Triviality of the Peripheral Point Spectrum

E B Davies

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Abstract

If $T_t = \mathrm{e}^{Zt}$ is a positive one-parameter contraction semigroup acting on $l^p(X)$ where X is a countable set and $1 \leq p < \infty$, then the peripheral point spectrum P of Z cannot contain any non-zero elements. The same holds for Feller semigroups acting on $L^p(X)$ if X is locally compact.

1 Introduction

Let $T_t = \mathrm{e}^{Zt}$ be a one-parameter contraction semigroup with generator Z acting on the separable Banach space $\mathcal{B}$. We say that $\theta \in \mathbf{R}$ lies in its peripheral point spectrum P if the linear subspace

$$\begin{aligned} \mathcal{L}_\theta &= \{f \in \mathrm{Dom}(Z) : Zf = i\theta f\} \\ &= \{f \in \mathcal{B} : T_t f = \mathrm{e}^{i\theta t} f \text{ for all } t \geq 0\}. \end{aligned}$$

is non-zero. Under the conditions of our paper it has long been known that P is cyclic in the sense that $\alpha \in P$ implies $\alpha \mathbf{Z} \subseteq P$. We will prove that P cannot contain any non-zero points under certain conditions. Our main results, Theorems 9 and 12, are contributions to the study of infinite-dimensional versions of the Perron-Frobenius theorems. See [1, 2] for references to the literature on this subject.

Although P may be uncountable, only a countable subset of P need be considered for our purposes.

Lemma 1 *There exist sequences $f_n \in \mathrm{Dom}(Z)$ and $\theta_n \in P$ such that $Zf_n = i\theta_n f_n$ and*

$$\mathcal{L} := \overline{\mathrm{lin}}\left(\bigcup\{\mathcal{L}_\theta : \theta \in P\}\right) = \overline{\mathrm{lin}}\{f_n : n \in \mathbf{N}\}$$

where $\overline{\text{lin}}$ *stands for the closed linear span. In particular every* $f \in \bigcup\{\mathcal{L}_\theta : \theta \in P\}$ *is the limit of finite linear combinations of* $\{f_n : n \in \mathbf{N}\}$.

Proof Since $\mathcal{B}$ is separable, $\mathcal{L}$ contains a countable dense subset $\{g_n : n = 1, 2, ...\}$. Each g_n is the limit of a sequence, each term of which is a finite linear combination of elements of $\bigcup\{\mathcal{L}_\theta : \theta \in P\}$. The number of such elements involved in these sums is countable.

Lemma 2 *There exists a sequence* $t(n) \to +\infty$ *such that*

$$\lim_{n\to\infty} \|T_{t(n)}f - f\| = 0 \tag{1}$$

for all $f \in \mathcal{L}$.

Proof Let $\{f_n\}$ and $\{\theta_n\}$ be the sequences constructed in Lemma 1 and let $\mathcal{L}_n = \text{lin}\{f_r : 1 \le r \le n\}$. For each n we will construct $t(n) \ge t(n-1) + 1$ such that

$$\|T_{t(n)}f - f\| \le 2^{-n}\|f\|$$

for all $f \in \mathcal{L}_n$. This implies (1) for all $f \in \mathcal{L}_m$ for any choice of m. The full statement of the lemma then uses the fact that T_t are all contractions.

Every $f \in \mathcal{L}_n$ may be written in the form $f = \sum_{r=1}^n \alpha_r f_r$. We then have

$$T_t f = \sum_{r=1}^{n} \alpha_r \mathrm{e}^{i\theta(r)t} f_r$$

for all $t \ge 0$. Since all norms on a finite-dimensional space are equivalent

$$\|(T_t - I)|_{\mathcal{L}_n}\| \le c_n \max\{|\mathrm{e}^{i\theta(r)t} - 1| : 1 \le r \le n\}.$$

The lemma follows by using the fact that the set

$$\left\{(\mathrm{e}^{i\theta(1)t}, ..., \mathrm{e}^{i\theta(n)t}) : t = 1, 2, ...\right\}$$

is a semigroup in the compact n-dimensional torus $\mathbf{T}^n$ where $\mathbf{T} = \{z : |z| = 1\}$. Every such semigroup has points arbitrarily close to the group identity.

Example 3 We normalize the Haar measure of $\mathbf{T}$ to have mass 1, and let $X = \mathbf{T}^{\mathbf{N}}$ with the usual product measure. Given $1 \le p < \infty$ define T_t on $\mathcal{B} = L^p(X)$ by $(T_t f)(x) = f(x_t)$, where $x_{t,m} = \mathrm{e}^{2\pi i t/m} x_m$ for all $m \in \mathbf{N}$. Then T_t is a group of isometries on $\mathcal{B}$. By taking Fourier transforms it may be shown that $\text{Spec}(Z) = i\mathbf{R}$. If we put $t(n) = n!$ then it is easy to show that

$$\lim_{n\to\infty} \|T_{t(n)}f - f\| = 0$$

for all $f \in \mathcal{B}$.

Theorem 4 *If*
$$\mathcal{M} = \{f \in \mathcal{B} : \lim_{n\to\infty} \|T_{t(n)}f - f\| = 0\}$$
then T_t acts as a one-parameter group *of isometries on $\mathcal{M}$.*

Proof If $f \in \mathcal{M}$ and $t \geq 0$ then it follows from

$$\|T_{t(n)}(T_t f) - (T_t f)\| \leq \|T_{t(n)}f - f\|$$

that $T_t f \in \mathcal{M}$.

If $t \geq 0$ then
$$\|T_{t(n)}f\| = \|T_{t(n)-t}T_t f\| \leq \|T_t f\|$$
for all $t(n) \geq t$. If $f \in \mathcal{M}$ then by letting $n \to \infty$ we see that $\|f\| \leq \|T_t f\|$. Therefore T_t is an isometry when restricted to the subspace $\mathcal{M}$.

It follows from the above that for any $t \geq 0$, $\mathcal{M}_t = T_t(\mathcal{M})$ is a closed linear subspace of $\mathcal{M}$. Moreover $T_s(\mathcal{M}_t) \subseteq \mathcal{M}_t$ for all $s \geq 0$. If $f \in \mathcal{M}$ then putting $s = t(n) - t$ we deduce that

$$f = \lim_{n\to\infty} T_{t(n)-t}(T_t f) \in \mathcal{M}_t.$$

Therefore $\mathcal{M}_t = \mathcal{M}$ for all $t \geq 0$. The operators T_t are defined for $t < 0$ by $T_t = (T_{-t})^{-1}$.

2 Positive Semigroups

We now assume that $\mathcal{B} = L^p_{\mathbf{C}}(X, \mathrm{d}x)$ where $1 \leq p < \infty$ and $\mathrm{d}x$ is a countably additive σ-finite measure on X. In order to pass back and forth between the real and complex spaces the following standard proposition is needed.

Proposition 5 *Let $1 \leq p, q \leq \infty$ and let $A_{\mathbf{R}} : L^p_{\mathbf{R}}(X, \mathrm{d}x) \to L^q_{\mathbf{R}}(X, \mathrm{d}x)$ be a positive linear operator with complex-linear extension $A_{\mathbf{C}}$. Then*

$$|A_{\mathbf{C}}(f + ig)| \leq A_{\mathbf{R}}(|f + ig|)$$

for all $f, g \in L^p_{\mathbf{R}}(X, \mathrm{d}x)$. Hence $\|A_{\mathbf{C}}\| = \|A_{\mathbf{R}}\|$.

Proof Given $\theta \in \mathbf{R}$ we have

$$\begin{aligned} |(A_{\mathbf{R}}f)\cos(\theta) + (A_{\mathbf{R}}g)\sin(\theta)| &= |A_{\mathbf{R}}(f\cos(\theta) + g\sin(\theta))| \\ &\leq A_{\mathbf{R}}(|f\cos(\theta) + g\sin(\theta)|) \\ &\leq A_{\mathbf{R}}(|f + ig|). \end{aligned}$$

Let $u, v, w : X \to \mathbf{R}$ be functions in the classes of $A_{\mathbf{R}}f, A_{\mathbf{R}}g, A_{\mathbf{R}}(|f+ig|)$. Then we have shown that

$$|u(x)\cos(\theta) + v(x)\sin(\theta)| \le w(x)$$

for all x not in some null set $N(\theta)$. If $\{\theta_n\}_{n=1}^{\infty}$ is a countable dense subset of $[-\pi, \pi]$ then

$$|u(x) + iv(x)| = \sup_{1\le n<\infty} |u(x)\cos(\theta_n) + v(x)\sin(\theta_n)| \le w(x)$$

for all x not in the null set $\bigcup_{n=1}^{\infty} N(\theta_n)$. This implies the first statement of the theorem, from which the second follows immediately.

We now assume that T_t is a contraction semigroup on $\mathcal{B}$ which commutes with complex conjugation and which is positive when restricted to $L^p_{\mathbf{R}}(X, \mathrm{d}x)$. This latter space is a lattice with respect to the operations

$$\begin{aligned} f \vee g &= \max\{f, g\} = \tfrac{1}{2}(f+g) + \tfrac{1}{2}|f-g|, && (2)\\ f \wedge g &= \min\{f, g\} = \tfrac{1}{2}(f+g) - \tfrac{1}{2}|f-g|. && (3) \end{aligned}$$

The strict monotonicity of the norm in $L^p_{\mathbf{R}}(X, \mathrm{d}x)$ for $1 \le p < \infty$ is of critical importance for our next theorem and hence for the rest of the paper.

Theorem 6 *The subspace $\mathcal{M}$ is closed under complex conjugation and its real part $\mathcal{N}$ is a closed linear sublattice of $L^p_{\mathbf{R}}(X, \mathrm{d}x)$. Moreover the operators T_t are lattice isomorphisms when restricted to $\mathcal{N}$ for all $t \in \mathbf{R}$.*

Proof We use the fact, proved in Proposition 5, that $|T_t(f)| \le T_t(|f|)$ for all $f \in L^p_{\mathbf{C}}(X, \mathrm{d}x)$ and $t \ge 0$. If $f \in \mathcal{M}$ and $t \ge 0$ then

$$\|\,|f|\,\| \ge \|T_t(|f|)\| \ge \|\,|T_t(f)|\,\| = \|T_t f\| = \|f\| = \|\,|f|\,\|.$$

It follows that $\|T_t(|f|)\| = \|\,|T_t(f)|\,\|$, and combining this with $|T_t(f)| \le T_t(|f|)$ and $1 \le p < \infty$, we deduce that $|T_t(f)| = T_t(|f|)$. This finally implies that

$$\|T_{t(n)}(|f|) - |f|\,\| = \|\,|T_{t(n)}(f)| - |f|\,\| \le \|T_{t(n)}(f) - f\| \to 0$$

as $n \to \infty$. Therefore $|f| \in \mathcal{M}$.

This establishes that $\mathcal{N}$ is a closed linear sublattice of $L^p_{\mathbf{R}}(X, \mathrm{d}x)$. The identity $T_t(|f|) = |T_t(f)|$ for all $f \in \mathcal{M}$ and $t \ge 0$ implies the same equality for all $t \in \mathbf{R}$. Hence T_t restricted to $\mathcal{N}$ is a lattice isomorphism for all $t \in \mathbf{R}$ by (2) and (3).

Lemma 7 *There exists a Borel set E in X and $e \in \mathcal{N}_+$ such that* $\mathrm{supp}(e) = E$ *and* $\mathrm{supp}(g) \subseteq E$ *for all $g \in \mathcal{N}$. In particular* $\mathrm{supp}(f_\theta) \subseteq E$ *for all $\theta \in P$. The set E is invariant in the sense that if $g \in \mathcal{B}$ and* $\mathrm{supp}(g) \subseteq E$ *then* $\mathrm{supp}(T_t(g)) \subseteq E$ *for all $t \ge 0$.*

Proof Let e_n be a countable dense subset of $\mathcal{N}$ and let E be the support of

$$e := \sum_{n=1}^{\infty} 2^{-n}|e_n|/\|e_n\|.$$

If $g \in \mathcal{N}$ then by approximating g by terms of the sequence e_n we see that $\mathrm{supp}(g) \subseteq E$. In particular $\mathrm{supp}(T_t e) \subseteq E$ for all $t \geq 0$. This implies that E is an invariant set.

Our next proposition, which is an l^p version of the Stone-Weierstrass theorem, depends on making some further definitions. Let $\mathcal{W}$ be any closed linear sublattice of $l^p(X, w)$ where $1 \leq p < \infty$ and $w(x)$ is a positive weight for every x in the countable set X and $\sum_{x \in X} w(x) < \infty$. Suppose that $1 \in \mathcal{W}$. We define an equivalence relation on X by putting $x \sim y$ if $f(x) = f(y)$ for all $f \in \mathcal{W}$ and let $\tilde{X}$ denote the set of all equivalence classes.

Proposition 8 *If $E \in \tilde{X}$ then its characteristic function χ_E lies in $\mathcal{W}$. Moreover*

$$\mathcal{W} = \overline{\mathrm{lin}}\{\chi_E : E \in \tilde{X}\}.$$

Proof If $f \in \mathcal{W}$ then f is constant on any class u, and we write $f(u)$ to denote this constant value.

Let v be some chosen class. If u is a distinct class then there exists $g_u \in \mathcal{W}$ such that $g_u(u) > g_u(v)$. On putting $h_u = (g_u - g_u(v)1) \vee 0$ we see that $h_u \in \mathcal{W}_+$, $h_u(u) > 0$ and $h_u(v) = 0$. If $n(\cdot)$ is some enumeration of $\tilde{X}$ then

$$k = \sum_{\{u : u \neq v\}} 2^{-n(u)} h_u / \|h_u\|_p$$

lies in $\mathcal{W}_+$ and satisfies $k(v) = 0$ and $k(u) > 0$ for all $u \neq v$. If $p_n = 1 \wedge (nk)$ then p_n increases monotonically with n and converges in l^p norm to $1 - \chi_v$ by the dominated convergence theorem. Hence $\chi_v \in \mathcal{W}$.

The final statement of the proposition follows from the fact that if $f \in \mathcal{W}_+$ then

$$f = \sum_{u \in \tilde{X}} f(u)\chi_u$$

where this sum converges monotonically and in l^p norm by virtue of the dominated convergence theorem.

Theorem 9 *If T_t is a positive one-parameter contraction semigroup acting on $l^p(X)$ where X is a countable set, then $T_t|_{\mathcal{N}} = I$. In particular the peripheral point spectrum P cannot contain any non-zero elements.*

Proof We henceforth assume that the set E of Lemma 7 equals X; equivalently we restrict attention to the invariant subspace $l^p(E)$. We define the positive weight w on X by $w(x) = e(x)^p$ where e is the function defined in Lemma 7. We then define the isometry $U : l^p(X, w) \to l^p(X)$ by $Ug = eg$ and transfer T_t to $l^p(X, w)$ by putting $\tilde{T}_t = U^{-1}T_tU$. On suppressing the tilde, this enables us to assume that $1 \in \mathcal{N}$ and that X has finite measure.

We next apply Proposition 8. We say that $h \in \mathcal{N}_+$ lies on an extreme ray of $\mathcal{N}_+$ if $0 \leq g \leq h$ and $g \in \mathcal{N}_+$ implies that $g = \lambda h$ for some λ such that $0 \leq \lambda \leq 1$. It is immediate that h lies on an extreme ray if and only if $h = \lambda\chi_E$ where $\lambda \geq 0$ and $E \in \tilde{X}$. Since T_t are lattice automorphisms they permute the extreme rays of $\mathcal{N}_+$ and hence induce permutations $\pi(t)$ of $\tilde{X}$.

Given an equivalence class E, we see that $T_t(\chi_E)$ depends continuously on t, but it is also a multiple of $\chi_{\pi(t)E}$. Therefore $T_t(\chi_E) = \chi_E$ for all $t \in \mathbf{R}$ and $T_t = I$ on $\mathcal{N}$. This implies that $T_t = I$ on $\mathcal{M}$ and hence that $\theta = 0$ for all $\theta \in P$.

3 Feller Semigroups

In this section we suppose that X is a locally compact Hausdorff space with a countable basis to its topology, and that $\mathrm{d}x$ is a regular Borel measure on X with support equal to X. We suppose that $T_t = \mathrm{e}^{Zt}$ is a positive one-parameter semigroup on $\mathcal{B} = L^p(X, \mathrm{d}x)$, where $1 \leq p < \infty$. We finally suppose that T_t has the Feller property: that is $T_t f \in C(X)$ for all $f \in \mathcal{B}$ and $t > 0$, where $C(X)$ is the space of all continuous functions on X. Although elements of $L^p(X, \mathrm{d}x)$ are only defined up to null sets, our support hypothesis implies that if an element of $L^p(X, \mathrm{d}x)$ can be represented by a continuous function, then that continuous representative is unique. If X is discrete and $\mathrm{d}x$ is the counting measure then the Feller property is automatic, so the theorems of this section contain the earlier results as special cases.

It follows from our assumptions that any eigenfunction of Z lies in $C(X)$. Theorem 4 states that $T_t(\mathcal{M}) = \mathcal{M}$ for all $t > 0$, so $\mathcal{M}$ is both a closed subspace of $L^p(X, \mathrm{d}x)$ and also a subspace of $C(X)$. The closed graph theorem implies that within $\mathcal{M}$ convergence in L^p norm implies locally uniform convergence. If $f \in C(X)$ we put

$$\mathrm{supp}(f) = \{x : f(x) \neq 0\},$$

this being an open set.

Lemma 10 *The real part $\mathcal{N}$ of $\mathcal{M}$ is a linear sublattice of $C(X)$. There exists an invariant open set $U \subseteq X$ and $e \in \mathcal{N}_+$ such that* $\mathrm{supp}(e) = U$ *and* $\mathrm{supp}(g) \subseteq U$ *for all $g \in \mathcal{N}$.*

Proof This is a result of combining the above observations with Theorem 6 and Lemma 7.

The set U is empty if and only if $\mathcal{M} = 0$, in which case the peripheral point spectrum is empty; we assume that this is not the case. From this point onwards we restrict attention to the action of T_t on the invariant subspace $L^p(U, \mathrm{d}x)$, which contains all of the eigenfunctions associated with the peripheral point spectrum P. This is most conveniently done by putting $X = U$.

We define the isometry $V : L^p(X, e(x)^p \mathrm{d}x) \to L^p(X, \mathrm{d}x))$ by $Vg = eg$ where e is the positive continuous function of Lemma 10, and transfer T_t to $L^p(X, e(x)^p \mathrm{d}x)$ by putting $\tilde{T}_t = V^{-1} T_t V$. On suppressing the tilde, this enables us to assume that $1 \in \mathcal{N}$ and that X has finite measure.

As before we define an equivalence relation on X by putting $x \sim y$ if $f(x) = f(y)$ for all $f \in \mathcal{N}$.

Lemma 11 *Every equivalence class E is both open and closed and has positive measure. Moreover $\chi_E \in \mathcal{N}$.*

Proof We modify the proof of Proposition 8. If $a \in E$ then

$$E = \bigcap_{f \in \mathcal{N}} \{x : f(x) = f(a)\}$$

so E is a closed set. If $x \notin E$ then there exists $g_x \in \mathcal{N}$ such that $g_x(x) > g_x(a)$. Putting $h_x = (g_x - g_x(a)1) \vee 0 \in \mathcal{N}_+$ and $U_x = \{y : h_x(y) > 0\}$ we see that

$$\bigcup_{x \notin E} U_x = X \backslash E.$$

The Lindelöf property states that there exists a sequence $x(n) \notin E$ such that

$$\bigcup_{n=1}^{\infty} U_{x(n)} = X \backslash E.$$

It follows that if

$$k = \left\{ \sum_{n=1}^{\infty} 2^{-n} h_{x(n)} / \|h_{x(n)}\|_p \in \mathcal{N}_+ \right\} \wedge 1 \tag{4}$$

then $k(x) = 0$ for all $x \in E$ and $0 < k(x) \le 1$ for all $x \notin E$. Finally $p_n = 1 \wedge (nk)$ increases monotonically as n increases and

$$\lim_{n=1} p_n = 1 - \chi_E. \tag{5}$$

Therefore $\chi_E \in \mathcal{N}$.

The limits in (4) and (5) were taken in L^p norm. However, they involve monotone sequences and converge pointwise (everywhere, not almost everywhere) to the stated limits. In particular $k = \lim_{m\to\infty} k_m$ where

$$k_m = \left\{ \sum_{n=1}^{m} 2^{-n} h_{x(n)}/\|h_{x(n)}\|_p \right\} \wedge 1 \in \mathcal{N}_+$$

increase monotonically and are bounded above by 1. Since all of the functions lie in $\mathcal{N} \subseteq C(X)$, L^p norm convergence implies locally uniform convergence. We deduce that χ_E is continuous and hence that E is open. Since E is non-empty its measure must be positive.

Theorem 12 *If T_t is a one-parameter positive semigroup with the Feller property acting on $L^p(X, \mathrm{d}x)$ for some $1 \leq p < \infty$ then $T_t|_{\mathcal{N}} = 1$. Hence the peripheral point spectrum P cannot contain any non-zero points.*

Proof Without loss of generality we may restrict to the open set U and apply the isometry V as described before Lemma 11. The remainder of the proof follows the argument of Theorem 9.

4 Irreducibility

Let T_t be a positive one-parameter contraction semigroup acting on $\mathcal{B} = L^p_{\mathbf{R}}(X, \mathrm{d}x)$ where $1 \leq p < \infty$ and $\mathrm{d}x$ is a countably additive σ-finite measure on X.

Theorem 13 *Suppose that $f \in \mathcal{B}_+$ has support S and $T_t f = f$ for all $t > 0$. If $E \subseteq X$ is an invariant Borel set for the semigroup T_t then so is $S\backslash E$.*

Proof Since S is an invariant set we may restrict attention to the action of T_t in the subspace $L^p(S, \mathrm{d}x)$. Putting $g_t = T_t(\chi_E f)$ we observe that $0 \leq g_t \leq f$ and $\mathrm{supp}(g_t) \subseteq E$. Hence $0 \leq g_t \leq g_0$ for all $t \geq 0$. Putting $h_t = T_t(\chi_{S\backslash E} f)$ we deduce that

$$h_t = T_t(f - g_0) \geq f - g_0 = h_0.$$

Sine T_t are contractions and the norm is strictly monotone we deduce that $h_t = h_0$. This implies that $\mathrm{supp}(h_0)$, i.e. $S\backslash E$, is invariant.

Theorem 14 *Let X be a locally compact Hausdorff space with a countable basis to its topology, and let $\mathrm{d}x$ be a regular Borel measure on X with support equal to X. Let T_t be a positive one-parameter semigroup with the Feller property acting on $\mathcal{B} = L^p(X, \mathrm{d}x)$, where $1 \leq p < \infty$. Let f be a positive continuous function in $\mathcal{B}$ satisfying $T_t f = f$ for all $t > 0$. If X is connected then T_t is irreducible.*

Proof If E is an invariant Borel set in X and $g = \chi_E f$ then the proof of Theorem 13 establishes that $T_t g = g$ for all $t > 0$. If we let g be the unique continuous function in the class of the 'same' element of L^p then $g^2 = fg$ almost everywhere implies $g^2 = fg$ everywhere. The continuous function $h = g/f$ satisfies $h^2 = h$ so the support F of h is both open and closed. Since X is connected either $F = X$ or $F = \emptyset$. We finally observe that $E = F$ up to alteration on a null set.

References

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Department of Mathematics
King's College
Strand
London
WC2R 2LS
UK

E.Brian.Davies@kcl.ac.uk