

p -adic numbers, LTCC 2010

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- (a) Let $c > 0$ and let $f \in \text{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$. Then it's almost trivial that f is continuous, but let's show it in some detail.

Fix an $M > 0$ such that $|f(x) - f(y)| \leq M \cdot |x - y|^c$ for all $x, y \in \mathbb{Z}_p$. Given $\varepsilon > 0$ choose $\delta = (\varepsilon/M)^{1/c}$. Now if $|x - y| < \delta$ then $|f(x) - f(y)| \leq M \cdot |x - y|^c < M \cdot \delta^c = \varepsilon$. This shows that f is uniformly continuous, hence continuous.

- (b) Let $f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$ be a locally constant function, and let $c > 0$. We want to show that $f \in \text{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$.

For every $x \in \mathbb{Z}_p$ there exists an $r_x > 0$ such that the restriction of f to the open ball $B_{<r_x}(x)$ is constant. Since \mathbb{Z}_p is compact, there exist finitely many points $x_1, \dots, x_k \in \mathbb{Z}_p$ such that $\mathbb{Z}_p = B_{<r_{x_1}}(x_1) \cup \dots \cup B_{<r_{x_k}}(x_k)$. Let $r = \min\{r_1, \dots, r_k\}$. Let $M = r^{-c} \cdot \sup_{x \in \mathbb{Z}_p} |f(x)|$. We claim that then $|f(x) - f(y)| \leq M \cdot |x - y|^c$ for all $x, y \in \mathbb{Z}_p$, and thus $f \in \text{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$. To see this we distinguish two cases.

Case $|x - y| < r$. Then $x \in B_{r_{x_i}}(x_i)$ for some i . It follows that $y \in B_{r_{x_i}}(x_i)$ (because $|y - x_i| \leq \max\{|y - x|, |x - x_i|\} < r_{x_i}$). Hence $f(x) = f(y)$ and thus $|f(x) - f(y)| = 0 \leq M \cdot |x - y|^c$.

Case $|x - y| \geq r$. Then

$$|f(x) - f(y)| \leq \max\{|f(x)|, |f(y)|\} \leq \sup_{x \in \mathbb{Z}_p} |f(x)| = M \cdot r^c \leq M \cdot |x - y|^c.$$

- (c) Let $f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$ be the function given by

$$f(a_0 + a_1p^1 + a_2p^2 + a_3p^3 + \dots) = a_0 + a_1p^{1!} + a_2p^{2!} + a_3p^{3!} + \dots$$

where $a_0, a_1, a_2, a_3, \dots \in \{0, 1, 2, \dots, p-1\}$ (it was shown in the course that every element of \mathbb{Z}_p can be written uniquely in this form).

Let $x, y \in \mathbb{Z}_p$ with $|x - y| = p^{-k}$ for some $k \in \mathbb{Z}_{\geq 0}$. Write $x = a_0 + a_1p + \dots$ and $y = b_0 + b_1p + \dots$. Then $a_0 = b_0, a_1 = b_1, \dots, a_{k-1} = b_{k-1}$ and $a_k \neq b_k$. It follows that $f(x) - f(y) = (a_k - b_k)p^{k!} + (a_{k+1} - b_{k+1})p^{(k+1)!} + \dots$, hence $|f(x) - f(y)| = p^{-k!}$.

Now let $c > 0$. Choose a constant M such that $M \geq p^{ck - k!}$ for all $k \in \mathbb{Z}_{\geq 0}$ (such an M exists because $ck - k! \rightarrow -\infty$ as $k \rightarrow \infty$). If $x, y \in \mathbb{Z}_p$, then $|x - y| = p^{-k}$ for some $k \in \mathbb{Z}_{\geq 0}$, and hence $|f(x) - f(y)| = p^{-k!} \leq M \cdot p^{-ck} = M \cdot |x - y|^c$. This shows that $f \in \text{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$.

The function f is not locally constant because it is not constant in any neighbourhood of 0 (since $f(0) = 0$ but $f(p^n) = p^{n!} \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$).

- (d) Let $e_0, e_1, e_2, \dots \in \mathbb{Z}_{\geq 0}$ be any (not strictly) monotonically increasing sequence satisfying $e_k \rightarrow \infty$ as $k \rightarrow \infty$ and $kc - e_k \rightarrow \infty$ as $k \rightarrow \infty$ for every $c > 0$. For example one can take $e_k = \lceil \sqrt{k} \rceil$. Define a function $f : \mathbb{Z}_p \rightarrow \mathbb{Q}_p$ by

$$f(a_0 + a_1p^1 + a_2p^2 + a_3p^3 + \dots) = a_0 + a_1p^{e_1} + a_2p^{e_2} + a_3p^{e_3} + \dots$$

where $a_0, a_1, a_2, a_3, \dots \in \{0, 1, 2, \dots, p-1\}$.

Let $x, y \in \mathbb{Z}_p$ with $|x - y| = p^{-k}$ for some $k \in \mathbb{Z}_{\geq 0}$. Write $x = a_0 + a_1p + \dots$ and $y = b_0 + b_1p + \dots$. Then $a_0 = b_0, a_1 = b_1, \dots, a_{k-1} = b_{k-1}$ and $a_k \neq b_k$. It follows that $f(x) - f(y) = (a_k - b_k)p^{e_k} + (a_{k+1} - b_{k+1})p^{e_{k+1}} + \dots$, hence $|f(x) - f(y)| \leq p^{-e_k}$. Since $e_k \rightarrow \infty$ as $k \rightarrow \infty$, this easily implies that f is uniformly continuous and hence continuous.

Finally let $c > 0$. We claim that $f \notin \text{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$. To show this it suffices to show that $\frac{|f(x) - f(y)|}{|x - y|^c}$ is not bounded for $x, y \in \mathbb{Z}_p$. But this is clear because if $x = p^k$ and $y = 0$ then

$$\frac{|f(x) - f(y)|}{|x - y|^c} = \frac{|p^{e_k} - 0|}{|p^k - 0|^c} = p^{kc - e_k}$$

and $kc - e_k \rightarrow \infty$ as $k \rightarrow \infty$ by assumption.