Solutions to the Exam, April 2010

(a) Let \( c > 0 \) and let \( f \in \text{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p) \). Then it’s almost trivial that \( f \) is continuous, but let’s show it in some detail.

Fix an \( M > 0 \) such that \( |f(x) - f(y)| \leq M \cdot |x - y|^c \) for all \( x, y \in \mathbb{Z}_p \).

Given \( \varepsilon > 0 \) choose \( \delta = (\varepsilon / M)^{1/c} \). Now if \( |x - y| < \delta \) then \( |f(x) - f(y)| \leq M \cdot |x - y|^c < M \cdot \delta^c = \varepsilon \). This shows that \( f \) is uniformly continuous, hence continuous.

(b) Let \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) be a locally constant function, and let \( c > 0 \). We want to show that \( f \in \text{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p) \).

For every \( x \in \mathbb{Z}_p \) there exists an \( r_x > 0 \) such that the restriction of \( f \) to the open ball \( B_{r_x}(x) \) is constant. Since \( \mathbb{Z}_p \) is compact, there exist finitely many points \( x_1, \ldots, x_k \in \mathbb{Z}_p \) such that \( \mathbb{Z}_p = B_{r_{x_1}}(x_1) \cup \cdots \cup B_{r_{x_k}}(x_k) \).

Let \( r = \min\{r_1, \ldots, r_k\} \). Let \( M = r^{-c} \cdot \sup_{x \in \mathbb{Z}_p} |f(x)| \). We claim that then \( |f(x) - f(y)| \leq M \cdot |x - y|^c \) for all \( x, y \in \mathbb{Z}_p \), and thus \( f \in \text{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p) \).

To see this we distinguish two cases.

Case \( |x - y| < r \). Then \( x \in B_{r_x}(x_i) \) for some \( i \). It follows that \( y \in B_{r_x}(x_i) \) (because \( |y - x_i| \leq \max\{|y - x|, |x - x_i|\} < r_x \)). Hence \( f(x) = f(y) \) and thus \( |f(x) - f(y)| = 0 \leq M \cdot |x - y|^c \).

Case \( |x - y| \geq r \). Then \( |f(x) - f(y)| \leq \max\{|f(x)|, |f(y)|\} \leq \sup_{x \in \mathbb{Z}_p} |f(x)| = M \cdot r^c \leq M \cdot |x - y|^c \).

(c) Let \( f : \mathbb{Z}_p \to \mathbb{Q}_p \) be the function given by
\[
f(a_0 + a_1p + a_2p^2 + a_3p^3 + \ldots) = a_0 + a_1p + a_2p^2 + a_3p^3 + \ldots
\]
where \( a_0, a_1, a_2, a_3, \ldots \in \{0, 1, 2, \ldots, p - 1\} \) (it was shown in the course that every element of \( \mathbb{Z}_p \) can be written uniquely in this form).

Let \( x, y \in \mathbb{Z}_p \) with \( |x - y| = p^{-k} \) for some \( k \in \mathbb{Z}_{\geq 0} \). Write \( x = a_0 + a_1p + \ldots \) and \( y = b_0 + b_1p + \ldots \). Then \( a_0 = b_0, a_1 = b_1, \ldots, a_{k-1} = b_{k-1} \) and \( a_k \neq b_k \). It follows that \( f(x) - f(y) = (a_k - b_k)p^k + (a_{k+1} - b_{k+1})p^{k+1} + \ldots \), hence \( |f(x) - f(y)| = p^{-k} \).

Now let \( c > 0 \). Choose a constant \( M \) such that \( M \geq p^{ck-k!} \) for all \( k \in \mathbb{Z}_{\geq 0} \) (such an \( M \) exists because \( ck - k! \rightarrow -\infty \) as \( k \rightarrow \infty \)). If \( x, y \in \mathbb{Z}_p \), then \( |x - y| = p^{-k} \) for some \( k \in \mathbb{Z}_{\geq 0} \), and hence \( |f(x) - f(y)| = p^{-k!} \leq M \cdot p^{-ck} = M \cdot |x - y|^c \). This shows that \( f \in \text{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p) \).
The function $f$ is not locally constant because it is not constant in any
neighbourhood of 0 (since $f(0) = 0$ but $f(p^n) = p^n \neq 0$ for all $n \in \mathbb{Z}_{\geq 0}$).

(d) Let $e_0, e_1, e_2, \cdots \in \mathbb{Z}_{\geq 0}$ be any (not strictly) monotonically increasing
sequence satisfying $e_k \to \infty$ as $k \to \infty$ and $kc - e_k \to \infty$ as $k \to \infty$
for every $c > 0$. For example one can take $e_k = \lfloor \sqrt{k} \rfloor$. Define a function
$f : \mathbb{Z}_p \to \mathbb{Q}_p$ by
\[
f(a_0 + a_1p^1 + a_2p^2 + a_3p^3 + \ldots) = a_0 + a_1p^{e_1} + a_2p^{e_2} + a_3p^{e_3} + \ldots
\]
where $a_0, a_1, a_2, a_3, \cdots \in \{0, 1, 2, \ldots, p - 1\}$.

Let $x, y \in \mathbb{Z}_p$ with $|x - y| = p^{-k}$ for some $k \in \mathbb{Z}_{\geq 0}$. Write $x = a_0 + a_1p +
\ldots$ and $y = b_0 + b_1p + \ldots$. Then $a_0 = b_0, a_1 = b_1, \ldots, a_{k-1} = b_{k-1}$ and
$a_k \neq b_k$. It follows that $f(x) - f(y) = (a_0 - b_0)p^{e_k} + (a_{k+1} - b_{k+1})p^{e_{k+1}} + \ldots$, hence $|f(x) - f(y)| \leq p^{-e_k}$. Since $e_k \to \infty$ as $k \to \infty$, this easily implies
that $f$ is uniformly continuous and hence continuous.

Finally let $c > 0$. We claim that $f \not\in \text{Lip}_c(\mathbb{Z}_p, \mathbb{Q}_p)$. To show this it
suffices to show that $\frac{|f(x) - f(y)|}{|x - y|^c}$ is not bounded for $x, y \in \mathbb{Z}_p$. But this is
clear because if $x = p^k$ and $y = 0$ then
\[
\frac{|f(x) - f(y)|}{|x - y|^c} = \frac{|p^{e_k} - 0|}{|p^k - 0|^c} = p^{kc - e_k}
\]
and $kc - e_k \to \infty$ as $k \to \infty$ by assumption.