# BRST quantization in the canonical setting

Alice Rogers

*King's College London, Department of Mathematics, Strand, London WC2R 2LS, Great Britain*

**Abstract.** After an introduction to the canonical BRST quantization procedure for a first class constrained system on a symplectic manifold, the modification required for a system with reducible symmetry is described. A topological model which exhibits this method is discussed.

**Keywords:** BRST, Hamiltonian, symplectic, ghost, reducible symmetry, topological, equivariant
**PACS:** 02.40.-k,03.65.-w,03.65.Ca,11.15.-q

## 1. INTRODUCTION

The motivation for the work described in this lecture is the desire to better understand path integral and functional integral methods for quantum theories with symmetry. The generic functional integral for a theory has the form

$$\int_{\text{Fields/Symmetry}} \mathscr{D}\phi \exp(-S(\phi(\cdot))) \tag{1}$$

where $\phi$ denotes the fields and $S$ the action of the theory. Such integrals are extremely powerful heuristic tools, which have led to many new and deep results in both fundamental physics and mathematics. However in general these integrals are not mathematically well defined objects, so that results obtained are not fully justified by these methods. In the case of simple quantum mechanical systems the path integrals used can be derived in a mathematically watertight way from the canonical formalism. However when a system posses symmetry the measure on the much more complicated space of paths modulo symmetries is harder to construct.

In this lecture the canonical BRST quantization procedure, which provides a route for constructing these measures in a mathematically rigorous way, is described. The construction is extended to a doubly reduced system corresponding to a system with reducible symmetry, which leads to an equivariant BRST operator. The method is illustrated by a topological model.

## 2. CLASSICAL DYNAMICS AND BRST

Canonical quantization is based on classical Hamiltonian dynamics. The ingredients of a typical system are a $2n$-dimensional symplectic manifold $\mathscr{N}$ with symplectic form $\omega$ and a Hamiltonian $H : \mathscr{N} \to \mathbb{R}$ which determines the time development of the system.

Such a system has a symmetry if $H$ is invariant under some action

$$\mathscr{N} \times G \to \mathscr{N}$$

CP956, *XXVI Workshop on Geometrical Methods in Physics*
edited by P. Kielanowski, A. Odzijewicz, M. Schlichenmaier, and T. Voronov

of a Lie group $G$ on the symplectic manifold $\mathcal{N}$ where this action is Hamiltonian, that is, it preserves the symplectic form and additionally there is a *moment map*

$$\Phi : \mathcal{N} \to \mathfrak{g}^* \tag{2}$$

(where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of $G$) which obeys certain conditions.

These conditions are most easily stated in terms of the transpose of the moment map, which is a function taking values in the space $\mathscr{F}(\mathcal{N})$ of smooth functions on $\mathcal{N}$ defined by

$$\begin{array}{rclcrcl} T:\mathfrak{g} & \to & \mathscr{F}(\mathcal{N}) & & \pi & \mapsto & T_\pi \\ T_\pi(y) & = & \langle \Phi(y), \pi \rangle & \forall y \in \mathcal{N}, \pi \in \mathfrak{g}\,. & & & \end{array} \tag{3}$$

This map, known as the constraint map, relates to the constraints which appear when passing from the Lagrangian to the Hamiltonian picture using the Legendre transformation. Suppose that

$$\{\pi_a | a = 1, \ldots, m\} \tag{4}$$

is a basis of $\mathfrak{g}$; then the corresponding constraint functions are

$$\{T_a \equiv T_{\pi_a} | a = 1, \ldots, m\}\,. \tag{5}$$

The conditions which the function $T$ must satisfy are that

$$\mathscr{L}_{\underline{\pi}} f = \{T_\pi, f\} \quad \text{and} \quad T_\pi(yg) = T_{\mathrm{Ad}_g \pi}(y) \quad \forall y \in \mathcal{N}, g \in G \, \text{and} \, \pi \in \mathfrak{g} \tag{6}$$

where $\underline{\pi}$ is the vector field on $\mathcal{N}$ corresponding to $\pi \in \mathfrak{g}$ under the group $G$ action and $\mathscr{L}$ denotes the Lie derivative. The second of these conditions implies that

$$\{T_\pi, T_{\pi'}\} = T_{[\pi, \pi']} \tag{7}$$

or, in terms of the basis (4),

$$\{T_a, T_b\} = C^c_{ab} T_c\,, \tag{8}$$

where $C^c_{ab}$ are the structure constants of $\mathfrak{g}$ in this basis.

The symmetry of this system leads to some redundancy; this is removed by passing to the *reduced phase space* which is the space where the dynamics of the system takes place. This reduced phase space is defined in two stages: first, define the subset $C$ of $\mathcal{N}$ by the condition

$$C = \Phi^{-1}(0)\,, \tag{9}$$

so that $C$ is the subset of $\mathcal{N}$ on which every function of the form $T_\pi, \pi \in \mathfrak{g}$ vanishes. Then, by (6), the action of $G$ on $\mathcal{N}$ reduces to an action of $G$ on $C$. The reduced phase space, denoted $\mathcal{N} /\!\!/ G$, is defined by

$$\mathcal{N} /\!\!/ G = C/G\,. \tag{10}$$

When the $G$ action is sufficiently nice $C$ is a manifold of dimension $2n - m$ (where $m$ is the dimension of $G$) and $\mathcal{N} /\!\!/ G$ is a manifold of dimension $2n - 2m$ which by a theorem of Marsden and Weinstein [1] has a natural symplectic structure with symplectic form $\nu$ determined by

$$\iota^*\omega = \pi^*\nu \tag{11}$$

where $\iota : C \to \mathscr{N}$ is inclusion and $\pi : C \to C/G = \mathscr{N}/\!\!/G$ is the natural projection. This symplectic form corresponds to the Dirac bracket, a modified Poisson bracket introduced by Dirac to handle systems with symmetry [2, 3]. Details of this Marsden-Weinstein reduction process, which is an elegant process for handling the classical Hamiltonian dynamics of a system with symmetry, can be found in various texts, for example in the book of Woodhouse [4].

It is often the case that while the original phase space $\mathscr{N}$ is a relatively simple symplectic manifold (typically, the cotangent bundle of a manifold, with canonical symplectic structure) admitting a quantization, the corresponding reduced phase space is rather complicated. The BRST method is one approach to quantizing such systems, providing a quantization scheme which is well adapted to functional integral methods whenever this was the case with the original phase space. The formulation of BRST cohomology in the canonical setting was first given by Henneaux [23] and by McMullan [18], providing a powerful development of the BFV construction of the vacuum generating functional of a gauge theory [20, 25, 19, 22, 21]. The BRST construction was expressed in a more abstract mathematical setting by Kostant and Sternberg [24] and by Stasheff [16, 17]. This construction will now be described.

The starting point is a cohomological representation of the space of classical observables $\mathscr{F}(\mathscr{N})$, which is achieved in two stages.

First, an odd superderivation

$$\delta : \Lambda^q(\mathfrak{g}) \otimes \mathscr{F}(\mathscr{N}) \to \Lambda^{q-1} \otimes \mathscr{F}(\mathscr{N})$$

is defined by setting

$$\delta(\pi \otimes 1) = 1 \otimes T_\pi\,, \qquad \delta(1 \otimes f) = 0 \tag{12}$$

so that $\delta^2 = 0$.

It can be seen that the cohomology $\mathrm{H}^0_\delta$ at degree zero is isomorphic to $\mathscr{F}(C)$ (the space of smooth functions on $C$). This follows because $Z^0_\delta$, the set of degree zero cycles of $\delta$, is simply $1 \otimes \mathscr{F}(\mathscr{N})$ while $B^0_\delta$, the set of degree zero boundaries of $\delta$, consists of sums of terms of the form $1 \otimes f\,T_\pi$ where $f \in \mathscr{F}(\mathscr{N})$ and $\pi \in \mathfrak{g}$. Let $\mathscr{F}_0 = \{f | f \in \mathscr{F}, f(y) = 0 \,\forall\, y \in C\}$, that is, the set of smooth function on $\mathscr{N}$ which vanish on $C$. It can be argued that any function in $\mathscr{F}_0$ can be expressed as a sum of terms of the form $f\,T_\pi$, so that $B^0_\delta$ can be identified with $\mathscr{F}_0$. Also, since any function in $\mathscr{F}(C)$ can be extended to a smooth function on $\mathscr{N}$, $\mathscr{F}(C) \cong \mathscr{F}(\mathscr{N})/\mathscr{F}_0$. As a result

$$\begin{aligned} \mathrm{H}^0_\delta &\cong \mathscr{F}(\mathscr{N})/\mathscr{F}_0 \\ &\cong \mathscr{F}(C)\,. \end{aligned} \tag{13}$$

The next step is to introduce a further odd superderivation

$$\mathrm{d} : \Lambda^p(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}) \otimes \mathscr{F}(\mathscr{N}) \to \Lambda^{p+1}(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}) \otimes \mathscr{F}(\mathscr{N})$$

by setting

$$\begin{aligned} \mathrm{d}(\eta\otimes 1\otimes 1) &= \mathrm{d}\eta\otimes 1\otimes 1 \\ \mathrm{d}(1\otimes\pi_a\otimes 1) &= \eta^b\otimes\pi_c\otimes C^c_{ab} \\ \mathrm{d}(1\otimes 1\otimes f) &= \eta^b\otimes 1\otimes\{T_b, f\}\,. \end{aligned} \tag{14}$$

Here for brevity the basis (4) of $\mathfrak{g}$ has been used, together with the dual basis $\{\eta^a | a = 1,\ldots,m\}$ of $\mathfrak{g}^*$. An intrinsic definition of d is of course possible [24].

For each pair of non-negative integers $p,q$ the space $\Lambda^p(\mathfrak{g}^*)\otimes\Lambda^q(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N})$ carries a $G$ action, built from the Hamiltonian $G$ action on $\mathscr{N}$, the adjoint action on $\mathfrak{g}$ and the coadjoint action on $\mathfrak{g}^*$. The degree of an element in the complex $\Lambda(\mathfrak{g}^*)\otimes\Lambda(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N})$ on which d acts is defined to be the degree in $\Lambda(\mathfrak{g}^*)$, so that the degree zero cohomology of d is isomorphic to the set of $G$ invariant elements of $(\Lambda(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N}))$. This can be seen by noting that the action of d on an element $k$ of $\Lambda(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N})\otimes 1$ is given by

$$\mathrm{d}k = \eta^a\pi_a(k) \tag{15}$$

where $\pi_a(k)$ denotes the result of the action of $\pi_a\in\mathfrak{g}$ on $k$, while the set of zero boundaries contains the single element 0.

If one extends $\delta$ to the space $\Lambda^p(\mathfrak{g}^*)\otimes\Lambda^q(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N})$ by defining $\delta(\eta^a\otimes 1\otimes 1) = 0$ then $\delta$ commutes with the $G$ action. Additionally $\delta$ commutes with d. Hence the space $\mathrm{H}^0_\mathrm{d}\left(\mathrm{H}^0_\delta\left(\Lambda(\mathfrak{g}^*)\otimes\Lambda(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N})\right)\right)$ is well defined, and satisfies

$$\mathrm{H}^0_\mathrm{d}\left(\mathrm{H}^0_\delta\left(\Lambda(\mathfrak{g}^*)\otimes\Lambda(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N})\right)\right)\cong\mathscr{F}(C)^G \tag{16}$$

so that

$$\mathscr{F}(\mathscr{N}/\!\!/G)\cong\mathrm{H}^0_\mathrm{d}\left(\mathrm{H}^0_\delta\left(\Lambda(\mathfrak{g}^*)\otimes\Lambda(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N})\right)\right)\,. \tag{17}$$

We thus have a cohomological characterization of the space of observables on the reduced phase space $\mathscr{N}/\!\!/G$, but this is not yet quite in the form required. We have a double complex

$$\begin{array}{ccc} \Lambda^p(\mathfrak{g}^*)\otimes\Lambda^q(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N}) & \overset{\delta}{\longrightarrow} & \Lambda^p(\mathfrak{g}^*)\otimes\Lambda^{q-1}(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N}) \\ \mathrm{d}\downarrow & & \\ \Lambda^{p+1}(\mathfrak{g}^*)\otimes\Lambda^q(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N}) & & \end{array}$$

which can be combined into a single complex with differential

$$\mathrm{D} = \mathrm{d} + (-1)^p\delta \tag{18}$$

which raises the total degree (defined as $p-q$) by one and satisfies

$$\mathrm{D}^2 = 0\,. \tag{19}$$

Also, by a spectral sequence argument,

$$\mathrm{H}^0_\mathrm{D}\left(\Lambda(\mathfrak{g}^*)\otimes\Lambda(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N})\right)\cong\mathrm{H}^0_\mathrm{d}\left(\mathrm{H}^0_\delta\left(\Lambda(\mathfrak{g}^*)\otimes\Lambda(\mathfrak{g})\otimes\mathscr{F}(\mathscr{N})\right)\right) \tag{20}$$

so that we now have the observables of the system expressed in terms of the cohomology of a single complex.

A final feature of this complex is vital for its exploitation in quantization; this is that the complex can be represented as the space of functions on a symplectic supermanifold $\mathscr{S}\mathscr{N}$, and there is a function $\Omega$ on this supermanifold which represents by Poisson bracket the action of the differential D. An account of the theory of supermanifolds, including the example required here, may be found in [5]. A rough account is to say that $\mathscr{S}\mathscr{N}$ has local coordinates $p_i, x^i, \pi_a, \eta^a, i = 1, \ldots, n, a = 1, \ldots, m$ where $p_i, x^i$ are even coordinates corresponding to the coordinates of $\mathscr{N}$ while $\pi_a, \eta^a$ are odd coordinates which correspond to the anticommuting generators of $\Lambda(\mathfrak{g})$ and $\Lambda(\mathfrak{g}^*)$ respectively. More formally, $\mathscr{S}\mathscr{N}$ is the supermanifold built from the trivial $\mathfrak{g} \times \mathfrak{g}^*$ bundle over $\mathscr{N}$. The symplectic form on $\mathscr{S}\mathscr{N}$ is

$$\omega_s = \omega + \mathrm{d}\pi_a \wedge \mathrm{d}\eta^a$$

where $\omega$ is the symplectic form on $\mathscr{N}$. It may then be verified that the function

$$\Omega = \eta^a T_a + \tfrac{1}{2} C^c_{ab} \eta^a \eta^b \pi_c \tag{21}$$

has the required property.

The situation described here is the simplest, most idealized, situation which can arise. In many examples (including that described in section 5) the group action is local, in the sense described in [6], and the corresponding symplectic supermanifold has a twisted product structure.

## 3. BRST QUANTIZATION AND GAUGE FIXING

The 'classical' BRST operator which has been constructed can readily be quantized, using the quantization for the original $2n$-dimensional phase space $\mathscr{N}$. The observables for the BRST quantization take the form

$$A = \sum_{\underline{a},\underline{b}} A^{\underline{b}}_{\underline{a}} \eta^{\underline{a}} \pi_{\underline{b}} \tag{22}$$

where $\underline{a} = a_1 ... a_{k_{\underline{a}}}$ is a multi index consisting of $k_{\underline{a}}$ terms with $1 \leq a_1 < \cdots < a_{k_{\underline{a}}} \leq m$ (and $\underline{b}$ similarly), and summation is carried out over all possible multi indices including the empty multi index. Also each $A^{\underline{b}}_{\underline{a}}$ is an observable on $\mathscr{N}$, while $\eta^{\underline{a}} = \eta^{a_1} \ldots \eta^{a_{k_{\underline{a}}}}$ and $\pi_{\underline{b}} = \pi_{b_1} \ldots \pi_{b_{k_{\underline{b}}}}$. The observables $\eta^a, a = 1, \ldots, m$ are known as ghosts, with $\pi_b, b = 1, \ldots, m$ conjugate ghost momenta. Quantization is carried out by taking states of the form

$$f = \sum_{\underline{a}} f_{\underline{a}} \eta^{\underline{a}} \tag{23}$$

where each $f_{\underline{a}}$ is a state for the quantization of $\mathscr{N}$. The inner product is defined on the space of states by

$$\langle f, g \rangle = \sum_{\underline{a}} \langle f_{\underline{a}}, g_{\underline{a}} \rangle . \tag{24}$$

Action of the observables $\eta^a$ and $\pi_b$ on such states is defined by setting $\eta^a$ to act as a multiplication operator while

$$\pi_b f = -i\frac{\partial}{\partial\eta^b} f\,, \tag{25}$$

so that $\eta^a$ and $\pi_b$ obey canonical anticommutation relations

$$[\eta^a, \pi_b] = -i\delta^a_b\,. \tag{26}$$

In this quantization scheme the BRST function $\Omega$ has a quantization as an operator $Q$ on the space of states which satisfies $Q^2 = 0$. The space of physical observables consists of observables $A$ of degree zero which satisfy $[A, Q] = 0$, modulo observables of the form $[B, Q]$ for some quantized function $B$. The corresponding space of physical states consists of states $f$ which satisfy $Qf = 0$, modulo states of the form $Qg$ for some state $g$. It may be readily verified that physical observables act on the space of physical states in a well-defined way.

The BRST quantization scheme which has been described readily allows path integral quantization, using the extension of stochastic calculus to spaces of paths on supermanifolds described in [7, 5]. The path integral approach is powerful because it naturally calculates supertraces, and these can be shown to correspond to traces over the space of physical states [3]. In general the objects to be calculated in a quantum theory can be expressed as traces of operators on the space of physical states. There is however an analytic difficulty which may occur. The proof that a supertrace gives a trace over cohomology classes involves reordering of infinite sums, a procedure which is not necessarily valid unless the sums involved are absolutely convergent. A procedure known as gauge-fixing is required; this adds to the Hamiltonian of the theory a cohomologically trivial term of the form $[Q, \chi]$ where $\chi$ is an operator of degree $-1$, chosen so that it regularises the trace. Further details of this procedure may be found in [8, 5].

## 4. EQUIVARIANT BRST AND REDUCIBLE SYMMETRY

There is a class of symmetry referred to as 'reducible symmetry' which is important in many physical models, but has not been extensively studied in the mathematical quantization literature, an exception being the paper of Fisch, Henneaux, Stasheff and Teitelboim [9]. In the Lagrangian setting this manifests itself as a set of symmetries whose generators are not linearly independent, so that effectively there appear to be generators of a group $G$ but with some that are missing. This picture is not very clear, and corresponds in the Hamiltonian setting to linearly independent constraints (in number equal to the dimension of the group $G$) whose Poisson bracket algebra does not close. Loosely speaking the situation is one where the system has a symmetry under a group $G$ up to action by a subgroup $H$.

One scenario to which this corresponds is when as before we have a Hamiltonian action of a Lie group $G$ on the phase space $\mathscr{N}$, but when the constraints derived from the Lagrangian of the theory take the form

$$T_a - \langle v, \pi_a \rangle = 0 \tag{27}$$

where $v$ is an arbitrary element of $\mathfrak{h}^*$, the dual of the Lie algebra $\mathfrak{h}$ of an abelian subgroup $H$ of $G$ with certain properties. The required properties are that there is a subspace $\mathfrak{k}$ of the Lie algebra $\mathfrak{g}$ of $G$ which satisfies

$$\mathfrak{h} \oplus \mathfrak{k} = \mathfrak{g} \quad \text{and} \quad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}. \tag{28}$$

(An example is when $G$ is semisimple and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.) For simplicity, we will work in a specific basis $\{\pi_a, a = 1, \dots, m\}$ of $\mathfrak{g}$ such that $\{\pi_\alpha, \alpha = 1, \dots, l = \dim H\}$ is a basis of $\mathfrak{h}$ while $\{\pi_r, r = l+1, \dots, m\}$ is a basis of $\mathfrak{k}$. As a consequence of (28) the only non-zero structure constants in this basis are those of the form $C^s_{\alpha r}$, $C^s_{tr}$ or $C^\alpha_{tr}$, where the index convention used is that Roman letters from the latter part of the alphabet denote basis elements in $\mathfrak{k}$, Greek letters denote basis elements in $\mathfrak{h}$ (while Roman letters from the start of the alphabet denote basis elements from either part of $\mathfrak{g}$).

The $2n$-dimensional phase space $\mathscr{N}$ is now extended to the $2(n+l)$-dimensional space $\mathscr{N}' = \mathscr{N} \times T^*H$ on which there is the natural symplectic form

$$\omega + \mathrm{d}v_\alpha \wedge \mathrm{d}w^\alpha \tag{29}$$

where $w^\alpha, v_\alpha$ are local coordinates at a point $(h, v)$ in $T^*H$ with $w^\alpha$ local coordinates of $h \in H$ and $v_\alpha$ the component of $v \in \mathfrak{h}^*$ in the basis $\{\mathrm{d}w^\alpha, \alpha = 1, \dots, l\}$. The extra coordinates $w^\alpha, v_\alpha$, introduced without justification, are simply constrained to be zero. The constraints of the system can now be written

$$v_\alpha - T_\alpha = 0\,, w^\alpha = 0\,, v_\alpha = 0\,,\ \alpha = 1, \dots, l \qquad T_r = 0\,,\ r = l+1, \dots, m. \tag{30}$$

Of course the Poisson bracket algebra of this set of constraints does not close, however it is now possible to understand these constraints. The $2l$ constraints $v_\alpha - T_\alpha, w^\alpha, \alpha = 1, \dots, l$ are a second class set which correspond to a reduction of $\mathscr{N}'$ to $\mathscr{N}' /\!\!/ H$ under the $H$ action generated by $v_\alpha - T_\alpha, \alpha = 1, \dots, l$. This is an action which allows the Dirac construction of the symplectic form on the reduced phase space. The constraints $w^\alpha = 0, \alpha = 1, \dots, l$ pick out one point in each $H$ orbit, so that the reduced phase space $\mathscr{N}' /\!\!/ H$ is explicitly represented as the subset of $\mathscr{N}'$ where $w^\alpha = 0$ and $v_\alpha - T_\alpha = 0$ for $\alpha = 1, \dots, l$. The Poisson bracket $\{,\}_D$ on the reduced phase space is defined in terms of that on $\mathscr{N}'$ by

$$\{f, g\}_D = \{f, g\} - \{f, v_\alpha - T_\alpha\} \{g, w^\alpha\} + \{g, v_\alpha - T_\alpha\} \{f, w^\alpha\}\,. \tag{31}$$

(In the physics literature these brackets are known as Dirac brackets. They take the form given above because $\left\{w^\alpha, v_\beta - T_\beta\right\} = \delta^\alpha_\beta$.)

The remaining constraints $v_\alpha = 0, \alpha = 1, \dots, l, T_r = 0, r = l+1, \dots, m$ form a closed algebra under Poisson bracket on the reduced phase space $\mathscr{N}' /\!\!/ H$, corresponding to a Hamiltonian $G$ action on this space. This leads to a further reduced phase space $(\mathscr{N}' /\!\!/ H) /\!\!/ G$ which is morally $(\mathscr{N} /\!\!/ H) /\!\!/ (G/H)$.

This two-stage reduction of $\mathscr{N}' = \mathscr{N} \times T^*H$ can be implemented in one step by the action on $\mathscr{N}'$ of the semidirect product $G \rtimes H$ corresponding to the action of $H$ on $G$ by inverse conjugation. An explicit basis for the Lie algebra of $G \rtimes H$ may be denoted

$\{\pi_\alpha, \pi_r, \lambda_\alpha\}$, $\alpha = 1, \ldots, l, r = 1 + l, \ldots, m$ with $\pi_\alpha, \pi_r$ as before and the extra nontrivial bracket

$$[\lambda_\alpha \, \pi_r] = -C^s_{\alpha r} \pi_s \,. \tag{32}$$

The constraint functions generating this action (referred to the same basis) are

$$T_\alpha \,, \quad T_r \quad \text{and} \quad v_\alpha - T_\alpha. \tag{33}$$

An extensive discussion of two stage reduction has been given by Marsden, Misiolek, Ortega, Perlmutter and Ratiu in [13, 14].

The BRST cohomology corresponding to this reduction can be constructed using the methods of the preceding section. This leads to a symplectic supermanifold of dimension $(2n + 2l, 2m + 2l)$. Denoting the ghosts corresponding to the basis elements $\pi_a, \lambda_\alpha, a = 1, \ldots, m$ and $\alpha = 1, \ldots, l$ by $\eta^a$ and $\theta^\alpha$ respectively, and the corresponding ghost momenta by $\pi_a, \rho_\alpha$, the BRST operator takes the form

$$Q = \eta^a T_a + \theta^\alpha (v_\alpha - T_\alpha) - \tfrac{1}{2} \eta^a \eta^b C^c_{ab} \pi_c + \theta^\alpha \eta^r C^s_{\alpha r} \pi_s \,. \tag{34}$$

Using techniques borrowed from equivariant de Rham theory, this cohomology can be recast in a number of useful forms. (A very nice account of equivariant de Rham theory may be found in the book of Guillemin and Sternberg [27].) It can be seen that the BRST operator splits into two canonically defined parts

$$Q = Q_G + Q_H \tag{35}$$

with

$$Q_G = \eta^a T_a - \tfrac{1}{2} \eta^a \eta^b C^c_{ab} \pi_c \quad \text{(as before)} \quad \text{and} \quad Q_H = -\theta^\alpha (\mathscr{L}_\alpha - v_\alpha) \,. \tag{36}$$

where

$$\mathscr{L}_\alpha = \left[ \eta^a T_a - \tfrac{1}{2} \eta^a \eta^b C^c_{ab} \pi_c, \pi_\alpha \right] = [Q_G, \pi_\alpha] = T_\alpha - C^s_{\alpha r} \eta^r \pi_s \,. \tag{37}$$

Now the cohomology of $Q_H$ is acyclic, that is, it has trivial cohomology except at degree zero, and consists of states $f$ which satisfy

$$\rho_\alpha f = 0 \quad \text{and} \quad (\mathscr{L}_\alpha - v_\alpha) f = 0 \,. \tag{38}$$

(In the terminology of equivariant de Rham theory, such states are called ‘basic’.) Making the Kalkman transformation [10], that is, conjugating operators by $\exp(-\theta^\alpha \pi_\alpha)$, gives the BRST operator in the form

$$Q = Q_G + \theta^\alpha v_\alpha \,. \tag{39}$$

After this transformation the basic conditions (38) corresponding to solving the $Q_H$ cohomology become

$$\rho_\alpha f = \pi_\alpha f \quad \text{and} \quad (\mathscr{L}_\alpha - v_\alpha) f = 0 \,. \tag{40}$$

Using further techniques from equivariant de Rham theory it can be shown that equivalent formulations of the cohomology are given by

$$Q = Q_G + u^\alpha \rho_\alpha \tag{41}$$

acting on $\Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathscr{F}\mathscr{N} \otimes \Lambda(\mathfrak{h}^*) \otimes S(\mathfrak{h}^*)$, where $u^\alpha$ is the even coordinates for $\mathfrak{h}^*$, with the basic conditions (38) now taking the form

$$\mathscr{L}_\alpha = 0\,, \quad \text{and} \quad \pi_\alpha + \rho_\alpha = 0\,. \tag{42}$$

This equivalence is proved by recognizing that the $W^*$ algebra $\mathscr{F}(T^*H) \otimes \Lambda(\mathfrak{h}^*) \otimes \Lambda(\mathfrak{h})$ can be exchanged for the $W^*$ algebra $\Lambda(\mathfrak{h}^*) \otimes S(\mathfrak{h}^*)$, that is, for the Weyl algebra of $H$. (Full details may be found in [6].) This form of the BRST cohomology is in close analogy to the Weyl model of equivariant de Rham cohomology (for the group $H$), with the operator $Q_G$ playing the role of the exterior derivative. It will thus be referred to as equivariant BRST. The Kalkman transformation in this setting implements the Mathai-Quillen transformation, leading to an an analogue of the Cartan model. Explicitly, conjugation by $\exp(\theta^\alpha \pi_\alpha)$ gives BRST operator in the form

$$Q_G + \theta^\alpha \mathscr{L}_\alpha + u^\alpha \rho_\alpha - u^\alpha \pi_\alpha, \tag{43}$$

with basic condition $\mathscr{L}_\alpha = 0, \rho_\alpha = 0$ which is the same as the cohomology of

$$Q_{Cartan} = Q_G - u^\alpha \pi_\alpha \tag{44}$$

on $\mathscr{L}_\alpha$ invariant elements of $\Lambda(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}) \otimes \mathscr{F}(\mathscr{N}) \otimes S(\mathfrak{h}^*)$.

## 5. EXAMPLE: A MODEL WITH REDUCIBLE SYMMETRY

An example of a model whose BRST cohomology is equivariant in this way will now be described. The field of the model is a path $x : I \to \mathscr{M}$ where $I$ is the interval $[0,T]$ of the real line and $\mathscr{M}$ is an $n$-dimensional Riemannian manifold with metric $g$ which carries an isometric circle action. The Killing vector which generates this action is denoted $X$, and the one form dual to this vector under the metric is denoted $\tilde{X}$. It will be useful to introduce local coordinates $x^i, i = 1, \ldots, n$ on $\mathscr{M}$, so that $X$ has components $X^i$ satisfying

$$X = X^i \frac{\partial}{\partial x^i} \tag{45}$$

and $\tilde{X}$ has components $X_i$ satisfying

$$X = X_i \, \mathrm{d}x^i\,. \tag{46}$$

The action of the theory is

$$S(x(\cdot)) = \int_0^T v\, x^* \tilde{X} \tag{47}$$

where $v$ is a constant. This action is invariant under infinitesimal changes in $x$ of the form $x^i(t) \to x^i(t) + \varepsilon^i(t)$ provided that $\varepsilon$ satisfies

$$\varepsilon^i X_i = 0\,. \tag{48}$$

In local coordinates the action has the form

$$S(x(\cdot)) = \int_0^T v\, X_i(x(t))\, \dot{x}^{\,i}(t)\, \mathrm{d}t \tag{49}$$

from which it may readily be seen that under (imaginary time) Legendre transformation the system has constraints

$$T_i = p_i - ivX_i \tag{50}$$

with Poisson brackets

$$\left\{T_i, T_j\right\} = 2iv\mathscr{D}_i X_j\,. \tag{51}$$

These constraints thus do not have a closing algebra - as could be anticipated from the linear relationship (48) obeyed by infinitesimal parameters $\varepsilon^i$ of the symmetry in the Lagrangian setting, which shows that the symmetry is reducible.

Using the notation of the previous sections, the original phase space $\mathscr{N}$ of this model is the cotangent bundle $T^*\mathscr{M}$ which is a $2n$-dimensional symplectic manifold with standard symplectic form

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}x^i\,. \tag{52}$$

(Here local coordinates $(x^i, p_i) i = 1, \ldots, n$ at $(x, p) \in T^*\mathscr{M}$ consist of the local coordinates $x^i$ of $x \in \mathscr{M}$ together with components $p_i$ of $p$ in the basis $\left\{\mathrm{d}x^i\right\}$.) The full (unreduced) symmetry is the diffeomorphism group, with (locally defined) moment map corresponding to constraints $T_i \equiv p_i = 0, i = 1, \ldots, n$, so that in this case

$$Q_G = \eta^i p_i\,. \tag{53}$$

It will be seen below that states for this system can be identified with differential forms, in such a way that $Q_G$ corresponds to the exterior derivative d.

The symplectic supermanifold for standard BRST quantization of the full symmetry is the cotangent bundle of the super configuration space described below; it is a $(2n, 2n)$-dimensional supermanifold with even local coordinates $x^i, p_i$ and odd local coordinates $\eta^i, \pi_i$. (In each case the index $i$ runs from 1 to $n$, the dimension of $\mathscr{M}$.) The anticommuting coordinates $\eta^i$ are the ghosts corresponding to the constraints $T_i$, while the coordinates $\pi_i$ are the corresponding conjugate momenta. The super configuration space (whose functions provide the states of the quantized system) is the supermanifold $\mathrm{S}(\mathscr{M}, T\mathscr{M})$ built from the tangent bundle of $\mathscr{M}$. It has even local coordinates $x^i$ and odd local coordinates $\eta^i, i = 1, \ldots, n$, with coordinate patches corresponding to those on $\mathscr{M}$ and changes of the coordinates $x^i, \eta^i \to x'^i, \eta'^i$ on overlapping coordinate patches defined by setting $x^i \to x'^i(x)$ as on $\mathscr{M}$ and

$$\eta'^i = \frac{\partial x'^i}{\partial x^j}\,\eta^j \quad . \tag{54}$$

Functions on this configuration space are naturally identified with differential forms on $\mathscr{M}$, with the odd coordinate $\eta^i$ corresponding to the differential $\mathrm{d}x^i$.

The super phase space is the cotangent bundle of this super configuration space; the coordinates $x^i, p_i, \eta^i$ and $\pi_i$ used are not however standard cotangent space coordinates. These coordinates are chosen so that $x^i$ and $\eta^i$ transform as above while

$$p'_i = \frac{\partial x^j}{\partial x'^i} p_j, \quad \text{and} \quad \pi'_i = \frac{\partial x^j}{\partial x'^i} \pi_j \quad . \tag{55}$$

In using these coordinates we have implicitly made a non-canonical identification of the cotangent bundle of the super configuration space $\mathrm{S}(\mathscr{M}, T\mathscr{M})$ with the direct sum of the supermanifolds $\mathrm{S}(\mathscr{M}, T\mathscr{M})$ and $\mathrm{S}(\mathscr{M}, T^*\mathscr{M})$, using the Levi-Civita connection corresponding to the metric $g$ on $\mathscr{M}$. This allows the rather simple coordinate transformation rules specified above. While these coordinates are not Darboux coordinates, the symplectic form has a natural geometric structure:

$$\begin{aligned} \omega &= \mathrm{d}\left(p_i \wedge \mathrm{d}x^i + \pi_i \wedge \mathscr{D}\eta^i\right) \\ &= \mathrm{d}p_i \wedge \mathrm{d}x^i + \mathscr{D}\pi_i \wedge \mathscr{D}\eta^i - \frac{1}{2} R_{ijk}{}^l \eta^k \pi_l \mathrm{d}x^i \wedge \mathrm{d}x^j, \end{aligned} \tag{56}$$

where the Levi-Civita connection corresponding to the Riemannian metric $g$ has been used, with Christoffel symbols $\Gamma_{ij}{}^k$ and curvature tensor components $R_{ijk}{}^l$, so that

$$\mathscr{D}\eta^i = \mathrm{d}\eta^i + \Gamma_{jk}{}^i \eta^k \mathrm{d}x^j \quad \text{and} \quad \mathscr{D}\pi_i = \mathrm{d}\pi_i - \Gamma_{ji}{}^k \pi_k \mathrm{d}x^j \quad . \tag{57}$$

The corresponding Poisson brackets are:

$$\begin{array}{lllllllll} \{p_i, x^j\} &=& \delta_i^j, & \{p_i, p_j\} &=& R_{ijk}{}^l \pi_l \eta^k, & \{\pi_i, \eta^j\} &=& \delta_i^j, \\ \{p_i, \eta^j\} &=& \Gamma_{il}{}^j \eta^l, & \{p_j, \pi_l\} &=& -\Gamma_{jl}{}^i \pi_i, & & & \end{array} \tag{58}$$

the remainder being zero. Under quantization the BRST function $\eta^i p_i$ becomes (up to a factor) $\eta^i \frac{\partial}{\partial x^i}$ which is equivalent to d when states are identified with forms on $\mathscr{M}$.

Equipped with this symplectic supermanifold, the equivariant BRST cohomology of the system with action (47) can be formulated in several ways as described above. Using (44) gives the second of the two supersymmetric models described by Witten in [11], with BRST operator taking the form

$$Q = \mathrm{d} - iuX^i \pi_i \,. \tag{59}$$

The techniques described in [7, 5] allow rigorous construction of the path integrals used in quantizing this model. Further details may be found in [15].

## 6. CONCLUSION

In this lecture the theory of canonical BRST quantization for models with reducible symmetry has been described. The motivation is to construct rigorous functional integrals for these models. So far this has been achieved for quantum mechanical models. To extend these methods to field theory the covariant Hamiltonian approach to BRST developed by Hrabak [12] seems particularly likely to be a useful technique.

# REFERENCES


1. J. Marsden, and A. Weinstein, *Rep. Math. Phys.* **5**, 121–130 (1974).
2. P. Dirac, *Can. J. Math.* **2**, 129 (1951).
3. M. Henneaux, and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, 1992.
4. N. M. J. Woodhouse, *Geometric Quantization*, OUP, 1997, 2nd edn.
5. A. Rogers, *Supermanifolds: theory and application*, World Scientific, 2007.
6. A. Rogers, *J. Phys. A: Math. Theor.* **40**, 4649–4663 (2007).
7. A. Rogers, *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **6**, 83–102 (2003).
8. A. Rogers, *Class. Quantum Grav.* **17**, 389–397 (2000).
9. J. Fisch, M. Henneaux, J. Stasheff, and C. Teitelboim, *Comm. Math. Phys.* **120**, 379 (1989).
10. J. Kalkman, *Comm. Math. Phys.* **153**, 447–463 (1992).
11. E. Witten, *Journal of Differential Geometry* **17**, 661–692 (1982).
12. S. Hrabak, *preprint* **math-ph/9904026** (1999).
13. M. Perlmutter J.E. Marsden, G. Misiolek and T. Ratiu. Symplectic reduction for semidirect products and central extensions. *Journal of Differential Geometry and its Applications*, 9(1-2):173–212, 1998.
14. J.E. Marsden, G. Misiolek, J.P. Ortega, M. Perlmutter, and T. Ratiu. *Hamiltonian reduction by stages*. Springer Lecture Notes in Mathematics, to appear, 2004.
15. A. Rogers. Gauge fixing and equivariant cohomology. *Class. Quantum Grav.*, 22:4083–4094, 2005.
16. J D Stasheff. Constrained Poisson algebras and strong homotopy representations,. *Bull. Amer. Math. Soc.*, 19:287–290, 1988.
17. J.D. Stasheff. Homological reduction of constrained Poisson algebras. *J. Diff. Geom.*, 45:221–240, 1997.
18. D. McMullan. Constraints and BRS symmetry. Imperial College Preprint TP/83-84/21, 1984.
19. E.S. Fradkin and T.E. Fradkina. Quantization of relativistic systems with boson and fermion first- and second-class constraints. *Phys. Lett. B72*, B72:343, 1978.
20. E.S. Fradkin and G.A. Vilkovisky. Quantization of relativistic systems with constraints. *Phys. Lett.*, B55:224, 1975.
21. E.S. Fradkin and G.A. Vilkovisky. Quantization of relativistic systems with constraints: equivalence of canonical and covariant formalisms in quantum theory of gravitational field. *CERN Preprint*, CERN–TH–2332, 1977.
22. I.A. Batalin and E.S. Fradkin. Operator quantization and abelization of dynamical systems subject to first class constraints. *Revista del Nuovo Cimento*, 9:1–48, 1986.
23. M. Henneaux. Hamiltonian form of the path integral for theories with a gauge freedom. *Phys. Rep.*, 126:1, 1985.
24. B. Kostant and S. Sternberg. Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. *Annals of Physics*, 176:49–113, 1987.
25. I. A. Batalin and G.A. Vilkovisky. Relatavistic *S*-matrix of dynamical systems with boson and fermion constraints. *Phys. Lett.*, B69:309, 1977.
26. I. A. Batalin and G.A. Vilkovisky. Gauge algebra and quantization. *Phys. Lett.*, 102B:27–31, 1981.
27. V.W. Guillemin and S. Sternberg. *Supersymmetry and equivariant de Rham theory*. Springer, 1991.