement of the proposition and Δ^* is the diagonal matrix with elements $\Delta^*_{ii} = \kappa_i(a_i + b) - \kappa_i(a_i)$. Writing $F^*_t[b]$ for the matrix with elements $\mathbb{E}^*_{0,i}[\mathrm{e}^{bX_t} \mathbf{1}_{Z_t = j}]$ and differentiating (46) with respect to t, we arrive at the matrix differential equation

$$F^{*\prime}_t[b] = F^*_t[b](G^* + \Delta^*) \qquad F^*_0[b] = I,$$

where $\prime$ denotes the time derivative. Solving this system shows that the characteristic matrix of X under $\mathbb{P}^*$ is given by $K^*[b] = G^* + \Delta^*$. See [1] for a proof of (44) and (45); The rest of the statements of (i) and (ii) directly follow from Proposition 3 in [24]. □

Proof of Proposition 2(ii) From Proposition 3 it follows that the a_i have been chosen in such a way that, under $\mathbb{P}^*$ X^i has cumulant satisfying $\kappa^*_i(1) = \kappa_i(a_i + 1) - \kappa_i(a_i) = r_i$, so that, the characteristic matrix K^* of $(X - \int_0^\cdot r(Z_s)\mathrm{d}s, Z)$ under $\mathbb{P}^*$, satisfies $K^*[1] = G^*$ and it follows that $\mathbf{1}$ is an eigenvector of $K^*[1]$ with eigenvalue 0. Setting $g \equiv 1$ in (43) and taking $a = 0$, it follows that the process $\mathrm{e}^{X_t - \int_0^t r(Z_s)\mathrm{d}s}$ is a martingale under $\mathbb{P}^*$. □

A.3 Wiener-Hopf factorisation

Proof of Theorem 2(ii) Now we turn to the proof of the uniqueness of the Wiener-Hopf factorisation. To that end, let (Z^+, G^+, Z^-, G^-) be a Wiener-Hopf factorisation and define the function $\widetilde{f}$ as in (19), but replacing η^+ and Q^+ by Z^+ and G^+ respectively. Since (Z^+, G^+) satisfies the first equation of (16), it follows from an application of Itô's lemma, that $\widetilde{f}(Y_t, A_t)$ is a local martingale that is bounded on $\{t \leq \tau^+_\ell\}$, so that, invoking Doob's Optional Stopping Theorem, it holds that

$$\begin{aligned} \widetilde{f}(j, x) &= \mathbb{E}_{j,x}[\widetilde{f}(Y_{t \wedge \tau^+_\ell}, A_{t \wedge \tau^+_\ell})] \\ &= \mathbb{E}_{j,x}[\widetilde{f}(\widetilde{Y}^+_\ell, A_{\tau^+_\ell}) I_{(\tau^+_\ell < \infty)}] + \lim_{t \to \infty} \mathbb{E}_{j,x}[\widetilde{f}(Y_t, A_t) I_{(\tau^+_\ell = \infty)}]. \end{aligned} \qquad (47)$$

By definition of $\widetilde{f}$ and the absence of positive jumps of A, the first expectation in (47) is equal to $f(j, x)$. Note that second term in (47) is zero if Q is transient or Q is recurrent and $\sup_t A_t = +\infty$. Indeed, in the latter case, τ^+_ℓ is finite a.s., whereas in the former case $\mathbb{P}_{x,i}(Y_t \in E)$ converges to zero. Thus $f = \widetilde{f}$ for all h and we deduce that $G^+ = Q^+$ and $Z^+ = \eta^+$. Similarly, one can show that $G^- = Q^-$ and $Z^- = \eta^-$ and the uniqueness is proved. □

Proof of Theorem 2(iii) Assume that Q is recurrent but $A_t \to -\infty$. As Q^+ inherits the irreducibility property of Q, it follows from the Perron-Frobenius theorem, that the matrix Q^+ has a probability vector μ as left-eigenvector with its largest eigenvalue. Since the quadruple $(\eta^+, Q^+, \eta^-, Q^-)$ satisfies (16), it is straightforward to check that this remains the case if we replace (η^+, Q^+) by $(\eta^+(I - \mathbf{1}\mu) + \mu, Q^+(I - \mathbf{1}\mu))$. We are left to show that these are the only two factorisations of (A, Y). As in the proof of Theorem 2(ii), it follows that any factorisation quadruple of (A, Y) must contain η^- and Q^-.

Letting (η^+, G^+) and $\widetilde{f}(j, x)$ be as in the proof of Theorem 2, we distinguish between the cases that G is recurrent or transient. In the latter case $\widetilde{f}(j, x)$ tends to zero if $x \to -\infty$ and we deduce from (47) that $f = \widetilde{f}$ and thus $G = Q^+$ and $Z^+ = \eta^+$. In the former case, we note that, as G^+ inherits the irreducibility property of Q, it has a unique invariant distribution ν given by the left-eigenvector of G with eigenvalue 0. Thus $\widetilde{f}(j, x)$ converges to $e'_j \mathbf{1}\nu h = \nu h$ as $x \to \infty$. The right-hand side of (47) is thus equal to

$$\begin{aligned}\widetilde{f}(j, x) &= f(j, x) + \mathbb{P}_{j,x}(\tau_k^+ = \infty)\nu h \\ &= f(j, x) + \left(1 - e'_j W^+ \exp(Q^+(k - x))\mathbf{1}\right) \nu h. \qquad (48)\end{aligned}$$

By differentiation of (48) with respect to x, we deduce that $G = Q^+(I - \mathbf{1}\nu)$. In particular, it follows that ν is a left-eigenvector of Q^+. Since the Perron-Frobenius eigenvector is the unique eigenvector with the largest eigenvalue, it follows that $\mu = \nu$ and then also that $Z^+ = \eta^+(I - \mathbf{1}\mu) + \mu$, which completes the proof. $\square$

A.4 First passage under state-dependent levels

Proof of Theorem 3 Appealing to the strong Markov property, it follows that, for $x > k_1$,

$$v(x, i) = \mathbb{E}_{x,i}[v(k_1, Y_{\tau^-}) I_{(\tau^- < \zeta)}] = \mathbb{E}_{x,i}[h_1^-(Y_{\tau^-}) I_{(\tau^- < \zeta)}]$$

where $\tau^- = \tau_{k_1}^-$, and for $k_j < x < k_{j-1}$,

$$\begin{aligned}v(x, i) &= \mathbb{E}_{x,i}\left[v(k_{j-1}, Y_\tau) I_{(\tau<\zeta, A_\tau = k_{j-1})}\right] \\ &+ \mathbb{E}_{x,i}\left[v(k_j, Y_\tau) I_{(\tau<\zeta, A_\tau = k_j)}\right] + \mathbb{E}_{x,i}\left[v(A_{\zeta-}, Y_{\zeta-}) I_{(\zeta<\tau)}\right].\end{aligned}$$

Invoking results from Proposition 1 yields that (33) is valid, for some vectors $h_j^-, h_j^+, h_j^\dagger$. To finish the proof we have to show that the stated form of these vectors is correct. We start with noting that, by the structure of the process (A, Y), it holds that $v(k_j, j) = \mathrm{e}^{bk_j}$ and $v(k_j, \ell) = \mathrm{e}^{bk_j} e'_\ell (sI - T_j^-)^{-1} t_j^-$ for $\ell \in E_j^-$. Further, we claim that $v(\cdot, i)$ is continuous. Indeed, note that in view of the Markov property it follows that for $\ell \in E_m^-$, $m \in E^0$, it holds that

$$v(z, \ell) = \int_0^\infty v(z + y, m) e'_\ell \mathrm{e}^{T_m^- y} t_m^- \mathrm{d}y, \qquad (49)$$

so that it in particular follows that $v(\cdot, \ell)$ is continuously differentiable on (k_m, ∞). Similarly, it follows that $v(\cdot, \ell) \in C^1(k_m, \infty)$ for $\ell \in E_m^+$. The continuity of

$v(\cdot,\ell)$ for $\ell \in E^0$ follows directly from its definition. As a consequence it follows that the equations (35) hold true. Let $\ell > j, \ell \in E^0$ and consider $v(x,\ell)$ for $x \in [k_j-\epsilon, k_j+\epsilon]$. By a Feynman-Kac argument it follows that, on $x \in [k_j-\epsilon, k_j+\epsilon]$ for $\epsilon > 0$ small enough (such that $k_j - \epsilon > k_\ell$), $v(\cdot,\ell)$ is equal to the unique C^2 solution of the ODE

$$\frac{\sigma^2(\ell)}{2} f'' + \mu(\ell) f' - c(\ell) f = g \qquad f(k_j \pm \epsilon) = v(k_j \pm \epsilon, \ell)$$

for some continuous function g and some constant $c(\ell)$. In particular, $v(\cdot,\ell)$ is continuously differentiable in k_j and it follows that (34) holds true. □

A.5 American put

Proof of Theorem 1 The proof of this result follows a standard approach for solving perpetual American option pricing problems. As argued above the optimal stopping time must be of the form (7). Therefore the value function is given by V_{k^*} for some vector $k^* \in (-\infty, \log K)^N$. The vector k^* can subsequently be found by optimisation. At this point we note that the condition (5) implies that for the embedding $s(i)^2/2 + m(i) < -q_{ii}$ is satisfied for all $i \in E$, so that we can apply Theorem 3. Since, for fixed (x,i), $k \mapsto V_k(x,i)$ is continuously differentiable it follows that k^* satisfies

$$\left.\frac{\partial V_k}{\partial k_j}(\mathrm{e}^x, i)\right|_{k=k^*} = 0 \qquad \text{for all } (x,i),\ j = 1,\ldots,N. \tag{50}$$

Consider next the finite difference $[V_k(\mathrm{e}^{k_j+h}, j) - V_k(\mathrm{e}^{k_j}, j)]/h$ and note that it is equal to the sum

$$\frac{V_k(\mathrm{e}^{k_j+h}, j) - V_{k+h}(\mathrm{e}^{k_j+h}, j)}{h} + \frac{V_{k+h}(\mathrm{e}^{k_j+h}, j) - V_k(\mathrm{e}^{k_j}, j)}{h}. \tag{51}$$

Letting $h \downarrow 0$, it follows in view of (50), that the first term on the left-hand side converges to zero, while the second term, being equal to $[(K-\mathrm{e}^{k_j+h})-(K-\mathrm{e}^{k_j})]/h$, converges to $-\mathrm{e}^{k_j}$. Thus we see that the smooth fit equations (9) hold true. By a martingale argument it also follows that $V_{k^*} = W$ for any solution $k^* \in (-\infty, \log K)^N$ of (9). □

Proof of Lemma 1 Suppose first that $\alpha \neq [\mu_1 + \sqrt{\mu_1^2 + 2(r_1+q_1)\sigma_1^2}]/\sigma_1^2$. From the definitions of $g(\theta), F_1(\theta)$, and $F_2(\theta)$, we have that $g(+\infty) = +\infty$, $g(-\infty) = -\infty$, $g(0) = \alpha[(q_1+r_1)(q_2+r_2) - q_1q_2] > 0$. Note that $F_1(\theta)$ has two different real roots $\theta_{0,1} > 0 > \theta_{0,2}$ with $\theta_{0,2} = -[\mu_1 + \sqrt{\mu_1^2 + 2(r+q_1)\sigma_1^2}]/\sigma_1^2$, we then have $\theta_{0,2} \neq -\alpha$. Also, $g(\theta_{0,1}) = -q_1q_2(\alpha+\theta_{0,1}) < 0$ because $q_1, q_2, \alpha, \theta_{0,1} > 0$. Therefore we further have that

$$g(\theta_{0,2}) = \begin{cases} -q_1q_2(\alpha+\theta_{0,2}) < 0, & \text{if} \quad \theta_{0,2} > -\alpha \\ -q_1q_2(\alpha+\theta_{0,2}) > 0, & \text{if} \quad \theta_{0,2} < -\alpha. \end{cases}$$

$$\lim_{\theta\to-\alpha} g(\theta) = \begin{cases} \lambda\alpha F_1(-\alpha; r) > 0, & \text{if} \quad \theta_{0,2} > -\alpha \\ \lambda\alpha F_1(-\alpha; r) < 0, & \text{if} \quad \theta_{0,2} < -\alpha. \end{cases}$$

Appealing to the the intermediate value theorem completes the proof. Since Q^- is a generator matrix, it is negative semi-definite and the final assertion follows.

If $\alpha = [\mu_1 + \sqrt{\mu_1^2 + 2(r_1 + q_1)\sigma_1^2}]/\sigma_1^2$, $\theta = -\alpha$ is a root. By a similar reasoning as above applied to $h(\theta) = g(\theta)/(\theta + \alpha)$, noting that $h(-\alpha) < 0$, it can be shown that h has four distinct roots (two positive and two negative). $\square$

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