Corrections to
‘Spectral Theory and Differential Operators’

E.B. Davies

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I am collecting errors and misprints for this web-page. Please send any you find to me at E.Brian.Davies@kcl.ac.uk

p7,-10 In general the domain of $A^*$ might not be dense, so it may not be an operator as defined on page 2. This possibility is dealt with in Lemma 1.2.1.

p7,-4 Delete the word ‘and’.

p7, In Lemma 1.2.1 I should also have stated that if $A$ is closable then the adjoint of $A$ is equal to the adjoint of its closure $\overline{A}$. This follows immediately by inspecting the proof of the lemma.

p15 The end of the proof of Theorem 1.2.10 has a gap, which may be filled as follows.

For any $\phi, \psi \in \mathcal{H}$ the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) := \langle (z - H)^{-1}\phi, \psi \rangle$$

is entire with

$$|f(z)| \leq \frac{c}{|\text{Im} (z)|}$$  \hspace{1cm} (1)

for all $z \notin \mathbb{R}$. We now need to prove that this implies that $f$ vanishes identically. This being so we derive the contradiction that $(z - H)^{-1} = 0$ for all $z \in \mathbb{C}$.

**Theorem 1** Let $f$ be an entire function satisfying (1). Then $f$ is identically zero.

**Proof** Let $a \in \mathbb{R}$ and let $n$ be a large positive integer. Define

$$g(z) = f(z) \frac{n^2 - (z - a)^2}{n^2}$$

and let $\gamma$ be the circle with centre $a$ and radius $n$. The second factor on the RHS of the above equation is bounded by 2 on and inside $\gamma$ and vanishes at the
points $z = a \pm n$, which lie on $\gamma$. An application of Cauchy’s integral formula for derivatives yields

$$|f'(a)| = |g'(a)| \leq \frac{1}{n} \max\{|g(z)| : z \in \gamma\}.$$ 

Although the function $g$ depends upon $n$ and $a$, it is bounded on $\gamma$ with a bound $b$ that is independent of these parameters. From the bound

$$|f'(a)| \leq b/n$$

valid for all $a$ and $n$, we deduce that $f'(a) = 0$ for all $a \in \mathbb{R}$. This implies that $f^{(m)}(a) = 0$ for all $a \in \mathbb{R}$ and $m \geq 1$, then that $f$ is constant and finally that $f = 0$.

Another proof can be based upon the following facts. If $H$ has empty spectrum then $H^{-1}$ is a bounded self-adjoint operator with spectrum equal to $\{0\}$. By applying results about such operators we obtain the contradiction $H^{-1} = 0$.

I could have pointed out that the operator series $\sum_{n=0}^{\infty} C^n$ is norm convergent if $\|C\| < 1$. Since the sum of the series is invertible with inverse $I - C$, its range is equal to $\mathcal{H}$.

Exercise 1.13 should refer to the notation of Exercise 1.3, not 1.9.

$\chi_m$ should be the characteristic function of $\{x : m < |x| < 2m\}$

It is not entirely obvious that $v \in L$ for the cyclic subspace $L$ this follows from

$$v = \lim_{n \to +\infty} in(I - H)^{-1}v.$$ 

$L^2(S \times \mathbb{N}, d\mu)$

$h \cdot U(\xi) \in L^2$

$= wr_w(H)U^{-1}(\xi) - U^{-1}(\xi)$

$\mathbb{N}$ not $N$.

This is correct, but what is actually proved is the equally valid equality in which $A$ and $B$ are interchanged. Both yield the estimate of the lemma equally easily.

$C^\infty$ not $C^\infty$.

How the publisher managed to convert comments on Lemma 3.2.3 into a new “lemma”, given that they were sent a latex document, will always remain a mystery.

Lemma 3.4.2

In the definition of uniform ellipticity, I should also have required that $|a_\alpha(x)| \leq c$ for some $c$, all $\alpha$ and all $x \in \mathbb{R}^N$.

Theorems 2.5.3 and 3.5.3

$(|y|^2 + 1)^{-\alpha}$
p66,11 $(|y|^2 + \lambda)^{-\alpha}$
p66,-6 $(|y|^2 + 1)^{-\alpha}$
p86,-8 Replace sentence by: It follows that $A$ is one-one and we may regard $\mathcal{H}_1$ as a subspace of $\mathcal{H}$. This establishes that $Q$ is closable, and we may then apply Theorem 4.4.2 to its closure.
p91,7 the norm of (4.4.2)
p101,13/14 Exercises 5.2 and 5.3
p103,-13 Replace $c_1$ by $(2\pi)^{N/2}$.
p104,12 an 14 $|x-y|^2$ not $(x-y)^2$
p104,-4 Exercise 5.6
p107,8 state explicitly that $s \in \mathbb{R}$
p112,10 constants missing on the RHS here and below
p113,-5 This should be
\[
\int_{\Omega} |\Delta f|^2 d^Nx \geq \int_{\Omega} \{-\overline{f}\Delta f - f\Delta \overline{f} - |f|^2\} d^Nx
\]
\[
= \int_{\Omega} \{2|\nabla f|^2 - |f|^2\} d^Nx
\]
p114,-13 $C$ should not be boldface.
p115, diagram $\phi(x) = \cos(x)$
p119,-15 form of (6.1.1)
p125,3 The proof of this uses the bound
\[
|p(x)| \leq c \text{diam}(K_s)
\]
for all $x \in \mathbb{R}^N$, which may be proved from page 124, line-2 by taking any $y$ to be any point in the boundary of $K_s$.
p125,126 In five places here equations contain $N + ...$ where they could contain $1 + ...$. The equations are, therefore, correct as they stand, but there is no good reason for the change.
p127,-11 Delete repeated ‘that’.
p127,-5 Corollary 4.2.3

p 128 Finish the proof of Corollary 6.2.2 with:
exists and is positive. The corollary follows by combining the inequality
\[
M(\lambda_n - 1) < n \leq M(\lambda_n)
\]
with the asymptotic expression for $M(\lambda)$.
p133,12 Script capital H not italic.
P133, proof of Theorem 6.2.6 Replace \( \hat{\psi} \) by \( \psi \) and \( \psi \) by \( \hat{\psi} \) everywhere in this proof.

p143, proof of Theorem 7.1.4 It may be helpful to observe that differential operators of the form (7.1.2) can be characterized in a coordinate-free manner as those operators of the form \( b_0I + M \) where \( b_0 \in C^\infty(\overline{\Omega}) \) and \( M : C^\infty(\overline{\Omega}) \rightarrow C^\infty(\overline{\Omega}) \) satisfies the derivation property

\[
M(fg) = M(f)g + fM(g).
\]

p137,-7 is equal to

p138, Exercise 6.3 \( \partial \phi / \partial x_r \)

p153, Exercise 7.1 This requires \( N \geq 2 \)

p176,8 the two eigenvalues

Professor E B Davies
Department of Mathematics
King’s College
Strand
London WC2R 2LS
England