p-adic numbers, LTCC 2010

Manuel Breuning
King's College London

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5. Differentiation

Let p be a prime number and let $(K, |\ |)$ be a complete extension of $(\mathbb{Q}_p, |\ |)$ (e.g. $K = \mathbb{Q}_p$ or $K = \mathbb{C}_p$). In this chapter we discuss the definition and some basic properties of (strictly) differentiable functions $X \to K$ where $X \subseteq K$.

5.1. **Differentiability and strict differentiability.** We say that a non-empty subset X of K has *no isolated points* if for every $a \in X$ and every neighbourhood U of a in X the set $U \setminus \{a\}$ is non-empty.

Definition 5.1. Let X be a non-empty subset of K without isolated points, and let $f : X \to K$ be a function.

(1) We say that f is *differentiable at a point* $a \in X$ (with derivative $f'(a)$) if the limit $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.
(2) We say that f is *differentiable* (on X) if f is differentiable at every point $a \in X$. In this case the derivative f' is again a function $X \to K$.

Exercise 5.2. Show that if $f : X \to K$ is differentiable at a point $a \in X$ then f is continuous at a.

A function $f : X \to K$ is called *locally constant* if every point $a \in X$ has a neighbourhood U in X such that the restriction of f to U is a constant function. Clearly every locally constant function is differentiable with derivative 0. The following example shows that the converse of this statement is not true.

Example 5.3. For every $n \in \mathbb{N}$ let $B_n = \{x \in \mathbb{Z}_p : |x - p^n| < p^{-2n}\} \subset \mathbb{Z}_p$. Note that the balls B_n are pairwise disjoint. Define a function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ ($\subset \mathbb{Q}_p$) by

$$f(x) = \begin{cases} p^{2n} & \text{if } x \in B_n, \\ 0 & \text{if } x \in \mathbb{Z}_p \setminus \bigcup_{n\in\mathbb{N}} B_n. \end{cases}$$

We claim that

(1) f is differentiable with $f' = 0$,
(2) f is not locally constant.

It is easy to see that on $\mathbb{Z}_p \setminus \{0\}$ the function f is locally constant. Hence if $a \in \mathbb{Z}_p \setminus \{0\}$ then f is differentiable at a with derivative $f'(a) = 0$. Furthermore f is differentiable at the point 0 with derivative $f'(0) = 0$, because for $x \neq 0$ we have

$$\left|\frac{f(x) - f(0)}{x - 0}\right| = \begin{cases} |p^{2n}|/|p^n| = p^{-n} & \text{if } x \in B_n, \\ 0 & \text{if } x \in \mathbb{Z}_p \setminus \bigcup_{n\in\mathbb{N}} B_n \end{cases}$$

and hence $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0$. Finally, f is not locally constant on $\mathbb{Z}_p$ because $f(0) = 0$ but in every neighbourhood of 0 there exists a point x with $f(x) \neq 0$.

Definition 5.4. Let X be a non-empty subset of K without isolated points, and let $f : X \to K$ be a function.

(1) We say that f is *strictly differentiable at a point* $a \in X$ if the difference quotient

$$\Phi f(x, y) = \frac{f(x) - f(y)}{x - y}$$

has a limit as $(x, y) \to (a, a)$, $x \neq y$.

(2) We say that f is *strictly differentiable* (on X) if f is strictly differentiable at every point $a \in X$.

Some authors (e.g. [Schikhof]) use the expression *continuously differentiable* instead of strictly differentiable.

Lemma 5.5. (1) *If f is strictly differentiable at a point $a \in X$ then f is differentiable at a and $f'(a) = \lim_{(x,y)\to(a,a)} \Phi f(x,y)$.*

(2) *If f is strictly differentiable on X then f is differentiable on X and the function $f' : X \to K$ is continuous.*

Proof. Everything is clear except for the continuity of f'. Let $a \in X$ and $\varepsilon > 0$. We must show that there exists a neighbourhood U of a in X such that $|f'(a) - f'(b)| < \varepsilon$ for all $b \in U$. Since f is strictly differentiable in a, there exists an open neighbourhood U of a in X such that $|f'(a) - \Phi f(x,y)| < \varepsilon$ for all $(x,y) \in U \times U$ with $x \neq y$. Now let $b \in U$. Then since f is strictly differentiable in b, the point b has a neighbourhood $V \subseteq U$ such that $|f'(b) - \Phi f(x,y)| < \varepsilon$ for all $(x,y) \in V \times V$ with $x \neq y$. Fix $y \in V \setminus \{b\}$. Then

$$\begin{aligned} |f'(a) - f'(b)| &= |f'(a) - \Phi f(b,y) + \Phi f(b,y) - f'(b)| \\ &\leq \max\{|f'(a) - \Phi f(b,y)|, |\Phi f(b,y) - f'(b)|\} \\ &< \varepsilon \end{aligned}$$

as required. ☐

By Lemma 5.5(2), every strictly differentiable function is differentiable with continuous derivative. However the following example shows that the converse of this statement is false.

Example 5.6. Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be the function from Example 5.3. We have already seen that f is differentiable and f' is continuous (because $f' = 0$). However f' is not strictly differentiable at the point 0, i.e. the limit $\lim_{(x,y)\to(0,0)} \Phi f(x,y)$ does not exist. Indeed, taking the sequence $(x_n, y_n) = (p^n, 0)$ (which converges to $(0,0)$) gives the limit $\lim_{n\to\infty} \Phi f(x_n, y_n) = 0$ (as seen in Example 5.3), but taking the sequence $(x_n, y_n) = (p^n, p^n - p^{2n})$ (which also converges to $(0,0)$) gives the limit

$$\lim_{n\to\infty} \Phi f(x_n, y_n) = \lim_{n\to\infty} \frac{p^{2n} - 0}{p^n - (p^n - p^{2n})} = 1.$$

Exercise 5.7. Let X and Y be non-empty subsets of K without isolated points. Let $f : X \to K$ and $g : Y \to K$ be functions such that $f(X) \subseteq Y$. Show that if f is strictly differentiable at a point $a \in X$ and g is strictly differentiable at the point $f(a)$, then $g \circ f$ is strictly differentiable at a with derivative $(g \circ f)'(a) = g'(f(a))f'(a)$.

5.2. **Local invertibility of strictly differentiable functions.**

Lemma 5.8. *Let X be a non-empty subset of K without isolated points. Let $f : X \to K$ be strictly differentiable at a point $a \in X$. If $f'(a) \neq 0$ then there exists a neighbourhood U of a in X such that*

$$|f(x) - f(y)| = |f'(a)| \cdot |x - y|$$

for all $x, y \in U$. In particular, f is injective on U.

Proof. Since $f'(a) \neq 0$ and $\Phi f(x,y) \to f'(a)$ as $(x,y) \to (a,a)$ (with $x \neq y$), there exists a neighbourhood U of a in X such that $|\Phi f(x,y) - f'(a)| < |f'(a)|$ for all $x, y \in U$ with $x \neq y$. But this implies that $|\Phi f(x,y)| = |f'(a)|$ (because otherwise Lemma 2.4 would give the contradiction $\max\{|\Phi f(x,y)|, |f'(a)|\} = |\Phi f(x,y) - f'(a)| < |f'(a)|$). After multiplying by $|x-y|$ we obtain $|f(x) - f(y)| = |f'(a)| \cdot |x-y|$ for all $x, y \in U$ with $x \neq y$. But clearly this equality is also true if $x, y \in U$ and $x = y$. ☐

Example 5.9. The lemma is not true if strictly differentiable is replaced by differentiable. For example, if $g : \mathbb{Z}_p \to \mathbb{Z}_p$ is defined by $g(x) = f(x) + x$ where $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is the function from Example 5.3, then g is differentiable at 0 with derivative $g'(0) = 1 \neq 0$. However g is not injective on any neighbourhood of 0 because $g(p^n) = p^n + p^{2n} = g(p^n + p^{2n})$ for all $n \in \mathbb{N}$.

Theorem 5.10. *Let $X \subseteq K$ be non-empty and open. Let $f : X \to K$ be strictly differentiable at a point $a \in X$. If $f'(a) \neq 0$ then for all sufficiently small $r > 0$ the function f maps the closed ball $B_{\leq r}(a)$ bijectively onto the closed ball $B_{\leq |f'(a)|r}(f(a))$, and the local inverse $g : B_{\leq |f'(a)|r}(f(a)) \to B_{\leq r}(a)$ is strictly differentiable at $f(a)$ with $g'(f(a)) = f'(a)^{-1}$.*

Before proving Theorem 5.10 we recall Banach's contraction theorem.

Theorem 5.11 (Banach's contraction theorem)**.** *Let (X, d) be a non-empty complete metric space. Let $F : X \to X$ be a contraction (i.e. there exists a constant $0 < \tau < 1$ such that $d(F(x), F(y)) \leq \tau d(x, y)$ for all $x, y \in X$). Then F has precisely one fixed point (i.e. there exists precisely one $b \in X$ such that $F(b) = b$).*

For a proof of Banach's contraction theorem see e.g. [Schikhof, Appendix A.1].

Proof of Theorem 5.10. Fix a constant τ with $0 < \tau < 1$. Since $\Phi f(x, y) \to f'(a)$ as $(x, y) \to (a, a)$ (with $x \neq y$), there exists a neighbourhood U of a in X such that

$$\left| \frac{f(x) - f(y)}{x - y} - f'(a) \right| \leq \tau |f'(a)|$$

for all $x, y \in U$ with $x \neq y$. For all sufficiently small $r > 0$ we have $B_{\leq r}(a) \subseteq U$. We claim that for such r the map f maps $B_{\leq r}(a)$ bijectively onto $B_{\leq |f'(a)|r}(f(a))$.

As in the proof of Lemma 5.8 we have $\left| \frac{f(x)-f(y)}{x-y} \right| = |f'(a)|$ for all $x, y \in B_{\leq r}(a)$ with $x \neq y$. By choosing $y = a$ this implies that $|f(x) - f(a)| = |f'(a)| \cdot |x - a|$ for all $x \in B_{\leq r}(a)$. Hence $f(x) \in B_{\leq |f'(a)|r}(f(a))$ for all $x \in B_{\leq r}(a)$ as required. Furthermore the equality $|f(x) - f(y)| = |f'(a)| \cdot |x - y|$ also implies that f is injective on $B_{\leq r}(a)$.

Now let $c \in B_{\leq |f'(a)|r}(f(a))$. Define a function F by $F(x) = x - (f(x) - c)/f'(a)$. If $x \in B_{\leq r}(a)$ then $|f(x) - c| \leq \max\{|f(x) - f(a)|, |f(a) - c|\} \leq |f'(a)|r$ and hence

$$|F(x) - a| = |x - a - (f(x) - c)/f'(a)| \leq \max\{|x - a|, |f(x) - c|/|f'(a)|\} \leq r.$$

This shows that F is a map $B_{\leq r}(a) \to B_{\leq r}(a)$. Furthermore F is a contraction because for any $x, y \in B_{\leq r}(a)$ we have

$$\begin{aligned} |F(x) - F(y)| &= \left| x - y - \frac{f(x) - f(y)}{f'(a)} \right| \\ &= \frac{|x - y|}{|f'(a)|} \cdot \left| f'(a) - \frac{f(x) - f(y)}{x - y} \right| \\ &\leq \frac{|x - y|}{|f'(a)|} \cdot \tau |f'(a)| = \tau |x - y|. \end{aligned}$$

Now the space $B_{\leq r}(a)$ is complete because it is closed in the complete space K. Hence by Banach's contraction theorem the map F has a fixed point $b \in B_{\leq r}(a)$. But clearly $F(b) = b$ implies $f(b) = c$. This shows that $B_{\leq |f'(a)|r}(f(a)) \subseteq f(B_{\leq r}(a))$ as required.

Let $g : B_{\leq |f'(a)|r}(f(a)) \to B_{\leq r}(a)$ be the inverse of f. From $|f(x) - f(y)| = |f'(a)| \cdot |x - y|$ for all $x, y \in B_{\leq r}(a)$ we obtain $|f'(a)|^{-1} \cdot |t - u| = |g(t) - g(u)|$ for all $t, u \in B_{\leq |f'(a)|r}(f(a))$. This implies that g is continuous. Now

$$\Phi g(t, u) = \frac{g(t) - g(u)}{f(g(t)) - f(g(u))} = \big(\Phi f(g(t), g(u))\big)^{-1}.$$

If $(t,u) \to (f(a), f(a))$ with $t \neq u$, then $(g(t), g(u)) \to (g(f(a)), g(f(a)) = (a,a)$. Hence we see that $\lim_{(t,u)\to(f(a),f(a))} \Phi g(t,u)$ exists and is equal to $f'(a)^{-1}$. $\square$

5.3. **Further results on strictly differentiable functions.** Finally we mention some further results without proof.

Theorem 5.12. *Let* $f : \mathbb{Z}_p \to K$ *be a continuous function with Mahler expansion* $f = \sum_{n=0}^{\infty} a_n \binom{\cdot}{n}$. *Then* f *is strictly differentiable if and only if* $\lim_{n\to\infty} n|a_n| = 0$.

For a proof see [Schikhof, §53]. Recall that if $a \in 1 + M \subset K$, i.e. $|a-1| < 1$, and $x \in \mathbb{Z}_p$ then a^x is given by the Mahler series

$$a^x = \sum_{n=0}^{\infty} (a-1)^n \binom{x}{n}$$

(cf. §4.3). Since $\lim_{n\to\infty} n|a-1|^n = 0$, it follows from the theorem that the function $x \mapsto a^x$ is strictly differentiable.

Theorem 5.13. *Let* X *be a non-empty subset of* K *without isolated points, and let* $f : X \to K$ *be a continuous function. Then there exists a strictly differentiable function* $F : X \to K$ *such that* $F' = f$.

For a proof of this theorem and many more results on (strictly) differentiable functions see [Schikhof].

References

[Schikhof] W.H. Schikhof, *Ultrametric calculus. An introduction to p-adic analysis*, Cambridge University Press, 1984.