EXTENSIONS OF RANK ONE (φ, Γ)-MODULES AND CRYSTALLINE REPRESENTATIONS

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ABSTRACT. Let K be a finite unramified extension of $\mathbf{Q}_p$. We parametrize the (φ, Γ)-modules corresponding to reducible two-dimensional $\overline{\mathbf{F}}_p$-representations of G_K and characterize those which have reducible crystalline lifts with certain Hodge-Tate weights.

CONTENTS

1. Introduction

Buzzard, Diamond and Jarvis [BDJ] have formulated a generalization of Serre's conjecture for mod p Galois representations over totally real fields unramified at p. To give a recipe for weights, certain distinguished subspaces of local Galois cohomology groups in characteristic p are defined in terms of the existence of "crystalline lifts" to characteristic zero. More precisely, let K be a finite unramified extension of $\mathbf{Q}_p$ with residue field k, $\mathbf{F}$ a finite extension of $\mathbf{F}_p$ containing k, $\psi : G_K \to \mathbf{F}^\times$ a character, and denote by S the set of embeddings of k in $\mathbf{F}$. For each $J \subset S$, they define a subspace (or in certain cases two subspaces) of $H^1(G_K, \mathbf{F}(\psi))$ which we denote L_J (or $L_J^\pm$); with certain exceptions these subspaces have dimension $|J|$ (see Remark 7.7 below for the relation between our notation and that of [BDJ]). The definition of these subspaces in terms of crystalline lifts is somewhat indirect, making it hard for example to compare the spaces L_J for different J. The aim of this paper is to describe them more explicitly using Fontaine's theory of (φ, Γ)-modules. In particular, we prove that if ψ is *generic*, as defined in §5.2, then the subspaces are well-behaved with respect to J in the following sense:

Theorem 1.1. *If ψ is generic and $\psi|_{I_K} \neq \chi^{\pm 1}$ where χ is the mod p cyclotomic character, then $L_J = \oplus_{\tau \in J} L_{\{\tau\}}$.*

We remark that Theorem 1.1 has been proved independently by Breuil [Bre09, Prop. A.3] using different methods. We also treat the case where $\psi|_{I_K} = \chi^{\pm 1}$; see Theorem 7.8 below for the statement.

We also give a complete description of the spaces L_J (and $L_J^\pm$) in terms of (φ, Γ)-modules when K is quadratic, without the assumption that ψ is generic. In particular, we prove the following theorem which exhibits cases where the spaces L_J are not well-behaved as in Theorem 1.1:

Theorem 1.2. *Suppose that $[K : \mathbf{Q}_p] = 2$, ψ is ramified and $S = \{\tau, \tau'\}$. Then $L_{\{\tau\}} = L_{\{\tau'\}}$ if and only if $\psi|_{I_K} = \omega_2^i$ for some fundamental character ω_2 of niveau 2 and some integer $i \in \{1, \ldots, p-1\}$.*

This is part of Theorem 7.12 below; see also Theorem 7.15 for the case when ψ is unramified.

The paper is organized as follows: In §2 we review preliminary facts on p-adic representations and (φ, Γ)-modules, and set up the category of étale (φ, Γ)-modules (corresponding to $\mathbf{F}[G_K]$-modules) in which we will be working. In §3 we give a parametrization of rank one objects in the category, and identify them as reductions of crystalline characters of G_K using results of Dousmanis [Dou07]. In §4 we construct bases for the space of extensions of rank ones. (In a different but related direction, see [Her98, Her01, Liu07] for computation of p-adic Galois cohomology via (φ, Γ)-modules.) In §5 we introduce the notion of bounded extensions, motivated by the theory of Wach modules which characterizes those (φ, Γ)-modules corresponding to crystalline representations (see [Wac96, Wac97, Ber02, Ber04b]), and use this to define subspaces $V_J^{(\pm)}$ which we compute in the generic and quadratic cases. In §6 we treat certain exceptional cases excluded from §§4,5. In §7 we relate the spaces $L_J^{(\pm)}$ and $V_J^{(\pm)}$ in the generic and quadratic cases and prove our main results. We remark that a difficulty arises from the fact that the integral Wach module functor is not right exact; to overcome this we derive sufficient conditions for exactness which may be of independent interest.

This paper grew out of the first author's Brandeis Ph.D. thesis [Cha06] written under the supervision of the second author. The thesis already contains most of the key technical results in §§4,5. The authors are grateful to Laurent Berger for helpful conversations and correspondence. The second author is grateful to the Isaac Newton Institute for its hospitality in the final stages of writing this paper. The research was supported by NSF grant #0300434.

2. Generalities on p-adic representations

In this section we summarize (and expand a bit upon) basic facts on p-adic representations, crystalline representations, (φ,Γ)-modules and Wach modules. We will give references for details and proofs along the way. For excellent general introductions to the theory, see [Ber04a] and [FO].

Let p be a rational prime and fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. If K is a finite extension of $\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$, G_K denotes the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ and K_0 denotes the absolutely unramified subfield of K. Let $\chi : G_K \to \mathbf{Z}_p^\times$ be the cyclotomic character and let $\bar{\cdot} : \mathbf{Z}_p \to \mathbf{F}_p$ be the reduction modulo p, so that $\overline{\chi} = \bar{\cdot}\circ\chi : G_K \to \mathbf{F}_p^\times$ is the mod p cyclotomic character. We set $K_n = K(\mu_{p^n}) \subset \overline{\mathbf{Q}}_p$ for $n \geq 1$, and get a tower of fields

$$K = K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots \subset K_\infty \subset \overline{\mathbf{Q}}_p$$

where $K_\infty = \cup_{n\geq 1} K_n$. We define H_K to be the kernel of χ, i.e., $H_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$ and set $\Gamma_K = G_K/H_K = \mathrm{Gal}(K_\infty/K)$. In many cases where there is no confusion, we will simply write Γ for Γ_K suppressing K. We set $\Gamma_n = \Gamma_{K,n} = \mathrm{Gal}(K_\infty/K_n)$ for $n \geq 1$.

2.1. **Fontaine's rings.** Here we give a summary of the constructions of some of the rings introduced by Fontaine that we will be using. See [Col99, CC98, Fon94a] for more details. Let $\mathbf{C}_p$ denote the p-adic completion of $\overline{\mathbf{Q}}_p$ and v_p the p-adic valuation normalized by $v_p(p) = 1$. The set

$$\widetilde{\mathbf{E}} = \varprojlim_{x\mapsto x^p} \mathbf{C}_p = \{x = (x^{(0)}, x^{(1)}, \ldots) \mid x^{(i)} \in \mathbf{C}_p, (x^{(i+1)})^p = x^{(i)}\}$$

together with the addition and the multiplication defined by

$$(x+y)^{(i)} = \lim_{j\to\infty} (x^{i+j} + y^{i+j})^{p^j} \text{ and } (xy)^{(i)} = x^{(i)}y^{(i)}$$

is an algebraically closed field of characteristic p, complete for the valuation $v_{\mathbf{E}}$ defined by $v_{\mathbf{E}}(x) = v_p(x^{(0)})$. We endow $\widetilde{\mathbf{E}}$ a Frobenius φ and the action of $G_{\mathbf{Q}_p}$ by

$$\varphi((x^{(i)})) = ((x^{(i)})^p) \text{ and } g((x^{(i)})) = (g(x^{(i)}))$$

if $g \in G_{\mathbf{Q}_p}$ We denoted the ring of integers of $\widetilde{\mathbf{E}}$ by $\widetilde{\mathbf{E}}^+$, which is stable by the actions of φ and $G_{\mathbf{Q}_p}$. Let $\varepsilon = (1, \varepsilon^{(1)}, \ldots, \varepsilon^{(i)}, \ldots)$ be an element of $\widetilde{\mathbf{E}}$ such that $\varepsilon^{(1)} \neq 1$, so that $\varepsilon^{(i)}$ is a primitive p^i-th root of unity for all $i \geq 1$. Then $v_{\mathbf{E}}(\varepsilon - 1) = p/(p-1)$ and $\mathbf{E}_{\mathbf{Q}_p}$ is defined to be the subfield $\mathbf{F}_p((\varepsilon - 1))$ of $\widetilde{\mathbf{E}}$. We define $\mathbf{E}$ to be the separable closure of $\mathbf{E}_{\mathbf{Q}_p}$ in $\widetilde{\mathbf{E}}$ and $\widetilde{\mathbf{E}}^+$ (resp. $\mathfrak{m}_{\mathbf{E}}$) to be the ring of integers (resp. the maximal ideal of $\mathbf{E}^+$). The field $\mathbf{E}$ is stable by the action of $G_{\mathbf{Q}_p}$ and we have $\mathbf{E}^{H_{\mathbf{Q}_p}} = \mathbf{E}_{\mathbf{Q}_p}$. The theory of the field of norms allows one to show that $\mathbf{E}_K = \mathbf{E}^{H_K}$ is a finite separable extension of $\mathbf{E}_{\mathbf{Q}_p}$ of degree $|H_{\mathbf{Q}_p}/H_K| = [K_\infty : \mathbf{Q}_p(\mu_{p^\infty})]$, which also allows one to identify $\mathrm{Gal}(\mathbf{E}/\mathbf{E}_K)$ with H_K. The ring of integers of $\mathbf{E}_K$ is denoted by $\mathbf{E}_K^+$.

Let $\widetilde{\mathbf{A}} = W(\widetilde{\mathbf{E}})$ be the ring of Witt vectors with coefficients in $\widetilde{\mathbf{E}}$ and let $\widetilde{\mathbf{B}} = \widetilde{\mathbf{A}}[1/p] = \mathrm{Fr}(\widetilde{\mathbf{A}})$. Then $\widetilde{\mathbf{B}}$ is a complete discrete valuation field with ring of valuation $\widetilde{\mathbf{A}}$ and residue field $\widetilde{\mathbf{E}}$. If $x \in \widetilde{\mathbf{E}}$, $[x]$ denotes Teichmuller representative of x in $\widetilde{\mathbf{A}}$. Then every element of $\widetilde{A}$ can be written uniquely in the form $\sum_{i\geq 0} p^i[x_i]$ and that of $\widetilde{\mathbf{B}}$ in the form $\sum_{i\gg\infty} p^i[x_i]$. We endow $\widetilde{\mathbf{A}}$ with the topology which makes the map $x \mapsto (x_i)_{i\in\mathbf{N}}$ a homeomorphism $\tilde{\mathbf{A}} \to \tilde{\mathbf{E}}^{\mathbf{N}}$ where $\tilde{\mathbf{E}}$ is endowed with the product topology ($\tilde{\mathbf{E}}$ is endowed with the topology defined by the valuation $\mathrm{v}_{\mathbf{E}}$). We endow $\tilde{\mathbf{B}} = \cup_{i\in\mathbf{N}} p^{-i}\tilde{\mathbf{A}}$ the topology of inductive limit. The action of $G_{\mathbf{Q}_p}$ on $\tilde{\mathbf{E}}$ extends continuously to an action of $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ which commutes with Frobenius φ. Let $\pi = [\varepsilon] - 1$. Define $\mathbf{A}_{\mathbf{Q}_p}$ to be the closure of $\mathbf{Z}_p[\pi, \pi^{-1}]$ in $\tilde{\mathbf{A}}$. Then

$$\mathbf{A}_{\mathbf{Q}_p} = \{\sum_{i\in\mathbf{Z}} a_n\pi^i |\, a_i \in \mathbf{Z}_p, a_i \to 0 \text{ as } i \to -\infty\}$$

and $\mathbf{A}_{\mathbf{Q}_p}$ is a complete discrete valuation ring with residue field $\mathbf{E}_{\mathbf{Q}_p}$. As

$$\varphi(\pi) = (1+\pi)^p - 1 \text{ and } \gamma(\pi) = (1+\pi)^{\chi(g)} - 1 \text{ if } g \in G_{\mathbf{Q}_p},$$

the ring $\mathbf{A}_{\mathbf{Q}_p}$ and its field of fractions $\mathbf{B}_{\mathbf{Q}_p} = \mathbf{A}_{\mathbf{Q}_p}[1/p]$ are stable by φ and by the action of $G_{\mathbf{Q}_p}$. Let $\mathbf{B}$ be the closure of the maximal unramified extension of $\mathbf{B}_{\mathbf{Q}_p}$ contained in $\widetilde{\mathbf{B}}$, and set $\mathbf{A} = \mathbf{B} \cap \tilde{\mathbf{A}}$, so that we have $\mathbf{B} = \mathbf{A}[1/p]$. Then $\mathbf{A}$ is a complete discrete valuation ring with field of fractions $\mathbf{B}$ and residue field $\mathbf{E}$. The ring $\mathbf{A}$ and the field $\mathbf{B}$ are stable by φ and $G_{\mathbf{Q}_p}$. If K is a finite extension of $\mathbf{Q}_p$, we define $\mathbf{A}_K = \mathbf{A}^{H_K}$ and $\mathbf{B}_K = \mathbf{B}^{H_K}$, which makes $\mathbf{A}_K$ a complete discrete valuation ring with residue field $\mathbf{E}_K$ and the field of fractions $\mathbf{B}_K = \mathbf{A}_K[1/p]$. When $K = \mathbf{Q}_p$, the two definition of $\mathbf{A}_K$ and $\mathbf{B}_K$ coincide. If F is a finite extension of K, then $\mathbf{B}_F$ is an unramified extension of $\mathbf{B}_K$ of degree $[F_\infty : K_\infty]$. If the extension F/K Galois, then the extensions $\widetilde{\mathbf{B}}_F/\widetilde{\mathbf{B}}_K$ and $\mathbf{B}_F/\mathbf{B}_K$ are also Galois with the Galois group

$$\mathrm{Gal}(\widetilde{\mathbf{B}}_F/\widetilde{\mathbf{B}}_K) = \mathrm{Gal}(\mathbf{B}_F/\mathbf{B}_K) = \mathrm{Gal}(\mathbf{E}_F/\mathbf{E}_K) = \mathrm{Gal}(F_\infty/K_\infty) = H_K/H_F.$$

In particular, if K is a finite unramified extension of $\mathbf{Q}_p$, we have

$$\mathbf{A}_K = \{\sum_{n\in\mathbf{Z}} a_n\pi^n |\, a_n \in \mathcal{O}_K, a_n \to 0 \text{ as } n \to -\infty\}$$

with φ acting as the Frobenius and Γ acting trivially on $\mathcal{O}_K$.

The homomorphism $\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbf{C}_p}$, $\sum_{n\geq 0} p^n[x_n] \mapsto \sum_{n\geq 0} p^n x_n^{(0)}$ is surjective and its kernel is a principal ideal generated by $\omega = \pi/\varphi^{-1}(\pi)$. We extend θ to a homomorphism $\widetilde{\mathbf{B}}^+ = \widetilde{\mathbf{A}}^+[1/p] \to \mathbf{C}_p$ and we set $\mathbf{B}_{\mathrm{dR}}^+$ to be the ring $\varprojlim \widetilde{\mathbf{B}}^+/(\ker\theta)^n$. Then θ extends by continuity to a homomorphism $\mathbf{B}_{\mathrm{dR}}^+ \to \mathbf{C}_p$. This makes $\mathbf{B}_{\mathrm{dR}}^+$ a discrete valuation ring with maximal ideal $\ker\theta$ and residue field $\mathbf{C}_p$. The action of $G_{\mathbf{Q}_p}$ on $\widetilde{\mathbf{B}}^+$ extends by continuity to a continuous action of $G_{\mathbf{Q}_p}$ on $\mathbf{B}_{\mathrm{dR}}^+$. The series $\log[\varepsilon] = \sum_{n\geq 1}(-1)^{n-1}\pi^n/n$ converges in $\mathbf{B}_{\mathrm{dR}}^+$ to an element t, which is a generator of $\ker\theta$ on which $\sigma \in G_{\mathbf{Q}_p}$ act via the formula $\sigma(t) = \chi(\sigma)t$. We set $\mathbf{B}_{\mathrm{dR}} = \mathbf{B}_{\mathrm{dR}}^+[t^{-1}] = \mathrm{Fr}\mathbf{B}_{\mathrm{dR}}^+$, and $\mathbf{B}_{\mathrm{dR}}$ comes with a decreasing, separated and exhaustive filtration $\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}} := t^i\mathbf{B}_{\mathrm{dR}}^+$ for $i \in \mathbf{Z}$. Let $\mathbf{A}_{\mathrm{cris}} = \{x = \sum_{n\geq 0} a_n\frac{\omega^n}{n!} \in \mathbf{B}_{\mathrm{dR}}^+ |\, a_n \in \widetilde{\mathbf{A}}^+, a_n \to 0\}$. Then $\mathbf{B}_{\mathrm{cris}}^+ = \mathbf{A}_{\mathrm{cris}}[1/p]$ is a subring of $\mathbf{B}_{\mathrm{dR}}^+$ stable by $G_{\mathbf{Q}_p}$ and contains t and the action of φ on $\widetilde{\mathbf{B}}^+$ extends by continuity to an action of $\mathbf{B}_{\mathrm{cris}}^+$. We have $\varphi(t) = pt$ and we define $\mathbf{B}_{\mathrm{cris}}$ to be the subring $\mathbf{B}_{\mathrm{cris}}^+[1/t]$ of $\mathbf{B}_{\mathrm{dR}}$. We see that $\mathbf{B}_{\mathrm{cris}}$ also comes with a filtration $\mathrm{Fil}^i\mathbf{B}_{\mathrm{cris}} := t^i\mathbf{B}_{\mathrm{cris}}^+$.

2.2. **Crystalline representations.** Let K be a finite extension of $\mathbf{Q}_p$, and let K_0 denote its maximal absolutely unramified subfield.

Definition 2.1. A *p-adic representation of* G_K is a finite dimensional $\mathbf{Q}_p$-vector space together with a linear and continuous action of G_K. A $\mathbf{Z}_p$*-representation of* G_K is a $\mathbf{Z}_p$-module of finite type with a $\mathbf{Z}_p$-linear and continuous action of G_K. A *mod p representation of* G_K is a finite dimensional $\mathbf{F}_p$-vector space with a linear and continuous action of G_K.

Remark 2.2. A $\mathbf{Z}_p$-representation T of G_K which is torsion-free over $\mathbf{Z}_p$ is naturally identified with a (G_K*-stable*) *lattice* of the p-adic representation $V := \mathbf{Q}_p \otimes_{\mathbf{Z}_p} T$ of G_K.

If B is a topological $\mathbf{Q}_p$-algebra endowed with a continuous action of G_K and if V is a p-adic representation of G_K, we define $\mathbf{D}_B(V) := (B \otimes_{\mathbf{Q}_p} V)^{G_K}$, which is naturally a module over B^{G_K}. If, in addition, B is G_K-regular (i.e., B is a domain, $(\mathrm{Fr}B)^{G_K} = B^{G_K}$, and every $b \in B - \{0\}$ such that $\mathbf{Q}_p b$ is stable under G_K-action is a unit), then the map

$$\alpha_V : B \otimes_{B^{G_K}} \mathbf{D}_B(V) \to B \otimes_{\mathbf{Q}_p} V$$

defined by $b \otimes v \mapsto 1 \otimes bv$ is an injection. In particular, we have

$$\dim_{B^{G_K}} \mathbf{D}_B(V) \le \dim_{\mathbf{Q}_p} V.$$

(If B is G_K-regular, B^{G_K} is forced to be a field.)

Definition 2.3. If B is G_K-regular, we say that a p-adic representation V of G_K is B*-admissible* if $\dim_{B^{G_K}} \mathbf{D}_B(V) = \dim_{\mathbf{Q}_p} V$. We say that V is *crystalline* if it is $\mathbf{B}_{\mathrm{cris}}$-admissible and that V is *de Rham* if $\mathbf{B}_{\mathrm{dR}}$-admissible.

Remark 2.4. We have the following equivalent conditions:

(1) $\dim_{B^{G_K}} \mathbf{D}_B(V) = \dim_{\mathbf{Q}_p} V$,
(2) α_V is an isomorphism,
(3) $B \otimes_{\mathbf{Q}_p} V \simeq B^{\dim_{\mathbf{Q}_p} V}$ as $B[G_K]$-modules.

If V is a p-adic representation of G_K, $\mathbf{D}_{\mathrm{dR}}(V) := (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ is naturally a filtered K-vector space. More precisely, it is a finite dimensional K vector space with a decreasing, separated and exhaustive filtration $\mathrm{Fil}^i\mathbf{D}_{\mathrm{dR}}(V) := (\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ of K-subspaces for $i \in \mathbf{Z}$. If V is de Rham, a *Hodge-Tate weight* of V is defined to be an integer $h \in \mathbf{Z}$ such that $\mathrm{Fil}^h\mathbf{D}_{\mathrm{dR}}(V) \ne \mathrm{Fil}^{h+1}\mathbf{D}_{\mathrm{dR}}(V)$ with multiplicity $\dim_K \mathrm{Fil}^h\mathbf{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^{h+1}\mathbf{D}_{\mathrm{dR}}(V)$. So there are $\dim_{\mathbf{Q}_p} V$ Hodge-Tate weights of V counting multiplicities. We remark that some authors (e.g., [Ber04a]) define a Hodge-Tate weight of V as $-h$ for h as above.

Definition 2.5. A *filtered* φ*-module over* K is a (finite dimensional) K_0-vector space D together with a σ-semilinear bijection $\varphi : D \to D$ and a $\mathbf{Z}$-indexed filtration on $D_K := D \otimes_{K_0} K$ of K-subspaces which is decreasing, separated and exhaustive.

The reason for the above definition is that if V is a p-adic representation of G_K, then $\mathbf{D}_{\mathrm{cris}}(V) := (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ is a filtered φ-module over K. More precisely, the Frobenius on $\mathbf{B}_{\mathrm{cris}}$ induces a Frobenius map $\varphi : \mathbf{D}_{\mathrm{cris}}(V) \to \mathbf{D}_{\mathrm{cris}}(V)$ and the filtration on $\mathbf{B}_{\mathrm{dR}}$ induces a filtration $\mathrm{Fil}^i\mathbf{D}_{\mathrm{cris}}(V) := D_K \cap (\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ on $\mathbf{D}_{\mathrm{cris}}(V)$. Moreover, $\mathbf{D}_{\mathrm{cris}}(V)$ has finite dimension over K_0 and φ is bijective on $\mathbf{D}_{\mathrm{cris}}(V)$. We get a functor

$$\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}_{\mathbf{Q}_p} G_K \to \mathrm{MF}_K^{\varphi}$$

from the category of p-adic representations of G_K to the category of filtered φ-modules over K.

If D is a filtered φ-module over K of finite dimension $d \geq 1$, then $\wedge^d D$ is a filtered φ-module of dimension 1. If $e \in \wedge^d_{K_0} D - \{0\}$ and $\varphi(e) = \lambda e$ then $\mathrm{val}(\lambda)$ is independent of choice of e and we define $t_N(D) := v_p(\lambda)$. Also, we define $t_H(D) = t_H(D_K)$ to be the largest integer such that $\mathrm{Fil}^{t_H(D)}(\wedge^d_K D_K)$ is nonzero, i.e. $\mathrm{Fil}^i(\wedge^d_K D_K) = \wedge^d_K D_K$ for $i \leq t_H(D)$ and $\mathrm{Fil}^i(\wedge^d_K D_K) = 0$ for $i > t_H(D)$.

Definition 2.6. Let D be a filtered φ-module over K. We say that D is *weakly admissible* if $\dim_{K_0} D < \infty$, φ is bijective on D, $t_H(D) = t_N(D)$ and $t_H(D') \leq t_N(D')$ for every subobject D' of D. We say that D is *admissible* if $D \simeq \mathbf{D}_{\mathrm{cris}}(V)$ for some p-adic representation V of dimension $\dim_{K_0} D$.

One can show that if V is a crystalline representation of G_K, then $\mathbf{D}_{\mathrm{cris}}(V)$ is weakly admissible. The converse is true by important work of Colmez and Fontaine.

Theorem 2.7. [CF00] *Every weakly admissible filtered φ-module over K is admissible.*

In sum, we have an equivalence of categories

$$\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}^{cris}_{\mathbf{Q}_p} G_K \to \mathrm{MF}_K^{\varphi, w.a.}$$

between crystalline representations of G_K and weakly admissible filtered φ-modules over K with a quasi-inverse given by $V_{\mathrm{cris}}(\cdot) := (\mathrm{Fil}^0(\cdot))^{\varphi=0}$.

2.3. (φ, Γ)**-modules.**

Definition 2.8. A (φ, Γ)-*module over* $\mathbf{A}_K$ (resp. $\mathbf{B}_K$, $\mathbf{E}_K$) is a $\mathbf{A}_K$-module of finite type (resp. finite dimensional vector space over $\mathbf{B}_K$, $\mathbf{E}_K$) endowed with a semilinear and continuous action of Γ_K and with a semilinear map φ which commutes with the action of Γ_K. We say that a (φ, Γ)-module M over $\mathbf{A}_K$ (resp. $\mathbf{E}_K$) is *étale* if $\varphi(M)$ generates M over $\mathbf{A}_K$ (resp. $\mathbf{E}_K$). A (φ, Γ)-module M over $\mathbf{B}_K$ is *étale* if M contains an $\mathbf{A}_K$-lattice which is stable under φ and is étale.

Remark 2.9. The (étale) (φ, Γ)-modules over $\mathbf{A}_K$ killed by p are precisely the (étale) (φ, Γ)-modules (defined) over $\mathbf{E}_K$.

If T is a $\mathbf{Z}_p$-representation of G_K, we define $\mathbf{D}(T) = (\mathbf{A} \otimes_{\mathbf{Z}_p} T)^{H_K}$. Then $\mathbf{D}(T)$ is naturally a module over $\mathbf{A}_K$ of finite type. The Frobenius φ on $\mathbf{A}$ induces a Frobenius map $\varphi : \mathbf{D}(T) \to \mathbf{D}(T)$ and the residual action of Γ_K on $\mathbf{D}(T)$ commutes with φ. Also, one can check that $\mathbf{D}(T)$ étale over $\mathbf{A}_K$. Conversely, if M is an étale (φ, Γ)-module over $\mathbf{A}_K$ we define $\mathbf{T}(M) = (\mathbf{A} \otimes_{\mathbf{A}_K} M)^{\varphi=1}$, which is a $\mathbf{Z}_p$-representation of G_K.

Theorem 2.10. [Fon91] *The functor $T \mapsto \mathbf{D}(T)$ defines an equivalence of categories*

$$\mathbf{D} : \mathrm{Rep}_{\mathbf{Z}_p} G_K \to \mathrm{M}_{\mathbf{A}_K}^{\varphi,\Gamma,et}$$

between $\mathbf{Z}_p$-representations and étale (φ, Γ)-modules over $\mathbf{A}_K$ with $\mathbf{T}$ as a quasi-inverse. It induces, by inverting p, an equivalence of categories

$$\mathbf{D} : \mathrm{Rep}_{\mathbf{Q}_p} G_K \to \mathrm{M}_{\mathbf{B}_K}^{\varphi,\Gamma,et}$$

between p-adic representations and étale (φ, Γ)-modules over $\mathbf{B}_K$ with $M \mapsto \mathbf{V}(M) := (\mathbf{B} \otimes_{\mathbf{B}_K} D)^{\varphi=1}$ as a quasi-inverse. Moreover, if T is a $\mathbf{Z}_p$-representation and V a p-adic representation of G_K, then

$$\mathrm{rank}_{\mathbf{Z}_p} T = \mathrm{rank}_{\mathbf{A}_K} \mathbf{D}(T),$$
$$\dim_{\mathbf{Q}_p} V = \dim_{\mathbf{B}_K} \mathbf{D}(V).$$

When we restrict the equivalence to the p-torsion objects we get the following.

Corollary 2.11. *The functor $T \mapsto \mathbf{D}(T)$ defines an equivalence of categories between mod p representations of G_K and étale (φ, Γ_K)-modules over $\mathbf{E}_K$.*

Now we introduce coefficients to representations of G_K and (φ, Γ)-modules to extend Theorem 2.10 and Corollary 2.11. We assume K is absolutely unramified (of degree f over $\mathbf{Q}_p$) and let F be a finite extension of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_F$, uniformizer ϖ_F and residue field $\mathbf{F}$. Consider the ring $\mathbf{A}_{K,F} := \mathcal{O}_F \otimes_{\mathbf{Z}_p} \mathbf{A}_K$ with the actions of φ and Γ_K extended to $\mathbf{A}_{K,F}$ by linearity, i.e. φ acts as $1 \otimes \varphi$ and $\gamma \in \Gamma_K$ as $1 \otimes \gamma$. We assume there is an embedding $\tau_0 : K \hookrightarrow F$, which we fix once and for all, and put $\tau_i = \tau_0 \circ \varphi^i$ where φ is the Frobenius on K. We denote by S the set of all embeddings $K \hookrightarrow F$ and fix the identification $S = \mathbf{Z}/f\mathbf{Z}$ via the map $\tau_i \mapsto i$. We can then identify $\mathbf{A}_{K,F}$ with $\mathbf{A}_{\mathbf{Q}_p,F}^S$ via the isomorphism defined by $a \otimes b\pi^n \mapsto (a\tau(b) \otimes \pi^n)_\tau$. Note that

$$\mathbf{A}_{\mathbf{Q}_p,F} = \{\sum_{n \in \mathbf{Z}} a_n \pi^n |\, a_n \in \mathcal{O}_F, a_n \to 0 \text{ as } n \to -\infty\},$$

and the actions of φ and $\gamma \in \Gamma_K$ on $\mathbf{A}_{\mathbf{Q}_p,F}^S$ become

$$\varphi(g_0(\pi), g_1(\pi), \ldots, g_{f-1}(\pi)) = (g_1(\varphi(\pi)), \ldots, g_{f-1}(\varphi(\pi)), g_0(\varphi(\pi^))),$$
$$\gamma(g_0(\pi), g_1(\pi), \ldots, g_{f-1}(\pi)) = (g_0(\gamma(\pi)), g_1(\gamma(\pi)), \ldots, g_{f-1}(\gamma(\pi))).$$

We similarly define $\mathbf{B}_{K,F} = F \otimes_{\mathbf{Q}_p} \mathbf{B}_K$ and $\mathbf{E}_{K,F} = \mathbf{F} \otimes_{\mathbf{F}_p} \mathbf{E}_K$ and endow them with actions of φ and Γ. Note that $\mathbf{B}_{K,F} = \mathbf{A}_{K,F}[1/p]$ and $\mathbf{E}_{K,F} = \mathbf{A}_{K,F}/\varpi_F \mathbf{A}_{K,F}$. Again identifying S with the set of embeddings $k \to \mathbf{F}$, we have the isomorphism $\mathbf{E}_{K,F} = \mathbf{F}((\pi))^S$ with the actions of φ and Γ_K given by the same formulas as above.

Definition 2.12. An *$\mathcal{O}_F$-representation* of G_K is a finitely generated $\mathcal{O}_F$-module with a continuous $\mathcal{O}_F$-linear action of G_K. A *(φ, Γ_K)-module over $\mathbf{A}_{K,F}$* is a finitely generated $\mathbf{A}_{K,F}$-module M endowed with commuting semilinear actions of Γ_K and φ. A (φ, Γ_K)-module M over $\mathbf{A}_{K,F}$ is *étale* if $\varphi(M)$ generates M over $\mathbf{A}_K$, or equivalently over $\mathbf{A}_{K,F}$.

We write $\mathrm{Rep}_{\mathcal{O}_F} G_K$ for the category of $\mathcal{O}_F$-representations of G_K, and $\mathrm{M}_{\mathbf{A}_{K,F}}^{\varphi,\Gamma,et}$ for that of étale (φ, Γ_K)-modules over $\mathbf{A}_{K,F}$. We use analogous definitions and notation for representations of G_K over F and $\mathbf{F}$, and (φ, Γ_K)-modules over $\mathbf{B}_{K,F}$ and $\mathbf{E}_{K,F}$. The category of étale (φ, Γ_K)-modules over $\mathbf{E}_{K,F}$ is the main category we will be working in. Theorem 2.10 and Corollary 2.11 immediately yield the following:

Corollary 2.13. *The functor* $\mathbf{D}$ *induces equivalences of categories* $\mathrm{Rep}_{\mathcal{O}_F} G_K \to \mathrm{M}^{\varphi,\Gamma,et}_{\mathbf{A}_{K,F}}$, $\mathrm{Rep}_F G_K \to \mathrm{M}^{\varphi,\Gamma,et}_{\mathbf{B}_{K,F}}$ *and* $\mathrm{Rep}_{\mathbf{F}} G_K \to \mathrm{M}^{\varphi,\Gamma,et}_{\mathbf{E}_{K,F}}$.

For each embedding $\tau : K \hookrightarrow \mathbf{F}$, let $e_\tau : \mathbf{A}_{K,F} \to \mathbf{A}_{\mathbf{Q}_p,F}$ denote the projection to the τ-component, defined by $a \otimes b\pi^i \mapsto a\tau(b)\pi^i$. If M is a (φ,Γ)-module over $\mathbf{A}_{K,F}$, then $M = \prod_{\tau\in S} e_\tau M$, each $e_\tau M$ inherits an action of Γ, and φ induces semilinear morphisms $e_{\tau\circ\varphi} M \to e_\tau M$ compatible with the action of Γ. We use the same notation for (φ,Γ)-modules over $\mathbf{B}_{K,F}$ and $\mathbf{E}_{K,F}$.

Lemma 2.14. *If* M *is an étale* (φ,Γ)*-module over* $\mathbf{A}_{K,F}$*, then the following are equivalent:*

1. $\mathbf{T}(M)$ *is free over* $\mathcal{O}_F$ *of rank* d;
2. M *is free over* $\mathbf{A}_K$ *of rank* $d[F : \mathbf{Q}_p]$;
3. M *is free over* $\mathbf{A}_{K,F}$ *of rank* d.

If M *is an étale* (φ,Γ)*-module over* $\mathbf{B}_{K,F}$ *(resp.* $\mathbf{E}_{K,F}$*), then* M *is free over* $\mathbf{B}_{K,F}$ *(resp.* $\mathbf{E}_{K,F}$*) of rank* $\dim_F \mathbf{T}(M)$ *(resp.* $\dim_{\mathbf{F}} \mathbf{T}(M)$*).*

Proof. Suppose that M is étale over $\mathbf{A}_{K,F}$. Then multiplication by p is injective on M if and only if it is injective on $\mathbf{T}(M)$. Thus M is torsion-free, and hence free, over $\mathbf{A}_K$ if and only if $\mathbf{T}(M)$ is free over $\mathcal{O}_F$. Since the $\mathbf{A}_K$-rank of M coincides with the $\mathbf{Z}_p$ rank of $\mathbf{T}(M)$, the first two conditions are equivalent.

If M is free of rank d over $\mathbf{A}_{K,F}$, then it is clearly free of rank $d[F : \mathbf{Q}_p]$ over $\mathbf{A}_K$. Conversely suppose that M is free over $\mathbf{A}_K$. Then each $e_\tau M$ is torsion-free, hence free, over the discrete valuation ring $\mathbf{A}_{\mathbf{Q}_p,F}$. We need only show that each $e_\tau M$ has the same rank. Since M is étale, the maps

$$e_{\tau\circ\varphi} M \otimes_{A_{\mathbf{Q}_p,F},\varphi} \to e_\tau M$$

are surjective, so we have $\mathrm{rank} e_{\tau_i} M \le \mathrm{rank} e_{\tau_{i+1}} M$ for all $i \in \mathbf{Z}/f\mathbf{Z}$. The equivalence between the last two conditions follows.

The assertions for étale (φ,Γ)-modules over $\mathbf{B}_{K,F}$ and $\mathbf{E}_{K,F}$ are similar, but simpler since $\mathbf{B}_K$ and $\mathbf{E}_K$ are fields. □

Finally, there are tensor products and exact sequences in the various categories of étale (φ,Γ)-modules, compatible via $\mathbf{D}$ with tensor products and exact sequences in the corresponding categories of representations of G_K.

2.4. **Wach modules.** As étale (φ,Γ)-modules classify all p-adic representations, including of course crystalline ones, it is an interesting problem to determine when a p-adic representation V is crystalline in terms of its (φ,Γ)-module $\mathbf{D}(V)$. In principle, we may as well expect that the (φ,Γ)-module corresponding to a crystalline representation "should" enjoy some "nice" properties. Let $\mathbf{A}^+ = \mathbf{A} \cap \widetilde{\mathbf{A}}^+ = \mathbf{B} \cap \widetilde{\mathbf{A}}^+$ and $\mathbf{B}^+ = \mathbf{A}^+[1/p]$. If K is a finite unramified extension of $\mathbf{Q}_p$, we set $\mathbf{A}^+_K = (\mathbf{A}^+)^{H_K} = \mathcal{O}_K[[\pi]] \subset \mathbf{A}_K$ and $\mathbf{B}^+_K = (\mathbf{B}^+)^{H_K} = \mathbf{A}^+_K[p^{-1}] \subset \mathbf{B}_K$.

Definition 2.15. Let K be a finite unramified extension of $\mathbf{Q}_p$. We say that a $\mathbf{Z}_p$-adic representation T (resp. p-adic representation V) of G_K, is *of finite height* if there exists a basis of $\mathbf{D}(T)$ (resp. $\mathbf{D}(V)$) such that the matrices describing the action of φ and the action of Γ_K are defined over $\mathbf{A}^+_K$ (resp. $\mathbf{B}^+_K$).

Colmez [Col99] proved that every crystalline representation is necessarily of finite height. The converse is not true in general and there are representations of finite height which are not crystalline. However, Wach [Wac96, Wac97] proved that finitude of height together with a certain condition (existence of a certain $\mathbf{A}_K^+$-sumbodule of the corresponding (φ,Γ)-module) implies crystallinity. Berger [Ber02, Ber04b] then refined the results of Wach and Colmez to define natural objects called "Wach modules." Berger's results are summarized in the following.

Definition 2.16. Suppose $a \le b \in \mathbf{Z}$. A *Wach module* over $\mathbf{A}_K^+$ (resp. $\mathbf{B}_K^+$) with weights in $[a,b]$ is a free $\mathbf{A}_K^+$-module (resp. $\mathbf{B}_K^+$-module) N of finite rank, endowed with an action of Γ_K which becomes trivial modulo π, and also with a Frobenius map $\varphi : N[1/\pi] \to N[1/\pi]$ which commutes with the action of Γ_K and such that $\varphi(\pi^{-a}N) \subset \pi^{-a}N$ and $\pi^{-a}N/\varphi(\pi^{-a}N)$ is killed by $(\varphi(\pi)/\pi)^{b-a}$.

Theorem 2.17. [Ber04b]

(1) *A p-adic representation V is crystalline with Hodge-Tate weights in $[a,b]$ if and only if $\mathbf{D}(V)$ contains a Wach module $\mathbf{N}(V)$ of rank $\dim_{\mathbf{Q}_p} V$ with weights in $[a,b]$. The functor $V \mapsto \mathbf{N}(V)$ defines an equivalence of categories between crystalline representations of G_K and Wach modules over $\mathbf{B}_K^+$, compatible with tensor products, duality and exact sequences.*
(2) *For a given crystalline representation V, the map $T \mapsto \mathbf{N}(T) := \mathbf{N}(V) \cap \mathbf{D}(T)$ induces a bijection between G_K-stable lattices of V and Wach modules over $\mathbf{A}_K^+$ which are $\mathbf{A}_K^+$-lattices contained in $\mathbf{N}(V)$. Moreover $\mathbf{D}(T) = \mathbf{A}_K \otimes_{\mathbf{A}_K^+} \mathbf{N}(T)$.*
(3) *If V is a crystalline representation of G_K, and if we endow $\mathbf{N}(V)$ with the filtration $\mathrm{Fil}^i\mathbf{N}(V) = \{x \in \mathbf{N}(V) | \varphi(x) \in q^i\mathbf{N}(V)\}$, then we have an isomorphism $\mathbf{D}_{\mathrm{cris}}(V) \to \mathbf{N}(V)/\pi\mathbf{N}(V)$ of filtered φ-modules (with the induced filtration on $\mathbf{N}(V)/\pi\mathbf{N}(V)$).*

Remark 2.18. If $0 \to V_1 \to V \to V_2 \to 0$ is an exact sequence of crystalline representations of G_K, then

$$0 \to \mathbf{N}(V_1) \to \mathbf{N}(V) \to \mathbf{N}(V_2) \to 0$$

is an exact sequence of $\mathbf{B}_K^+$-modules. However $\mathbf{N}$ does not define an exact functor from G_K-stable lattices to $\mathbf{A}_K^+$-modules; indeed it fails to be right exact. We return to this point in more detail in §7.

Again by introducing an action of F to the categories, we get an analogous equivalence of categories between crystalline F-representations and Wach modules over $\mathbf{B}_{K,F}^+ := F \otimes_{\mathbf{Q}_p} \mathbf{B}_K^+$. Here, by a crystalline F-representation we mean a finite dimensional F-vector space with a continuous action of G_K which is crystalline considered as a $\mathbf{Q}_p$-linear representation (i.e., forgetting F-structure). Similarly, for a fixed crystalline F-representation of G_K, we have a corresponding equivalence of categories between G_K-stable $\mathcal{O}_F$-lattices and Wach modules over $\mathbf{A}_{K,F}^+ := \mathcal{O}_F \otimes_{\mathbf{Z}_p} \mathbf{A}_K^+$.

Corollary 2.19. *Let $k \in \mathbf{Z}_{\ge 0}$. An F-representation V of G_K is crystalline with Hodge-Tate weights in $[0,k]$ (i.e., positive crystalline) if and only if there exists a $\mathbf{B}_{K,F}^+$-module N free of rank $d := \dim_F(V)$ contained in $\mathbf{D}(V)$ such that*

(1) *the Γ-action preserves N and is trivial on $N/\pi N$, and*
(2) *$\varphi(N) \subset N$ and $N/\varphi^*(N)$ is killed by q^k.*

Moreover, if N is given a filtration by

$$\mathrm{Fil}^i(N) := \{x \in N | \varphi(x) \in q^i N\}$$

for $i \geq 0$, then we have an isomorphism

$$\mathbf{D}_{\mathrm{cris}}(V) \simeq N/\pi N$$

of filtered φ-modules over $F \otimes_{\mathbf{Q}_p} K$ where $N/\pi N$ is endowed with induced filtration.

A standard argument (cf. Lemma 2.14) shows that an F-representation V of G_K is crystalline if and only if the filtered φ-module $\mathbf{D}_{\mathrm{cris}}(V) = (\mathbf{B}_{\mathrm{cris}} \otimes_{\mathbf{Q}_p} V)^{G_K}$ is free of rank $\dim_F V$ over $F \otimes_{\mathbf{Q}_p} K$. We have a decomposition $\mathbf{D}_{\mathrm{cris}}(V) = \oplus_{\tau:K\hookrightarrow F} e_\tau \mathbf{D}_{\mathrm{cris}}(V)$ where $e_\tau \mathbf{D}_{\mathrm{cris}}(V)$ denotes the filtered K-vector space $\mathbf{D}_{\mathrm{cris}}(V) \otimes_{K\otimes_{\mathbf{Q}_p} F, e_\tau} F$ with the filtration given by $\mathrm{Fil}^i e_\tau \mathbf{D}_{\mathrm{cris}}(V) := e_\tau \mathrm{Fil}^i \mathbf{D}_{\mathrm{cris}}(V)$. A *labeled Hodge-Tate weight* with respect to the embedding $\tau : K \hookrightarrow F$ is an integer $h \in \mathbf{Z}$ such that $\mathrm{Fil}^h e_\tau \mathbf{D}_{\mathrm{cris}}(V) \neq \mathrm{Fil}^{h+1} e_\tau \mathbf{D}_{\mathrm{cris}}(V)$, counted with multiplicity

$$\dim_F \mathrm{Fil}^h e_\tau \mathbf{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^{h+1} e_\tau \mathbf{D}_{\mathrm{cris}}(V).$$

Lemma 2.20. *If N is a Wach module over over $\mathbf{A}^+_{K,F}$ (resp. $\mathbf{B}^+_{K,F}$), then N is free over $\mathbf{A}^+_{K,F}$ (resp. $\mathbf{B}^+_{K,F}$).*

Proof. We just give the proof for Wach modules over $\mathbf{A}^+_{K,F}$; the case of $\mathbf{B}^+_{K,F}$ can be deduced from this or proved similarly.

As in the proof of Lemma 2.14, we decompose $\mathbf{A}^+_{K,F} = \prod_{\tau\in S} \mathcal{O}_F[[\pi]]$ and $N = \prod_{\tau \in S} e_\tau N$ accordingly. It suffices to prove the $e_\tau N$ are free over $\mathcal{O}_F[[\pi]]$, all of the same rank. On the one hand, we know by Lemma 2.14 that $\mathbf{A}_{K,F} \otimes_{\mathbf{A}^+_{K,F}} N$ is free of rank d over $\mathbf{A}_{K,F}$, where d is the rank of the corresponding $\mathcal{O}_F$-representation T. It follows that $\mathbf{A}_{\mathbf{Q}_p,F} \otimes_{\mathcal{O}_F[[\pi]]]} e_\tau N$ is free of rank d over $\mathbf{A}_{\mathbf{Q}_p,F}$ for each τ. On the other hand, the $\mathcal{O}_F \otimes_{\mathbf{Z}_p} \mathcal{O}_K$-module $N/\pi N$ is a lattice in $D_{\mathrm{crys}}(\mathbf{Q}_p \otimes_{\mathbf{Z}_p} T)$, which is free of rank d over $F \otimes_{\mathbf{Q}_p} K$. It follows that each $e_\tau N/\pi e_\tau N$ is free of rank d over $\mathcal{O}_F$. By Nakayama's Lemma, we conclude that $e_\tau N$ is free of rank d over $\mathcal{O}_F[[\pi]]$. □

3. Rank ones

In this section we give a parametrization of rank one étale (φ, Γ)-modules over $\mathbf{E}_{K,F}$ (with a view toward parametrizing their extensions) and then identify them with the reduction modulo p of Wach modules of rank one over $\mathbf{A}^+_{K,F}$.

3.1. A parametrization.

Denote by $\mathrm{val}_\pi : \mathbf{F}((\pi)) \to \mathbf{Z}$ the valuation normalized by $\mathrm{val}_\pi(\pi) = 1$, and let $\lambda_\gamma \in \mathbf{F}_p[[\pi]]$ be the unique $\frac{p^f-1}{p-1}$-th root of $\frac{\gamma(\pi)}{\overline{\chi}(\gamma)\pi}$ which is $\equiv 1 \mod \pi$, if $\gamma \in \Gamma$.

Proposition 3.1. *For any $C \in \mathbf{F}^\times$ and any $\vec{c} = (c_0, \dots, c_{f-1}) \in \mathbf{Z}^S$, letting $M = \mathbf{E}_{K,f} e$ with*

$$\varphi(e) = Pe = (C\pi^{(p-1)c_0}, \pi^{(p-1)c_1}, \dots, \pi^{(p-1)c_{f-1}})\, e,$$

$$\gamma(e) = G_\gamma e = (\lambda_\gamma^{\sum_0 \vec{c}}, \lambda_\gamma^{\sum_1 \vec{c}}, \dots, \lambda_\gamma^{\sum_{f-1} \vec{c}})\, e,$$

where $\Sigma_l = \Sigma_l \vec{c} = \sum c_i p^j$ summing over $0 \leq i, j \leq f-1$, $i - j \equiv l \mod f$, defines an étale (φ, Γ)-module of rank one over $\mathbf{E}_{K,F}$. Conversely, for any given rank one M we can choose a basis e so that the module $M = \mathbf{E}_{K,F} e$ with the action of φ

and Γ given as above for some C and some $\vec{c}$. Two such modules M and M' are isomorphic if and only if $C = C'$ and $\Sigma_0\vec{c} \equiv \Sigma_0\vec{c'} \mod p^f - 1$.

Proof. To show that the given formula actually defines a (φ,Γ)-module we need to verify that $P\varphi(G_\gamma) = G_\gamma\gamma(P)$ and $G_{\gamma\gamma'} = G_\gamma\gamma(G_{\gamma'})$. The first identity holds as

$$\begin{aligned}\varphi(G_\gamma)/G_\gamma &= (\lambda_\gamma^{p\Sigma_1-\Sigma_0},\dots,\lambda_\gamma^{p\Sigma_0-\Sigma_{f-1}})\\ &= (\lambda_\gamma^{c_0(p^f-1)},\dots,\lambda_\gamma^{c_{f-1}(p^f-1)})\\ &= \left(\left(\frac{\gamma(\pi)}{\pi}\right)^{c_0(p-1)},\dots,\left(\frac{\gamma(\pi)}{\pi}\right)^{c_{f-1}(p-1)}\right) = \gamma(P)/P.\end{aligned}$$

To prove the second identity, as Γ acting componentwise, we need to show that $\lambda_{\gamma\gamma'} = \lambda\gamma(\lambda)$. But note that $(\lambda_\gamma\gamma(\lambda_{\gamma'}))^{\frac{p^f-1}{f-1}} = \frac{\gamma(\pi)}{\pi}\overline{\chi}(\gamma)\,\gamma\left(\frac{\gamma'(\pi)}{\pi}\overline{\chi}(\gamma')\right) = \frac{\gamma\gamma'(\pi)}{\pi}\overline{\chi}(\gamma\gamma')$ and $\lambda_\gamma\gamma(\lambda_{\gamma'}) \equiv 1 \mod \pi$. The claim follows from uniqueness of λ_γ's. Note also that the function $\gamma \mapsto \lambda_\gamma$ is continuous since it is the composite of $\gamma(\pi)/\overline{\chi}(\gamma)\pi$ with the inverse of the continuous bijective function $x \mapsto x^{(p^f-1)/(p-1)}$ on the compact Hausdorff space $1 + \pi\mathbf{F}_p[[\pi]]$; it follows that the Γ-action we have just defined is continuous.

We now prove that any rank one module can be written in this form. Suppose we are given a rank one module $M = \mathbf{E}_{K,F}e$ such that $\varphi(e) = (h_0(\pi),\dots,h_{f-1}(\pi))$ and $\gamma(e) = (g_0(\pi),\dots,g_{f-1}(\pi))$. Note that if $u \in \mathbf{E}_{K,F}^\times$, by a change of basis $e' = ue$ we get $\varphi(e') = (\varphi(u)/u)\varphi(e)$ and $\gamma(e') = (\gamma(u)/u)\gamma(e)$. If $u = (\pi^j,\dots,\pi^j)$, then $\varphi(u)/u = (\pi^{(p-1)j},\dots,\pi^{(p-1)j})$. So we can assume that $h_i(\pi) \in \mathbf{F}[[\pi]]$ by large enough $j > 0$. We can "shift" between components by appropriate change of basis: if $u = (1,\dots,1,u_i(\pi),1,\dots,1)$, then $\varphi(u)/u = (1,\dots,1,u_{i-1}(\pi^p),u_i(\pi)^{-1},1,\dots,1)$. By successive changes of basis we can make it into a form where $\varphi(e) = (h(\pi),1,\dots,1)$ with $h(\pi) \in \mathbf{F}[[\pi]]$. We can have, for some choice of e, $\varphi(e) = (C\pi^v,1,\dots,1)$ for $C \in \mathbf{F}^\times$ and $v \geq 0$ as

$$\frac{\varphi(u(\pi),u(\pi^{p^{f-1}}),\dots,u(\pi^{p^2}))}{(u(\pi),u(\pi^{p^{f-1}}),\dots,u(\pi^{p^2}))} = (u(\pi^{p^f})/u(\pi),1\dots,1)$$

and the map $1 + \pi\mathbf{F}[[\pi]] \to 1 + \pi\mathbf{F}[[\pi]]$, $u(\pi) \mapsto u(\pi^{p^f})/u(\pi)$, is surjective: as the map is multiplicative and $1+\pi\mathbf{F}[[\pi]]$ is complete π-adically, it suffices to prove that for any $s \geq 1$ and $\alpha \in \mathbf{F}^\times$, $1 + \alpha\pi^s t(\pi)$ is in the image for some $t(\pi) \in \mathbf{F}[[\pi]]^\times$, and indeed $1 - \alpha\pi^s \mapsto (1 - \alpha\pi^{sp^f})/(1 - \alpha\pi^s) \equiv 1 + \alpha\pi^s \mod \pi^{s+1}$.

To show that $(p-1)|v$, we note that $\varphi\gamma(e) = \gamma\varphi(e)$ if and only if

$$\frac{\varphi(g_0,\dots,g_{f-1})}{(g_0,\dots,g_{f-1})} = \frac{\gamma(C\pi^v,1,\dots,1)}{(C\pi^v,1,\dots,1)}$$

where $G_\gamma = (g_0,\dots,g_{f-1})$. This is equivalent to

$$\left(\frac{g_1(\pi^p)}{g_0(\pi)},\dots,\frac{g_0(\pi^p)}{g_{f-1}(\pi)}\right) = \left(\left(\frac{\gamma(\pi)}{\pi}\right)^v,1,\dots,1\right),$$

which implies that $(\gamma(\pi)/\pi)^v = g_0(\pi^{p^f})/g_0(\pi) \equiv 1 \mod \pi$. If $\delta \in \Gamma$ is such that $\delta\Gamma_1$ generates $\Gamma/\Gamma_1 \simeq \mu_{p-1}$ then $\delta(\pi)/\pi \equiv \chi(\delta) \mod \pi$. Thus $\delta(\pi)/\pi$ has order $p-1$ modulo π so that $p-1|v$ and $\varphi(e) = (C\pi^{(p-1)w},1,\cdots,1)$ where $(p-1)w = v$.

To determine the corresponding action of $\gamma \in \Gamma$, we note that $\varphi\gamma(e) = \gamma\varphi(e)$ if and only if

$$\left(\frac{g_1(\pi^p)}{g_0(\pi)}, \dots, \frac{g_0(\pi^p)}{g_{f-1}(\pi)}\right) = \left(\left(\frac{\gamma(\pi)}{\pi}\right)^{(p-1)w}, 1, \dots, 1\right)$$

if and only if $g_0(\pi^{p^f})/g_0(\pi) = (\gamma(\pi)/\pi)^{(p-1)w} = (\gamma(\pi)/\pi\overline{\chi}(\gamma))^{(p-1)w}$ (the order of $\overline{\chi}(\gamma)$ being $p-1$) and $g_1(\pi) = g_2(\pi^p), \dots, g_{f-2}(\pi) = g_{f-1}(\pi^p), g_{f-1}(\pi) = g_0(\pi^p)$. Thus, to get g_i's satisfying the above identity we just need to define $g_0(\pi)$ such that $g_0(\pi^{p^f}) = (\gamma(\pi)/\pi\overline{\chi}(\gamma))^{(p-1)w}$. If we set $g_0(\pi) = \alpha_\gamma\lambda_\gamma(\pi)^w$ with $\alpha_\gamma \in \mathbf{F}^\times$ (as $\lambda \in \mathbf{F}_p[[\pi]]$) we have $g_0(\pi)^{p^f-1} = g_0(\pi^{p^f})/g_0(\pi) = \lambda_\gamma(\pi^{p^f})^w/\lambda_\gamma(\pi)^w = \lambda_\gamma(\pi)^{w(p^f-1)} = (\gamma(\pi)/\pi\overline{\chi}(\gamma))^{(p-1)w}$. Conversely, we see that $g_0(\pi) = \alpha_\gamma\lambda_\gamma(\pi)^w$, $\alpha_\gamma \in \mathbf{F}^\times$ gives all the solutions of the equation $g_0(\pi)^{p^f-1} = (\gamma(\pi)/\pi\overline{\chi}(\gamma))^{(p-1)w}$ counting the number of solutions. By the identity $G_{\gamma\gamma'} = G_\gamma\gamma(G'_\gamma)$ the map $\gamma \mapsto \alpha_\gamma$ must define a character $\Gamma \to \mathbf{F}^\times$, from which we conclude that $\alpha_\gamma = \overline{\chi}(\gamma)^j$ for some $0 \le j \le p^f-1$. The following consideration shows that we can rescale $g_i(\pi) = \overline{\chi}(\gamma)^j\lambda_\gamma(\pi^{p^i})$'s to become $1 \mod \pi$: if $u = (\pi, \pi^{p^{f-1}}, \pi^{p^{f-2}}, \dots, \pi^p)$, then

$$\begin{aligned}
\frac{\varphi(u)}{u} &= (\pi^{p^f-1}, 1, \dots, 1),\\
\frac{\gamma(u)}{u} &= \left(\frac{\gamma(\pi)}{\pi}, \left(\frac{\gamma(\pi)}{\pi}\right)^{p^{f-1}}, \left(\frac{\gamma(\pi)}{\pi}\right)^{p^{f-2}}, \dots, \left(\frac{\gamma(\pi)}{\pi}\right)^{p}\right)\\
&\equiv (\overline{\chi}(\gamma), \dots, \overline{\chi}(\gamma)) \mod \pi.
\end{aligned}$$

Thus, we now can assume that $M = \mathbf{E}_{K,F}e$ with

$$\begin{aligned}
\varphi(e) &= (C\pi^{(p-1)w}, 1, \dots, 1)e,\\
\gamma(e) &= (\lambda_\gamma(\pi)^w, \lambda_\gamma(\pi^p)^w, \dots, \lambda_\gamma(\pi^{p^{f-1}})^w)e
\end{aligned}$$

where $0 \le w < p^f - 1$. Write $w = c_0 + c_1p + \cdots + c_{f-1}p^{f-1}$ with $0 \le c_i \le p-1$. By $e' = ue$ with $u = (1, \pi^{(p-1)(c_1+c_2p+\cdots+c_{f-1}p^{p-2})}, 1, \dots, 1)$ we get

$$\varphi(e') = (C\pi^{(p-1)c_0}, \pi^{(p-1)(c_1+c_2p+\cdots+c_{f-1}p^{p-2})}, 1, \dots, 1).$$

Doing this successively gives $\varphi(e) = (C\pi^{(p-1)c_0}, \pi^{(p-1)c_1}, \dots, \pi^{(p-1)c_{f-1}})$ for some basis e. It's easily checked that those changes of basis that maintain $G_\gamma \equiv (1, \dots, 1)$ mod π are $e' = ue$ such that $u = (u_0, \dots, u_{f-1})$ with $(p-1)|\operatorname{val}_\pi(u_i)$ and that the corresponding action of $\gamma \in \Gamma$ is given by $\gamma(e) = (\lambda_\gamma^{\sum_0 \vec{c}}, \lambda_\gamma^{\sum_1 \vec{c}}, \dots, \lambda_\gamma^{\sum_{f-1} \vec{c}})$.

Finally, we suppose that M is isomorphic to $M' = \mathbf{E}_{K,F}e'$ with

$$\begin{aligned}
\varphi(e') &= Pe' = (C'\pi^{(p-1)c'_0}, \pi^{(p-1)c'_1}, \dots, \pi^{(p-1)c'_{f-1}})\, e',\\
\gamma(e') &= G_\gamma e' = (\lambda_\gamma^{\sum_0 \vec{c'}}, \lambda_\gamma^{\sum_1 \vec{c'}}, \dots, \lambda_\gamma^{\sum_{f-1} \vec{c'}})\, e'
\end{aligned}$$

and determine when the two are isomorphic. After appropriate changes of bases we can assume that

$$\begin{aligned}
\varphi(e) &= Pe = (C\pi^{(p-1)\Sigma_0\vec{c}}, 1, \dots, 1)e,\\
\varphi(e') &= P'e' = (C'\pi^{(p-1)\Sigma_0\vec{c'}}, 1, \dots, 1)e'.
\end{aligned}$$

Those $u=(u_0,\cdots,u_{f-1})\in\mathbf{E}_{K,F}^{\times}$ such that $P'=(\varphi(u)/u)P$ and $(p-1)|\operatorname{val}_\pi(u_i)$ are precisely given by $u=(u_0(\pi),u_0(\pi^{p^{f-1}}),\dots,u_0(\pi^p))$ with $u_0(\pi)=\alpha\pi^{(p-1)j}$ for some $\alpha\in\mathbf{F}^\times$ and $j\in\mathbf{Z}$, in which case we have $\varphi(u)/u=(\pi^{(p-1)(p^f-1)j},1,\dots,1)$. Thus, we conclude that M and M' are isomorphic if only if $C=C'$ and $\sum c_ip^i\equiv\sum c_i'p^i \mod p^f-1$. Étaleness is clear. □

Thus defined module is denoted by $M_{C\vec{c}}=M_{C(c_0,\dots,c_{f-1})}$. We also put

$$\kappa_\varphi(M_{C\vec{c}})=\kappa_\varphi(C,\vec{c})=(C\pi^{(p-1)c_0},\pi^{(p-1)c_1},\dots,\pi^{(p-1)c_{f-1}})$$
$$\kappa_\gamma(M_{C\vec{c}})=\kappa_\gamma(C,\vec{c})=(\lambda_\gamma^{\sum_0\vec{c}},\lambda_\gamma^{\sum_1\vec{c}},\dots,\lambda_\gamma^{\sum_{f-1}\vec{c}})$$

and write Σ_l for $\Sigma_l\vec{c}$ where c_i's are understood.

Remark 3.2. Any rank one étale (φ,Γ)-module can be written uniquely in the form of Proposition 3.1 with $0\le c_i\le p-1$ and at least one $c_i<p-1$.

3.2. **Lifts in characteristic zero.** We now construct rank one Wach modules over $\mathbf{A}_{K,F}^+$ following Dousmanis [Dou07, §2] and check that these reduce modulo ϖ_F to the (φ,Γ)-modules $M_{C\vec{c}}$ over $\mathbf{E}_{K,F}$.

Let $q_1=q=\varphi(\pi)/\pi,q_n=\varphi^{n-1}(q)\in\mathbf{Z}_p[[\pi]]$ and let $\Lambda_f=\prod_{j\ge0}q_{1+jf}/p,\Lambda_\gamma=\frac{\Lambda_f}{\gamma(\Lambda_f)}\in\mathbf{Q}[[\pi]]$. One then has that $\Lambda_f\in1+\pi\mathbf{Q}_p[[\pi]]$ and $\Lambda_\gamma\in1+\pi\mathbf{Z}_p[[\pi]]$.

Suppose we want to construct a rank one Wach module $N=\mathbf{A}_{K,F}^+e$ such that

$$\varphi(e)=(\tilde{C}q^{c_0},q^{c_1},\dots,q^{c_{f-1}})e,$$
$$\gamma(e)=(g_0(\pi),\dots,g_{f-1}(\pi))e$$

if $\gamma\in\Gamma$, where $\tilde{C}\in\mathcal{O}_F^\times$ is any lift of $C\in\mathbf{F}^\times$ and each $g_i(\pi)=g_{\gamma,i}(\pi)\in\mathcal{O}_F[[\pi]]$ depends on $\gamma\in\Gamma$. Commutativity of the actions of φ and Γ amounts to the following identities:

$$\gamma(q)^{c_0}g_0(\pi)=q^{c_0}\varphi(g_1(\pi)),$$
$$\gamma(q)^{c_1}g_1(\pi)=q^{c_1}\varphi(g_2(\pi)),$$
$$\cdots$$
$$\gamma(q)^{c_{f-2}}g_{f-2}(\pi)=q^{c_{f-2}}\varphi(g_{f-1}(\pi)),$$
$$\gamma(q)^{c_{f-1}}g_{f-1}(\pi)=q^{c_{f-1}}\varphi(g_0(\pi)).$$

Thus, we are looking for a solution $g_i(\pi)$ for each γ of the equation

$$g_0(\pi)=\left(\frac{q}{\gamma(q)}\right)^{c_0}\varphi\left(\frac{q}{\gamma(q)}\right)^{c_1}\varphi^2\left(\frac{q}{\gamma(q)}\right)^{c_2}\cdots\varphi^{f-1}\left(\frac{q}{\gamma(q)}\right)^{c_{f-1}}\varphi^f(g_0(\pi)).$$

It is straightforward to check that

$$g_0(\pi)=\Lambda_\gamma^{c_0}\varphi(\Lambda_\gamma)^{c_1}\varphi^2(\Lambda_\gamma)^{c_2}\cdots\varphi^{f-1}(\Lambda_\gamma)^{c_{f-1}}$$

gives the unique solution which is $\equiv 1$ modulo π, and that the remaining $g_i(\pi)$'s are uniquely determined by

$$g_1(\pi) = \left(\frac{q}{\gamma(q)}\right)^{c_1} \varphi\left(\frac{q}{\gamma(q)}\right)^{c_2} \cdots \varphi^{f-1}\left(\frac{q}{\gamma(q)}\right)^{c_{f-1}} \varphi^{f-1}(g_0(\pi)),$$
$$\cdots$$
$$g_{f-2}(\pi) = \left(\frac{q}{\gamma(q)}\right)^{c_{f-2}} \varphi\left(\frac{q}{\gamma(q)}\right)^{c_{f-1}} \varphi^2(g_0(\pi)),$$
$$g_{f-1}(\pi) = \left(\frac{q}{\gamma(q)}\right)^{c_{f-1}} \varphi(g_0(\pi)).$$

Dousmanis [Dou07, §6] shows that $N = \mathbf{A}^+_{K,F}e$ endowed with the actions of φ and Γ described above defines a Wach module over $\mathbf{A}^+_{K,F}$ which we denote by $N_{\tilde{C}\vec{c}}$. Furthermore, $(N_{\tilde{C}\vec{c}}/\pi N_{\tilde{C}\vec{c}}) \otimes_{\mathbf{A}^+_{K,F}} \mathbf{B}_{K,F}$ is a filtered φ-module corresponding to a positive character $G_K \to \mathbf{F}^\times$ with labeled Hodge-Tate weights $(c_{f-1}, c_0, c_1, \ldots, c_{f-2})$. One checks the following by direct computation.

Proposition 3.3. *We have an isomorphism $M_{C\vec{c}} \simeq N_{\tilde{C}\vec{c}} \otimes_{\mathbf{A}^+_{K,F}} \mathbf{E}_{K,F}$ of (φ,Γ)-modules over $\mathbf{E}_{K,F}$.*

Combined with Lemma 3.8 of [BDJ], we obtain the following:

Corollary 3.4. *If $\psi : G_K \to \mathbf{F}^\times$ is the character defined by the action on $\mathbf{V}(M_{C\vec{c}})$, then $\psi|_{I_K} = \prod_{\tau\in S} \omega_\tau^{-c_{\tau\circ\varphi^{-1}}}$ where ω_τ is the fundamental character associated to τ.*

4. Bases for the space of extensions

We will assume $p > 2$ for the rest of the paper except in §6.3 and §7. We fix a topological generator η of the pro-cyclic group $\Gamma = \Gamma_K$, and set $\xi = \eta^{p-1}$, so that ξ topologically generates Γ_1.

Given $C \in \mathbf{F}^\times$ and $\vec{c} = (c_0, \ldots, c_{f-1}) \in \{0, 1, \ldots, p-1\}^S$ with some $c_i < p-1$, we are going to parametrize the space of extension classes $\mathrm{Ext}^1(M_0, M_{C\vec{c}})$ in the category of étale (φ,Γ)-modules over $\mathbf{E}_{K,F}$. We start by noticing that there is an $\mathbf{F}$-linear isomorphism

$$\mu : \mathrm{Ext}^1(M_0, M_{C\vec{c}}) \to H/H_0,$$

where H is the subgroup of $\mathbf{E}_{K,F} \times \mathbf{E}^\Gamma_{K,F}$ consisting of elements $\mu(E) = (\mu_\varphi, (\mu_\gamma)_{\gamma\in\Gamma})$ such that $\gamma \mapsto \mu_\gamma$ is continuous and satisfies

$$(\dagger)\ (\kappa_\varphi\varphi - 1)(\mu_\gamma) = (\kappa_\gamma\gamma - 1)(\mu_\varphi)\ \forall\gamma \in \Gamma,$$
$$(\ddagger)\ \mu_{\gamma\gamma'} = \kappa_\gamma\gamma(\mu_{\gamma'}) + \mu_\gamma\ \forall\gamma,\gamma' \in \Gamma,$$

and $H_0 = \{(\kappa_\varphi\varphi(b) - b, (\kappa_\gamma\gamma(b) - b)_{\gamma\in\Gamma})|\, b \in \mathbf{F}((\pi))^S\} \subset H$ where $\kappa_\varphi = \kappa_\varphi(C,\vec{c})$ and $\kappa_\gamma = \kappa_\gamma(C,\vec{c})$. We call elements of H *cocyles* and those of H_0 *coboundaries*. We will also refer to the corresponding extensions as cocycles and as coboundaries respectively. The map μ is defined as follows: choose a basis of a given extension E and write the action of φ and $\gamma \in \Gamma$ in the matrices $P = \begin{pmatrix} \kappa_\varphi & \mu_\varphi \\ 0 & 1 \end{pmatrix}$ and $G_\gamma = \begin{pmatrix} \kappa_\gamma & \mu_\gamma \\ 0 & 1 \end{pmatrix}$, and then define $\mu(E) := (\mu_\varphi, (\mu_\gamma)_{\gamma\in\Gamma})$. It is straightforward to check that the matrices P and G_γ's define an extension if and only if $\mu(E) \in H$, and that a change of basis for an extension E corresponds to adding an element of H_0 to $\mu(E)$.

By Corollary 2.13, we get an isomorphism $\mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}) \simeq H^1(K, \mathbf{F}(\psi))$ where $\psi : G_K \to \mathbf{F}^\times$ is the character defined by the action on $\mathbf{V}(M_{C\vec{c}})$.

Lemma 4.1. *Via Corollary 2.13, $M_{\vec{0}}$ corresponds to the trivial character and $M_{\overrightarrow{p-2}}$ to the mod p cyclotomic character.*

Proof. The assertion is clear for the trivial character. The mod p cyclotomic character factors as $G_K \to \mathbf{Z}_p^\times \to \mathbf{F}_p^\times \hookrightarrow \mathbf{F}^\times$ where the arrow in the middle is the reduction mod p. If $T = \mathbf{Z}_p(1)$, its Wach module is given by $\mathbf{N}(\mathbf{Z}_p(1)) = \mathbf{A}_K^+ e$ where $\varphi(e) = \frac{\pi}{\varphi(\pi)} e$ and $\gamma(e) = \frac{\chi(\gamma)\pi}{\gamma(\pi)} e$ if $\gamma \in \Gamma$ (cf. [Ber04b, Appendice A]). Thinking modulo p and considering trivial action of $\mathbf{F}$ we see that the étale (φ,Γ)-module over $\mathbf{E}_{K,F}$ corresponding to mod p cyclotomic character is given by $M = \mathbf{E}_{K,F} e$ with $\varphi(e) = \pi^{1-p} e = (\pi^{1-p}, \ldots, \pi^{1-p})e$. By a change of basis $e' = ue$ with $u = (\pi^{p-1}, \ldots, \pi^{p-1})$, we get $M \simeq M_{\overrightarrow{p-2}}$. □

Since

$$\dim_{\mathbf{F}} H^1(K, \mathbf{F}(\psi)) = \begin{cases} f+1 & \text{if} \quad \psi = 1 \,\mathrm{or}\, \overline{\chi} \\ f & \text{if} \quad \psi \notin \{1, \overline{\chi}\}, \end{cases}$$

we have

$$\dim_{\mathbf{F}} \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}) = \begin{cases} f+1 & \text{if } C=1, \text{ and } \vec{c} = \vec{0} \text{ or } \vec{c} = \overrightarrow{p-2} \\ f & \text{otherwise.} \end{cases}$$

We are about to define a system of linearly independent elements $B_0, \ldots, B_{f-1}$ in the space of extension classes $\mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}})$. Once constructed, these B_i's form a basis for $\mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}})$ except for the two cases where $C = 1, \vec{c} = \vec{0}$ or $C = 1, \vec{c} = \overrightarrow{p-2}$, for which a separate treatment will be given in §6. Thanks to the isomorphism μ we only need to define $\mu_\varphi(B_i)$ and $\mu_\gamma(B_i)$'s satisfying the desired properties (†) and (‡). According to whether the parameter c_i is equal to $p-1$ or not the extension B_i is constructed in a slightly different manner.

4.1. **Construction of B_i when $c_i < p-1$.**

Lemma 4.2. *Let $\delta \in \Gamma$ be such that $\delta\Gamma_1$ generates $\Gamma/\Gamma_1 \simeq \mathbf{F}_p^\times$ and let $\Sigma, s \in \mathbb{Z}$. If $\Sigma + s(p^f-1)/(p-1)$ is divisible by p^v but not by p^{v+1} for some $v \in \mathbf{Z}$, then*

$$\lambda_\delta^\Sigma \delta(\pi^s) - \pi^s \in (\overline{\chi}(\delta)^s - 1)\pi^s + \overline{s_v} \frac{\overline{\chi}(\delta)^s(\overline{\chi}(\delta)-1)}{2} \pi^{s+p^v} + \pi^{s+2p^v} \mathbf{F}_p[[\pi^{p^v}]],$$

where $\Sigma + s(p^f-1)/(p-1) = \sum_{j \geq v} s_j p^j$.

Proof. It's easy to see that

$$\lambda_\delta = \frac{\delta(\pi)}{\overline{\chi}(\delta)\pi} = \overline{\chi}(\delta)^{-1} \sum_{j=1}^{d_0-1} \overline{\frac{d_0!}{j!(d_0-j)!}} \pi^{j-1} = 1 + \sum_{j=2}^{d_0-1} \overline{\frac{d_0!}{d_0 j!(d_0-j)!}} \pi^{j-1}$$

in $\mathbf{F}_p[[\pi]]/\pi^{p-1}$, where $\chi(\delta) = \sum_{j \geq 0} d_j p^j \in \mathbf{Z}_p^\times$. Noting that, if $s \in \mathbf{Z}$,

$$(\lambda_\delta^\Sigma \delta - 1)(\pi^s) = \left(\overline{\chi}(\delta)^s \lambda_\delta^\Sigma \left(\frac{\delta(\pi)}{\overline{\chi}(\delta)\pi} \right)^s - 1 \right) \pi^s,$$

the result follows as

$$\begin{aligned}
\overline{\chi}(\delta)^s\lambda_\delta^\Sigma\left(\frac{\delta(\pi)}{\overline{\chi}(\delta)\pi}\right)^s - 1 &= \overline{\chi}(\delta)^s\lambda_\delta^{\Sigma+s(p^f-1)/(p-1)} - 1\\
&= \overline{\chi}(\delta)^s\lambda_\delta^{\sum_{j\ge v} s_jp^j} - 1\\
&= \overline{\chi}(\delta)^s\lambda_\delta(\pi^{p^v})^{s_v}\lambda_\delta(\pi^{p^{v+1}})^{s_{v+1}}\cdots - 1\\
&\equiv (\overline{\chi}(\delta)^s - 1) + \overline{\chi}(\delta)^s\left(\left(1+\frac{\overline{\chi}(\delta)-1}{2}\pi^{p^v}\right)^{s_v} - 1\right)\\
&\equiv (\overline{\chi}(\delta)^s - 1) + \overline{s_v}\frac{\overline{\chi}(\delta)^s(\overline{\chi}(\delta)-1)}{2}\pi^{p^v} \mod \pi^{2p^v}
\end{aligned}$$

and

$$\overline{\chi}(\delta)^s\lambda_\delta^\Sigma\left(\frac{\delta(\pi)}{\overline{\chi}(\delta)\pi}\right)^s - 1 - \left((\overline{\chi}(\delta)^s - 1) + \overline{s_v}\frac{\overline{\chi}(\delta)^s(\overline{\chi}(\delta)-1)}{2}\pi^{p^v}\right) \in \pi^{2p^v}\mathbf{F}_p[[\pi^{p^v}]].$$

□

We note the following lemma, whose straightforward proof we omit:

Lemma 4.3. *If $n \ge 1$, $\gamma \in \Gamma_n$ and $\chi(\gamma) \equiv 1 + zp^n \mod p^{n+1}$, then $\lambda_\gamma \equiv 1 + z\pi^{p^n} + z\pi^{p^{n+1}} \mod \pi^{2p^n-1}$.*

Lemma 4.4. *Let $\chi(\xi) \equiv 1 + zp \mod p^2$ with $0 < z \le p-1$ and let $\Sigma, s \in \mathbf{Z}$. If $\Sigma + s(p^f-1)/(p-1)$ is divisible by p^v but not by p^{v+1} for some $v \in \mathbf{Z}$, then*

$$\lambda_\xi^\Sigma\xi(\pi^s) - \pi^s \in \overline{s_v z}(\pi^{s+(p-1)p^v} + \pi^{s+p^{v+1}}) + \pi^{s+2p^v(p-1)}\mathbf{F}_p[[\pi^{p^v}]],$$

where $\Sigma + s(p^f-1)/(p-1) = \sum_{j\ge v} s_jp^j$.

Proof. By Lemma 4.3, we have

$$\lambda_\xi \equiv \frac{\xi(\pi)}{\pi\overline{\chi}(\xi)} \equiv 1 + z\pi^{p-1} + z\pi^p \mod \pi^{2p-1}.$$

Noting that, if $s \in \mathbf{Z}$,

$$(\lambda_\xi^\Sigma\xi - 1)(\pi^s) = \left(\lambda_\xi^\Sigma\left(\frac{\xi(\pi)}{\pi\overline{\chi}(\xi)}\right)^s - 1\right)\pi^s,$$

the result follows as

$$\begin{aligned}
\lambda_\xi^\Sigma\left(\frac{\xi(\pi)}{\pi\overline{\chi}(\xi)}\right)^s - 1 &= \lambda_\xi^{\Sigma+s(p^f-1)/(p-1)} - 1\\
&= \lambda_\xi(\pi^{p^v})^{s_v}\lambda_\xi(\pi^{p^{v+1}})^{s_{v+1}}\cdots - 1\\
&\equiv \lambda_\xi(\pi^{p^v})^{s_v} - 1 \mod \pi^{(p-1)p^{v+1}}\\
&\equiv \overline{s_v z}(\pi^{(p-1)p^v} + \pi^{p^{v+1}}) \mod \pi^{2(p-1)p^v}
\end{aligned}$$

and

$$\lambda_\xi^\Sigma\left(\frac{\xi(\pi)}{\pi\overline{\chi}(\xi)}\right)^s - 1 - \overline{s_v z}(\pi^{(p-1)p^v} + \pi^{p^{v+1}}) \in \pi^{2(p-1)p^v}\mathbf{F}_p[[\pi^{p^v}]].$$

□

Let's assume $c_i < p-1$ and construct an extension $B_i \in \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}})$. Suppose for the moment that we have successfully defined B_i with $\mu_\varphi(B_i)$ of the form $(0,\dots,0,H_i(\pi),0,\dots,0)$, $H_i(\pi)$ being the ith component. For each $\gamma\in\Gamma$, by the condition (†) there should exist $\mu_\gamma(B_i) = (G_0(\pi),\dots,G_{f-1}(\pi))$ such that

$$(\kappa_\varphi\varphi - 1)(\mu_\gamma(B_i)) = (\kappa_\gamma\gamma - 1)(\mu_\varphi(B_i)),$$

i.e.,

$$\begin{aligned}&(C\pi^{(p-1)c_0}G_1(\pi^p) - G_0(\pi), \pi^{(p-1)c_1}G_2(\pi^p) - G_1(\pi), \dots, \pi^{(p-1)c_{f-1}}G_0(\pi^p) - G_{f-1}(\pi))\\ &= (0,\dots,0,(\lambda_\gamma^{\Sigma_i}\gamma - 1)(H_i(\pi)),0,\dots,0).\end{aligned}$$

This is true if and only if

$$\begin{aligned}(C\pi^{(p-1)\Sigma_i}\Phi - 1)(G_i(\pi)) &= (\lambda_\gamma^{\Sigma_i}\gamma - 1)(H_i(\pi)),\\ G_{i+1}(\pi) &= \pi^{(p-1)c_{i+1}}G_{i+2}(\pi^p),\\ &\cdots\\ G_{f-1}(\pi) &= \pi^{(p-1)c_{f-1}}G_0(\pi^p),\\ G_0(\pi) &= C\pi^{(p-1)c_0}G_1(\pi^p),\\ G_1(\pi) &= \pi^{(p-1)c_1}G_2(\pi^p),\\ &\cdots\\ G_{i-1}(\pi) &= \pi^{(p-1)c_{i-1}}G_i(\pi^p),\end{aligned}$$

where $\Phi(G(\pi)) = G(\pi^{p^f})$. Except for the case $C=1$, $\vec{c}=\vec{0}$, the map $C\pi^{(p-1)\Sigma_i}\Phi - 1$ defines a bijection $\mathbf{F}[[\pi]] \to \mathbf{F}[[\pi]]$. So the trick is to find $H_i(\pi)$ so that we have $(\lambda_\gamma^{\Sigma_i}\gamma - 1)(H_i(\pi)) \in \mathbf{F}[[\pi]]$. The corresponding $G_i(\pi)$ and so $\mu_\gamma(B_i)$'s are automatically and uniquely determined by the bijectivity. Moreover since the bijection $C\pi^{(p-1)\Sigma_i}\Phi - 1$ on the compact Hausdorff space $\mathbf{F}[[\pi]]$ is continuous, so is its inverse, and it follows that $\gamma \mapsto \mu_\gamma(B_i)$ is continuous.

To find such $H_i(\pi)$, we observe via Lemma 4.2 that

$$\mathrm{val}_\pi(\lambda_\eta^{\Sigma_i}\eta - 1)(\pi^{1-p}) = 2-p \text{ and } \mathrm{val}_\pi(\lambda_\eta^{\Sigma_i}\eta - 1)(\pi^s) = s \text{ if } 2-p \le s \le -1.$$

Then there exist $\epsilon_{2-p},\dots,\epsilon_{-1} \in \mathbf{F}_p$ such that

$$\left(\lambda_\eta^{\Sigma_i}\eta - 1\right)(\pi^{1-p} + \epsilon_{2-p}\pi^{2-p} + \ldots + \epsilon_{-1}\pi^{-1}) \in \mathbf{F}[[\pi]].$$

We set

$$H_i(\pi) = \pi^{1-p} + h_i(\pi) = \pi^{1-p} + \epsilon_{2-p}\pi^{2-p} + \ldots + \epsilon_{-1}\pi^{-1}$$

and claim that

$$(\lambda_\gamma^{\Sigma_i}\gamma - 1)(H_i(\pi)) \in \mathbf{F}_p[[\pi]]$$

for all $\gamma\in\Gamma$. Note that by Corollary 4.3 we have $\lambda_{\gamma_1} \equiv 1 \mod \pi^{p-1}$, so that $(\lambda_{\gamma_1}\gamma_1 - 1)(\pi^s) \equiv 0 \mod \pi^{p-1}$ for all $1-p \le s \le -1$ if $\gamma_1 \in \Gamma_1$. Since any given $\gamma\in\Gamma$ can be written as $\gamma = \eta^m\gamma_1$ where $m \in \mathbf{N}_{\ge 0}$ and $\gamma_1\in\Gamma_1$, we have

$$(\lambda_\gamma^{\Sigma_i}\gamma - 1)(H_i(\pi)) \in \mathbf{F}_p[[\pi]]$$

by the following.

Lemma 4.5. *Let Σ and v be integers and $H(\pi) \in \mathbf{F}((\pi))$. For any $\gamma,\gamma' \in \Gamma$, if the valuations (in π) of $(\lambda_\gamma^\Sigma\gamma - 1)(H(\pi))$ and $(\lambda_{\gamma'}^\Sigma\gamma' - 1)(H(\pi))$ are $\ge v$, so is that of $(\lambda_{\gamma\gamma'}^\Sigma\gamma\gamma' - 1)(H(\pi))$.*

Proof. If both $\lambda^{\Sigma}_{\gamma}\gamma(H(\pi)) - H(\pi)$ and $\lambda^{\Sigma}_{\gamma'}\gamma'(H(\pi)) - H(\pi)$ are in $\pi^v\mathbf{F}[[\pi]]$, then

$$\begin{aligned}(\lambda^{\Sigma}_{\gamma\gamma'}\gamma\gamma' - 1)(H(\pi)) &= \left(\frac{\gamma\gamma'(\pi)}{\pi\overline{\chi}(\gamma\gamma')}\right)^{\frac{p-1}{p^f-1}\Sigma}\gamma\gamma'(H(\pi)) - H(\pi)\\ &= \left(\gamma\left(\frac{\gamma'(\pi)}{\pi\overline{\chi}(\gamma')}\right)\frac{\gamma(\pi)}{\pi\overline{\chi}(\gamma)}\right)^{\frac{p-1}{p^f-1}\Sigma}\gamma(\gamma'(H(\pi))) - H(\pi)\\ &= \lambda^{\Sigma}_{\gamma}\gamma\left(\lambda^{\Sigma}_{\gamma'}\gamma'(H(\pi)) - H(\pi)\right) + \lambda^{\Sigma}_{\gamma}\gamma(H(\pi)) - H(\pi) \in \pi^v\mathbf{F}[[\pi]].\end{aligned}$$

□

So far we have defined $\mu_\varphi = \mu_\varphi(B_i)$ and $\mu_\gamma = \mu_\gamma(B_i)$ satisfying the condition (†), and need to verify the condition (‡). It is easily checked that if $\gamma, \gamma' \in \Gamma$, both $\mu_{\gamma\gamma'}$ and $\mu'_{\gamma\gamma'} = \kappa_\gamma\gamma(\mu_{\gamma'}) + \mu_\gamma$ satisfy (†). Since when we fix μ_φ the solution of (†) for $\gamma\gamma'$ is unique (by the bijectivity of the map $C\pi^{\Sigma_i}\Phi - 1$), we must have (‡) $\mu_{\gamma\gamma'} = \kappa_\gamma\gamma(\mu_{\gamma'}) + \mu_\gamma$.

4.2. **Construction of B_i when $c_i = p-1$.**

Proposition 4.6. *If $c_i = p-1$ and $c_{i+1} \neq p-2$, we have*

$$(\lambda^{\Sigma_i}_{\eta}\eta - 1)\left(\pi^{1-p^2} + h''_i(\pi) + \epsilon(\pi^{2-2p} + h'_i(\pi) + h_i(\pi))\right) \in \mathbf{F}[[\pi]]$$

for some $\epsilon \in \mathbf{F}^\times$ and for some Laurent polynomials $h''_i(\pi) = \sum_{s=1}^{p-2}\epsilon''_s\pi^{1-p^2+sp}$, $h'_i(\pi) = \sum_{s=1}^{p-2}\epsilon'_s\pi^{2-2p+s}$, $h_i(\pi) = \sum_{s=1}^{p-2}\epsilon_s\pi^{1-p+s} \in \mathbf{F}[[\pi]][1/\pi]$.

Proof. By Lemma 4.2 we have,

$$(\lambda^{\Sigma_i}_{\eta}\eta - 1)(\pi^{1-p^2}) \in \mathbf{F}^\times\pi^{1-p^2+p} + \pi^{1-p^2+2p}\mathbf{F}[[\pi^p]]$$

and

$$(\lambda^{\Sigma_i}_{\eta}\eta - 1)(\pi^{1-p^2+sp}) - 1 \in \mathbf{F}^\times\pi^{1-p^2+sp} + \sum_{j=1}^{p-s}\mathbf{F}\pi^{1-p^2+(s+j)p} + \mathbf{F}[[\pi]]$$

for $1 \le s \le p-2$. Then there exist $\epsilon''_s, \nu' \in \mathbf{F}$ such that

$$(\lambda^{\Sigma_i}_{\eta}\eta - 1)\left(\pi^{1-p^2} + \sum_{s=1}^{p-2}\epsilon''_s\pi^{1-p^2+sp}\right) \in \nu'\pi^{1-p} + \mathbf{F}[[\pi]].$$

Similarly, for some $\epsilon'_s, \epsilon_s, \nu \in F$ we have

$$(\lambda^{\Sigma_i}_{\eta}\eta - 1)\left(\pi^{2-2p} + \sum_{s=1}^{p-2}\epsilon'_s\pi^{2-2p+s} + \sum_{s=1}^{p-2}\epsilon_s\pi^{1-p+s}\right) \in \nu\pi^{1-p} + \mathbf{F}[[\pi]].$$

Put $h''_i(\pi) = \sum_{s=1}^{p-2}\epsilon''_s\pi^{2-p^2+sp}, h'_i(\pi) = \sum_{s=1}^{p-2}\epsilon'_s\pi^{2-2p+s}, h_i(\pi) = \sum_{s=1}^{p-2}\epsilon_s\pi^{1-p+s}$.

The point then is to show that both ν and ν' are nonzero so that

$$(\lambda^{\Sigma_i}_{\eta}\eta - 1)\left(\pi^{1-p^2} + h''_i(\pi)\right) + \epsilon\,(\lambda^{\Sigma_i}_{\eta}\eta - 1)\left(\pi^{2-2p} + h'_i(\pi) + h_i(\pi)\right) \in \mathbf{F}[[\pi]]$$

where $\epsilon = -\nu'/\nu \in \mathbf{F} - \{0\}$. So let us prove nonvanishing of ν' and ν. Suppose $\nu' = 0$ so that $\mathrm{val}_\pi(\lambda^{\Sigma_i}_{\eta}\eta - 1)\left(\pi^{1-p^2} + h''_i(\pi)\right) \ge 0$. By Lemma 4.5, recalling $\xi = \eta^{p-1}$, it follows that $\mathrm{val}_\pi(\lambda^{\Sigma_i}_{\xi}\xi - 1)\left(\pi^{1-p^2} + h''_i(\pi)\right) \ge 0$, However, by Lemma 4.4

we have $\mathrm{val}_\pi(\lambda_\eta^{\Sigma_i}\eta-1)\left(\pi^{1-p^2}+h_i''(\pi)\right)=1-p$. Thus ν' cannot be zero. Similarly, we get $\nu\neq 0$. □

Proposition 4.7. *If $c_i=p-1$, and $r\in\{0,\dots,f-1\}$ is such that $c_{i+1}=\cdots=c_{i+r}=p-2$ and $c_{i+r+1}\neq p-2$, we have*

$$(\lambda_\eta^{\Sigma_i}\eta-1)\left(\pi^{1-p^{r+2}}+\sum_{j=0}^{r+1}h_i^{(j)}+\sum_{j=0}^{r}\epsilon^{(j)}h_i'^{(j)}\right)\in\mathbf{F}[[\pi]]$$

for some Laurent polynomials

$$h_i^{(j)}(\pi)=\sum_{s=1}^{p-2}\epsilon_s^{(j)}\pi^{1-p^{j+1}+sp^j}\ (0\leq j\leq r+1),$$

$$h_i'^{(j)}(\pi)=\pi^{1+p^j-2p^{j+1}}+\sum_{s=1}^{p-2}\epsilon_s'^{(j)}\pi^{1+p^j-2p^{j+1}+sp^j}\ (0\leq j\leq r)$$

in $\mathbf{F}[1/\pi]$ with $\epsilon_1^{(r+1)},\epsilon^{(r)}\neq 0$.

Proof. By Lemma 4.2 (with $v=r+1, s_v=c_{i+r+1}+2$) we get

$$(\lambda_\eta^{\Sigma_i}\eta-1)\pi^{1-p^{r+2}}\in\mathbf{F}^\times\pi^{1-p^{r+2}+p^{r+1}}+\pi^{1-p^{r+2}+2p^{r+1}}\mathbf{F}[[\pi^{p^{r+1}}]],$$

and

$$(\lambda_\eta^{\Sigma_i}\eta-1)\pi^{1-p^{r+2}+sp^{r+1}}\in\mathbf{F}^\times\pi^{1-p^{r+2}+sp^{r+1}}+\sum_{t=1}^{p-s-1}\mathbf{F}\pi^{1-p^{r+2}+(s+t)p^{r+1}}+\mathbf{F}[[\pi]]$$

for $1\leq s\leq p-2$, so that there exist $\epsilon_1^{(r+1)},\dots,\epsilon_{p-2}^{(r+1)},\nu^{(r+1)}\in\mathbf{F}$ such that

$$(\lambda_\eta^{\Sigma_i}\eta-1)\left(\pi^{1-p^{r+2}}+\sum_{s=1}^{p-2}\epsilon_s^{(r+1)}\pi^{1-p^{r+2}+sp^{r+1}}\right)\in\nu^{(r+1)}\pi^{1-p^{r+1}}+\mathbf{F}[[\pi]].$$

We set $h_i^{(r+1)}=\sum_{s=1}^{p-2}\epsilon_s^{(r+1)}\pi^{1-p^{r+2}+sp^{r+1}}$.

Again by Lemma 4.2, we get

$$(\lambda_\eta^{\Sigma_i}\eta-1)\pi^{1-2p^{r+1}+p^r}\in\mathbf{F}^\times\pi^{1-2p^{r+1}+2p^r}+\pi^{1-2p^{r+1}+3p^r}\mathbf{F}[[\pi^{p^r}]],$$

and

$$(\lambda_\eta^{\Sigma_i}\eta-1)\pi^{1-2p^{r+1}+(1+s)p^r}\in\mathbf{F}^\times\pi^{1-2p^{j+1}+(1+s)p^j}+\sum_{t=1}^{p-s-2}\mathbf{F}\pi^{1-2p^{r+1}+(1+s+t)p^r}+\pi^{1-p^{r+1}}\mathbf{F}[[\pi^{p^r}]]$$

for $1\leq s\leq p-2$, so that there exist $\epsilon_1'^{(r)},\dots,\epsilon_{p-2}'^{(r)},\nu'^{(r)}\in\mathbf{F}$ such that

$$(\lambda_\eta^{\Sigma_i}\eta-1)\left(\sum_{s=0}^{p-2}\epsilon_s'^{(r)}\pi^{1+p^r-2p^{r+1}+sp^r}\right)\in\nu'^{(r)}\pi^{1-p^{r+1}}+\pi^{1-p^{r+1}+p^r}\mathbf{F}[[\pi^{p^r}]],$$

where we have set $\epsilon_0^{'(r)}=1$.

As in the proof of Proposition 4.6, one can show that both $\nu^{(r+1)}$ and $\nu^{'(r)}$ are not zero, so that

$$(\lambda_\eta^{\Sigma_i}\eta-1)\left(\pi^{1-p^{r+2}}+h_i^{(r+1)}+\epsilon^{(r)}h_i^{'(r)}\right)\in\pi^{1-p^{r+1}+p^r}\mathbf{F}[[\pi^{p^r}]]$$

where $\epsilon^{(r)} = -\nu^{(r+1)}/\nu'^{(r)} \neq 0$. Then again

$$(\lambda_\eta^{\Sigma_i}\eta - 1)\left(\pi^{1-p^{r+2}} + h_i^{(r+1)} + \epsilon^{(r)} h_i'^{(r)} + h_i^{(r)}\right) \in \mathbf{F}\pi^{1-p^r} + \mathbf{F}[[\pi]]$$

for some $h_i^{(r)} = \sum_{s=1}^{p-2} \epsilon_s^{(r)} \pi^{1-p^{r+1}+sp^r}$. Iterating the process proves the proposition. □

When $c_i = p-1, c_{i+1} = \cdots = c_{i+r} = p-2, c_{i+r+1} \neq p-2$ we define

$$\mu_\varphi(B_i) = (0, \ldots, 0, H_i(\pi), 0, \ldots, 0)$$

where

$$H_i(\pi) = \pi^{1-p^{r+2}} + \sum_{j=1}^{r+1} h_i^{(j)}(\pi) + \sum_{j=0}^{r} \epsilon^{(j)} h_i'^{(j)}(\pi)$$

is the ith component. By Proposition 4.7, we get $(\lambda_\gamma^{\Sigma_i}\gamma - 1)(\mu_\varphi(B_i)) \in \mathbf{F}[[\pi]]$ and then $\mu_\gamma(B_i)$ is determined by bijectivity of the map $C\pi^{(p-1)\Sigma_i}\Phi - 1 : \mathbf{F}[[\pi]] \to \mathbf{F}[[\pi]]$. The condition (‡) is checked in an analogous fashion as in §4.1.

Remark 4.8. The cocycle B_i for the case $c_i = p-1$, $c_{i+1} = \cdots = c_{i+r} = p-2, c_{i+r+1} \neq p-2$ is cohomologous to a cocycle B_i' defined by

$$\begin{aligned}
\mu_\varphi(B_i') = &\left(\epsilon^{(0)}\pi^{2-2p}\sum_{s=0}^{p-2}\epsilon_s'^{(0)}\pi^s + \pi^{1-p}\sum_{s=1}^{p-2}\epsilon_s^{(0)}\pi^s\right)e_i \\
&+ \left(\epsilon^{(1)}\pi^{3-3p}\sum_{s=0}^{p-2}\epsilon_s'^{(1)}\pi^s + \pi^{2-2p}\sum_{s=1}^{p-2}\epsilon_s^{(1)}\pi^s\right)e_{i+1} \\
&+ \cdots \\
&+ \left(\epsilon^{(r)}\pi^{3-3p}\sum_{s=0}^{p-2}\epsilon_s'^{(r)}\pi^s + \pi^{2-2p}\sum_{s=1}^{p-2}\epsilon_s^{(r)}\pi^s\right)e_{i+r} \\
&+ \left(\pi^{2-2p}\sum_{s=0}^{p-2}\epsilon_s^{(r+1)}\pi^s\right)e_{i+r+1},
\end{aligned}$$

where $\epsilon_0'^{(0)} = \epsilon_0'^{(1)} = \cdots = \epsilon_0'^{(r)} = \epsilon_0^{(r+1)} = 1$ and $\epsilon^{(r)} \neq 0$. See Lemma 5.5 for the proof in the case $f = 2$.

4.3. **Linear independence of B_i's.** Throughout this section we assume $C \neq 1$ if $\vec{c} = \vec{0}$, so that $C\pi^{(p-1)\Sigma_i} : \mathbf{F}[[\pi]] \to \mathbf{F}[[\pi]]$ defines a valuation-preserving bijection for all $i \in S$. From the constructions in §§4.1, 4.2 we have at hand the extensions $B_0, \ldots, B_{f-1} \in \mathrm{Ext}^1(M_0, M_{C\vec{c}})$ such that, if $i \in S$,

$$\mu_\varphi(B_i) = (0, \ldots, 0, H_i(\pi), \ldots, 0)$$

has ith component

$$H_i(\pi) = \pi^{1-p^{r+2}} + \sum_{j=0}^{r+1} h_i^{(j)}(\pi) + \sum_{j=0}^{r} \epsilon^{(j)} h_i'^{(j)}(\pi),$$

where if $c_i \neq p-1$, then we set $r = -1$ and $h_i^{(0)}(\pi) = h_i(\pi)$ was defined in §4.1, and if $c_i = p-1$, then r is the least non-negative integer such that $c_{i+r+1} \neq p-2$ and $h_i^{(j)}$, $h_i^{(j)}$ and $\epsilon'^{(j)}$ were defined in §4.2.

To prove linear independence of B_i's, suppose that $\beta_0 B_0 + \cdots + \beta_{f-1}B_{f-1}$ is equivalent to the trivial extension for some $\beta_0,\dots,\beta_{f-1}\in \mathbf{F}$. We want to show that $\beta_0 = \cdots = \beta_{f-1} = 0$. By cyclic nature of indexing, it is enough to show that $\beta_{f-1} = 0$. Since $\beta_0\mu_\varphi(B_0) + \cdots + \beta_{f-1}\mu_\varphi(B_{f-1}) = (\beta_0 H_0(\pi),\dots,\beta_{f-1}H_{f-1}(\pi))$ defines a coboundary, by "adding some coboundary," i.e., by making an appropriate change of basis for the extension, we see that

$$\begin{aligned}&(\beta_0 H_0(\pi) + \beta_1 C\pi^{(p-1)c_0}H_1(\pi^p) + \cdots + \beta_{f-1}C\pi^{(p-1)\sum_{j=0}^{f-2}c_jp^j}H_{f-1}(\pi^{p^{f-1}}),0,\dots,0)\\ =&(C\pi^{(p-1)c_0}b_1(\pi^p) - b_0(\pi), \pi^{(p-1)c_1}b_2(\pi) - b_1(\pi),\dots,\pi^{(p-1)c_{f-1}}b_0(\pi^p) - b_{f-1}(\pi))\end{aligned}$$

for some $(b_0(\pi),\dots,b_{f-1}(\pi))\in \mathbf{F}((\pi))^S$. It follows that

$$\begin{aligned}&\beta_0 H_0(\pi) + \beta_1 C\pi^{(p-1)c_0}H_1(\pi^p) + \cdots + \beta_{f-1}C\pi^{(p-1)\sum_{j=0}^{f-2}c_jp^j}H_{f-1}(\pi^{p^{f-1}})\\ =&(C\pi^{(p-1)\Sigma_0}\Phi - 1)(b_0(\pi))\end{aligned}$$

and

$$\begin{aligned}b_1(\pi) &= \pi^{(p-1)c_1}b_2(\pi^p),\\ b_2(\pi) &= \pi^{(p-1)c_2}b_3(\pi^p),\\ &\cdots\\ b_{f-2}(\pi) &= \pi^{(p-1)c_{f-2}}b_{f-1}(\pi^p)\\ b_{f-1}(\pi) &= \pi^{(p-1)c_{f-1}}b_0(\pi^p).\end{aligned}$$

As the map $C\pi^{(p-1)\Sigma_0}\Phi - 1 : \mathbf{F}[[\pi]] \to \mathbf{F}[[\pi]]$ is a bijection, we get a congruence

$$\begin{aligned}&\beta_0 H_0(\pi) + \beta_1 C\pi^{(p-1)c_0}H_1(\pi^p) + \cdots + \beta_{f-1}C\pi^{(p-1)\sum_{j=0}^{f-2}c_jp^j}H_{f-1}(\pi^{p^{f-1}})\\ \equiv&(C\pi^{(p-1)\Sigma_0}\Phi - 1)(b(\pi)) \mod \mathbf{F}[[\pi]]\end{aligned}$$

for some $b(\pi) = b_{-s}\pi^{-s} + \sum_{j=1}^{s-1} b_{-s+j}\pi^{-s+j} \in \mathbf{F}[1/\pi]$ with $s>0$ and $b_{-s}\neq 0$. Suppose $\beta_{f-1}\neq 0$ and we will get contradictions.

First assume $c_{f-1} = p-1, c_f = \cdots c_{f-1+r} = p-2, c_{f+r}\neq p-2$ with $r>0$, in which case we have

$$H_{f-1}(\pi) = \pi^{1-p^{r+2}} + \sum_{j=0}^{r+1} h_i^{(j)}(\pi) + \sum_{j=0}^{r}\epsilon^{(j)}h_i'^{(j)}(\pi).$$

One checks that the lowest degree term (in π) of the LHS of the congruence is

$$\beta_{f-1}C\pi^{(p-1)\sum_{j=0}^{f-2}c_jp^j}\pi^{(1-p^{r+2})p^{f-1}},$$

so that the valuation of the LHS is $(p-1)(\sum_{j=0}^{f-2}c_jp^j - (1+p+\cdots+p^{r+1})p^{f-1})$. On the other hand, we have three possibilities for the RHS: $(p-1)\Sigma_0 - sp^f < -s$, $-s < (p-1)\Sigma_0 - sp^f$ and $(p-1)\Sigma_0 - sp^f = -s$.

If $(p-1)\Sigma_0 - sp^f < -s$, the leading term of the RHS is $b_{-s}C\pi^{(p-1)\Sigma_0}\pi^{-sp^f}$ and we have $s = (p-1)(2+p+\cdots+p^r)$ and $\beta_{f-1} = b_{-s}$. Now the term

$$\beta_{f-1}C\pi^{(p-1)\sum_{j=0}^{f-2}c_jp^j}\epsilon^{(r)}\pi^{(1+p^r-2p^{r+1})p^{f-1}}$$

is alive on the LHS and must match a term on the RHS. Considering possible matching valuations on the RHS we get either

$$(p-1)\sum_{j=0}^{f-2}c_jp^j + (1+p^r-2p^{r+1})p^{f-1} = -t$$

or

$$(p-1)\sum_{j=0}^{f-2} c_j p^j + (1+p^r-2p^{r+1})p^{f-1} = (p-1)\Sigma_0 - tp^f$$

for some $0 < t < s = (p-1)(2+p+\cdots+p^r)$. The former equation contradicts the inequality $t < s$ and the latter implies that $t = 2p^r - p^{r-1} + p - 2$. Since $p^f \not| t + (p-1)\Sigma_0$, there must be a term of degree $-t$ on the LHS. However if $m < r$, then the leading term of $\pi^{(p-1)\sum_{j=0}^{m-1} c_j p^j} H_m(\pi^{p^m})$ has degree $> -t$, and if $m \geq r$, then its terms cannot be congruent to $-t \bmod p^r$, and we again arrive at a contradiction. If $-s < (p-1)\Sigma_0 - sp^f$, the leading term of the RHS is $-b_{-s}\pi^{-s}$. Then $(p-1)|s$ and $s(p^f-1)/(p-1) < \Sigma_0 < p^f - 1$, so that $s < 1$, which is impossible. Lastly, if $(p-1)\Sigma_0 - sp^f = -s$, working modulo powers of p, we get $s = c_0 = \cdots = c_{f-1} = p-1$, a contradiction.

Now we may assume that $\beta_j = 0$ for all $j \in S$ such that $c_j = p-1$ and $c_{j+1} = p-2$. Suppose now that $c_{f-1} = p-1$ and $c_0 \neq p-2$. We then proceed to show that $\beta_{f-1} = 0$ by induction on m where $m \geq 1$ is such that $c_{f-m-1} \neq p-1$ and $c_{f-m} = c_{f-m+1} = \cdots c_{f-1} = p-1$. We may thus assume that $\beta_{f-m} = \cdots = \beta_{f-2} = 0$ if $m \geq 2$. The argument used in the case $r > 0$ then goes through with the following two changes: 1) the induction hypothesis is used to show that the term

$$\beta_{f-1} C \pi^{(p-1)\sum_{j=0}^{f-2} c_j p^j} \epsilon^{(0)} \pi^{(2-2p)p^{f-1}}$$

is alive on the LHS, and 2) the equality

$$(p-1)\sum_{j=0}^{f-2} c_j p^j + (2-2p)p^{f-1} = (p-1)\Sigma_0 - tp^f$$

immediately gives a contradiction without considering more terms.

Now we may assume that $\beta_j = 0$ for all $j \in S$ such that $c_j = p-1$, and suppose $c_{f-1} < p-1$. The leading term of the RHS then is

$$\beta_{f-1} C \pi^{(p-1)\sum_{j=0}^{f-2} c_j p^j} \pi^{(1-p)p^{f-1}}.$$

If $(p-1)\Sigma_0 - sp^f < -s$, $sp^f = (p-1)(\Sigma_0 - \sum_{j=0}^{f-2} c_j p^j + p^{f-1}) = (p-1)(c_{f-1}+1)p^{f-1}$, ans so $p|(c_{f-1}+1)$, which is impossible as $0 \leq c_{f-1} < p-1$. If $(p-1)\Sigma_0 - sp^f \geq -s$, then $-s \leq (p-1)(\sum_{j=0}^{f-2} c_j p^j - p^{f-1}) \leq 1-p$, contradicting that $s(p^f-1)/(p-1) \leq \Sigma_0 < p^f - 1$.

This completes the proof that the B_i are linearly independent, hence form a basis for $\mathrm{Ext}^1(M_0, M_{C\vec{c}})$ (unless $C = 1, \vec{c} = \vec{0}$ or $C = 1, \vec{c} = \overline{p-2}$).

5. The space of bounded extensions

In this section we define bounded extensions, which we will later relate to extensions arising from crystalline representations.

5.1. Bounded extensions.

Definition 5.1. Suppose $A, B \in \mathbf{F}^\times$ and $0 \leq a_i, b_i \leq p$ with exactly one of a_i or b_i is zero for each $i \in S$. We say that an extension (class) $E \in \mathrm{Ext}^1(M_{A\vec{a}}, M_{B\vec{b}})$ is *bounded* if there exists a basis for E in which the defining matrices P and G_γ satisfy the following:

(1) $P = \begin{pmatrix} \kappa_\varphi(B,\vec{b}) & * \\ 0 & \kappa_\varphi(A,\vec{a}) \end{pmatrix}$ and $G_\gamma = \begin{pmatrix} \kappa_\gamma(B,\vec{b}) & * \\ 0 & \kappa_\gamma(A,\vec{a}) \end{pmatrix}$ if $\gamma \in \Gamma$,
(2) $P \in \mathrm{M}_2(\mathbf{F}[[\pi]]^S)$,
(3) $G_\gamma - I_2 \in \pi\mathrm{M}_2(\mathbf{F}[[\pi]]^S)$ if $\gamma \in \Gamma_1$.

Bounded extensions form a subspace, denoted by ${\mathrm{Ext}^1}_{bdd}(M_{A\vec{a}}, M_{B\vec{b}})$, of the full space $\mathrm{Ext}^1(M_{A\vec{a}}, M_{B\vec{b}})$ of extensions.

Lemma 5.2. *The condition (3) can be replaced by a weaker condition (3′) $G_\xi - I_2 \in \pi\mathrm{M}_2(\mathbf{F}[[\pi]]^S)$.*

Proof. If $\gamma, \gamma' \in \Gamma_1$, then

$$G_{\gamma\gamma'} = G_\gamma \gamma G_{\gamma'} \equiv \begin{pmatrix} 1 & \mu_\gamma(E) + \gamma\mu_{\gamma'}(E) \\ 0 & 1 \end{pmatrix} \mod \pi$$

by Corollary 4.3. So, if $G_\xi = I_2 \mod \pi$, we have by induction that $G_{\xi^n} \equiv I_2$ for all $n \geq 1$. If $\gamma \in \Gamma_n$ for sufficiently large $n \geq 1$ we get, by continuity of the action of Γ, that $G_\gamma - I_2 \in \pi\mathrm{M}_2(\mathbf{F}[[\pi]]^S)$. □

We now describe a way to analyze extensions systematically and to check for boundedness. Given $J \subset S$ and $n \in \mathbf{Z}/(p^f-1)\mathbf{Z}$ we can always find a_i, b_j for $i \in J, j \in S - J$ with $1 \leq a_i, b_j \leq p$ such that

$$n \equiv \sum_{j \notin J} b_j p^j - \sum_{i \in J} a_i p^i \mod p^f - 1.$$

The congruence has a unique solution if $n \not\equiv n_J \mod p^f - 1$, and has two solutions if $n \equiv n_J \mod p^f - 1$ where $n_J := \sum_{i\in J} p^{i+1} - \sum_{i \notin J} p^i$ (cf. [BDJ, §3]). To compute the solutions explicitly in the double solution case suppose $n \equiv n_J \mod p^f - 1$ and we have two solutions a_i, b_j and a'_i, b'_j. Then $\Sigma := \sum_{j\notin J}(b_j - b'_j)p^j - \sum_{i \in J}(a_i - a'_i)p^i \equiv 0 \mod p^f - 1$. Note that $|\Sigma| \leq p^f - 1$, as $|a_i - a'_i|, |b_j - b'_j| \leq p - 1$, so that $\Sigma = 0$ or $\pm(p^f - 1)$. If $0 = \Sigma = \sum_{i \in S} d_i p^i$, where $d_i = a'_i - a_i$ if $i \in J$ and $d_j = b_j - b'_j$ if $j \notin J$. Noting $|d_i| \leq p - 1$, computations modulo powers of p shows $\vec{d} = \vec{0}$. If $\Sigma = \pm(p^f - 1)$, we have solutions $a_i = p, b_i = 1$ and $a'_i = 1, b'_i = p$ $(i \in J, j \in S - J)$. Conversely, given a_i, b_i (for $i \in J, j \in S - J$ with $1 \leq a_i, b_j \leq p$) there exists unique $\vec{c} = (c_0, \dots, c_{f-1}) \in \{0, 1, \dots, p-1\}^S$ with some $c_i < p - 1$ such that $\sum_{i=0}^{f-1} c_i p^i \equiv \sum_{i \notin J} b_i p^i - \sum_{i \in J} a_i p^i$.

Now fix $J \subset S$, $C \in \mathbf{F}^\times$ and $\vec{c} \in \{0, 1, \dots, p-1\}^S$ with some $c_i < p - 1$. If $\Sigma_0 \vec{c} \not\equiv n_J \mod p^f - 1$, we can solve the congruence $\Sigma_0 \vec{c} \equiv \sum_{i \notin J} b_i p^i - \sum_{i \in J} a_i p^i$ mod $p^f - 1$ with unique solution, and get an isomorphism

$$\mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}) \simeq \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{d}}),$$

where $d_i = -a_i$ if $i \in J$ and $d_j = b_j$ if $j \notin J$. The isomorphism is (not canonical but) well-defined up to $\mathrm{Aut} M_{C\vec{c}} = \mathbf{F}^\times$ and the valuation of entries of the matrices defining (φ, Γ)-module extensions are invariant, which suffices our needs. Tensoring $M_{A\vec{a}}$ to the sub and the quotient of the extension we get an isomorphism

$$\iota : \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}) \to \mathrm{Ext}^1(M_{A\vec{a}}, M_{B\vec{b}})$$

where $CA = B$ and $a_i = 0$ if $i \notin J$ and $b_i = 0$ if $i \in J$. In fact, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}) & \longrightarrow & \mathrm{Ext}^1(M_{A\vec{a}}, M_{B\vec{b}}) \\ \downarrow & & \downarrow \\ H/H_0 & \longrightarrow & H'/H'_0. \end{array}$$

The vertical arrows are μ's, and the bottom arrow is induced by

$$(\mu_\varphi, (\mu_\gamma)_{\gamma\in\Gamma}) \mapsto (\kappa_\varphi(A,\vec{a})\langle\vec{c}\rangle_J \mu_\varphi, (\kappa_\gamma(A,\vec{a})\langle\vec{c}\rangle_J \mu_\gamma)_{\gamma\in\Gamma}),$$

where the isomorphism $\mathbf{E}_{K,F}e = M_{C\vec{c}} \simeq M_{C\vec{d}} = \mathbf{E}_{K,F}e'$ is defined by the change of basis $e = \langle\vec{c}\rangle_J e'$ with $\langle\vec{c}\rangle_J \in \mathbf{E}_{K,F}^\times$. We define

$$V_J = \iota^{-1}(\mathrm{Ext}^1{}_{bdd}(M_{A\vec{a}}, M_{B\vec{b}})) \subset \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}),$$

so that $\dim_{\mathbf{F}} V_J = \dim_{\mathbf{F}} \mathrm{Ext}^1{}_{bdd}(M_{A\vec{a}}, M_{B\vec{b}})$.

If $\Sigma_0\vec{c} \equiv n_J \mod p^f - 1$, we can assume that $n_J = \sum_{i\notin J} b_i p^i - \sum_{i\in J} a_i p^i$ and $n_J + 1 - p^f = \sum_{i\notin J} b'_i p^i - \sum_{i\in J} a'_i p^i$ where a_i, b_j and a'_i, b'_j are two solutions. Then we have $a_i = p, b_j = 1$ and $a'_i = 1, b'_j = p$ (for $i \in J$ and $j \in S - J$).

Similarly to the case of unique solution, we have isomorphisms

$$\iota_+ : \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}) \to \mathrm{Ext}^1(M_{A\vec{a}}, M_{B\vec{b}}),$$
$$\iota_- : \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}) \to \mathrm{Ext}^1(M_{A\vec{a'}}, M_{B\vec{b'}})$$

and define

$$V_J^+ = \iota_+^{-1}(\mathrm{Ext}^1{}_{bdd}(M_{A\vec{a}}, M_{B\vec{b}})) \subset \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}),$$
$$V_J^- = \iota_-^{-1}(\mathrm{Ext}^1{}_{bdd}(M_{A\vec{a'}}, M_{B\vec{b'}})) \subset \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}).$$

In the next two subsections we will compute explicitly the spaces of bounded extensions in the generic case and in the case $f = 2$.

5.2. **Generic case.** For each $i \in S$, let $e_i : \mathbf{E}_{K,F} = \mathbf{F}((\pi))^S \to \mathbf{F}((\pi))$ denote the projection $(g_0, \ldots, g_{f-1}) \mapsto g_i$.

Proposition 5.3. *If $0 < c_i < p - 1$ for all $i \in S$, then*

(1) $V_{\{i\}} = \mathbf{F}B_{i+1}$ *for all* $i \in S$;
(2) $V_J = \oplus_{i\in J} V_{\{i\}}$ *if* $J \subset S$;
(3) $V_S^+ = V_S^- = \mathrm{Ext}^1(M_{\vec{0}}, M_{C\overrightarrow{p-2}})$ *if* $C \neq 1, \vec{c} = \overrightarrow{p-2}$.

Remark 5.4. Proposition 5.3 does not say anything about the cyclotomic case $C = 1, \vec{c} = \overrightarrow{p-2}$, which will be treated in §6.1.

Proof. First the case $J \neq \emptyset$. We may assume that $f - 1 \in J$; even though we do not have complete symmetry due to the presence of the constant C, we will see that the argument goes independently of which component C lies in. As $0 < c_i < p - 1$ for all $i \in S$ we have $\frac{p^f-1}{p-1} \leq \Sigma_0\vec{c} \leq (p-2)\frac{p^f-1}{p-1}$. We claim that the congruence

$$\Sigma_0\vec{c} \equiv \sum_{i\notin J} b_i p^i - \sum_{i\in J} a_i p^i \mod p^f - 1,$$

has a unique solution a_i, b_j ($i \in J, j \notin J$) such that $1 \leq a_i, b_j \leq p$, except when $J = S$ and $\vec{c} = \overrightarrow{p-2}$. If there were another distinct solution, we have either $\Sigma_0\vec{c} = \sum_{j\notin J} p^{j+1} - \sum_{j\in J} p^j$ or $\Sigma_0\vec{c} = \sum_{j\notin J} p^{j+1} - \sum_{j\in J} p^j + p^f - 1$. The former is impossible since modulo p we have $c_0 \equiv -1$ if $0 \in J$ and $c_0 \equiv 0$ if $j \notin J$, and thus

$c_0 = p-1$ or 0, contradicting the assumption. In the latter, we have $0 \in J$ and $c_0 = p-2$. Computations modulo p^2 show that $1 \in J$ and $c_1 = p-2$. By induction we get $J = S$ and $c_i = p-2$ for all $i \in S$. Thus, unless $J = S$, $\vec{c} = \overrightarrow{p-2}$, we have unique a_i, b_j $(i \in J, j \notin J)$ satisfying the equation $\sum_{i=0}^{f-1} c_i p^i = \sum_{i\notin J} b_i p^i - \sum_{i\in J} a_i p^i + p^f - 1$. Letting $u = (\pi^{(p-1)\delta_{f-1J}}, \pi^{(p-1)\delta_{0J}}, \dots, \pi^{(p-1)\delta_{f-2J}})$ with $\delta_{iJ} = 1$ if $i \in J$ and $\delta_{iJ} = 0$ otherwise, one checks that $\mathbf{E}_{K,F}e = M_{C\vec{c}} \simeq M_{C\vec{d}} = \mathbf{E}_{K,F}e'$ via the change of basis $e = ue'$ where $d_i = b_i$ if $i \notin J$ and $d_i = -a_i$ if $i \in J$, and that $\langle \vec{c} \rangle_J = u$. Note that

$$\begin{aligned} \frac{\varphi(\langle \vec{c} \rangle_J)}{\langle \vec{c} \rangle_J} &= \frac{(\pi^{(p-1)p\delta_{0J}}, \pi^{(p-1)p\delta_{1J}}, \dots, \pi^{(p-1)p\delta_{f-1J}})}{(\pi^{(p-1)\delta_{f-1J}}, \pi^{(p-1)\delta_{0J}}, \dots, \pi^{(p-1)\delta_{f-2J}})} \\ &= (\pi^{(p-1)(p\delta_{0J}-\delta_{f-1J})}, \pi^{(p-1)(p\delta_{1J}-\delta_{0J})}, \dots, \pi^{(p-1)(p\delta_{f-1J}-\delta_{f-2J})}) \end{aligned}$$

and that

$$(p\delta_{0J}-\delta_{f-1J}) + p(p\delta_{1J}-\delta_{0J}) + \cdots + p^{f-1}(p\delta_{f-1J}-\delta_{f-2J}) = (p^f-1)\delta_{f-1J} = p^f - 1.$$

Recall that we have a basis $B_0, \dots, B_{f-1}$ for $\mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}})$ such that

$$\begin{aligned} \mu_\varphi(B_i) &= (0, \dots, 0, \pi^{1-p} + h_i(\pi), 0, \dots, 0), \\ \mu_\xi(B_i) &= (G_0^{(i)}, \dots, G_{f-1}^{(i)}), \end{aligned}$$

where $h_i(\pi) \in \mathbf{F}\pi^{2-p} + \cdots + \mathbf{F}\pi^{-1}$ and

$$\begin{aligned} G_i^{(i)}(\pi) &= -\alpha_i + g_i(\pi) \\ G_{i-1}^{(i)}(\pi) &= \pi^{(p-1)c_{i-1}}(-\alpha_i + g_i(\pi^p)) \\ G_{i-2}^{(i)}(\pi) &= \pi^{((p-1)(c_{i-2}+c_{i-1}p)}(-\alpha_i + g_i(\pi^{p^2})) \\ &\cdots \\ G_0^{(i)}(\pi) &= \pi^{(p-1)(c_0+c_1p+\cdots+c_{i-1}p^{i-1})}(-\alpha_i + g_i(\pi^{p^i})) \\ G_{f-1}^{(i)}(\pi) &= \pi^{(p-1)(c_{f-1}+c_0p+c_1p^2+\cdots+c_{i-1}p^i)}(-\alpha_i + g_i(\pi^{p^{i+1}})) \\ &\cdots \\ G_{i+1}^{(i)}(\pi) &= \pi^{(p-1)(c_{i+1}+\cdots+c_{i-1}p^{f-2})}(-\alpha_i + g_i(\pi^{p^{f-1}})) \end{aligned}$$

with $\alpha_i = \overline{s_0 z} \in \mathbf{F}^\times$ as in Lemma 4.4.

To show that $\iota B_{i+1} \in \mathrm{Ext}^1(M_{A\vec{a}}, M_{B\vec{b}})$ is bounded if $i \in J$ is straightforward: As $\langle \vec{c} \rangle_J = (\pi^{(p-1)\delta_{f-1J}}, \pi^{(p-1)\delta_{0J}}, \dots, \pi^{(p-1)\delta_{f-2J}})$ and $\delta_{iJ} = 1$, we have

$$\begin{aligned} \mu_\varphi(\iota B_{i+1}) &= \kappa_\varphi(A, \vec{a}) \langle \vec{c} \rangle_J \mu_\varphi(B_{i+1}) \\ &= (0, \dots, 0, \pi^{(p-1)(a_{i+1}+1)}(\pi^{1-p} + h_{i+1}(\pi)), 0, \dots, 0) \in \mathbf{F}[[\pi]]^S \end{aligned}$$

with the nonzero entry in the $(i+1)$-component, and

$$\mu_\xi(\iota B_{i+1}) = \kappa_\xi(A, \vec{a}) \langle \vec{c} \rangle_J \mu_\gamma(B_{i+1}) \in \pi\mathbf{F}[[\pi]]^S$$

as $e_{i+1}\mu_\xi(\iota B_{i+1}) = \lambda_\xi^{(p-1)\Sigma_{j+1}\vec{a}} \pi^{p-1} G_{i+1}^{(i)}$ and $e_{j+1}\mu_\xi(\iota B_{i+1}) = \lambda_\xi^{(p-1)\Sigma_{j+1}\vec{a}} \pi^{(p-1)\delta_{jJ}} G_{j+1}^{(i)}$ is divisible by $\pi^{(p-1)c_{j+1}}$ if $j \neq i$.

Next we need to show that $E = \sum_{j=0}^{f-1} \beta_j B_j \in V_J$ implies $\beta_{i+1} = 0$ for all $i \notin J$. Suppose ιE is bounded, $i \notin J$ and $\beta_{i+1} \neq 0$. Then $\mu_\varphi \iota(E + B) \in \mathbf{F}[[\pi]]^S$ and

$\mu_\xi\iota(E+B) \in \pi\mathbf{F}[[\pi]]^S$ for some coboundary B, for which we let

$$\mu_\varphi(B) = (C\pi^{(p-1)c_0}b_1(\pi^p) - b_0, \pi^{(p-1)c_1}b_2(\pi^p) - b_1, \dots, \pi^{(p-1)c_{f-1}}b_0(\pi^p) - b_{f-1}),$$

$$\mu_\xi(B) = ((\lambda_\xi^{\Sigma_0\vec{c}}\xi - 1)b_0(\pi), (\lambda_\xi^{\Sigma_1\vec{c}}\xi - 1)b_1(\pi) \dots, (\lambda_\xi^{\Sigma_{f-1}\vec{c}}\xi - 1)b_{f-1}(\pi)).$$

for some $(b_0(\pi), \dots, b_{f-1}(\pi)) \in \mathbf{E}_{K,F}$. Note that $\kappa_\xi(A, \vec{a}) \equiv 1 = (1, \dots, 1) \mod \pi$ and $e_{i+1}\langle\vec{c}\rangle_J = \pi^{(p-1)\delta_{iJ}} = 1$. As $\mathrm{val}_\pi e_{i+1}\mu_\xi(E+B) = \mathrm{val}_\pi e_{i+1}\mu_\xi\iota(E+B) \geq 1$ while $\mathrm{val}_\pi e_{i+1}\mu_\xi(E) = 0$, the valuation of $e_{i+1}\mu_\xi(B) = (\lambda_\xi^{\Sigma_{i+1}}\xi - 1)b_{i+1}(\pi)$ has to be zero. Letting $s = \mathrm{val}_\pi b_{i+1}(\pi)$, Lemma 4.4 implies that $(\lambda_\xi^{\Sigma_{i+1}\vec{c}}\xi - 1)b_{i+1}(\pi) \in \pi\mathbf{F}[[\pi]]$ if $s \geq 0$ and $\mathrm{val}_\pi(\lambda_\xi^{\Sigma_{i+1}\vec{c}}\xi - 1)b_{i+1}(\pi) = s + (p-1)p^v$ if $s < 0$ and $\Sigma_{i+1}\vec{c} + s(p^f-1)/(p-1)$ is divisible by p^v but not p^{v+1}. Thus $\mathrm{val}_\pi b_{i+1}(\pi)$ must be negative and divisible by $p-1$. Looking at the i-th component, we have

$$\begin{aligned} e_i\mu_\varphi(\iota(E+B)) &= \pi^{(p-1)(a_i+\delta_{i-1J})}(e_i\mu_\varphi(E) + e_i\mu_\varphi(B)) \\ &= \pi^{(p-1)\delta_{i-1J}}(\pi^{1-p} + h_i(\pi) + \pi^{(p-1)c_i}b_{i+1}(\pi^p) - b_i(\pi)), \end{aligned}$$

whose valuation has to be non-negative. Since $(p-1)c_i + p\,\mathrm{val}_\pi b_{i+1}(\pi) < 1-p = \mathrm{val}_\pi(\pi^{1-p} + h_i(\pi))$, we get $\mathrm{val}_\pi b_i(\pi) = (p-1)c_i + p\,\mathrm{val}_\pi b_{i+1}(\pi)$. Cycling this through all $j \in S$ leads to $\mathrm{val}_\pi b_{i+1}(\pi) = (p-1)\Sigma_i\vec{c} + p^f\,\mathrm{val}_\pi b_{i+1}(\pi)$, so that $\mathrm{val}_\pi b_{i+1}(\pi) = -\frac{p-1}{p^f-1}\Sigma_i\vec{c} > 1-p$, which is a contradiction.

Now suppose $J = S, C \neq 1, \vec{c} = \overrightarrow{p-2}$. In this case we have two solutions $\vec{a} = \vec{p}, \vec{b} = \vec{0}$ and $\vec{a'} = \vec{1}, \vec{b'} = \vec{0}$ of the congruence and the corresponding isomorphisms

$$\begin{aligned} \iota_+ &: \mathrm{Ext}^1(M_{\vec{0}}, M_{C\overrightarrow{p-2}}) \to \mathrm{Ext}^1(M_{A\vec{p}}, M_{B\vec{0}}), \\ \iota_- &: \mathrm{Ext}^1(M_{\vec{0}}, M_{C\overrightarrow{p-2}}) \to \mathrm{Ext}^1(M_{A\vec{1}}, M_{B\vec{0}}). \end{aligned}$$

One shows that $V_J^+ = V_J^- = \mathrm{Ext}^1(M_{\vec{0}}, M_{C\overrightarrow{p-2}})$ by straightforward computations.

If $J = \emptyset$, the congruence equation has a unique solution unless $\vec{c} = \vec{1}$, in which case we have two solutions $\vec{a} = \vec{0}, \vec{b} = \vec{1}$ and $\vec{a'} = \vec{0}, \vec{b'} = \vec{p}$. The proof that $V_\emptyset = 0$ (when $\vec{c} \neq \vec{1}$) and $V_\emptyset^+ = V_\emptyset^- = 0$ (when $\vec{c} = \vec{1}$) is similar to the case $J \neq \emptyset$. □

5.3. **Case $f = 2$.** Throughout this section we assume that $0 \leq c_0, c_1 \leq p-1$ and not both $p-1$. If $\vec{c} = \vec{0}$ or $\overrightarrow{p-2}$, we further assume $C \neq 1$; the cases $\vec{c} = \vec{0}$ and $\vec{c} = \overrightarrow{p-2}$ when $C = 1$ are dealt with in §§6.1, 6.2. Before starting the verification, we list the basis elements in the form we will need.

Lemma 5.5. *If $f = 2$, we have the following description of basis elements, where*

$$g_i(\pi) \in 1 + \pi\mathbf{F}[[\pi]], \qquad g_i'(\pi) \in \mathbf{F}[[\pi]],$$
$$g_i^\bullet(\pi) \in \mathbf{F}^\times\pi^{1-p} + \pi^{2-p}\mathbf{F}[[\pi]], \quad g_i''(\pi) \in \pi^{2-2p}\mathbf{F}[[\pi]]$$

(with some choice of a coboundary B when necessary).

(a) *If $0 \leq c_0, c_1 < p-1$, $c_0 + c_1 > 0$, then*

$$\begin{aligned} \mu_\varphi(B_0) &= (\pi^{1-p} + h_0(\pi), 0), \\ \mu_\xi(B_0) &= (\alpha_0 g_0(\pi), \pi^{(p-1)c_1}\alpha_0 g_0(\pi^p)), \\ \mu_\varphi(B_1) &= (0, \pi^{1-p} + h_1(\pi)), \\ \mu_\xi(B_1) &= (C\pi^{(p-1)c_0}\alpha_1 g_1(\pi^p), \alpha_1 g_1(\pi)) \end{aligned}$$

for some $\alpha_0, \alpha_1 \in \mathbf{F}^\times$.

(b) *If* $c_0 = p-1$, $0 \le c_1 < p-2$, *then*

$$\begin{aligned}\mu_\varphi(B_0 - B) &\in (\epsilon^{(0)}\pi^{2-2p}, C^{-1}\pi^{2-2p}) + \pi^{3-2p}\mathbf{F}[[\pi]]^S,\\ \mu_\xi(B_0 - B) &= (g_0'(\pi), g_0^\bullet(\pi)),\\ \mu_\varphi(B_1) &= (0, \pi^{1-p} + h_1(\pi)),\\ \mu_\xi(B_1) &= (C\pi^{(p-1)c_0}\alpha_1 g_1(\pi^p), \alpha_1 g_1(\pi)).\end{aligned}$$

(c) *If* $c_0 = p-1$, $c_1 = p-2$, *then*

$$\begin{aligned}\mu_\varphi(B_0 - B) &\in (0, C^{-1}\epsilon^{(1)}\pi^{3-3p}) + \pi^{2-2p}\mathbf{F}[[\pi]] \times \pi^{4-3p}\mathbf{F}[[\pi]],\\ \mu_\xi(B_0 - B) &= (g_0^\bullet(\pi), g_0''(\pi)),\\ \mu_\varphi(B_1) &= (0, \pi^{1-p} + h_1(\pi)),\\ \mu_\xi(B_1) &= (C\pi^{(p-1)c_0}\alpha_1 g_1(\pi^p), \alpha_1 g_1(\pi)).\end{aligned}$$

(d) *If* $0 \le c_0 < p-2$, $c_1 = p-1$, *then*

$$\begin{aligned}\mu_\varphi(B_0) &= (\pi^{1-p} + h_0(\pi), 0),\\ \mu_\xi(B_0) &= (\alpha_0 g_0(\pi), \pi^{(p-1)c_1}\alpha_0 g_0(\pi^p)),\\ \mu_\varphi(B_1 - B) &\in (\pi^{2-2p}, \epsilon^{(0)}\pi^{2-2p}) + \pi^{3-2p}\mathbf{F}[[\pi]]^S,\\ \mu_\xi(B_1 - B) &= (g_1^\bullet(\pi), g_1'(\pi)).\end{aligned}$$

(e) *If* $c_0 = p-2$, $c_1 = p-1$, *then*

$$\begin{aligned}\mu_\varphi(B_0) &= (\pi^{1-p} + h_0(\pi), 0),\\ \mu_\xi(B_0) &= (\alpha_0 g_0(\pi), \pi^{(p-1)c_1}\alpha_1 g_1(\pi^p)),\\ \mu_\varphi(B_1 - B) &\in (\epsilon^{(1)}\pi^{3-3p}, 0) + \pi^{4-3p}\mathbf{F}[[\pi]] \times \pi^{2-2p}\mathbf{F}[[\pi]],\\ \mu_\xi(B_1 - B) &= (g_1''(\pi), g_1^\bullet(\pi)).\end{aligned}$$

(f) *If* $c_0 = c_1 = 0$ *(recall* $C \neq 1$ *is assumed), then*

$$\begin{aligned}\mu_\varphi(B_0) &= (\pi^{1-p} + h_0(\pi), 0),\\ \mu_\xi(B_0) &= (\beta_0 g_0(\pi), \beta_0 g_0(\pi^p)),\\ \mu_\varphi(B_1) &= (0, \pi^{1-p} + h_1(\pi)),\\ \mu_\xi(B_1) &= (C\beta_1 g_1(\pi^p), \beta_1 g_1(\pi))\end{aligned}$$

for some $\beta_0, \beta_1 \in \mathbf{F}^\times$.

Proof. Recall that

$$\begin{aligned}\mu_\varphi(B_0) &= (H_0(\pi), 0),\\ \mu_\xi(B_0) &= (G_0^{(0)}(\pi), G_1^{(0)}(\pi))\\ &= ((C\pi^{(p-1)\Sigma_0}\Phi - 1)^{-1}((\lambda_\xi^{\Sigma_0}\xi - 1)H_0(\pi)), \pi^{(p-1)c_1}G_0^{(0)}(\pi^p)),\\ \mu_\varphi(B_1) &= (0, H_1(\pi)),\\ \mu_\xi(B_1) &= (G_0^{(1)}(\pi), G_1^{(1)}(\pi))\\ &= (\pi^{(p-1)c_0}G_1^{(1)}(\pi^p), (C\pi^{(p-1)\Sigma_1}\Phi - 1)^{-1}((\lambda_\xi^{\Sigma_1}\xi - 1)(H_1(\pi)))),\end{aligned}$$

where

$$H_0(\pi) = \begin{cases} \pi^{1-p} + h_0 & \text{if } c_0 < p-1, \\ \pi^{1-p^2} + h_0^{(1)} + \epsilon^{(0)} h_0'^{(0)} + h_0^{(0)} & \text{if } c_0 = p-1, c_1 \neq p-2, \\ \pi^{1-p^3} + h_0^{(2)} + \epsilon^{(1)} h_0'^{(1)} + h_0^{(1)} + \epsilon^{(0)} h_0'^{(0)} + h_0^{(0)} & \text{if } c_0 = p-1, c_1 = p-2, \end{cases}$$

$$H_1(\pi) = \begin{cases} \pi^{1-p} + h_1 & \text{if } c_1 < p-1, \\ \pi^{1-p^2} + h_1^{(1)} + \epsilon^{(0)} h_1'^{(0)} + h_1^{(0)} & \text{if } c_1 = p-1, c_0 \neq p-2, \\ \pi^{1-p^3} + h_1^{(2)} + \epsilon^{(1)} h_1'^{(1)} + h_1^{(1)} + \epsilon^{(0)} h_1'^{(0)} + h_1^{(0)} & \text{if } c_1 = p-1, c_0 = p-2. \end{cases}$$

(a) Letting $\alpha_0 := -\overline{s_0 z} \in \mathbf{F}^\times$ as in Lemma 4.4 we have

$$L_0(\pi) := (\lambda_\xi^{\Sigma_0}\xi - 1)(\pi^{1-p} + h_0(\pi)) \equiv -\alpha_0 \mod \pi\mathbf{F}[[\pi]],$$

so that $e_0\mu_\xi(B_0) = (C\pi^{(p-1)\Sigma_0}\Phi - 1)^{-1}(L_0(\pi)) = \alpha_0 g_0(\pi)$ for some $g_0(\pi) \in 1 + \pi\mathbf{F}[[\pi]]$. Thus, $\mu_\xi(B_0) = (\alpha_0 g_0(\pi), \pi^{(p-1)c_1}\alpha_0 g_0(\pi^p))$. Similarly for B_1.

(b) By the very construction of $H_0(\pi)$, we have $L_0(\pi) := (\lambda_\xi^{\Sigma_0}\xi - 1)H_0(\pi) \in \mathbf{F}[[\pi]]$, so that $g_0'(\pi) := e_0\mu_\xi(B_0) = (C\pi^{(p-1)\Sigma_0}\Phi - 1)^{-1}(L_0(\pi)) \in \mathbf{F}[[\pi]]$ and $\mu_\xi(B_0) = (g_0'(\pi), \pi^{(p-1)c_1}g_0'(\pi^p))$. Now consider a coboundary B such that

$$\begin{aligned} \mu_\varphi(B) &= (C\pi^{(p-1)c_0}b_1(\pi^p) - b_0(\pi), \pi^{(p-1)c_1}b_0(\pi^p) - b_1(\pi)) \\ &= (\pi^{1-p^2} + h_0^{(1)}, -C^{-1}(\pi^{2-2p} + \widetilde{h}_0^{(1)}), \\ \mu_\xi(B) &= (0, (\lambda_\xi^{\Sigma_1}\xi - 1)b_1(\pi)), \end{aligned}$$

where $b_0(\pi) = 0, b_1(\pi) = C^{-1}(\pi^{2-2p} + \widetilde{h}_0^{(1)})$ with $\widetilde{h}_0^{(1)} := \sum_{s=1}^{p-2}\epsilon_s^{(1)}\pi^{2-2p+s}$. Then

$$\begin{aligned} \mu_\varphi(B_0 - B) &= \left(\epsilon^{(0)}h_0'^{(0)} + h_0^{(0)}, C^{-1}(\pi^{2-2p} + h_0^{(1)})\right) \\ &= \left(\epsilon^{(0)}(\pi^{2-2p} + \widehat{h}_0'^{(0)}) + h_0^{(0)}, C^{-1}(\pi^{2-2p} + h_0^{(1)})\right), \\ \mu_\xi(B_0 - B) &= (g_0'(\pi), g_0^\bullet(\pi)) \end{aligned}$$

where $\widehat{h}_0'^{(0)} := h_0'^{(0)} - \pi^{2-2p}$ and $g_0^\bullet(\pi) := \pi^{(p-1)c_1}g_0'(\pi^p) - (\lambda_\xi^{\Sigma_1}\xi - 1)b_1(\pi)$, whose valuation is $1-p$. Things on B_1 are the same as in (a).

(c) Again we have $L_0(\pi) := (\lambda_\xi^{\Sigma_0}\xi - 1)H_0(\pi) \in \mathbf{F}[[\pi]]$, $g_0'(\pi) := e_0\mu_\xi(B_0) = (C\pi^{(p-1)\Sigma_0}\Phi - 1)^{-1}(L_0(\pi)) \in \mathbf{F}[[\pi]]$ and $\mu_\xi(B_0) = (g_0'(\pi), \pi^{(p-1)c_1}g_0'(\pi^p))$. Considering the boundary B such that

$$\begin{aligned} \mu_\varphi(B) &= (C\pi^{(p-1)c_0}b_1(\pi^p) - b_0(\pi), \pi^{(p-1)c_1}b_0(\pi^p) - b_1(\pi)) \\ &= (\pi^{1-p^3} + \epsilon^{(1)}h_0'^{(1)} + h_0^{(1)} + h_0^{(2)} - C^{-1}\widetilde{h}_0^{(2)}, -C^{-1}(\epsilon^{(1)}\widetilde{h}_0'^{(1)} + \widetilde{h}_0^{(1)})), \\ \mu_\xi(B) &= ((\lambda_\xi^{\Sigma_0}\xi - 1)b_0(\pi), (\lambda_\xi^{\Sigma_1}\xi - 1)b_1(\pi)), \end{aligned}$$

where $b_0(\pi) = C^{-1}\widetilde{h}_0^{(2)}, b_1(\pi) = C^{-1}\left(\epsilon^{(1)}\widetilde{h}_0'^{(1)} + \widetilde{h}_0^{(1)}\right) + \pi^{(p-1)(p-2)}b_0(\pi^p)$ and $\widetilde{h}_0^{(2)} := \sum_{s=0}^{p-2}\epsilon_s^{(2)}\pi^{2-2p+s}$ (with $\epsilon_0^{(2)} = 1$), $\widetilde{h}_0'^{(1)} := \sum_{s=0}^{p-2}\epsilon_s'^{(1)}\pi^{3-3p+s}$ (with $\epsilon'^{(1)} = 1$), $\widetilde{h}_0^{(1)} := \sum_{s=1}^{p-2}\epsilon_s^{(1)}\pi^{2-2p+s}$, we have

$$\begin{aligned} \mu_\varphi(B_0 - B) &= \left(C^{-1}\widetilde{h}_0^{(2)} + \epsilon^{(0)}h_0'^{(0)} + h_0^{(0)}, C^{-1}(\epsilon^{(1)}\widetilde{h}_0'^{(1)} + \widetilde{h}_0^{(1)}\right)), \\ \mu_\xi(B_0 - B) &= (g_0^\bullet(\pi), g_0''(\pi)) \end{aligned}$$

where $g_0^\bullet(\pi) = g_0(\pi) + (\lambda_\xi^{\Sigma_0}\xi - 1)b_0(\pi)$ and $g_0''(\pi) = \pi^{(p-1)c_1}g_0(\pi^p) + (\lambda_\xi^{\Sigma_1}\xi - 1)b_1$. One checks that $\mathrm{val}_\pi g_0^\bullet(\pi) = 1-p$ and $\mathrm{val}_\pi g_0''(\pi) \geq 2-2p$.

(d), (e) are similar to (b), (c).

(f) Almost identical to (a) :

$$L_0(\pi) := (\xi - 1)(\pi^{1-p} + h_0(\pi)) \equiv \alpha_0 \mod \pi\mathbf{F}[[\pi]]$$

and

$$e_0\mu_\xi(B_0) = (C\Phi - 1)^{-1}(L_0(\pi)) = \beta_0 g_0(\pi)$$

for some $g_0(\pi) \in 1 + \pi\mathbf{F}[[\pi]]$ with $\beta_0 := (C-1)^{-1}\alpha_0$. Similarly for B_1. □

We will set $B_i' := B_i - B$ in pertaining cases.

Proposition 5.6. *If $f = 2$, then*

$$V_S = V_S^{\pm} = \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}})$$

with $\pm$ occuring when $\vec{c} = \overrightarrow{p-2}$.

Proof. By straightforward calculations one can check that both ιB_0 and ιB_1 are bounded in each of the following cases to consider:

- $0 \leq c_0, c_1 \leq p-2$, $1 \leq a_0, a_1 \leq p-1$, $\langle\vec{c}\rangle_J = (\pi^{p-1}, \pi^{p-1})$;
- $c_0 = p-1, 0 \leq c_1 < p-2$, $a_0 = 1, 1 \leq a_1 < p-1$, $\langle\vec{c}\rangle_J = (\pi^{p-1}, \pi^{2p-2})$;
- $0 \leq c_0 < p-2, c_1 = p-1$, $1 \leq a_0 < p-1, a_1 = 1$, $\langle\vec{c}\rangle_J = (\pi^{2p-2}, \pi^{p-1})$;
- $p-2 \leq c_0, c_1 \leq p-1$, $p-1 \leq a_0, a_1 \leq p$, $\langle\vec{c}\rangle_J = (\pi^{2p-2}, \pi^{2p-2})$;
- $c_0 = c_1 = p-2$, $a_0 = a_1 = p$, $\langle\vec{c}\rangle_J = (\pi^{2p-2}, \pi^{2p-2})$ (for V_J^+);
- $c_0 = c_1 = p-2$, $a_0 = a_1 = 1$, $\langle\vec{c}\rangle_J = (\pi^{p-1}, \pi^{p-1})$ (for V_J^-).

□

Proposition 5.7. *If $f = 2$, then*

$$V_\emptyset = V_\emptyset^{\pm} = 0$$

with $\pm$ occurring when $\vec{c} = \vec{1}$.

Proof. We have the following cases to consider:

- $1 \leq c_0, c_1 \leq p-1$,$1 \leq b_0, b_1 \leq p-1$, $\langle\vec{c}\rangle_J = (1, 1)$;
- $c_0 = 0, 2 \leq c_1 \leq p-1$,$b_0 = p, 1 \leq b_1 \leq p-2$, $\langle\vec{c}\rangle_J = (1, \pi^{1-p})$;
- $2 \leq c_0 \leq p-1, c_1 = 0$,$1 \leq b_0 \leq p-2, b_1 = p$, $\langle\vec{c}\rangle_J = (\pi^{1-p}, 1)$;
- $0 \leq c_0, c_1 \leq 1$, $p-1 \leq b_0, b_1 \leq p$, $\langle\vec{c}\rangle_J = (\pi^{1-p}, \pi^{1-p})$.

If E is a cocycle such that $\iota(E)$ is bounded, then there is a coboundary B associted to some $(b_0(\pi), b_1(\pi)) \in \mathbf{F}((\pi))^S$ such that $\iota(E+B)$ has $\mu_\varphi \in \mathbf{F}[[\pi]]^S$ and $\mu_\xi \in \pi\mathbf{F}[[\pi]]^S$. As $\kappa_\varphi(A, \vec{a}) \in (\mathbf{F}^\times)^S$ and $\langle\vec{c}\rangle = (\pi^{(1-p)\epsilon_0}, \pi^{(1-p)\epsilon_1})$ for some $\epsilon_j \in \{0,1\}$, we get $\mu_\varphi(E+B) \in \mathbf{F}[[\pi]]^S$ and $\mu_\xi(E+B) \in \pi\mathbf{F}[[\pi]]^S$.

First consider the case $0 \leq c_0, c_1 < p-1$ and $E = B_0 + \beta B_1$ for some $\beta \in \mathbf{F}^\times$. As $\mathrm{val}_\pi e_0\mu_\varphi(E) = 1-p$ and $\mathrm{val}_\pi e_1\mu_\varphi(E) \geq 1-p$, we have $\mathrm{val}_\pi(C\pi^{(p-1)c_0}b_1(\pi^p) - b_0(\pi)) = 1-p$ and $\mathrm{val}_\pi(\pi^{(p-1)c_1}b_0(\pi^p) - b_1(\pi)) \geq 1-p$. If $\mathrm{val}_\pi b_0(\pi) > 1-p$, then $(p-1)c_0 + p\mathrm{val}_\pi b_1(\pi) = 1-p$, which implies that $p|(c_0+1)$, contradicting $c_0 < p-1$. If $\mathrm{val}_\pi b_0(\pi) \leq 1-p$, then $(p-1)c_1 + p\mathrm{val}_\pi b_0(\pi) < 1-p$, which implies that $\mathrm{val}_\pi b_1(\pi) = (p-1)c_1 + p\mathrm{val}_\pi b_0(\pi) < 1-p$, which in turn implies $(p-1)c_0 + p\mathrm{val}_\pi b_1(\pi) < 1-p$, so that $\mathrm{val}_\pi b_0(\pi) = (p-1)c_0 + p\mathrm{val}_\pi b_1(\pi) = (p-1)\Sigma_0 + p^2\mathrm{val}_\pi b_0(\pi)$, yielding a contradiction. The proof that $\iota(B_1)$ is not bounded is the same.

Next suppose $c_0 = p-1$ and $0 < c_1 < p-2$. First consider the case $E = B_0' + \beta B_1$. As $\mathrm{val}_\pi e_1 \mu_\xi(E) = 1-p$, we have $\mathrm{val}_\pi(\lambda_\xi^{\Sigma_1}\xi - 1)b_1(\pi) = 1-p$, so that $\mathrm{val}_\pi b_1(\pi) \leq 2-2p$. Then $\mathrm{val}_\pi \pi^{(p-1)c_0} b_1(\pi^p) = (p-1)c_0 + p\mathrm{val}_\pi b_1(\pi) < (1-p)(1+p) < 2-2p = \mathrm{val}_\pi e_0 \mu_\varphi(E)$, and so $\mathrm{val}_\pi b_0(\pi) = \mathrm{val}_\pi \pi^{(p-1)c_0} b_1(\pi^p) = (p-1)c_0 + p\mathrm{val}_\pi b_1(\pi)$. Then again $\mathrm{val}_\pi \pi^{(p-1)c_1} b_0(\pi^p) = (p-1)\Sigma_1 + p^2 \mathrm{val}_\pi b_1(\pi) < 2-2p = \mathrm{val}_\pi e_1 \mu_\varphi(E)$, so that $\mathrm{val}_\pi b_1(\pi) = \mathrm{val}_\pi \pi^{(p-1)c_1} b_0(\pi^p) = (p-1)\Sigma_1 + p^2 \mathrm{val}_\pi b_1(\pi)$, or $\mathrm{val}_\pi b_1(\pi) = -\frac{p-1}{p^2-1}\Sigma_1 > 2-2p$, a contradiction. The proof that $\iota(B_1)$ is not bounded is the same as in the case $c_0 < p-1$.

If $c_0 = p-1, c_1 = p-2$, the proof is similar to the preceding case, except that we start by noting that $\mathrm{val}_\pi e_0 \mu_\xi(E) = 1-p$ if $E = B_0' + \beta B_1$.

The proof in the case that $c_1 = p-1$ is the same as the case $c_0 = p-1$. □

Proposition 5.8. *If $f = 2$, then*

$$V_{\{1\}} = \begin{cases} \mathbf{F}B_1 & \text{if } c_0 = p-1, \\ \mathbf{F}(\alpha_1 B_0 - \alpha_0 B_1) & \text{if } 0 < c_0 < p-1, c_1 = 0, \\ \mathbf{F}B_0 & 0 \leq c_0 < p-1, 0 < c_1 \leq p-1; \end{cases}$$
$$V_{\{1\}}^+ = \mathbf{F}(\beta_1 B_0 - \beta_0 B_1);$$
$$V_{\{1\}}^- = 0.$$

with $\pm$ occuring when $\vec{c} = \vec{0}$.

Proof. Unless $\vec{c} = \vec{0}$, $\vec{c}$ gives rise to unique $\vec{a} = (0, a_1), \vec{b} = (b_0, 0)$ with $1 \leq a_1, b_0 \leq p$. If $\vec{c} = \vec{0}$, we have $\vec{a} = (0,p), \vec{b} = (1,0)$ (for V_J^+) or $\vec{a} = (0,1), \vec{b} = (p,0)$ (for V_J^-). We always have $\langle \vec{c} \rangle = (\pi^{p-1}, 1)$ except when $\vec{c} = \vec{0}$, $b_0 = p$, $a_0 = 1$, in which case we have $\langle \vec{c} \rangle = (1, \pi^{1-p})$.

(1) Assume $c_0 = p-1$. It is strightforward to check that $\iota(B_1 + \beta B)$ is bounded for some $\beta \in \mathbf{F}^\times$ where B is a coboundary such that $\mu_\varphi(B) = (C\pi^{(p-1)c_0}\pi^{(1-p)p}, -\pi^{1-p})$ and $\mu_\xi(B) = (0, (\lambda_\xi^{\Sigma_1}\xi - 1)(\pi^{1-p}))$.

Suppose $\iota B_0'$ is bounded. There exists a coboundary B such that $\mu_\varphi \iota(B_0' + B) \in \mathbf{F}[[\pi]]^S$, $\mu_\xi \iota(B_0' + B) \in \pi\mathbf{F}[[\pi]]^S$, and so

$$\mu_\varphi(B_0') + (C\pi^{(p-1)c_0} b_1(\pi^p) - b_0(\pi), \pi^{(p-1)c_1} b_0(\pi^p) - b_1(\pi)) \in \pi^{1-p}\mathbf{F}[[\pi]] \times \pi^{(1-p)a_1}\mathbf{F}[[\pi]],$$
$$\mu_\xi(B_0') + ((\lambda_\xi^{\Sigma_0}\xi - 1)b_0(\pi), (\lambda_\xi^{\Sigma_1}\xi - 1)b_1(\pi)) \in \pi^{2-p}\mathbf{F}[[\pi]] \times \pi\mathbf{F}[[\pi]]$$

for some $b_0(\pi), b_1(\pi) \in \mathbf{F}((\pi))$. If $c_1 < p-2$, we have $\mathrm{val}_\pi(\lambda_\xi^{\Sigma_1}\xi - 1)b_1(\pi) = 1-p$ as $\mathrm{val}_\pi e_1 \mu_\xi(B_0') = 1-p$, so that $\mathrm{val}_\pi b_1(\pi) \leq 2-2p$. Then $\mathrm{val}_\pi(\pi^{(p-1)c_0} b_1(\pi^p)) = (p-1)c_0 + p\mathrm{val}_\pi b_1(\pi) < (1-p)(1+p) < \mathrm{val}_\pi e_0 \mu_\varphi(B_0')$, and so $\mathrm{val}_\pi b_0(\pi) = (p-1)c_0 + p\mathrm{val}_\pi b_1(\pi)$. Then again $\mathrm{val}_\pi \pi^{(p-1)c_1} b_0(\pi^p) = (p-1)c_1 + p\mathrm{val}_\pi b_0(\pi) = (p-1)\Sigma_1 + p^2 \mathrm{val}_\pi b_1(\pi) < (1-p)(1+p) < (1-p)a_1$, so that $\mathrm{val}_\pi b_1(\pi) = (p-1)\Sigma_1 + p^2 \mathrm{val}_\pi b_1(\pi)$, or $\mathrm{val}_\pi b_1(\pi) = -\frac{p-1}{p^2-1}\Sigma_1 > 1-p$, a contradiction. If $c_1 = p-2$, start with $\mathrm{val}_\pi(\lambda_\xi^{\Sigma_0}\xi - 1)b_0(\pi) = 1-p$ and the same argument as above (for the case $c_1 < p-2$) goes through.

(2) Assume $0 < c_0 < p-1, c_1 = 0$. Straightforward calculations show that $\mu_\varphi \iota B_0, \mu_\varphi \iota B_1 \in \mathbf{F}[[\pi]]^S$ but $\mu_\xi \iota B_0(\pi), \mu_\xi \iota B_1 \notin \pi\mathbf{F}[[\pi]]^S$. If, however, we take $\beta_1 B_0 - \beta_0 B_1$, it has μ_φ obviously in $\mathbf{F}[[\pi]]^S$ and $\mu_\xi = \kappa_\xi(A, \vec{a})(\pi^{p-1}\alpha_0\alpha_1(g_0(\pi) - C\pi^{(p-1)c_0} g_1(\pi^p)), \alpha_0\alpha_1(g_0(\pi^p) - g_1(\pi))) \in \pi\mathbf{F}[[\pi]]^S$.

Now suppose ιB_1 is bounded, and so we have, for some coboundary B, that $\mu_\varphi\iota(B_1+B)\in\mathbf{F}[[\pi]]^S$ and $\mu_\xi\iota(B_1+B)\in\pi\mathbf{F}[[\pi]]^S$, which implies

$$\mu_\varphi(B_1)+(C\pi^{(p-1)c_0}b_1(\pi^p)-b_0(\pi),\pi^{(p-1)c_1}b_0(\pi^p)-b_1(\pi))\in\pi^{1-p}\mathbf{F}[[\pi]]\times\pi^{(1-p)a_1}\mathbf{F}[[\pi]],$$
$$\mu_\xi(B_1)+((\lambda_\xi^{\Sigma_0}\xi-1)b_0(\pi),(\lambda_\xi^{\Sigma_1}\xi-1)b_1(\pi))\in\pi^{2-p}\mathbf{F}[[\pi]]\times\pi\mathbf{F}[[\pi]]$$

for some $b_0(\pi),b_1(\pi)\in\mathbf{F}((\pi))$. We have $\mathrm{val}_\pi(\lambda_\xi^{\Sigma_1}\xi-1)b_1(\pi)=0$ and so $\mathrm{val}_\pi b_1(\pi)\le 1-p$, so that $\mathrm{val}_\pi\pi^{(p-1)c_0}b_1(\pi^p)=(p-1)c_0+p\mathrm{val}_\pi b_1(\pi)<1-p$. Then $\mathrm{val}_\pi b_0(\pi)=(p-1)c_0+p\mathrm{val}_\pi b_1(\pi)$ and $\mathrm{val}_\pi\pi^{(p-1)c_1}b_0(\pi^p)=(p-1)\Sigma_1+p^2\mathrm{val}_\pi b_1(\pi)<(1-p)a_1$, so that $\mathrm{val}_\pi b_1(\pi)=\Sigma_1+p^2\mathrm{val}_\pi b_1(\pi)$, or $\mathrm{val}_\pi b_1(\pi)=-\frac{p-1}{p^2-1}\Sigma_1>1-p$, a contradiction.

(3) Assume $0\le c_0<p-1, 0<c_1\le p-1$. It is straightforward to check that ιB_0 is bounded:

$$\mu_\varphi\iota(B_0)=(A,\pi^{(p-1)a_1})(\pi^{p-1},1)(\pi^{1-p}+h_0(\pi))\in\mathbf{F}[[\pi]]^S,$$
$$\mu_\xi\iota(B_0)=\kappa_\xi(A,\vec{a})(\pi^{p-1},1)(\alpha_0 g(\pi),\pi^{(p-1)c_1}\alpha_0 g_0(\pi^p))\in\pi\mathbf{F}[[\pi]]^S$$

as $c_1>0$. Now suppose ιB_1 is bounded. There exists a coboundary B such that $\mu_\varphi\iota(B_1+B)\in\mathbf{F}[[\pi]]^S$, $\mu_\xi\iota(B_1+B)\in\pi\mathbf{F}[[\pi]]^S$, and so

$$\mu_\varphi(B_1+B)+(C\pi^{(p-1)c_0}b_1(\pi^p)-b_0(\pi),\pi^{(p-1)c_1}b_0(\pi^p)-b_1(\pi))\in\pi^{1-p}\mathbf{F}[[\pi]]\times\pi^{(1-p)a_1}\mathbf{F}[[\pi]],$$
$$\mu_\xi(B_1+B)+((\lambda_\xi^{\Sigma_0}\xi-1)b_0(\pi),(\lambda_\xi^{\Sigma_1}\xi-1)b_1(\pi))\in\pi^{2-p}\mathbf{F}[[\pi]]\times\pi\mathbf{F}[[\pi]].$$

If $c_1<p-1$,then the argument is the same as in case (2).

If $c_1=p-1, c_0<p-2$, then as $\mathrm{val}_\pi e_0\mu_\xi(B_1'+B)\ge 2-p$ and $\mathrm{val}_\pi e_0\mu_\xi(B_1')=1-p$, we have $\mathrm{val}_\pi e_0\mu_\xi(B)=\mathrm{val}_\pi(\lambda_\xi^{\Sigma_0}\xi-1)b_0(\pi)=1-p$, so that $\mathrm{val}_\pi b_0(\pi)\le 2-2p$. Then $\mathrm{val}_\pi\pi^{(p-1)c_1}b_0(\pi^p)=(p-1)c_1+p\mathrm{val}_\pi b_0(\pi)\le(1-p)(1+p)<\min(\mathrm{val}_\pi e_1\mu_\varphi B_1',(1-p)a_1)$, so that $\mathrm{val}_\pi b_1(\pi)=(p-1)c_1+p\mathrm{val}_\pi b_0(\pi)$. So $\mathrm{val}_\pi\pi^{(p-1)c_0}b_1(\pi^p)=(p-1)\Sigma_0+p^2\mathrm{val}_\pi b_0(\pi)<(1-p)(1+p)<\mathrm{val}_\pi e_0\mu_\varphi(B_1')$, which implies $\mathrm{val}_\pi b_0(\pi)=(p-1)\Sigma_0+p^2\mathrm{val}_\pi b_0(\pi)$, or $\mathrm{val}_\pi b_0(\pi)=-\frac{p-1}{p^2-1}\Sigma_0>1-p$, a contradiction.

If $c_1=p-1, c_0=p-2$, then as $\mathrm{val}_\pi e_1\mu_\xi(B_1'+B)\ge 1$ and $\mathrm{val}_\pi e_1\mu_\xi(B_1')=1-p$, we have $\mathrm{val}_\pi e_1\mu_\xi(B)=\mathrm{val}_\pi(\lambda_\xi^{\Sigma_1}\xi-1)b_1(\pi)=1-p$, so that $\mathrm{val}_\pi b_1(\pi)\le 2-2p$. Then $\mathrm{val}_\pi\pi^{(p-1)c_0}b_1(\pi^p)=(p-1)c_0+p\mathrm{val}_\pi b_1(\pi)\le(1-p)(1+p)<\mathrm{val}_\pi e_0\mu_\varphi B_1'$, so that $\mathrm{val}_\pi b_0(\pi)=(p-1)c_0+p\mathrm{val}_\pi b_1(\pi)$. So $\mathrm{val}_\pi\pi^{(p-1)c_1}b_0(\pi^p)=(p-1)\Sigma_1+p^2\mathrm{val}_\pi b_1(\pi)<(1-p)(1+p)<\mathrm{val}_\pi e_1\mu_\varphi(B_1')$, which implies $\mathrm{val}_\pi b_1(\pi)=(p-1)\Sigma_1+p^2\mathrm{val}_\pi b_1(\pi)$, or $\mathrm{val}_\pi b_1(\pi)=-\frac{p-1}{p^2-1}\Sigma_1>1-p$, a contradiction.

(4) Assume $c_0=c_1=0$, $b_0=1, a_1=p$. Straightforward calculations show that $\mu_\varphi\iota B_0(\pi),\mu_\varphi\iota B_1\in\mathbf{F}[[\pi]]^S$ but $\mu_\xi\iota B_0(\pi),\mu_\xi\iota B_1\notin\pi\mathbf{F}[[\pi]]^S$. If, however, we take $\beta_1 B_0-\beta_0 B_1$, it has μ_φ obviously in $\mathbf{F}[[\pi]]^S$ and

$$\mu_\xi=(\pi^{p-1}\beta_0\beta_1(g_0(\pi)-Cg_1(\pi^p)),\beta_0\beta_1(g_0(\pi^p)-g_1(\pi)))\in\pi\mathbf{F}[[\pi]]^S.$$

Now suppose ιB_1 is bounded, and so we have, for some coboundary B, that $\mu_\varphi\iota(B_1+B)\in\mathbf{F}[[\pi]]^S$ and $\mu_\xi\iota(B_1+B)\in\pi\mathbf{F}[[\pi]]^S$, which implies

$$\mu_\varphi(B_0)+(Cb_1(\pi^p)-b_0(\pi),b_0(\pi^p)-b_1(\pi))\in\pi^{1-p}\mathbf{F}[[\pi]]\times\pi^{(1-p)p}\mathbf{F}[[\pi]],$$
$$\mu_\xi(B_0)+((\xi-1)b_0(\pi),(\xi-1)b_1(\pi))\in\pi^{2-p}\mathbf{F}[[\pi]]\times\pi\mathbf{F}[[\pi]]$$

for some $b_0(\pi),b_1(\pi)\in\mathbf{F}((\pi))$. We have $\mathrm{val}_\pi(\xi-1)b_1(\pi)=0$ and so $\mathrm{val}_\pi b_1(\pi)\le 1-p$, so that $\mathrm{val}_\pi b_1(\pi^p)=p\mathrm{val}_\pi b_1(\pi)<1-p$. Then $\mathrm{val}_\pi b_0(\pi)=p\mathrm{val}_\pi b_1(\pi)$ and $\mathrm{val}_\pi b_0(\pi^p)=p^2\mathrm{val}_\pi b_1(\pi)<(1-p)p<\mathrm{val}_\pi e_0\mu_\varphi(B_1)$, giving $\mathrm{val}_\pi b_1(\pi)=0$ and a contradiction.

(5) Assume $c_0 = c_1 = 0$, $b_0 = p, a_1 = 1$. Suppose $\iota(B_0 + \beta B_1)$ is bounded for some $\beta \in \mathbf{F}$. There exist a coboundary B such that $\mu_\varphi \iota(B_0 + \beta B_1 + B) \in \mathbf{F}[[\pi]]^S$ and $\mu_\xi \iota(B_0 + \beta B_1 + B) \in \pi\mathbf{F}[[\pi]]^S$. As $\kappa_\varphi(A, \vec{a})\langle\vec{c}\rangle \in (\mathbf{F}^\times)^S$, we have

$$\mu_\varphi(B_0 + \beta B_1 + B) = \mu_\varphi(B_0 + \beta B_1) + (Cb_1(\pi^p) - b_0(\pi), b_0(\pi^p) - b_1(\pi)) \in \mathbf{F}[[\pi]]^S,$$

$$\mu_\xi(B_0 + \beta B_1 + B) = \mu_\xi(B_0 + \beta B_1) + ((\xi - 1)b_0(\pi), (\xi - 1)b_1(\pi)) \in \pi\mathbf{F}[[\pi]]^S$$

for some $b_0(\pi), b_1(\pi) \in \mathbf{F}((\pi))$. Note that

$$\mathrm{val}_\pi e_0 \mu_\varphi(B_0 + \beta B_1) = 1 - p \leq \mathrm{val}_\pi e_1 \mu_\varphi(B_0 + \beta B_1),$$
$$\mathrm{val}_\pi e_0 \mu_\xi(B_0 + \beta B_1) \geq 0, \mathrm{val}_\pi e_1 \mu_\xi(B_0 + \beta B_1) \geq 0.$$

Then $\mathrm{val}_\pi e_0 \mu\varphi(B) = \mathrm{val}_\pi(Cb_1(\pi^p) - b_0(\pi)) = 1 - p$, and we get either $\mathrm{val}_\pi b_0(\pi) = 1 - p < \mathrm{val}_\pi b_1(\pi^p)$ or $\mathrm{val}_\pi b_1(\pi^p) = \mathrm{val}_\pi b_0(\pi) < 1 - p$. In either case, we have $\mathrm{val}_\pi b_0(\pi^p) < 1 - p$, so $\mathrm{val}_\pi b_0(\pi^p) = \mathrm{val}_\pi b_1(\pi)$, giving a contradiction.

The same argument proves that ιB_1 is not bounded.

□

Similarly one proves the following.

Proposition 5.9. *If $f = 2$, then*

$$V_{\{0\}} = \begin{cases} \mathbf{F}B_0 & \text{if } c_1 = p - 1, \\ \mathbf{F}(C\alpha_1 B_0 - \alpha_0 B_1) & \text{if } c_0 = 0, 0 < c_1 < p - 1, \\ \mathbf{F}B_1 & 0 < c_0 \leq p - 1, 0 \leq c_1 < p - 1; \end{cases}$$
$$V^+_{\{0\}} = \mathbf{F}(C\beta_1 B_0 - \beta_0 B_1);$$
$$V^-_{\{0\}} = 0.$$

with $\pm$ occuring when $\vec{c} = \vec{0}$.

In proving Propositions 5.8 and 5.9 we have shown the following, which exhibits instances of coincidence of V_J's for distinct J's.

Corollary 5.10. *Suppose $f = 2$.*

(1) *If $c_0 = p - 1$, then $V_{\{1\}} = V_{\{0\}} = \mathbf{F}B_1$.*
(2) *If $c_1 = p - 1$, then $V_{\{1\}} = V_{\{0\}} = \mathbf{F}B_0$.*
(3) *If $0 \leq c_0, c_1 < p - 2$, then $V_{\{1\}}$ and $V_{\{0\}}$ are distinct and one-dimensional.*
(4) *If $c_0 = c_1 = 0$, then $V^+_{\{1\}}$ and $V^+_{\{0\}}$ are distinct and one-dimensional, and $V^-_{\{1\}} = V^-_{\{0\}} = 0$.*

6. Exceptional cases

6.1. **Cyclotomic character.** Assume $C = 1, \vec{c} = \overrightarrow{p-2}$, so that

$$\kappa_\varphi(C, \vec{c}) = (\pi^{(p-1)(p-2)}, \ldots, \pi^{(p-1)(p-2)}),$$
$$\kappa_\gamma(C, \vec{c}) = \left(\left(\frac{\gamma(\pi)}{\pi\overline{\chi}(\gamma)} \right)^{p-2}, \ldots, \left(\frac{\gamma(\pi)}{\pi\overline{\chi}(\gamma)} \right)^{p-2} \right)$$

if $\gamma \in \Gamma$. Recall that B_i's for all $i \in S$ have already been constructed in §4.1 and we just need to construct an additional basis element which we will denote B_{tr} (for *très ramifié*). Before we do this for arbitrary $f \geq 1$, let's first consider the situation where $f = 1$ (i.e., $K = \mathbf{Q}_p$) and $\mathbf{F} = \mathbf{F}_p$ as a foundation for the general construction.

Lemma 6.1. *Let $\delta\in\Gamma$ be such that $\delta\Gamma_1$ generates $\Gamma/\Gamma_1\simeq\mathbf{F}_p^\times$ and let $\chi(\xi)\equiv 1+zp \mod p^2$ with $0<z\le p-1$. If $s\in\mathbf{Z}$ is divisible by p^v but not by p^{v+1} for some $v\in\mathbf{Z}$, then*

$$\overline{\chi}(\delta)\delta(\pi^s)-\pi^s\in(\overline{\chi}(\delta)^{s+1}-1)\pi^s+\overline{s_v}\frac{\overline{\chi}(\delta)^{s+1}(\overline{\chi}(\delta)-1)}{2}\pi^{s+p^v}+\pi^{s+2p^v}\mathbf{F}_p[[\pi^{p^v}]],$$
$$\overline{\chi}(\xi)\xi(\pi^s)-\pi^s\in\overline{s_v z}(\pi^{s+(p-1)p^v}+\pi^{s+p^{v+1}})+\pi^{s+p^{v+1}(p-1)}\mathbf{F}_p[[\pi^{p^v}]],$$

where $s=\sum_{j\ge v}s_jp^j$.

Proof. Similar to Lemma 4.2 and 4.4. □

Lemma 6.2. *There exists $h'(\pi)\in\pi^{1-2p}+\pi^{2-2p}\mathbf{F}[[\pi]]$ such that*

$$(\overline{\chi}(\eta)\eta-1)(h'(\pi))\in\mathbf{F}(\pi^{-p}-\pi^{-1})+\pi\mathbf{F}[[\pi]].$$

Proof. By Lemma 6.1, there exist $\epsilon_{2-2p},\dots,\epsilon_{-1},\epsilon_0\in\mathbf{F}$ (unique if we set $\epsilon_{-p}=\epsilon_{-1}=0$) such that

$$(\overline{\chi}(\eta)\eta-1)(\pi^{1-2p}+\epsilon_{2-2p}\pi^{2-2p}+\cdots+\epsilon_{-1}\pi^{-1}+\epsilon_0)\in\mathbf{F}\pi^{-p}+\mathbf{F}\pi^{-1}+\pi\mathbf{F}[[\pi]].$$

Set $h'(\pi)=\pi^{1-2p}+\epsilon_{2-2p}\pi^{2-2p}+\cdots+\epsilon_{-1}\pi^{-1}+\epsilon_0$, so

$$(\overline{\chi}(\eta)\eta-1)(h'(\pi))\in\alpha\pi^{-p}+\beta\pi^{-1}+\pi\mathbf{F}[[\pi]]$$

for some $\alpha,\beta\in\mathbf{F}$. Writing $(\overline{\chi}(\xi)\xi-1)=\left(\sum_{i=0}^{p-2}\overline{\chi}(\eta)^i\eta^i\right)(\overline{\chi}(\eta)\eta-1)$ we find that

$$(\overline{\chi}(\xi)\xi-1)(h'(\pi))\in-(\alpha\pi^{-p}+\beta\pi^{-1})+\mathbf{F}[[\pi]].$$

On the other hand a direct computation shows that

$$(\overline{\chi}(\xi)\xi-1)(h'(\pi))\in z(\pi^{-p}-\pi^{-1})+\mathbf{F}[[\pi]]$$

where $z\in\mathbf{F}^\times$, so that $\alpha=\beta=-z$ and the lemma follows. □

Let $h'(\pi)$ be as in the lemma. Since $\varphi-1$ is bijective on $\pi\mathbf{F}[[\pi]]$, it follows that

$$(\overline{\chi}(\eta)\eta-1)(h'(\pi))\in(\varphi-1)(g'_\eta(\pi))$$

for a unique $g'_\eta\in-z^{-1}\pi^{-1}+\pi\mathbf{F}[[\pi]]$. We now extend the definition to construct elements $g'_\gamma(\pi)\in\pi^{-1}\mathbf{F}[[\pi]]$ for all $\gamma\in\Gamma$. We let

$$g'_{\eta^n}(\pi)=\sum_{i=0}^{n-1}\overline{\chi}(\eta)^i\eta^i(g'_\eta(\pi))$$

for $n\in\mathbf{N}$. If $\gamma'\in\Gamma_2$, then $(\overline{\chi}(\gamma')\gamma'-1)(h'(\pi))$ is in $\pi\mathbf{F}[[\pi]]$ and can therefore be written as $(\varphi-1)(g'_{\gamma'}(\pi))$ for a unique $g'_{\gamma'}(\pi)\in\pi\mathbf{F}[[\pi]]$. If $\eta^n\in\Gamma_2$, then $p(p-1)|n$ and the definitions coincide. Moreover, an arbitrary $\gamma\in\Gamma$ can be written as $\gamma'\eta^n$ for some $\gamma'\in\Gamma_2$ and $n\in\mathbf{N}$, and

$$g'_\gamma(\pi):=g'_{\gamma'}(\pi)+\gamma'(g'_{\eta^n}(\pi))$$

is independent of the choice of γ' and n.

One then checks that $\mu=\mu(B')=(h'(\pi),(g'_\gamma(\pi))_{\gamma'\in\Gamma})$ satisfies conditions (†) and (‡), so that we get an extension

$$0\to M_{\rm cyc}\to B'\to M_0\to 0$$

in the category of étale (φ,Γ)-modules over $\mathbf{E}_K$, where $M_{\rm cyc}=\mathbf{E}_Ke_1$ is a rank one defined by $\varphi(e_1')=e_1'$ and $\gamma(e_1')=\chi(\gamma)e_1'$ if $\gamma\in\Gamma$ (and, of course, $M_0=\mathbf{E}_Ke_0$ by

$\varphi(e_0) = e_0$ and $\gamma(e_0) = e_0$). Using the isomorphism $M_{\mathrm{cyc}} \simeq M_{p-2} = \mathbf{E}_K e_1$ defined by $e_1' = \pi^{2-p}e_1$ we get an extension

$$0 \to M_{p-2} \to B \to M_0 \to 0$$

where $\mu(B) = (\pi^{3(1-p)}h(\pi), (\pi^{1-p}g_\gamma(\pi))_{\gamma\in\Gamma})$ with $h(\pi) = \pi^{2p-1}h'(\pi)$, $g_\gamma(\pi) = \pi g_\gamma'(\pi)$.

Now we go back to the context of arbitrary $f \geq 1$, and define $\mu_\varphi(B_{\mathrm{tr}}) = (\pi^{3(1-p)}h(\pi), \dots, \pi^{3(1-p)}h(\pi))$ and $\mu_\gamma(B_{\mathrm{tr}}) = (\pi^{1-p}g_\gamma(\pi), \dots, \pi^{1-p}g_\gamma(\pi))$ for all $\gamma \in \Gamma$. It's straightforward to check that $B_{\mathrm{tr}} \in \mathrm{Ext}^1(M_{\vec{0}}, M_{\overrightarrow{p-2}})$.

Lemma 6.3. *The extensions $B_0, \dots, B_{f-1}, B_{\mathrm{tr}} \in \mathrm{Ext}^1(M_{\vec{0}}, M_{\overrightarrow{p-2}})$ are linearly independent, so that they form a basis.*

Proof. It suffices to show that B_{tr} is not contained in the span of B_i's. Suppose $B_{\mathrm{tr}} = \beta_0 B_0 + \cdots + \beta_{f-1}B_{f-1}$ for some $\beta_i \in \mathbf{F}$. Then $E := B_{\mathrm{tr}} - (\beta_0 B_0 + \cdots + \beta_{f-1}B_{f-1})$ is a coboundary, so that $\mu_\varphi(E) = (\pi^{(p-1)(p-2)}b_1(\pi^p) - b_0(\pi), \dots, \pi^{(p-1)(p-2)}b_0(\pi^p) - b_{f-1}(\pi))$ for some $b_i(\pi) \in \mathbf{F}((\pi))$. As

$$\mu_\varphi(B_{\mathrm{tr}}) = (\pi^{3(1-p)}h(\pi), \dots, \pi^{3(1-p)}h(\pi)),$$
$$\mu_\xi(B_{\mathrm{tr}}) = (\pi^{1-p}g_\xi(\pi), \dots, \pi^{1-p}g_\xi(\pi))$$

where $h(\pi), g_\xi(\pi) \in \mathbf{F}[[\pi]]^\times$, we have $\mathrm{val}_\pi e_i\mu_\varphi(E) = \mathrm{val}_\pi(\pi^{(p-1)(p-2)}b_{i+1}(\pi^p) - b_i(\pi)) = 3(1-p)$ for all $i \in S$. Letting $s_i = \mathrm{val}_\pi(b_i(\pi))$ for $i \in S$, we have $s_{f-1} \leq 3(1-p)$ or $(p-1)(p-2) + s_0 p = 3(1-p)$. Either case yields a contradiction. □

The determination of which linear combinations of $B_0, B_1, \dots, B_{f-1}$ are bounded is exactly as in the generic case. We now extend this to include B_{tr}.

Proposition 6.4. *Suppose that $f \geq 1$, $C = 1$, $\vec{c} = \overrightarrow{p-2}$ and let $A \in \mathbf{F}^\times$ be given.*

(1) *If $J = S$, then*
$$\iota B_{\mathrm{tr}} \in \mathrm{Ext}^1{}_{bdd}(M_{A\vec{p}}, M_{A\vec{0}}),$$
so that $V_S^+ = \mathrm{Ext}^1(M_{\vec{0}}, M_{\overrightarrow{p-2}})$.

(2) *$V_S^- = \oplus_{i\in S}\mathbf{F}B_i$, and if $J \neq S$, then $V_J = \oplus_{i\in J}\mathbf{F}B_{i+1}$.*

Proof. (1) Straightforward: as $\vec{a} = \vec{p}$ and $\langle\vec{c}\rangle = (\pi^{2(p-1)}, \dots, \pi^{2(p-1)})$, we have

$$\mu_\varphi(\iota B_{\mathrm{cyc}}) = (A\pi^{(p-1)^2}h(\pi), \pi^{(p-1)^2}h(\pi), \dots, \pi^{(p-1)^2}h(\pi)) \in \mathbf{F}[[\pi]]^S,$$

$$\mu_\xi(\iota B_{\mathrm{cyc}}) = (\lambda_\xi^{(p-2)\frac{p^f-1}{p-1}}, \dots, \lambda_\xi^{(p-2)\frac{p^f-1}{p-1}})(\pi^{p-1}g_\xi(\pi), \dots, \pi^{p-1}g_\xi(\pi)) \in \pi\mathbf{F}[[\pi]]^S.$$

(2) Let $E := \beta_0 B_0 + \cdots + \beta_{f-1}B_{f-1} + B_{\mathrm{tr}}$ for some $\beta_0, \dots, \beta_{f-1} \in \mathbf{F}$. We must show that in all other cases where $\iota : \mathrm{Ext}^1(M_{\vec{0}}, M_{\overrightarrow{p-2}}) \to \mathrm{Ext}^1(M_{A\vec{a}}, M_{A\vec{b}})$ was defined, we have that ιE is not bounded.

So suppose that ιE is bounded. Then there exists a coboundary B defined by $(b_0(\pi), \dots, b_{f-1}(\pi))$ such that $\mu_\varphi(\iota(E+B)) \in \mathbf{F}[[\pi]]^S$ and $\mu_\xi(\iota(E+B)) \in \pi\mathbf{F}[[\pi]]^S$. We have $e_i\langle\vec{c}\rangle = 1$ or π^{p-1} and $\mathrm{val}_\pi e_i\mu_\xi(\iota E) \leq 0$. It follows that $\mathrm{val}_\pi e_i\mu_\xi(B) = \mathrm{val}_\pi e_i\mu_\xi(E) = 1-p$, so by Lemma 4.4, we must have $s_i := \mathrm{val}_\pi(b_i(\pi)) \leq 2(1-p)$. Then $\mathrm{val}_\pi(\pi^{(p-1)c_i}b_i(\pi^p)) = (p-1)(p-2) + s_i p \leq (1-p)(p+2)$, so that $s_{i-1} = (p-1)(p-2) + s_i p$. Cycling this through indices leads to a contradiction.

□

6.2. **Trivial character.** In this subsection, we assume that $C = 1, \vec{c} = \vec{0}$, so that $\kappa_\varphi(C,\vec{c}) = \kappa_\gamma(C,\vec{c}) = (1,\dots,1) \in \mathbf{F}((\pi))^S$.

Using Lemma 4.2 we can find $\epsilon_{2-p},\dots,\epsilon_{-1} \in \mathbf{F}$ such that

$$(\eta - 1)(\pi^{1-p} + \epsilon_{2-p}\pi^{2-p} + \cdots + \epsilon_{-1}\pi^{-1}) \in \mathbf{F}[[\pi]].$$

Set $H(\pi) = \pi^{1-p} + \epsilon_{2-p}\pi^{2-p} + \cdots + \epsilon_{-1}\pi^{-1}$. By Lemma 4.4, we get

$$(\xi - 1)(H(\pi)) \in \mathbf{F}^\times + \pi\mathbf{F}[[\pi]],$$

which implies, via Lemma 4.6, that

$$(\eta - 1)(H(\pi)) \in \nu + \pi\mathbf{F}[[\pi]]$$

for some $\nu \in \mathbf{F} - \{0\}$. Likewise we have

$$(\eta - 1)(H(\pi^p)) \in \nu + \pi\mathbf{F}[[\pi]],$$

so that

$$(\eta - 1)(-H(\pi^p) + H(\pi)) \in \pi\mathbf{F}[[\pi]].$$

Note that if $\gamma' \in \Gamma_2$, then $(\gamma' - 1)(H(\pi) \in \pi\mathbf{F}[[\pi]]$, and it follows that $(\gamma' - 1)(-H(\pi^p) + H(\pi)) \in \pi\mathbf{F}[[\pi]]$. Now for each $\gamma \in \Gamma$, writing $\gamma = \eta^n\gamma'$ where $\gamma' \in \Gamma_2$, we get by Lemma 4.5 that

$$(\gamma - 1)(-H(\pi^p) + H(\pi)) \in \pi\mathbf{F}[[\pi]].$$

As the map $g(\pi) \mapsto g(\pi^{p^f}) - g(\pi)$ defines a bijection $\pi\mathbf{F}[[\pi]] \to \pi\mathbf{F}[[\pi]]$, for each $\gamma \in \Gamma$ there exists a unique $g_\gamma(\pi) \in \pi\mathbf{F}[[\pi]]$ such that

$$g_\gamma(\pi^{p^f}) - g_\gamma(\pi) = (\gamma - 1)(-H(\pi^p) + H(\pi)),$$

or equivalently,

$$(\varphi - 1)(g_\gamma(\pi), g_\gamma(\pi^{p^{f-1}}), \dots, g_\gamma(\pi^p)) = (\gamma - 1)(-H(\pi^p) + H(\pi), 0, \dots, 0).$$

If we set

$$\mu_\varphi(B_0) = (-H(\pi^p) + H(\pi), 0, \dots, 0),$$
$$\mu_\gamma(B_0) = (g_\gamma(\pi), g_\gamma(\pi^{p^{f-1}}), \dots, g_\gamma(\pi^p)),$$

$\mu(B_0) = (\mu_\varphi(B_0), (\mu_\gamma(B_0))_{\gamma\in\Gamma})$ satisfies the condition (†) by the considerations above. We note that $\mu_\gamma(B_0)$ are uniquely determined so that they satisfy (†). As both $\mu_{\gamma\gamma'}(B_0)$ and $\mu'_{\gamma\gamma'} := \gamma(\mu_{\gamma'}(B_0)) + \mu_\gamma(B_0)$ satisfy (†) for $\gamma\gamma'$, they must coincide, so that (‡) is satisfied.

For each $1 \le i \le f-1$, we construct $B_i \in \mathrm{Ext}^1(M_{\vec{0}}, M_{\vec{0}})$ in a similar way, i.e., by setting

$$\mu_\varphi(B_i) = (0, \dots, 0, -H(\pi^p) + H(\pi), 0, \dots, 0)$$
$$\mu_\gamma(B_i) = (g_\gamma(\pi^{p^i}), \dots, g_\gamma(\pi^p), g_\gamma(\pi), g_\gamma(\pi^{p^{f-1}}), \dots, g_\gamma(\pi^{p^{i+1}})).$$

Remark 6.5. For each $0 \le i \le f-1$, consider the coboundary B''_i by

$$\mu_\varphi(B''_i) = (0, \dots, 0, H(\pi^p), -H(\pi), 0, \dots, 0)$$
$$\mu_\gamma(B''_i) = (0, \dots, 0, 0, (\gamma - 1)(H(\pi)), 0, \dots, 0),$$

where $H(\pi)$ is the i-th component and $-H(\pi)$ the $(i+1)$-th component of $\mu_\varphi(B''_i)$ and $(\gamma - 1)(H(\pi))$ is the $(i+1)$-th component of $\mu_\gamma(B''_i)$. Define $B'_i = B_i + B''_i$ for each $0 \le i \le f-1$. Then $\mathbf{F}B_i = \mathbf{F}B_i'$ in $\mathrm{Ext}^1(M_{\vec{0}}, M_{\vec{0}})$ for all $0 \le i \le f-1$, where we have

$$\mu_\varphi(B_i') = (0, \ldots, 0, H(\pi), -H(\pi), 0, \ldots, 0)$$
$$\mu_\gamma(B_i') = (g_\gamma(\pi^{p^i}), \ldots, g_\gamma(\pi^p), g_\gamma(\pi), g_\gamma(\pi^{p^{f-1}}) + (\gamma-1)H(\pi), \ldots, g_\gamma(\pi^{p^{i+1}})).$$

Next, we define B_{nr} (for *non-ramifié*) by setting

$$\mu_\varphi(B_{\mathrm{nr}}) = (1, 0, \ldots, 0),$$
$$\mu_\gamma(B_{\mathrm{nr}}) = (0, 0, \ldots, 0)$$

for all $\gamma \in \Gamma$. It is straightforward to check that this defines an extension $B_{\mathrm{nr}} \in \mathrm{Ext}^1(M_{\vec{0}}, M_{\vec{0}})$. We can "move" the 1 in μ_φ to any component, i.e., taking any of $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, 1)$ to be μ_φ defines the same cocycle class (up to coboundaries).

Set $B_{\mathrm{cyc}} = \sum_{i=0}^{f-1} B_i'$. Then we have

$$\mu_\varphi(B_{\mathrm{cyc}}) = (0, \ldots, 0),$$
$$\mu_\gamma(B_{\mathrm{cyc}}) = (g_\gamma'(\pi), \ldots, g_\gamma'(\pi))$$

for some $g_\gamma' \in \mathbf{F}[[\pi]]$. Since $(\varphi - 1)g_\gamma'(\pi) = 0$, we must in fact $g_\gamma'(\pi) = g_\gamma' \in \mathbf{F}$. In particular $g_\eta' = \nu$. Moreover $\gamma \mapsto g_\gamma'$ defines a homomorphism $\Gamma \to \mathbf{F}$. Thus if $\gamma = \eta^{n_\gamma}$ modulo Γ_2, then

$$\mu_\gamma(B_{\mathrm{cyc}}) = \nu \overline{n}_\gamma(1, \ldots, 1).$$

Lemma 6.6. *The extensions $B_{\mathrm{nr}}, B_0, \ldots, B_{f-1} \in \mathrm{Ext}^1(M_{\vec{0}}, M_{\vec{0}})$ are linearly independent, so that they form a basis.*

Proof. Suppose that $E = \beta B_{\mathrm{nr}} + \beta_0 B_0 + \cdots + \beta_{f-1} B_{f-1}$ is a coboundary. By adding some coboundary B we have

$$\begin{aligned} e_0\mu_\varphi(E+B) &= \beta + \beta_0(-H(\pi^p) + H(\pi)) + \beta_1(-H(\pi^{p^2}) + H(\pi^p))) + \cdots \\ &\quad + \beta_{f-2}(-H(\pi^{p^{f-1}}) + H(\pi^{p^{f-2}})) + \beta_{f-1}(-H(\pi^{p^f}) + H(\pi^{p^{f-1}})) \\ &= \beta + \beta_0 H(\pi) + (\beta_1 - \beta_0)H(\pi^p) + \cdots \\ &\quad + (\beta_{f-1} - \beta_{f-2})H(\pi^{p^{f-1}}) - \beta_{f-1}H(\pi^{p^f}) \\ &= (\Phi - 1)(\sum_{j \ge s} b_j \pi^j) \end{aligned}$$

for some $\sum_{j \ge s} b_j \pi^j \in \mathbf{F}((\pi))$. Equating constant terms gives $\beta = 0$. If $\beta_{f-1} \ne 0$, then $s = 1-p$, $\beta_0 H(\pi) = -(b_{1-p}\pi^{1-p} + \cdots + b_{-1}\pi^{-1})$ and $\beta_0 = \beta_1 = \cdots = \beta_{f-1}$. It follows that $E = \beta_{f-1}\sum_{i=0}^{f-1} B_i$ is cohomologous to $\beta_{f-1}B_{\mathrm{cyc}}$, and therefore that B_{cyc} is coboundary. Thus there exists $(b_0(\pi), \ldots, b_{f-1}(\pi)) \in \mathbf{F}((\pi))^S$ such that

$$(\varphi - 1)(b_0(\pi), \ldots, b_{f-1}(\pi)) = (0, \ldots, 0),$$
$$(\xi - 1)(b_0(\pi), \ldots, b_{f-1}(\pi)) = -\nu(1, \ldots, 1),$$

which is impossible as the former implies $b_0(\pi) = \cdots = b_{f-1}(\pi) \in \mathbf{F}$, so that $(\xi - 1)(b_0(\pi), \ldots, b_{f-1}(\pi)) = 0$. Thus, $\beta_{f-1} = 0$.

If $0 \le i \le f-2$ is the largest such that $\beta_i \ne 0$, then $\mathrm{val}_\pi(e_0\mu_\varphi(E+B)) = p^{i+1}(1-p)$, which leads to an easy contradiction. Thus, $\beta_i = 0$ for all $0 \le i \le f-2$. $\square$

We now assume $f=2$ and compute the spaces of bounded extensions. We then have the following cases to consider:

- $J=S$, $a_0=a_1=p-1$;
- $J=\{1\}$, $b_0=1$, $a_1=p$ (for V_J^+);
- $J=\{1\}$, $b_0=p$, $a_1=1$ (for V_J^-);
- $J=\{0\}$, $a_0=p$, $b_1=1$ (for V_J^+);
- $J=\{0\}$, $a_0=1$, $b_1=p$ (for V_J^-);
- $J=\emptyset$, $b_0=b_1=p-1$.

Proposition 6.7. *Suppose that $f=2$, $C=1$, $\vec{c}=\vec{0}$ and $A\in\mathbf{F}^\times$.*

(1) $V_S=\mathrm{Ext}^1(M_{\vec{0}},M_{\vec{0}})$;
(2) $V_{\{i\}}^+=\langle B_{\mathrm{nr}},B_i\rangle$ *for* $i=0,1$;
(3) $V_{\{i\}}^-=\langle B_{\mathrm{nr}}\rangle$ *for* $i=0,1$;
(4) $V_\emptyset=\{0\}$.

Proof. (1) We have $\langle\vec{c}\rangle=(\pi^{p-1},\pi^{p-1})$ and it is strightforward to check that ιB_0, ιB_1 and ιB_{nr} are bounded.

(2) Suppose $S=\{1\}$. Then $\langle\vec{c}\rangle=(\pi^{p-1},1)$ and it is straightforward to check that ιB_1 and ιB_{nr} are bounded. Therefore it suffices to prove that ιB_{cyc} is not bounded. So suppose that B is a coboundary such that $\iota(B_{\mathrm{cyc}}+B)$ has $\mu_\varphi\in\mathbf{F}[[\pi]]^S$ and $\mu_\xi\in\pi\mathbf{F}[[\pi]]^S$. Then

$$\mu_\varphi(B_{\mathrm{cyc}}+B)=\mu_\varphi(B)=(b_1(\pi^p)-b_0(\pi),b_0(\pi^p)-b_1(\pi))$$

for some $b_0(\pi),b_1(\pi)\in\mathbf{F}((\pi))$, and $(A\pi^{p-1},\pi^{p(p-1)})\mu_\varphi(B)\in\mathbf{F}[[\pi]]^S$. Letting $v_0=\mathrm{val}_\pi(b_0(\pi))$ and $v_1=\mathrm{val}_\pi(b_1(\pi))$, we see that $v_0\geq 1-p$ and $v_1\geq 0$. Therefore

$$\mu_\xi(B_{\mathrm{cyc}}+B)=(-\nu+(\xi-1)b_0(\pi),-\nu+(\xi-1)b_1(\pi)),$$

and since $-\nu+(\xi-1)b_1(\pi)$ has constant term $-\nu$, we arrive at a contradiction.

The case $S=\{0\}$ is the same.

(3) Suppose again that $S=\{1\}$. Now we have $\langle\vec{c}\rangle=(1,\pi^{1-p})$ and it is clear that ιB_{nr} is bounded. Therefore it suffices to prove that if $E=\beta_0B_0+\beta_1B_1$ with β_0,β_1 is such that ιE is bounded, then $\beta_0=\beta_1=0$. The argument in the proof of Lemma 6.6 shows that $\beta_0=\beta_1$, so we are reduced to proving that ιB_{cyc} is not bounded. The proof of this similar to part (2).

The case $S=\{0\}$ is the same.

(d) Now we have $\langle\vec{c}\rangle=(\pi^{1-p},\pi^{1-p})$, and if ιE is bounded then $\mu_\varphi(E+B)\in\pi^{p-1}\mathbf{F}[[\pi]]^S$ for some coboundary B. The proof of Lemma 6.6 then shows that E is cohomologous to a multiple of B_{cyc}, and the boundedness of ιB_{cyc} yields a contradiction as above. □

6.3. $p=2$**.** We assume $p=2$ throughout this section. Now Γ is not pro-cyclic; we write $\Gamma=\Delta\times\Gamma_2$ where $\Delta=\langle\eta\rangle$ with $\chi(\eta)=-1$, so Δ has order 2, and we choose a topological generator ξ of Γ_2.

Lemma 6.8. *We have $\lambda_\eta\equiv 1+\pi \bmod \pi^{2^f}\mathbf{F}[[\pi]]$. If $\gamma\in\Gamma_2$, then $\lambda_\gamma\equiv 1 \bmod \pi^3\mathbf{F}[[\pi]]$.*

Proof. The first assertion follows from the fact that

$$\lambda_\eta^{2^f-1}=\eta(\pi)/\pi=(1+\pi)^{-1}.$$

For the second assertion, note that if $\gamma \in \Gamma_2$, then $\chi(\gamma) \equiv 1 \bmod 4$, so $\gamma(\pi)/\pi \equiv 1 \bmod \pi^3\mathbf{F}[[\pi]]$. □

Let $C \in \mathbf{F}^\times$ and $\vec{c} = (c_0, \ldots, c_{f-1}) \in \{0,1\}^S$ with some $c_j = 0$ be given. First assume that $C \neq 1$ if $\vec{c} = \vec{0}$, so that $C\pi^{\Sigma_j \vec{c}}\Phi - 1 : \mathbf{F}[[\pi]] \to \mathbf{F}[[\pi]]$ defines a valuation-preserving bijection for all $j \in S$. As in the case $p > 2$, we we will define for each $i \in S$ an element $H_i(\pi) \in \mathbf{F}((\pi))$ such that

$$(\lambda_\gamma^{\Sigma_i \vec{c}}\gamma - 1)H_i(\pi) \in \mathbf{F}[[\pi]]$$

for all $\gamma \in \Gamma$. If $c_i = 0$, we let $H_i(\pi) = \pi^{-1}$; otherwise we use the following lemma:

Lemma 6.9. *Suppose that $c_i = 1$, and $r \in 0, \ldots, f-1$ is such that $c_{i+1} = \cdots = c_{i+r} = 0$ and $c_{i+r+1} = 1$. Let*

$$H_i(\pi) = \pi^{1-2^{r+2}} + \pi^{1+2^r-2^{r+2}}.$$

Then $(\lambda_\gamma^{\Sigma_i \vec{c}}\gamma - 1)H_i(\pi) \in \mathbf{F}[[\pi]]$ for all $\gamma \in \Gamma$.

Proof. Note that we can assume $f \geq 2$. We have

$$\lambda_\gamma^{\Sigma_i}\gamma\pi^{1-2^{r+2}} = \lambda_\gamma^{\Sigma_i}\left(\frac{\gamma(\pi)}{\pi}\right)^{1-2^{r+2}}\pi^{1-2^{r+2}} = \lambda_\gamma^{\Sigma_i+(2^f-1)(1-2^{r+2})}\pi^{1-2^{r+2}}.$$

Note that $\Sigma_i = 1$ if $r = f-1$ and $\Sigma_i \equiv 1 + 2^{r+1} \bmod 2^{r+2}$ otherwise. In either case we have $\Sigma_i + (2^f - 1)(1 - 2^{r+2}) \equiv 2^{r+1} \bmod 2^{r+2}$. It follows that

$$(\lambda_\gamma^{\Sigma_i}\gamma - 1)(\pi^{1-2^{r+2}}) \equiv (\lambda_\gamma^{2^{r+1}} - 1)\pi^{1-2^{r+2}} \bmod \mathbf{F}[[\pi]].$$

Similarly we find that

$$(\lambda_\gamma^{\Sigma_i}\gamma - 1)(\pi^{1+2^r-2^{r+2}}) \equiv (\lambda_\gamma^{2^r} - 1)\pi^{1+2^r-2^{r+2}} \bmod \mathbf{F}[[\pi]].$$

Lemma 6.8 gives $\lambda_\eta^{2^s} \equiv 1 + \pi^{2^s} \bmod \pi^{2^{s+f}}$ for $s \geq 0$, and it follows that

$$(\lambda_\eta^{2^{r+1}} - 1)\pi^{1-2^{r+2}} \equiv (\lambda_\eta^{2^r} - 1)\pi^{1+2^r-2^{r+2}} \equiv \pi^{1+2^{r+1}-2^{r+2}} \bmod \mathbf{F}[[\pi]].$$

Therefore the lemma holds for $\gamma = \eta$. We also get that $\lambda_\gamma^{2^s} \equiv 1 \bmod \pi^{3\cdot 2^s}$ for $\gamma \in \Gamma_2$, from which it follows that $(\lambda_\gamma^{2^{r+1}} - 1)\pi^{1-2^{r+2}}$ and $(\lambda_\gamma^{2^r} - 1)\pi^{1+2^r-2^{r+2}}$ are in $\mathbf{F}[[\pi]]$. The lemma therefore holds for $\gamma \in \Gamma_2$ as well, and we deduce from Lemma 4.5 that it holds for all $\gamma \in \Gamma$. □

By the bijectivity of $C\pi^{\Sigma_0 \vec{c}}\Phi - 1$, for each $\gamma \in \Gamma$ we have a unique $G_i(\pi) = G_{i,\gamma}(\pi) \in \mathbf{F}[[\pi]]$ such that $(C\pi^{\Sigma_0 \vec{c}}\Phi - 1)(G_i(\pi)) = (\lambda_\gamma^{\Sigma_i \vec{c}}\gamma - 1)(H_i(\pi))$. Then letting

$$\mu_\varphi(B_i) = (0, \ldots, 0, H_i(\pi), 0, \ldots, 0),$$
$$\mu_\gamma(B_i) = (G_0(\pi), \ldots, G_i(\pi), \ldots, G_{f-1}(\pi)),$$

where

$$\begin{aligned}
G_0(\pi) &= C\pi^{c_0+2c_1+\cdots+2^{i-1}c_{i-1}}G_i(\pi^{2^i}),\\
G_1(\pi) &= \pi^{c_1+2c_2+\cdots+2^{i-2}c_{i-1}}G_i(\pi^{2^{i-1}}),\\
&\cdots\\
G_{i-1}(\pi) &= \pi^{c_{i-1}}G_i(\pi^2),\\
G_{i+1}(\pi) &= C\pi^{c_{i+1}+2c_{i+2}\cdots+2^{f-2}c_{i-1}}G_i(\pi^{2^{f-1}})\\
&\cdots\\
G_{f-1}(\pi) &= C\pi^{c_{f-1}+2c_0+\cdots+2^i c_{i-1}}G_i(\pi^{2^{i+1}}),
\end{aligned}$$

gives rise to an extension $B_i \in \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}})$. By almost identical arguments to the case $p>2$, one finds that $B_0,\dots,B_{f-1}$ are linearly independent, so that they form a basis.

Now suppose $C=1$ and $\vec{c}=\vec{0}$. We can define, similarly to the $p>2$ case, $B_0,\dots,B_{f-2},B_{f-1}$ such that

$$\begin{aligned}
\mu_\varphi(B_0) &= (\pi^{-2}+\pi^{-1},0,\dots,0),\\
\mu_\varphi(B_1) &= (0,\pi^{-2}+\pi^{-1},0,\dots,0),\\
&\cdots\\
\mu_\varphi(B_{f-1}) &= (0,\dots,0,\pi^{-2}+\pi^{-1}).
\end{aligned}$$

As before each B_i is cohomologous to B_i' with

$$\mu_\varphi(B_i) = (0,\dots,0,\pi^{-1},\pi^{-1},0,\dots,0),$$

the non-zero entries being in the $i,i+1$ coordinates (unless $f=1$, in which case $\mu_\varphi(B_0)=0$). We again set $B_{\mathrm{cyc}}=\sum_{i=0}^{f-1}B_i'$, and define an extension B_{nr} by by setting

$$\begin{aligned}
\mu_\varphi(B_{\mathrm{nr}}) &= (1,0,\dots,0),\\
\mu_\gamma(B_{\mathrm{nr}}) &= (0,0,\dots,0)
\end{aligned}$$

for all $\gamma\in\Gamma$.

The difference now is that if $p=2$, then $\dim_{\mathbf{F}}\mathrm{Ext}^1(M_{\vec{0}},M_{\vec{0}})=f+2$, so we need one more basis element. We define B_{tr} by

$$\begin{aligned}
\mu_\varphi(B_{\mathrm{nr}}) &= (0,0,\dots,0),\\
\mu_\gamma(B_{\mathrm{nr}}) &= n_\gamma(1,1,\dots,1)
\end{aligned}$$

where $n_\gamma=0$ if $\gamma\in\Gamma_3\cup\eta\Gamma_3$, and $n_\gamma=1$ otherwise (so $\gamma\mapsto n_\gamma$ defines a homomorphism $\Gamma\to\mathbf{F}$). One checks as in the case $p>2$ that the elements $B_{\mathrm{nr}},B_0,B_1,\dots,B_{f-1},B_{\mathrm{tr}}$ are linearly independent, hence form a basis for $\mathrm{Ext}^1(M_{\vec{0}},M_{\vec{0}})$.

Finally we assume $f=2$ and compute the spaces of bounded extensions. There are three possibilites to consider:

(1) $\vec{c}=(0,1)$ or $(1,0)$;
(2) $\vec{c}=(0,0)$ and $C\neq 1$;
(3) $\vec{c}=(0,0)$ and $C=1$.

We omit the proofs of the following which are essentially the same as for $p>2$:

Proposition 6.10. *If $\vec{c}=(0,1)$ or $(1,0)$, then*

- *$V_S=\mathrm{Ext}^1(M_{\vec{0}},M_{C\vec{c}})$;*
- *if $\vec{c}=(0,1)$, then $V_{\{0\}}=V_{\{1\}}=\mathbf{F}B_0$;*

- *if* $\vec{c} = (1,0)$*, then* $V_{\{0\}} = V_{\{1\}} = \mathbf{F}B_1$*;*
- $V_\emptyset = 0$*.*

Proposition 6.11. *If* $\vec{c} = (0,0)$ *and* $C \in \mathbf{F}^\times$ *with* $C \neq 1$*, then*

- $V_S^+ = V_S^- = \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{0}})$*;*
- $V_{\{1\}}^+ = \mathbf{F}(B_0 + B_1)$*;*
- $V_{\{0\}}^+ = \mathbf{F}(CB_0 + B_1)$*;*
- $V_{\{1\}}^- = V_{\{0\}}^- = V_\emptyset^+ = V_\emptyset^- = 0$*.*

Proposition 6.12. *If* $\vec{c} = (0,0)$ *and* $C = 1$*, then*

- $V_S^+ = V_S^- = \mathrm{Ext}^1(M_{\vec{0}}, M_{\vec{0}})$*;*
- $V_{\{i\}}^+ = \mathbf{F}B_{\mathrm{nr}} \oplus \mathbf{F}B_i$ *for* $i = 0, 1$*;*
- $V_{\{i\}}^- = \mathbf{F}B_{\mathrm{nr}}$ *for* $i = 0, 1$*;*
- $V_\emptyset^+ = V_\emptyset^- = 0$*.*

Remark 6.13. With a view towards relating bounded extensions to crystalline ones, we would have liked $V_S^- = \mathbf{F}B_{\mathrm{nr}} \oplus \mathbf{F}B_0 \oplus \mathbf{F}B_1$ in the trivial case. This could have been achieved with a more restrictive definition of boundedness, requiring for example that $\mu_\gamma \in \pi^2\mathbf{F}[[\pi]]^S$ for $\gamma \in \Gamma_2$ if $p = 2$. However we opted instead for the definition we found most uniform and easiest to work with.

7. Crystalline ⇒ bounded

The paper [BDJ] formulates conjectures concerning weights of mod p Hilbert modular forms in terms of the associated local Galois representations $G_K \to \mathrm{GL}_2(\mathbf{F})$. When the local representation is reducible, i.e., of the form $\begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$, the set of weights is determined by the associated class in $H^1(G_K, \mathbf{F}(\chi_1\chi_2^{-1}))$, or more precisely whether the class lies in certain distinguished subspaces. These subspaces are defined in terms of reductions of crystalline extensions of crystalline characters. Our aim is to relate these to the spaces of bounded extensions we computed in the preceding sections. The idea is to show that Wach modules over $\mathbf{A}_{K,F}^+$ associated to crystalline extensions have bounded reductions. This is easily seen to be true when the Wach module itself is the extension of two Wach modules; the problem is that this is not always the case. Recall that Theorem 2.17 establishes an equivalence of categories between crystalline representations and Wach modules over $\mathbf{B}_K^+$. We note however that $\mathbf{N}$ does not define an exact functor from G_K-stable lattices to $\mathbf{A}_K^+$-modules.

Example 7.1. Let $K = \mathbf{Q}_p$ and $V = \mathbf{Q}_p(1-p) \oplus \mathbf{Q}_p$. The corresponding Wach module is $\mathbf{N}(V) = \mathbf{B}_{\mathbf{Q}_p}^+ e_1 \oplus \mathbf{B}_{\mathbf{Q}_p}^+ e_2$ with

- $\varphi(e_1) = q^{p-1}e_1$ and $\gamma(e_1) = (\gamma(\pi)/\chi(\gamma)\pi)^{p-1}e_1$ for $\gamma \in \Gamma$;
- φ and Γ acting trivially on e_2.

Let $f_1 = p^{-1}(e_1 - \pi^{p-1}e_2)$ and consider the $\mathbf{A}_{\mathbf{Q}_p}^+$-lattice $N = \mathbf{A}_{\mathbf{Q}_p}^+ f_1 \oplus \mathbf{A}_{\mathbf{Q}_p}^+ e_2$ in $\mathbf{N}(V)$. Then it is straightforward to check that N is a Wach module over $\mathbf{A}_{\mathbf{Q}_p}^+$, hence corresponds to a $G_{\mathbf{Q}_p}$-stable lattice T in V. Such a lattice necessarily fits into an exact sequence

$$0 \to \mathbf{Z}_p(1-p) \to T \to \mathbf{Z}_p \to 0$$

of $\mathbf{Z}_p$-representations of $G_{\mathbf{Q}_p}$, but there is no surjective morphism $\alpha : N \to \mathbf{A}^+_{\mathbf{Q}_p}$. Indeed the image would have to be generated over $\mathbf{A}^+_{\mathbf{Q}_p}$ by elements $\alpha(f_1)$ and $\alpha(e_2)$ satisfying $p\alpha(f_1) = \pi^{p-1}\alpha(e_2)$, and hence could not be free over $\mathbf{A}^+_{\mathbf{Q}_p}$. This example is somewhat special since V is split and T can also be written as an extension

$$0 \to \mathbf{Z}_p \to T \to \mathbf{Z}_p(1-p) \to 0,$$

which does correspond to an extension of Wach modules. However it illustrates the problem, which we shall see also occurs for lattices in non-split extensions of $\mathbf{Q}_p$-representations.

We will prove under certain hypotheses that the relevant extensions of $\mathbf{Z}_p$-representations do in fact correspond to extensions of Wach modules. In particular we will show this holds in the generic case, and in all but a few special cases when $f = 2$. As a result, we will be able to give a complete description of the distinguished subspaces in [BDJ] in terms of (φ,Γ)-modules in the generic case and the case $f = 2$.

7.1. **The extension lemma.** We first establish a general criterion for a Wach module over $\mathbf{A}^+_{K,F}$ to arise from an extension of two Wach modules. We consider extensions of crystalline representations of arbitrary dimension since it is no more difficult than the case of one-dimensional representations.

Suppose that we have an exact sequence

$$0 \to V_1 \to V \to V_2 \to 0$$

of crystalline $\mathbf{Q}_p$-representations of G_K with Hodge-Tate weights in $[0,b]$ for some $b \geq 0$. We shall identify V_1 with a subrepresentation of V. By Theorem 2.17, we have an exact sequence of corresponding Wach modules over $\mathbf{B}^+_K$:

$$0 \to M_1 \to M \to M_2 \to 0$$

where $M = \mathbf{N}(V)$, $M_1 = \mathbf{N}(V_1) = \mathbf{N}(V) \cap \mathbf{D}(V_1)$ and M_2 is the image of $\mathbf{N}(V)$ in $\mathbf{D}(V_2)$. Now suppose that T is a G_K-stable lattice in V. Letting $T_1 = T \cap V_1$ and $T_2 = T/T_1$, we have an exact sequence

$$0 \to T_1 \to T \to T_2 \to 0$$

of $\mathbf{Z}_p$-representations of G_K. Letting $N = \mathbf{N}(T) = M \cap \mathbf{D}(T)$ be the Wach module in $M = \mathbf{N}(V)$ corresponding to T, we see that $N_1 := N \cap M_1 = \mathbf{N}(T_1)$ since

$$N \cap M_1 = \mathbf{N}(T) \cap \mathbf{D}(V_1) = \mathbf{D}(T) \cap \mathbf{N}(V) \cap \mathbf{D}(V_1) = \mathbf{D}(T) \cap \mathbf{N}(V_1) = \mathbf{D}(T_1) \cap \mathbf{N}(V_1).$$

The quotient $N_2 := N/N_1$ is a finitely generated torsion-free $\mathbf{A}^+_K$-module with an action of φ and Γ such that $q^b N_2 \subset \varphi^*(N_2)$ and Γ acts trivially on $N_2/\pi N_2$. Furthermore $\mathbf{N}(T) \to \mathbf{N}(T_2)$ induces an injective homomorphism $N_2 \to \mathbf{N}(T_2)$ which becomes an isomorphism on tensoring with $\mathbf{B}^+_K$.

Letting $\mathbf{E}^+_K = \mathbf{A}^+_K/p\mathbf{A}^+_K$, $\overline{N} = N/pN$ and $\overline{N}_i = N_i/pN_i$, we know also that

$$\overline{N}[1/\pi] = \mathbf{E}_K \otimes_{\mathbf{E}^+_K} \overline{N} \quad \text{and} \quad \overline{N}_i[1/\pi] = \mathbf{E}_K \otimes_{\mathbf{E}^+_K} \overline{N}_i$$

for $i = 1, 2$ are the (φ,Γ)-modules over $\mathbf{E}_K$ corresponding to the reductions mod p of the corresponding G_K-stable lattices. Moreover $\overline{N}_1$ and $\overline{N}$ are free over $\mathbf{E}^+_K$ and the homomorphism $\overline{N}_1 \to \overline{N}$ is injective; we identify $\overline{N}_1$ with a submodule of $\overline{N}$.

Lemma 7.2. *The following are equivalent:*

(1) *the homomorphism* $\mathbf{N}(T) \to \mathbf{N}(T_2)$ *is surjective;*

(2) $N_2 = \mathbf{N}(T)/\mathbf{N}(T_1)$ *is free over* $\mathbf{A}^+_K$*;*
(3) $\overline{N}_1 = \overline{N} \cap \mathbf{D}(T_1/pT_1)$.

Proof. If $\mathbf{N}(T) \to \mathbf{N}(T_2)$ is surjective, then $N_2 \cong \mathbf{N}(T_2)$ is free over $\mathbf{A}^+_K$. Conversely if N_2 is free, then $\mathbf{N}(T)$ maps onto a Wach module over $\mathbf{A}^+_K$ in $\mathbf{N}(V_2)$, which by Theorem 2.17 is of the form $\mathbf{N}(T_2')$ for some G_K-stable lattice T_2' in V_2; moreover $\mathbf{N}(T_2') \subset \mathbf{N}(T_2)$ implies that $T_2' \subset T_2$. On the other hand, since $\mathbf{N}(T)$ maps to $\mathbf{N}(T_2')$, $D(T)$ maps to $\mathbf{D}(T_2')$, hence T maps to T_2', and therefore $T_2 = T_2'$.

Since $\mathbf{B}^+_K \otimes_{\mathbf{A}^+_K} N_2 \cong \mathbf{N}(V_2)$ is free of rank $d_2 := \dim_{\mathbf{Q}_p} V_2$ over B^+_K, it follows from Nakayama's Lemma that N_2 is free over $\mathbf{A}^+_K$ if and only if $N_2/pN_2 = \overline{N}/\overline{N}_1$ is free of rank d_2 over $\mathbf{E}^+_K$. Since $\overline{N}$ and $\overline{N}_1$ are free over $\mathbf{E}^+_K$ and the difference of their ranks is d_2, this in turn is equivalent to $\overline{N}/\overline{N}_1$ being torsion-free over $\mathbf{E}^+_K$, which in turn is equivalent to $\overline{N}_1 = \overline{N} \cap \overline{N}_1[1/\pi]$. □

Example 7.3. Returning to Example 7.1, note that since $e_1 - \pi^{p-1}e_2 \in pN$, we have $\pi^{p-1}\overline{e}_2 = -\overline{e}_1 \in \overline{N}$, so $\overline{e}_2 = -\pi^{1-p}\overline{e}_1 \in \overline{N}_1'$, where $\overline{N}_1' = \overline{N} \cap \overline{N}_1[1/\pi]$. Thus we find in this case that $\overline{N}_1 = \mathbf{F}_p[[\pi]]\overline{e}_1$, but $\overline{N}_1' = \pi^{1-p}\mathbf{F}_p[[\pi]]\overline{e}_1$, so the criterion of the lemma is not satisfied.

We remark that everything above holds with coefficients; in particular if

$$0 \to T_1 \to T \to T_2 \to 0$$

is an exact sequence of G_K-stable $\mathcal{O}_F$-lattices in crystalline representations, then the sequence

$$0 \to \mathbf{N}(T_1) \to \mathbf{N}(T) \to \mathbf{N}(T_2) \to 0$$

of $\mathbf{A}^+_{K,F}$-modules is exact if and only if

$$\mathbf{N}(T_1)/\varpi_F\mathbf{N}(T_1) = (\mathbf{N}(T)/\varpi_F\mathbf{N}(T)) \cap \mathbf{D}(T_1/\varpi_F T_1).$$

7.2. **Extensions of rank one modules.** We now specialize to the case where V_1 and V_2 are one-dimensional over F, with labelled Hodge-Tate weights $(b_{f-1}, b_0, \dots, b_{f-2})$ and $(a_{f-1}, a_0, \dots, a_{f-2})$ where each $a_i, b_i \geq 0$. Suppose that we have an exact sequence

$$0 \to V_1 \to V \to V_2 \to 0$$

of crystalline F-representations of G_K, and T is a G_K-stable $\mathcal{O}_F$-lattice in V. We thus have exact sequencea

$$0 \to T_1 \to T \to T_2 \to 0 \quad \text{and} \quad 0 \to \overline{T}_1 \to \overline{T} \to \overline{T}_2 \to 0$$

where each T_i is a G_K-stable $\mathcal{O}_F$-lattices in V_i and $\bar{\cdot}$ denotes reduction modulo ϖ_F. We let $N = \mathbf{N}(T)$ be the Wach module over $\mathbf{A}^+_{K,F}$ corresponding to T, and $\overline{N}$ its reduction modulo ϖ_F. Thus $\overline{N}$ is a free rank one $\mathbf{E}^+_{K,F}$-module with an action of φ and Γ such that Γ acts trivially modulo $\overline{N}/\pi\overline{N}$. Furthermore $\mathbf{E}_{K,F} \otimes_{\mathbf{E}^+_{K,F}} \overline{N} \cong \mathbf{D}(\overline{T})$ as (φ, Γ)-modules over $\mathbf{E}_{K,F}$. Letting $\overline{N}_1' = \mathbf{D}(\overline{T}_1) \cap \overline{N}$ and $\overline{N}_2' = \overline{N}/\overline{N}_1'$, we see that each $\overline{N}_i'$ is an $\mathbf{E}^+_{K,F}$-lattice in $\mathbf{D}(\overline{T}_i)$, stable under φ and Γ with Γ acting trivially modulo π.

From the classification of rank one (φ, Γ)-modules over $\mathbf{E}_{K,F}$, we know that $\mathbf{D}(\overline{T}_1) \cong M_{C\vec{c}} = \mathbf{E}_{K,F}e$ for some $C \in \mathbf{F}^\times$ and $\vec{c} \in \mathbf{Z}^S$. Under this isomorphism, $\overline{N}_1'$ corresponds to a submodule of the form $(\pi^{r_0}, \pi^{r_1}, \dots, \pi^{r_{f-1}})\mathbf{E}^+_{K,F}e$. Since Γ

acts trivially on $\mathbf{E}^+_{K,F}e/\pi\mathbf{E}^+_{K,F}e$ and on $\overline{N}'_1/\pi\overline{N}'_1$, we see that $(p-1)|r_i$ for $i=0,\dots,f-1$. It follows that

$$\varphi^*(\overline{N}'_1) = (\pi^{(p-1)b'_0},\dots,\pi^{(p-1)b'_{f-1}})\overline{N}'_1$$

for some $b'_0,\dots,b'_{f-1}$, all non-negative since $\overline{N}'_1$ is stable under φ. Similarly we have

$$\varphi^*(\overline{N}'_2) = (\pi^{(p-1)a'_0},\dots,\pi^{(p-1)a'_{f-1}})\overline{N}'_2$$

for some $a'_0,\dots,a'_{f-1}\geq 0$.

For the following lemma, recall that $\Sigma_j(\vec{c}) = \sum_{i=0}^{f-1} c_{i+j}p^i$ where c_k is defined for $k\in\mathbf{Z}$ by setting $c_k = c_{k'}$ if $k\equiv k' \bmod f$. We also define a partial ordering on $\mathbf{Z}^S$ by $\vec{c}\leq\vec{c}'$ if $c_i\leq c'_i$ for all i.

Proposition 7.4. *With the above notation, we have:*

(1) $\min(a_i,b_i)\leq a'_i\leq\max(a_i,b_i)$, $\min(a_i,b_i)\leq b'_i\leq\max(a_i,b_i)$ *and* $a'_i+b'_i = a_i+b_i$ *for* $i=0,\dots,f-1$*;*
(2) *If* $\vec{a}\leq\vec{b}$ *or* $\vec{b}\leq\vec{a}$*, then* $\{\vec{a},\vec{b}\}=\{\vec{a}',\vec{b}'\}$*;*
(3) $\Sigma_j(\vec{a}')\geq\Sigma_j(\vec{a})$, $\Sigma_j(\vec{b}')\leq\Sigma_j(\vec{b})$ $\Sigma_j(\vec{a}')\equiv\Sigma_j(\vec{a}) \bmod (p^f-1)$ *and* $\Sigma_j(\vec{b}')\equiv\Sigma_j(\vec{b}) \bmod (p^f-1)$ *for* $j=0,\dots,f-1$*;*
(4) $\vec{a}=\vec{a}'$ *if and only if* $\vec{b}=\vec{b}'$ *if and only if* $\mathbf{N}(T)\to\mathbf{N}(T_2)$ *is surjective.*

Proof. (1) We first prove that $a'_i+b'_i = a_i+b_i$ for $i=0,\dots,f-1$. The $\mathbf{A}^+_{K,F}$-module $\wedge^2_{\mathbf{A}^+_{K,F}}\mathbf{N}(T)$ inherits actions of φ and Γ making it a Wach module in $\wedge^2_{\mathbf{B}^+_{K,F}}\mathbf{N}(V)\cong\mathbf{N}(\wedge^2_F V)$, hence it corresponds to an $\mathcal{O}_F$-lattice in $\wedge^2_F V$. The same is true of $\mathbf{N}(T_1)\otimes_{\mathbf{A}^+_{K,F}}\mathbf{N}(T_2)$; since any two such lattices are scalar multiples of each other, it follows that the corresponding Wach modules over $\mathbf{A}^+_{K,F}$ are isomorphic, and hence that

$$\overline{N}'_1\otimes_{\mathbf{E}^+_{K,F}}\overline{N}'_2\cong\wedge^2_{\mathbf{E}^+_{K,F}}\overline{N}\cong(\mathbf{N}(T_1)/\varpi_F\mathbf{N}(T_1))\otimes_{\mathbf{E}^+_{K,F}}(\mathbf{N}(T_2)/\varpi_F\mathbf{N}(T_2))$$

as $\mathbf{E}^+_{K,F}$-modules. Moreover the isomorphisms are compatible with the action of φ, so $a'_i+b'_i=a_i+b_i$ for all i.

For the inequalities, suppose first that $\min(a_i,b_i)=0$ for each i. Then we know that $a'_i+b'_i\leq\max(a_i,b_i)$ for each i, and the result follows. The general case follows by twisting T by a character with the correct Hodge structure and $\mathbf{N}(T)$ by the corresponding Wach module.

(2) By twisting we can again reduce to the case where $\min(a_i,b_i)=0$ for each i. The condition $\vec{a}\leq\vec{b}$ or $\vec{b}\leq\vec{a}$ becomes $\vec{a}$ or $\vec{b}=\vec{0}$, and we must show that $\vec{a}'$ or $\vec{b}'=\vec{0}$. The result then follows from the equality $\vec{a}+\vec{b}=\vec{a}'+\vec{b}'$ proved in (1).

If $\vec{a}'\neq\vec{0}$ and $\vec{b}'\neq\vec{0}$, then $\varphi^f(\overline{N}'_i)\subset\pi\overline{N}'_i$ for $i=1,2$, so that $\varphi^{2f}(\overline{N})\subset\pi\overline{N}$. This means that φ is topologically nilpotent on N in the sense that $\varphi(N)\subset(\pi,\varpi_F)N$ for some $n>0$.

On the other hand, the F-representation V of G_K is *ordinary* in the sense that there is an exact sequence

$$0\to V_0\to V\to V/V_0\to 0$$

where V_0 is unramified, V/V_0 is positive crystalline, and each is one-dimensional over F. (If $\vec{b}=\vec{0}$, then take $V_0=V_1$; if $\vec{b}\neq\vec{0}$ and $\vec{a}=\vec{0}$, then the sequence $0\to V_1\to V\to V_2\to 0$ splits and we can take V_0 to be the image of V_2.) Since $\mathbf{D}_{\mathrm{crys}}(V_0)\subset$

$\mathbf{D}_{\mathrm{crys}}(V) \cong \mathbf{N}(V)/\pi\mathbf{N}(V)$ and $N/\pi N$ is a φ-stable lattice in $\mathbf{N}(V)/\pi\mathbf{N}(V)$, we see that there is an element $e_0 \in N/\pi N$ such that $e_0 \notin \varpi_F(N/\pi N)$ and $\phi(e_0) = ue_0$ for some $u \in (\mathcal{O}_F \otimes \mathcal{O}_K)^\times$. Choosing a lift $\tilde{e}_0 \in N$ of e_0, we have that $\varphi(\tilde{e}_0) \in u\tilde{e}_0 + (\pi, \varpi_F)N$, contradicting that φ is topologically nilpotent on N.

(3) Since $\overline{N}_1 = \mathbf{N}(T_1)/\varpi_F\mathbf{N}(T_1)$ is contained in $\overline{N}'_1$, we can write $\overline{N}_1 = (\pi^{t_0}, \pi^{t_1}, \ldots, \pi^{t_{f-1}})\overline{N}'_1$ for some integers $t_0, t_1, \ldots, t_{f-1} \geq 0$. We therefore have

$$\begin{aligned}\phi^*(\overline{N_1}) &= (\pi^{pt_1}, \pi^{pt_2}, \ldots, \pi^{pt_{f-1}}, \pi^{pt_0})\phi^*(\overline{N}'_1)\\ &= (\pi^{b'_0+pt_1}, \pi^{b'_1+pt_2}, \ldots, \pi^{b'_{f-2}+pt_{f-1}}, \pi^{b'_{f-1}+pt_0})\overline{N}'_1.\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}\phi^*(\overline{N_1}) &= (\pi^{b_0}, \pi^{b_1}, \ldots, \pi^{b_{f-1}})(\overline{N}_1)\\ &= (\pi^{b_0+t_0}, \pi^{b_1+t_1}, \ldots, \pi^{b_{f-1}+t_{f-1}})\overline{N}'_1.\end{aligned}$$

It follows that $\Sigma_j(\vec{b}) + \sum_{i=0}^{f-1} t_{i+j}p^i = \Sigma_j(\vec{b}') + \sum_{i=0}^{f-1} t_{i+j+1}p^{i+1}$, and therefore that $\Sigma_j(\vec{b}) = t_j(p^f - 1) + \Sigma_j(\vec{b}')$. The assertions concerning $\Sigma_j(\vec{b}')$ follow, and those concerning $\Sigma_j(\vec{a}')$ then follow using (1).

(4) We see from the proof of (3) that the hypotheses of Lemma 7.2 are satisfied if and only if $\overline{N}_1 = \overline{N}'_1$ if and only if $\vec{t} = 0$. On the other hand $\vec{b} = \vec{b}'$ if and only if $t_i = pt_{i+1}$ for $i = 0, \ldots, f-1$, which implies that $t_i = p^f t_i$ for $i = 0, \ldots, f-1$, hence is equivalent to $\vec{t} = \vec{0}$. That $\vec{a} = \vec{a}'$ if and only if $\vec{b} = \vec{b}'$ follows from (1). □

7.3. **Generic case.** In this section, we specialize to the generic case in the sense of §5.2, namely $0 < c_i < p-1$ for all i. Recall that if $J \subset S$, then there are integers a_i and b_i for $i \in S$ such that

- $1 \leq a_i \leq p$ if $i \in J$, and $a_i = 0$ if $i \notin J$;
- $1 \leq b_i \leq p$ if $i \notin J$, and $b_i = 0$ if $i \in J$;
- $\sum_{i\in S} b_ip^i - \sum_{i\in S} a_ip^i \equiv \sum_{i\in S} c_ip^i \bmod p^f - 1$.

Moreover the a_i and b_i are uniquely determined by $\vec{c}$ and J except in the case where we can take either $a_i = p$ for $i \in J$ and $b_i = 1$ for $i \notin J$, or $a_i = 1$ for $i \in J$ and $b_i = p$ for $i \notin J$.

Lemma 7.5. *Suppose that $0 < c_i < p-1$ for all i. Then $a_i < p$ and $b_i < p$ for all i unless $\vec{c} = \vec{1}$, $J = \emptyset$ and $\vec{b} = \vec{p}$, or $\vec{c} = \overrightarrow{p-2}$, $J = S$ and $\vec{a} = \vec{p}$. In particular, $\vec{a}$ and $\vec{b}$ are uniquely determined by $\vec{c}$ and J except in the above two cases where we can also have $\vec{b}$ or $\vec{a} = \vec{1}$ instead of $\vec{p}$.*

Proof. Suppose that $b_i = p$ and consider $\Sigma_i(\vec{c})$. We have $\Sigma_i(\vec{c}) \equiv \Sigma_i(\vec{b}) - \Sigma_i(\vec{a}) \bmod (p^f - 1)$ and

$$1 + p + \cdots + p^{f-1} \leq \Sigma_i(\vec{c}) \leq (p^f - 1) - (1 + p + \cdots + p^{f-1}).$$

If $\Sigma_i(\vec{b}) - \Sigma_i(\vec{a}) \in [0, p^f - 1)$, then $\Sigma_i(\vec{c}) = \Sigma_i(\vec{b}) - \Sigma_i(\vec{a}) \equiv 0 \bmod p$, so $c_i = 0$, giving a contradiction. If $\Sigma_i(\vec{b}) - \Sigma_i(\vec{a}) \in [1 - p^f, 0)$, then $\Sigma_i(\vec{c}) = p^f - 1 + \Sigma_i(\vec{b}) - \Sigma_i(\vec{a}) \equiv p - 1 \bmod p$, so $c_i = p - 1$, giving a contradiction. If $\Sigma_i(\vec{b}) - \Sigma_i(\vec{a}) \geq p^f - 1$, then $0 \leq \Sigma_i(\vec{b}) - \Sigma_i(\vec{a}) - (p^f - 1) \leq 1 + \cdots + p^{f-1}$, giving $\Sigma_i(\vec{c}) = 1 + \cdots + p^{f-1}$, so that $\vec{c} = \vec{1}$, $J = \emptyset$ and $\vec{b} = \vec{p}$. If $\Sigma_i(\vec{b}) - \Sigma_i(\vec{a}) \leq 1 - p^f$, then similar considerations give a contradiction. The proof in the case $a_i = p$ is similar (in fact, one can exchange $\vec{c}$ with $\overrightarrow{p-1} - \vec{c}$, J with its complement and $\vec{a}$ with $\vec{b}$), giving $\vec{c} = \overrightarrow{p-2}$, $J = S$ and $\vec{a} = \vec{p}$. □

Suppose that $V_1 = F(\chi_1)$ and $V_2 = F(\chi_2)$ where χ_1 and χ_2 are crystalline characters of G_K with labelled Hodge-Tate weights $(b_{f-1}, b_0, \ldots, b_{f-2})$ and $(a_{f-1}, a_0, \ldots, a_{f-2})$ respectively, V is an extension

$$0 \to V_1 \to V \to V_2 \to 0$$

of representations of G_K over F, and T is a G_K-stable $\mathcal{O}_F$-lattice in V. Letting $T_1 = T \cap V_1$ and $T_2 = T/T_1$, we have

$$0 \to T_1 \to T \to T_2 \to 0.$$

Lemma 7.6. *Suppose that $\vec{c} \in \mathbf{Z}^S$ is generic and $\vec{a}, \vec{b} \in \mathbf{Z}^S$ are as above. If V is crystalline, then*

$$0 \to \mathbf{N}(T_1) \to \mathbf{N}(T) \to \mathbf{N}(T_2) \to 0$$

is exact.

Proof. Since V is crystalline, there is a Wach module $N = \mathbf{N}(T)$ over $\mathbf{A}^+_{K,F}$ corresponding to T. Since $\vec{c}$ is generic, we have $\max(a_i, b_i) \le p-1$ for all i, unless $\{\vec{a}, \vec{b}\} = \{\vec{0}, \vec{p}\}$. If $\max(a_i, b_i) \le p-1$ for all i, then by Proposition 7.4 (1) and (3), we have

- $0 \le a'_i \le \max(a_i, b_i) \le p-1$ for all i, and
- $\sum_{i=0}^{f-1} a'_i p^i \equiv \sum_{i=0}^{f-1} a_i p^i \bmod (p^f - 1)$.

These conditions imply that $\vec{a} = \vec{a}'$ (unless $\{\vec{a}, \vec{a}'\} = \{\vec{0}, \overrightarrow{p-1}\}$, which would give $\{\vec{a}, \vec{b}\} = \{\vec{0}, \overrightarrow{p-1}\}$ and hence that $\vec{c} = \vec{0}$ is not generic). If $\{\vec{a}, \vec{b}\} = \{\vec{0}, \vec{p}\}$, then we instead use parts (2) and (3) of Proposition 7.4 to conclude that $\vec{a} = \vec{a}'$. Thus in either case, we conclude from part (4) of the proposition that $\mathbf{N}(T) \to \mathbf{N}(T_2)$ is surjective, and therefore the sequence of Wach modules is exact. $\square$

Now consider a character $\psi : G_K \to \mathbf{F}^\times$. By the classification of rank one (φ,Γ)-modules over $\mathbf{E}_{K,F}$, there is a unique pair $C \in \mathbf{F}^\times$, $\vec{c} \in \mathbf{Z}^S$ with $0 \le c_i \le p-1$ and some $c_i < p-1$, such that $\mathbf{D}(\mathbf{F}(\psi)) \cong M_{C\vec{c}}$. Suppose that $J \subset S$ and $\vec{a}, \vec{b} \in \mathbf{Z}^S$ satisfying the usual conditions, and that $A, B \in \mathbf{F}^\times$ with $BA^{-1} = C$. Recall then that we have defined a subspace $\mathrm{Ext}^1{}_{bdd}(M_{A\vec{a}}, M_{B\vec{b}})$ of $\mathrm{Ext}^1(M_{A\vec{a}}, M_{B\vec{b}})$ and an isomorphism

$$\iota : \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}) \to \mathrm{Ext}^1(M_{A\vec{a}}, M_{B\vec{b}}),$$

well-defined up to an element of $\mathbf{F}^\times$. We then define V_J as the preimage of $\mathrm{Ext}^1{}_{bdd}(M_{A\vec{a}}, M_{B\vec{b}})$. This space is independent of the choices of A and B such that $BA^{-1} = C$, but for certain J there are two choices for the pair $\vec{a}, \vec{b}$; we denote by V_J^+ the space gotten by taking $a_i = p$ for all $i \in J$ and $b_i = 1$ for all $i \notin J$, and by V_J^- the one gotten by taking $a_i = 1$ for all $i \in J$ and $b_i = p$ for all $i \notin J$.

We now also recall the definition of the subspaces of $H^1(G_K, \mathbf{F}(\psi))$ used in [BDJ], but we modify the notation from there to be more consistent with this paper. (For the translation between the notations, see the remark below.) For ψ, J, $\vec{a}$, $\vec{b}$ as above, we consider a crystalline lift $\tilde{\psi}_J : G_K \to F^\times$ of ψ with labeled Hodge-Tate weights $(h_{f-1}, h_0, \ldots, h_{f-2})$ where $h_i = -a_i$ if $i \in J$ and $h_i = b_i$ if $i \notin J$. Such a character $\tilde{\psi}_J$ is uniquely determined up to an unramified twist, which we specify by requiring that $\tilde{\psi}_J(g)$ be the Teichmüller lift of $\psi(g)$ for $g \in G_K$ corresponding via local class field theory to the uniformizer $p \in K^\times$. When $(\vec{a}, \vec{b})$ is not uniquely determined by J, we adopt the notation $\tilde{\psi}_J^{\pm}$ as usual. Recall that $H^1_f(G_K, F(\tilde{\psi}_J))$ denotes the space of cohomology classes corresponding to crystalline extensions

$$0 \to F(\tilde{\psi}) \to V \to F \to 0.$$

We then define the space L'_J as the image in $H^1(G_K, \mathbf{F}(\psi))$ of the preimage in $H^1(G_K, \mathcal{O}_F(\tilde{\psi}_J))$ of $H^1_f(G_K, F(\tilde{\psi}_J))$. We set $L_J = L'_J$ except in the following two cases:

- If ψ is cyclotomic, $J = S$ and $\vec{a} = \vec{p}$, we let $L_J = H^1(G_K, \mathbf{F}(\psi))$.
- If ψ is trivial and $J \neq S$, we let L_J be the span of L'_J and the unramified class.

As usual we disambiguate using the notation $L_J^{\pm}$. (In particular, the first exceptional case above applies to L_J^{+}.) We identify L_J (or $L_J^{\pm}$) with subspaces of $\mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}})$ via the isomorphisms

$$H^1(G_K, \mathbf{F}(\psi)) \cong \mathrm{Ext}^1{}_{\mathbf{F}[G_K]}(\mathbf{F}, \mathbf{F}(\psi)) \cong \mathrm{Ext}^1(\mathbf{D}(\mathbf{F}), \mathbf{D}(\mathbf{F}(\psi))) \cong \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{c}}),$$

the last of these given by an isomorphism $\mathbf{D}(\mathbf{F}(\psi)) \cong M_{C\vec{c}}$ which is unique up to an element of $\mathbf{F}^\times$.

Remark 7.7. The article [BDJ] (after Lemma 3.9) defines spaces $L_\alpha \subset H^1(G_K, \overline{\mathbf{F}}_p(\psi))$ for certain pairs (V, J) where $J \subset S$ and V is an irreducible representation of $\mathrm{GL}_2(k)$. The relation between the spaces is that $L_{(V,J')} = L_J \otimes_{\mathbf{F}} \overline{\mathbf{F}}_p$ where $J = \{\, i \,|\, i-1 \in J' \,\}$ and if $V \cong \otimes_{i\in S}\left(\det^{m_i} \otimes_k \mathrm{Sym}^{n_i-1} k^2 \otimes_{k,\tau_i} \overline{\mathbf{F}}_p\right)$, then we take $a_i = n_{i-1}$ if $i \in J$ and $b_i = n_{i-1}$ if $i \notin J$. (The space $L_{(V,J')}$ is in fact independent of $\vec{m}$, and when there are two choices of $\vec{n}$ compatible with ψ and J', the resulting spaces $L_{(V,J')}$ are gotten from $L_J^{\pm}$ in the evident way.)

We now prove our main result in the generic case.

Theorem 7.8. *Suppose that $\vec{c}$ is generic.*

1. *Suppose that $J \neq S$ (resp. $J \neq \emptyset$) if $\vec{c} = \overrightarrow{p-2}$ (resp. $\vec{c} \neq \vec{1}$). Then $V_J = L_J$, so $L_J = \oplus_{i\in J} L_{\{i\}}$.*
2. *If $\vec{c} = \overrightarrow{p-2}$ and $J = S$, then $V_J^{\pm} = L_J^{\pm}$, so $L_J^{-} = \oplus_{i\in J} L_{\{i\}}$ if $f > 1$.*
3. *If $\vec{c} = \vec{1}$ and $J = \emptyset$, then $V_J^{\pm} = L_J^{\pm} = \{0\}$.*

Proof. We first prove (1). Suppose that $x \in L_J$, so x is a class of extensions

$$0 \to M_{C\vec{c}} \to E \to M_{\vec{0}} \to 0$$

corresponding via $\mathbf{D}$ to a class of extensions of Galois representations

$$0 \to \mathbf{F}(\psi) \to \overline{T} \to \mathbf{F} \to 0.$$

The assumption that $x \in L_J$ means that there is an extension

$$0 \to \mathcal{O}_F(\tilde{\psi}_J) \to T \to \mathcal{O}_F \to 0$$

whose reduction mod ϖ_F is $\overline{T}$ and such that $F \otimes_{\mathcal{O}_F} T$ is crystalline. Let $\psi_2 : G_K \to F^\times$ be a crystalline character with labeled Hodge-Tate weights $(a_{f-1}, a_0, \ldots, a_{f-2})$ and let $\psi_1 = \tilde{\psi}_J \psi_2$. Then ψ_1 is crystalline with Hodge-Tate weights $(b_{f-1}, b_0, \ldots, b_{f-2})$ and we have an exact sequence

$$0 \to T_1 \to T(\psi_2) \to T_2 \to 0$$

where $T_i = \mathcal{O}_F(\psi_i)$ and $F\otimes_{\mathcal{O}_F} T(\psi_2)$ is crystalline. By Lemma 7.6, the corresponding sequence of Wach modules over $\mathbf{A}^+_{K,F}$

$$0 \to \mathbf{N}(T_1) \to N \to \mathbf{N}(T_2) \to 0$$

is exact. Reducing mod ϖ_F, we obtain an exact sequence of free $\mathbf{E}^+_{K,F}$-modules with commuting φ and Γ actions such that Γ acts trivially mod π. Tensoring with $\mathbf{E}_{K,F}$ yields an exact sequence

$$0 \to M_{B\vec{b}} \to E' \to M_{A\vec{a}} \to 0$$

of (φ,Γ)-modules, bounded with respect to a basis for $\overline{N}$. It follows that E' defines an element of $\mathrm{Ext}^1{}_{bdd}(M_{A\vec{a}}, M_{B\vec{b}})$. Moreover this exact sequence is gotten from the one defining x by twisting with $M_{A\vec{a}}$, so we have shown that $\iota(x)$ is bounded, and hence that $x \in V_J$. Thus $L_J \subset V_J$.

By Proposition 5.3 of this paper and Lemma 3.10 of [BDJ], we have that $\dim_{\mathbf{F}} V_J = |J| = \dim_{\mathbf{F}} L_J = |J|$; therefore $L_J = V_J$. The assertion that $L_J = \oplus_{i\in J} L_{\{i\}}$ then also follows from Proposition 5.3.

The proof of (2) and (3) is exactly the same as (1), except that for (2) in the cyclotomic case one uses Proposition 6.4. □

Remark 7.9. We see from the proof of the theorem that in the definition of L_J, $\tilde{\psi}_J$ can be replaced by its twist by any unramified character $G_K \to \mathcal{O}_F^\times$ with trivial reduction mod ϖ_F. This can also be proved using Fontaine-Laffaille theory.

However in the case where ψ is cyclotomic, $J = S$ and $\vec{a} = \vec{p}$, we defined L_J as $H^1(G_K, \mathbf{F}(\psi))$ rather than L'_J. In fact L'_J has codimension one and depends on the unramified twist, as the next proof shows.

As a further application, we show that in the generic case, bounded extensions "lift" to extensions of Wach modules.

Corollary 7.10. *Suppose that $\vec{c} \in \mathbf{Z}^S$ is generic and $\vec{a}, \vec{b} \in \mathbf{Z}^S$ are as above and that*

$$0 \to M_{B\vec{b}} \to E \to M_{A\vec{a}} \to 0$$

is a bounded extension of (φ,Γ)-modules over $\mathbf{E}_{K,F}$. In the case $A = B$, $\vec{c} = \overrightarrow{p-2}$ and $\vec{a} = \vec{p}$, assume F is ramified. Then the extension E arises by applying $\mathbf{E}_{K,F}\otimes_{\mathbf{A}^+_{K,F}}$ to an exact sequence over $\mathbf{A}^+_{K,F}$ of Wach modules of the form

$$0 \to \mathbf{N}(\psi_1) \to N \to \mathbf{N}(\psi_2) \to 0$$

where ψ_1 (resp. ψ_2) is a crystalline character with labeled Hodge-Tate weights $(b_{f-1}, b_0, \dots, b_{f-2})$ (resp. $(a_{f-1}, a_0, \dots, a_{f-2})$).

Proof. First assume we are not in the exceptional case where $A = B$, $\vec{c} = \overrightarrow{p-2}$ and $\vec{a} = \vec{p}$. Since the extension class defined by E is bounded, the equality $V_J = L_J$ of the preceding theorem shows that E arises by applying $\mathbf{D}$ to the reduction mod ϖ_F of a crystalline extension

$$0 \to \mathcal{O}_F(\psi_1) \to T \to \mathcal{O}_F(\psi_2) \to 0$$

where ψ_1 and ψ_2 have the required Hodge-Tate weights. Lemma 7.6 then gives the desired extension of Wach modules over $\mathbf{A}^+_{K,F}$.

Suppose now that $A = B$, $\vec{c} = \overrightarrow{p-2}$ and $\vec{a} = \vec{p}$. Consider the class $x := \iota(E) \in \mathrm{Ext}^1(M_{\vec{0}}, M_{\overrightarrow{p-2}}) \cong H^1(G_K, \mathbf{F}(\chi))$ where χ denotes the cyclotomic character. We claim that there is an unramified character $\mu : G_K \to \mathcal{O}_F^\times$ with trivial reduction mod ϖ_F so that x is in the image of $H^1(G_K, \mathcal{O}_F(\chi^p\mu))$. (This is essentially proved in Proposition 3.5 of [KW09] or Section 3.2.7 of [KW], but there it is assumed that x is très ramifié, so we recall the argument here.) The long exact sequence associated to

$$0 \to \mathcal{O}_F(\chi^p\mu) \xrightarrow{\varpi_F} \mathcal{O}_F(\chi^p\mu) \to \mathbf{F}(\chi) \to 0$$

shows that the image of $H^1(G_K, \mathcal{O}_F(\chi^p\mu))$ is the kernel of the connecting homomorphism

$$H^1(G_K, \mathbf{F}(\chi)) \to H^2(G_K, \mathcal{O}_F(\chi^p\mu)).$$

By Tate duality this is the space orthogonal to the image of the connecting homomorphism

$$H^0(G_K, (F/\mathcal{O}_F)(\chi^{1-p}\mu^{-1})) \to H^1(G_K, \mathbf{F})$$

arising from the dual short exact sequence. Letting α denote the homomorphism $G_K \to \mathbf{F}$ defined by $(\chi^{1-p} - 1)/p$, and β the unramified homomorphism sending Frob_K to 1, we find that the image of the connecting homomorphism is spanned by β if $\mu \not\equiv 1 \bmod p\mathcal{O}_F$ (which is possible as F is ramified over $\mathbf{Q}_p$) and by $\alpha + \lambda\beta$ if $\mu(\mathrm{Frob}_K) \equiv 1 + p\lambda \bmod p\varpi_F\mathcal{O}_F$. If $x \cup \beta = 0$ then we can take $\mu \not\equiv 1 \bmod p\mathcal{O}_F$, and if $x \cup \beta \neq 0$ then there is a unique λ so that $\lambda(x \cup \beta) = -x \cup \alpha$ and we choose μ accordingly. Now since $H^1(G_K, F(\chi^p\mu)) = H^1_f(G_K, F(\chi^p\mu))$, we see that E arises from the reduction of a crystalline extension of the required form, and the result again follows from Lemma 7.6. □

7.4. **$f = 2$.** In this section we will show that if $f = 2$, then $L_J = V_J$ (or $L_J^\pm = V_J^\pm$) unless $\vec{c} = \vec{0}$; in other words, the space of bounded extensions coincides with the one gotten from reductions of crystalline extensions of the corresponding weights unless the ratio of the characters is unramified. Furthermore, we give a complete description in this exceptional case.

Before treating the case $f = 2$, we note what happens in the case $f = 1$. The case $\vec{c} \neq \vec{0}$ is already treated by the results of the preceding section. Assume for the moment that $p > 2$. Then the proof goes through just the same if $\vec{c} = \vec{0}$ and $J = S = \{0\}$. Suppose then that $\vec{c} = \vec{0}$ and $J = \emptyset$. If $C \neq 1$, then $V_\emptyset = L_\emptyset = \{0\}$, so there is nothing to prove. If $C = 1$, then we have $V_\emptyset = \{0\}$, but $L_\emptyset = H^1(G_{\mathbf{Q}_p}, \mathbf{F})$. Indeed all such classes arise as reductions of lattices in representations of the form $\mathbf{Q}_p \oplus \mathbf{Q}_p(\chi^{1-p}\mu)$ with μ unramified; the corresponding Wach module is described just as in Example 7.1 and so does not give rise to a bounded extension. If $p = 2$, there are differences in the case $C = 1$ (see Remark 6.12). In that case

$$V_S^+ = V_S^- = L_S^+ = H^1(G_{\mathbf{Q}_2}, \mathbf{F}) = \langle B_{\mathrm{nr}}, B_{\mathrm{cyc}}, B_{\mathrm{tr}} \rangle$$

and $V_\emptyset^+ = V_\emptyset^- = \{0\}$, but $L_S^- = L_\emptyset^- = \langle B_{\mathrm{nr}}, B_{\mathrm{cyc}} \rangle$ and $L_\emptyset^+ = \langle B_{\mathrm{nr}}, B_{\mathrm{tr}} \rangle$. (For the explicit descriptions, note that the extensions of Galois representations are unramified twists of ones on which H_K acts trivially, and if H_K acts trivially on T, then $\mathbf{D}(T) = \mathbf{E}_K \otimes T$.)

We now turn our attention to $f = 2$. We maintain the notation of the preceding section, without the assumption that $\vec{c}$ is generic. In particular $J \subset S$ and $\vec{a}$, $\vec{b}$ satisfy the usual conditions, V_1 and V_2 are one-dimensional crystalline representaions with labelled Hodge-Tate weights (b_1, b_0) and (a_1, a_0), V is an extension of V_2 by V_1, T is a G_K-stable $\mathcal{O}_F$-lattice in V, $T_1 = T \cap V_1$ and $T_2 = T/T_1$. The refinement of Lemma 7.6 is the following:

Lemma 7.11. *Suppose that $f=2$ and $\vec{c}\neq\vec{0}$. If V is crystalline, then*

$$0\to\mathbf{N}(T_1)\to\mathbf{N}(T)\to\mathbf{N}(T_2)\to 0$$

is exact.

Proof. Since the generic case is covered by lemma 7.6, we can assume (interchanging embeddings if necessary) that $\vec{c}=(i,0)$ for some $i\in\{1,\ldots,p-2\}$ or $\vec{c}=(i,p-1)$ for some $i\in\{0,\ldots,p-2\}$. The cases where $J=\emptyset$ or $J=S$ are covered by the same argument (using parts (2), (3) and (4) of Proposition 7.4), as are the cases where $\vec{c}=(i,0)$ or $J=\{1\}$ (using parts (1), (3) and (4) of the proposition). We are thus left with the case where $\vec{c}=(i,p-1)$ for some $i\in\{0,\ldots,p-2\}$ and $J=\{0\}$, in which case $\vec{a}=(p-i,0)$ and $\vec{b}=(0,p)$. In the notation of Proposition 7.4, the possible values of $\vec{b}'$ are $(0,p)$ and $(1,0)$. To complete the proof, we must rule out the latter possibility, which we accomplish by considering the reduction of $\mathbf{N}(T)$ modulo p^2. From the exact sequence

$$0\to\mathbf{D}(T_1)\to\mathbf{D}(T)\to\mathbf{D}(T_2)\to 0$$

and the description in [Dou07] of rank one (φ,Γ)-modules recalled in §3, we see that there is a basis $\{e_1,e_2\}$ for $\mathbf{D}(T)$ over $\mathbf{A}_{K,F}$ in terms of which the matrices describing the actions of φ and $\gamma\in\Gamma$ are

$$P=\begin{pmatrix}(\tilde{B},q^p) & *\\ 0 & (\tilde{A}q^{p-i},1)\end{pmatrix}\quad\text{and}\quad G_\gamma=\begin{pmatrix}(\varphi(\Lambda_\gamma^p),\Lambda_\gamma^p) & *\\ 0 & (\Lambda_\gamma^{p-i},\varphi(\Lambda_\gamma)^{p-i})\end{pmatrix}$$

for some $\tilde{A},\tilde{B}\in\mathcal{O}_F^\times$. On the other hand, since V is crystalline, there is a basis $\{e_1',e_2'\}$ for $\mathbf{D}(T)$ over $\mathbf{A}_{K,F}$ in terms of which the matrices P' and G_γ' describing these actions lie in $\mathrm{GL}_2(\mathbf{A}_{K,F}^+)$, with $G_\gamma'\equiv I \bmod \pi\mathrm{M}_2(A_{K,F}^+)$. If we assume that further that $\vec{b}'=(1,0)$ (and so $\vec{a}'=(p-i-1,p)$), then we can choose e_1',e_2' that

$$\overline{P}'\equiv\begin{pmatrix}(B\pi^{p-1},1) & *\\ 0 & (A\pi^{(p-i-1)(p-1)},\pi^{p(p-1)})\end{pmatrix}\quad\text{and}\quad \overline{G}_\gamma'=\begin{pmatrix}(\lambda_\gamma,\lambda_\gamma^p) & *\\ 0 & (\lambda_\gamma^{p^2+p-i-1},\lambda_\gamma^{p^2-ip})\end{pmatrix},$$

where $\bar{\cdot}$ denotes reduction modulo ϖ_F. Since $\mathbf{D}(T)\cong\mathbf{A}_{K,F}\otimes_{\mathbf{A}_{K,F}^+}\mathbf{N}(T)$, we can write $(e_1',e_2')=(e_1,e_2)Q$ for some $Q\in\mathrm{GL}_2(A_{K,F})$, and then we have

$$P'=Q^{-1}P\varphi(T)\quad\text{and}\quad G_\gamma'=Q^{-1}G_\gamma\gamma(Q)\quad\text{for all }\gamma\in\Gamma.$$

Claim: *$Q\equiv RS \bmod p\mathbf{A}_{K,F}$ for some matrices $R=\begin{pmatrix}\alpha(q^{-1},1) & *\\ 0 & \beta(q,1)\end{pmatrix}\in \mathrm{GL}_2(\mathbf{A}_{K,F})$ with $\alpha,\beta\in\mathcal{O}_F^\times$, and $S\in I+\varpi_F\mathrm{M}_2(A_{K,F}^+)$.*

Since F may be ramified over $\mathbf{Q}_p$, we prove the claim by showing inductively that $Q\equiv R_mS_m \bmod \varpi_F^m\mathbf{A}_{K,F}$ for some matrices R_m,S_m of the prescribed form for $m=1,\ldots,e$ where $e=e(F/\mathbf{Q}_p)$.

To prove the statement for $m=1$, note that setting $R_0=\begin{pmatrix}(q^{-1},1) & 0\\ 0 & (q,1)\end{pmatrix}$ gives

$$\overline{R}_0^{-1}\overline{P}\varphi(\overline{R}_0)=\begin{pmatrix}(B\pi^{p-1},1) & *\\ 0 & (A\pi^{(p-i)(p-1)},\pi^{p(p-1)})\end{pmatrix}.$$

So if we write $R=R_0S_0$, then

$$\overline{S}_0\begin{pmatrix}(B\pi^{p-1},1) & *\\ 0 & (A\pi^{(p-i)(p-1)},\pi^{p(p-1)})\end{pmatrix}=\begin{pmatrix}(B\pi^{p-1},1) & *\\ 0 & (A\pi^{(p-i)(p-1)},\pi^{p(p-1)})\end{pmatrix}\varphi(\overline{S}_0).$$

It follows easily that $\overline{S}_0 = \begin{pmatrix} \bar\alpha & \bar\delta \\ 0 & \bar\beta \end{pmatrix}$ for some $\bar\alpha, \bar\beta \in \mathbf{F}^\times$, $\bar\delta \in \mathbf{E}_{K,F}$. Choosing lifts $\alpha, \beta \in \mathcal{O}_F^\times$ and $\delta \in \mathbf{A}_{K,F}$ and setting $R_1 = R_0 \begin{pmatrix} \alpha & \delta \\ 0 & \beta \end{pmatrix}$ gives the result for $m = 1$.

Suppose now that $m \in \{1, \ldots, e-1\}$ and that $Q \equiv R_m S_m \bmod \varpi_F^m \mathbf{A}_{K,F}$ with R_m, S_m of the prescribed form. Setting $Q_m = R_m^{-1} Q S_m^{-1}$, we have $Q_m = I + \varpi_F^m Q'_m$ for some $Q'_m \in \mathrm{M}_2(\mathbf{A}_{K,F})$. Define

$$\begin{array}{rlcl} & P_m = R_m^{-1} P \varphi(R_m), & & G_{\gamma,m} = R_m^{-1} G_\gamma \gamma(R_m), \\ & P'_m = S_m^{-1} P' \varphi(S_m) & \text{and} & G'_{\gamma,m} = S_m^{-1} G'_\gamma \gamma(S_m), \\ \text{so that} & P'_m = Q_m^{-1} P_m \varphi(Q_m) & \text{and} & G'_{\gamma,m} = Q_m^{-1} G'_{\gamma,m} \gamma(Q_m). \end{array}$$

Note that $P'_m \in \mathrm{M}_2(\mathbf{A}^+_{K,F})$, $G'_{\gamma,m} \in I + \pi \mathrm{M}_2(\mathbf{A}^+_{K,F})$, $P_m \equiv P'_m \bmod \varpi_F^m \mathrm{M}_2(\mathbf{A}_{K,F})$, $G_{\gamma,m} \equiv G'_{\gamma,m} \bmod \varpi_F^m \mathrm{M}_2(\mathbf{A}_{K,F})$,

$$P_m \equiv \begin{pmatrix} (\tilde B \pi^{p-1}, 1) & * \\ 0 & (\tilde A \pi^{(p-i-1)(p-1)}, \pi^{p(p-1)}) \end{pmatrix} \bmod p\mathrm{M}_2(\mathbf{A}_{K,F})$$

$$\text{and } G_{\gamma,m} \equiv \begin{pmatrix} (\lambda_\gamma, \lambda_\gamma^p) & * \\ 0 & (\lambda_\gamma^{p^2+p-i-1}, \lambda_\gamma^{p^2-ip}) \end{pmatrix} \bmod p\mathrm{M}_2(\mathbf{A}_{K,F}).$$

Note that since $m+1 \le e$, the last two congruences hold mod ϖ_F^{m+1}, and that $Q_m^{-1} \equiv I - \varpi_F^m Q'_m \bmod \varpi_F^{m+1} \mathrm{M}_2(\mathbf{A}_{K,F})$. It follows that

$$\begin{aligned} P'_m &\equiv (I - \varpi_F^m Q'_m) P_m (I + \varpi_F^m \varphi(Q'_m)) \\ &\equiv P_m + \varpi_F^m (P_m \varphi(Q'_m) - Q'_m P_m) \bmod \varpi_F^{m+1} \mathrm{M}_2(\mathbf{A}_{K,F}), \end{aligned}$$

and therefore that

$$\varpi_F^m (P_m \varphi(Q'_m) - Q'_m P_m) \equiv P'_m - P_m \equiv \varpi_F^m \begin{pmatrix} x & y \\ z & w \end{pmatrix} \bmod \varpi_F^{m+1} \mathrm{M}_2(\mathbf{A}_{K,F})$$

with $x, z, w \in \mathbf{A}^+_{K,F}$. Note that $\overline{P}_m = \overline{P}'$, so we have $\overline{P}' \varphi(\overline{Q}'_m) - \overline{Q}'_m \overline{P}' = \begin{pmatrix} \bar x & \bar y \\ \bar z & \bar w \end{pmatrix}$ for some $\bar x, \bar z, \bar w \in \mathbf{E}^+_{K,F}$, $\bar y \in \mathbf{E}_{K,F}$. Similarly we find that $\overline{G}'_\gamma \gamma(\overline{Q}'_m) - \overline{Q}'_m \overline{G}'_\gamma = \begin{pmatrix} \bar x_\gamma & \bar y_\gamma \\ \bar z_\gamma & \bar w_\gamma \end{pmatrix}$ for some $\bar x_\gamma, \bar z_\gamma, \bar w_\gamma \in \pi \mathbf{E}^+_{K,F}$, $\bar y_\gamma \in \mathbf{E}_{K,F}$.

Writing

$$\overline{Q}'_m = \begin{pmatrix} (r_0, r_1) & (s_0, s_1) \\ (t_0, t_1) & (u_0, u_1) \end{pmatrix}$$

with $r_0, r_1, \ldots, u_0, u_1 \in \mathbf{F}((\pi))$, the condition that $\bar z \in \mathbf{E}^+_{K,F}$ becomes $\pi^{p(p-1)} t_0(\pi)^p - t_1(\pi), A\pi^{(p-i-1)(p-1)} t_1(\pi^p) - B\pi^{p-1} t_0(\pi) \in \mathbf{F}[[\pi]]$, from which one deduces that $\mathrm{val}_\pi(t_0) \ge 1 - p$ and $\mathrm{val}_\pi(t_1) \ge 0$. The condition that $\bar z_\gamma \in \pi \mathbf{E}^+_{K,F}$ then becomes that $\gamma(t_0) \lambda_\gamma^{p^2+p-i-2} - t_0 \in \pi \mathbf{F}[[\pi]]$. Lemma 4.2 rules out the possibility that $1 - p < \mathrm{val}_\pi(t_0) < 0$, and Lemma 4.4 rules out the possibility that $\mathrm{val}_\pi(t_0) = 1 - p$. Therefore $(t_0, t_1) \in \mathbf{E}^+_{K,F}$. Since $\overline{P}' \in \mathrm{M}_2(\mathbf{E}^+_{K,F})$, the condition that $\bar x \in \mathbf{E}^+_{K,F}$ then becomes that $\pi^{p-1}(r_1(\pi^p) - r_0(\pi)), r_0(\pi^p) - r_1(\pi) \in \mathbf{F}[[\pi]]$, which implies that $(r_0, r_1) \in \mathbf{E}^+_{K,F}$. The condition that $\bar w \in \mathbf{E}^+_{K,F}$ becomes that $\pi^{(p-i-1)(p-1)}(u_1(\pi^p) - u_0(\pi)), \pi^{p(p-1)} u_0(\pi^p) - u_1(\pi) \in \mathbf{F}[[\pi]]$, which implies that $\mathrm{val}_\pi(u_0) \ge 1 - p$ and $\mathrm{val}_\pi(u_1) \ge i + 2 - p$. Since $(t_0, t_1) \in \mathbf{E}^+_{K,F}$ and

$\overline{G}'_\gamma \equiv I \bmod \pi \mathrm{M}_2(\mathbf{E}^+_{K,F})$, the condition that $\overline{w}_\gamma \in \pi\mathbf{E}^+_{K,F}$ becomes that $\gamma(u_i)-u_i \in \pi\mathbf{E}^+_{K,F}$ for $i=0,1$, so that Lemmas 4.2 and 4.4 again imply that $(u_0,u_1)\in\mathbf{E}^+_{K,F}$. We can thus lift $\overline{Q}'_m$ to a matrix $\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{M}_2(\mathbf{A}_{K,F})$ with $r,t,u \in \mathbf{A}^+_{K,F}$. Setting $R_{m+1} = R_m \begin{pmatrix} 1 & \varpi_F^m s \\ 0 & 1 \end{pmatrix}$ and $S_{m+1} = \begin{pmatrix} 1+\varpi_F^m r & 0 \\ \varpi_F^m t & 1+\varpi_F^m u \end{pmatrix} S_m$ then gives $Q \equiv R_{m+1}S_{m+1} \bmod \varpi_F^{m+1}\mathrm{M}_2(\mathbf{A}_{K,F})$ with R_{m+1}, S_{m+1} of the prescribed form, and completes the proof of the claim.

To derive a contradiction from the claim, we proceed as in the proof of the induction step above, but with $m=e$ and working modulo ϖ_F^{m+1}. More precisely, we define Q_e, Q'_e, P_e, $G_{\gamma,e}$, P'_e and $G'_{\gamma,e}$ as above; the difference now is that the congruences satisfied by P_e and $G_{\gamma,e}$ modulo p are not satisfied modulo $p\varpi_F$. In particular, the upper-left hand entry of P_e is $(\tilde{A}q, q^p/\varphi(q))$, and a straightforward calculation shows that

$$\frac{q^p}{\varphi(q)} \equiv 1 + p(g(\pi^{-p}) - g(\pi^{-1}) + f(\pi)) \bmod p^2\mathbf{A}_{\mathbf{Q}_p}$$

where $g(X) = \sum_{i=1}^{p-1}(-X)^i/i$ and $f(\pi)\in\mathbf{A}^+_{\mathbf{Q}_p}$. As before, we have $\overline{P}'\varphi(\overline{Q}'_e) - \overline{Q}'_e\overline{P}' = \begin{pmatrix} \bar{x} & \bar{y} \\ \bar{z} & \bar{w} \end{pmatrix}$ with $\bar{z}\in\mathbf{E}^+_{K,F}$ since P_e is upper-triangular, but now $\bar{x} \in (0, c(\bar{g}(\pi^{-p}) - \bar{g}(\pi^{-1}))) + \mathbf{E}^+_{K,F}$ for some $c\in\mathbf{F}^\times$ (the reduction of p/ϖ_F^e). Similarly we have $\overline{G}'_\gamma\gamma(\overline{Q}'_e) - \overline{Q}'_e\overline{G}'_\gamma = \begin{pmatrix} \bar{x}_\gamma & \bar{y}_\gamma \\ \bar{z}_\gamma & \bar{w}_\gamma \end{pmatrix}$ with $\bar{z}_\gamma \in \pi\mathbf{E}^+_{K,F}$. So just as before we get $(t_0,t_1)\in\mathbf{E}^+_{K,F}$, but this implies that $\pi^{p-1}(r_1(\pi^p) - r_0(\pi)) \in \mathbf{F}[[\pi]]$ and $r_0(\pi^p) - r_1(\pi) \in c(\bar{g}(\pi^{-p}) - \bar{g}(\pi^{-1})) + \mathbf{F}[[\pi]]$, which leads to a contradiction and completes the proof of the lemma. □

Theorem 7.12. *Suppose that $f=2$ and $\vec{c}\neq\vec{0}$. Then $V_J = L_J$ (or $V_J^\pm = L_J^\pm$) for all $J\subset S$. In particular $L_{\{0\}} = L_{\{1\}}$ if and only if $\vec{c} = (i,p-1)$ or $(p-1,i)$ for some $i\in\{1,\dots,p-2\}$.*

Proof. The proof of the first assertion is exactly the same as for Theorem 7.8. The second then follows from the corresponding result for V_J in §5.3. □

Theorem 1.2 of the introduction now follows in view of Corollary 3.4.

Remark 7.13. Again we see that in the definition of L_J, $\tilde{\psi}_J$ can be replaced by its twist by any unramified character with trivial reduction; the cases where some a_i or b_i is p (with $J=\{0\}$ or $\{1\}$) are outside the range of Fontaine-Laffaille theory.

Note also that the case where we had to work the hardest in the proof of Lemma 7.11 is precisely the one where $L_{\{0\}} = L_{\{1\}}$.

By the same proof as Corollary 7.10, we obtain:

Corollary 7.14. *Suppose that $f=2$ and $\vec{c}\neq\vec{0}$ and $\vec{a},\vec{b}\in\mathbf{Z}^S$ are as above and that*

$$0 \to M_{B\vec{b}} \to E \to M_{A\vec{a}} \to 0$$

is a bounded extension of (φ,Γ)-modules over $\mathbf{E}_{K,F}$. In the case $A=B$, $\vec{c} = \overrightarrow{p-2}$ and $\vec{a}=\vec{p}$, assume F is ramified. Then the extension E arises by applying

$\mathbf{E}_{K,F}\otimes_{\mathbf{A}^+_{K,F}}$ *to an exact sequence over* $\mathbf{A}^+_{K,F}$ *of Wach modules of the form*

$$0 \to \mathbf{N}(\psi_1) \to N \to \mathbf{N}(\psi_2) \to 0$$

where ψ_1 *(resp.* ψ_2*) is a crystalline character with labeled Hodge-Tate weights* (b_1, b_0) *(resp.* (a_1, a_0)*).*

We now say what we can in the case $\vec{c} = \vec{0}$. First note that the proof of Lemma 7.11 goes through in the following cases:

- $J = S$, in which case $\vec{a} = \overrightarrow{p-1}$ (or $\vec{2}$ if $p = 2$);
- $J = \{0\}$, $\vec{a} = (p, 0)$, $\vec{b} = (0, 1)$ (the + case);
- $J = \{1\}$, $\vec{a} = (0, p)$, $\vec{b} = (1, 0)$ (the + case).

The proof of Theorem 7.12 goes through in these cases unless $J = S$, $p = 2$, $\vec{a} = 1$, $C = 1$ where we get $L^-_S \subset V^-_S$, but $\dim L^-_S = 3 \neq \dim V^-_S = 4$ (see Remark 6.12). In this case however we know that L^-_S consists of the peu ramifiée extensions. To compute the corresponding (φ, Γ)-modules, note that $V^+_{\{0\}} = L^+_{\{0\}}$ contains the classes arising from reductions of Galois stable lattices in $F(\mu\psi^2) \oplus F(\psi^\sigma)$ where $\psi : G_K \to \mathcal{O}_F^\times$ is a crystalline character with labeled Hodge-Tate weights $(0, 1)$, σ is the non-trivial element of $\mathrm{Gal}(K/\mathbf{Q}_2)$, and $\mu : G_K \to \mathcal{O}_F^\times$ is an unramified character with trivial reduction mod ϖ_F. These classes correspond to homomorphisms $G_K \to \mathbf{F}$ whose restriction to inertia is a multiple of the reduction of $1/2(\psi^\sigma\psi^{-2} - 1)|_{I_K}$. One can compute these explicitly using class field theory and check that they are peu ramifiée. It follows that $L^+_{\{0\}} \subset L^-_S$, and similarly $L^+_{\{1\}} \subset L^-_S$, so that $L^-_S = \langle B_{\mathrm{nr}}, B_0, B_1\rangle$.

If $J = \emptyset$, we have $V_\emptyset = \{0\}$, and $L_\emptyset = \{0\}$ unless $C = 1$. If $C = 1$, one can compute the extensions and associated (φ, Γ)-modules explicitly since they are unramified twists of representations on which H_K acts trivially. If $p \neq 2$, one gets $L_\emptyset = \langle B_{\mathrm{nr}}, B_{\mathrm{cyc}}\rangle$. If $p = 2$, one gets $L^+_\emptyset = \langle B_{\mathrm{nr}}, B_{\mathrm{cyc}}\rangle$ (with $\vec{b} = \vec{1}$) and $L^-_\emptyset = \langle B_{\mathrm{nr}}, B_{\mathrm{tr}}\rangle$ (with $\vec{b} = \vec{2}$).

The most interesting is the $-$ case when $S = \{0\}$ or $\{1\}$. For example if $S = \{0\}$, $\vec{a} = (p, 0)$ and $\vec{b} = (0, 1)$, the proof of Lemma 7.11 breaks down, but we see that if the associated sequence of Wach modules is not exact, then $\vec{a}' = (0, 1)$ and $\vec{b}' = (p, 0)$, so the extension of (φ, Γ)-modules associated to $\overline{T}$ is in $V^+_{\{1\}}$. Since $V^-_{\{0\}} \subset V^+_{\{1\}}$, it follows that $L^-_{\{0\}} \subset V^+_{\{1\}}$, and dimension counting implies equality. We therefore have that $V^-_{\{0\}}$ is contained in $L^-_{\{0\}} = V^+_{\{1\}} = L^+_{\{1\}}$. with codimension one. Similarly $V^-_{\{1\}}$ is contained in $L^-_{\{1\}} = V^+_{\{0\}} = L^+_{\{0\}}$ with codimension one.

Putting everything together we get:

Theorem 7.15. *Suppose that* $f = 2$, $\vec{c} = \vec{0}$.

(1) *If* $C \neq 1$, *then:*
  - *if* $p > 2$ *then* $L_S = V_S = \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{0}})$;
  - *if* $p = 2$ *then* $L^\pm_S = V^\pm_S = \mathrm{Ext}^1(M_{\vec{0}}, M_{C\vec{0}})$;
  - $V^-_{\{0\}} = V^-_{\{1\}} = \{0\}$, *and* $L^-_{\{0\}} = L^+_{\{1\}} = V^+_{\{1\}} \neq V^+_{\{0\}} = L^+_{\{0\}} = L^-_{\{1\}}$;
  - *if* $p > 2$ *then* $L_\emptyset = V_\emptyset = \{0\}$;
  - *if* $p = 2$ *then* $L^\pm_\emptyset = V^\pm_\emptyset = \{0\}$.

(2) *If* $C = 1$, *then:*
  - *if* $p > 2$ *then* $L_S = V_S = \mathrm{Ext}^1(M_{\vec{0}}, M_{\vec{0}})$;

- *if* $p=2$ *then* $L_S^+ = V_S^{\pm} = \mathrm{Ext}^1(M_{\vec{0}}, M_{\vec{0}})$ *and* $L_S^- = \langle B_{\mathrm{nr}}, B_0, B_1\rangle$*;*
- $V_{\{0\}}^- = V_{\{1\}}^- = \langle B_{\mathrm{nr}}\rangle$*,* $L_{\{0\}}^- = L_{\{1\}}^+ = V_{\{1\}}^+ = \langle B_{\mathrm{nr}}, B_1\rangle$*, and* $L_{\{1\}}^- = L_{\{0\}}^+ = V_{\{0\}}^+ = \langle B_{\mathrm{nr}}, B_0\rangle$*;*
- *if* $p>2$ *then* $V_\emptyset = \{0\}$ *and* $L_\emptyset = \langle B_{\mathrm{nr}}, B_{\mathrm{cyc}}\rangle$*;*
- *if* $p=2$ *then* $V_\emptyset^{\pm} = \{0\}$*,* $L_\emptyset^+ = \langle B_{\mathrm{nr}}, B_{\mathrm{cyc}}\rangle$ *and* $L_\emptyset^- = \langle B_{\mathrm{nr}}, B_{\mathrm{tr}}\rangle$*.*

Note that the strict inequality $V_{\{0\}}^- \subset L_{\{0\}}^-$ implies the existence of *non-split* crystalline extensions $0 \to F(\psi_1) \to V \to F(\psi_2) \to 0$ with Galois stable $\mathcal{O}_F$-lattices T such that the corresponding sequence of Wach modules over $\mathbf{A}_{K,F}^+$ is not exact (with ψ_1 and ψ_2 of labeled Hodge-Tate weights $(p,0)$ and $(0,1)$ respectively).

As in Remark 7.13, we see that the definitions of L_J are independent of the choice of unramified twist, unless $C=1$, $J=\emptyset$ and F is ramified, in which case twisting by an unramified character which is trivial mod ϖ_F but not mod p would give $L'_J = L_J = \langle B_{\mathrm{nr}}\rangle$.

Finally we remark that the proof of Corollary 7.14 goes through when $\vec{c} = \vec{0}$ except in the following two cases where $C=1$:

- If $p=2$ and $\vec{a} = \vec{1}$, then only classes in L_S^- lift (see Remark 6.12).
- If $\vec{a} = (1,0)$ and $\vec{b} = (0,p)$ (or $\vec{a} = (0,1)$ and $\vec{b} = (p,0)$), then we have not determined whether B_{nr} lifts.

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