p-adic numbers, LTCC 2010

Manuel Breuning
King's College London

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(a) It is clear that $C(X,K)$ is a vector space over K and that $\| \ \|$ is a norm on $C(X,K)$. The only thing that remains to be shown is that $C(X,K)$ is complete. Let $f_1, f_2, f_3, \ldots$ be a Cauchy sequence in $C(X,K)$. We must show that there exists a continuous function $f : X \to K$ such that $\lim_{i\to\infty} f_i = f$ in $C(X,K)$. For every $x \in X$, the sequence $f_1(x), f_2(x), f_3(x), \ldots$ is a Cauchy sequence in K (because $|f_i(x) - f_j(x)| \le \|f_i - f_j\|$), hence we can define a function $f : X \to K$ by $f(x) = \lim_{i\to\infty} f_i(x)$.

Let $\varepsilon > 0$, and let $N \in \mathbb{N}$ be such that $\|f_i - f_j\| < \varepsilon$ for all $i, j \ge N$. Then for all $x \in X$ and $i \ge N$ we have

$$\begin{aligned} |f(x) - f_i(x)| &\le \max\{|f(x) - f_j(x)|, |f_j(x) - f_i(x)|\} \\ &\le \max\{|f(x) - f_j(x)|, \|f_j - f_i\|\} \\ &< \varepsilon \end{aligned}$$

(the last inequality holds if one chooses a sufficiently large j). Now once we have shown that f is continuous, it also follows that $\|f - f_i\| \le \varepsilon$ for all $i \ge N$, and hence $\lim_{i\to\infty} f_i = f$.

It remains to show the continuity of f. Let $x \in X$ and $\varepsilon > 0$. Since f_N is continuous there exists $\delta > 0$ such that $|x - y| < \delta \Longrightarrow |f_N(x) - f_N(y)| < \varepsilon$ (where N is as above). It follows that for $|x - y| < \delta$ we have

$$\begin{aligned} |f(x) - f(y)| &\le \max\{|f(x) - f_N(x)|, |f_N(x) - f_N(y)|, |f_N(y) - f(y)|\} \\ &< \varepsilon, \end{aligned}$$

so f is continuous.

(b) It is clear that $\mathrm{LC}(X,K)$ is a vector subspace of $C(X,K)$, so we only need to show that $\mathrm{LC}(X,K)$ is dense in $C(X,K)$. Let $f \in C(X,K)$ and let $\varepsilon > 0$. We must show that there exists a locally constant function $c : X \to K$ such that $\|f - c\| < \varepsilon$.

If $k_1, k_2 \in K$ then the open balls $B_{<\varepsilon}(k_1)$ and $B_{<\varepsilon}(k_2)$ are either equal or disjoint. Hence

$$\{B_{<\varepsilon}(k) : k \in K\}$$

is a cover of K consisting of disjoint open and closed sets. Therefore

$$\mathcal{X} = \{f^{-1}(B_{<\varepsilon}(k)) : k \in K\}$$

is a cover of X consisting of disjoint open and closed sets. For each nonempty set $U \in \mathcal{X}$ we fix a point $a_U \in U$. We then define $c : X \to K$ by

$$c(x) = f(a_U) \quad \text{if} \quad x \in U.$$

Then c is locally constant and $\|f - c\| < \varepsilon$ (because if $x \in U \in \mathcal{X}$ then $|f(x) - c(x)| = |f(x) - f(a_U)| < \varepsilon$ since $f(x), f(a_U) \in f(U) \subseteq B_{<\varepsilon}(k)$ for some $k \in K$).

(c) First assume that a is a p-power root of unity, i.e. $a^{p^n} = 1$ for some $n \in \mathbb{N}$. If $x, y \in \mathbb{Z}_p$ with $|x - y| \le p^{-n}$ then $x - y = p^n z$ for some $z \in \mathbb{Z}_p$. Hence $a^x = a^{y + p^n z} = a^y (a^{p^n})^z = a^y$. This shows that the function $x \mapsto a^x$ is locally constant.

Conversely assume that the function $x \mapsto a^x$ is locally constant. Then there exists a neighbourhood U of 0 in $\mathbb{Z}_p$ such that the function $x \mapsto a^x$ is constant on U. We have $p^n \in U$ for sufficiently large $n \in \mathbb{N}$, hence $a^{p^n} = a^0 = 1$ for some $n \in \mathbb{N}$.