Newman's proof of Wiener's Theorem

by E B Davies

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Since Wiener's original proof of his theorem about periodic functions whose Fourier series are absolutely summable, a variety of quite different proofs have been devised. Some of these are described in [2, Sect. B.9.4]. In this paper we rewrite the beautiful proof of Newman [1] in a slightly more general form, so that it may be readily applied to a variety of related problems. Although we use the language of commutative Banach algebras, we do not use Gelfand's representation theory, but construct the inverse directly using completely elementary methods.

Theorem 1 *Let X be a compact Hausdorff space and let $\mathcal{A}$ be a subalgebra of $C(X)$ that contains the constants. Suppose that $\mathcal{A}$ is a Banach algebra with respect to a norm $\|\cdot\|$, and that $\mathcal{D}$ is a dense subset of $\mathcal{A}$. Let k, c be positive constants and suppose that for every $g \in \mathcal{D}$ satisfying $|g(x)| \geq \sigma$ for some $\sigma > 0$ and all $x \in X$, g is invertible in $\mathcal{A}$ and*

$$\|g^{-n}\| \leq c_g n^k c^n \sigma^{-n} \tag{1}$$

for all positive integers n. Then every $f \in \mathcal{A}$ which is invertible in $C(X)$ is also invertible in $\mathcal{A}$ and the norm of its inverse is effectively computable.

Proof If $f \in \mathcal{A}$ and $|z| > \|f\|$ then $(z - f)$ is invertible in $\mathcal{A}$ and therefore also invertible in $C(X)$. This implies that $\|f\|_\infty \leq \|f\|$ for all $f \in \mathcal{A}$.

If $f \in \mathcal{A}$ and $|f(x)| \geq \sigma > 0$ for all $x \in X$, let $g \in \mathcal{D}$ satisfy $\|g - f\| < \delta\sigma$; we put $\delta = \{2(1 + c)\}^{-1}$. This implies that $|g(x)| \geq (1 - \delta)\sigma > 0$ for all $x \in X$. Therefore g is invertible in $\mathcal{A}$ and

$$\|g^{-n}\| \leq c_g n^k c^n (1 - \delta)^{-n} \sigma^{-n}$$

for all positive integers n. The inverse of f in $\mathcal{A}$ is given by the formula

$$f^{-1} = \sum_{n=0}^{\infty} (g - f)^n g^{-n-1}.$$

This is norm convergent in $\mathcal{A}$, and hence also uniformly convergent in $C(X)$, with

$$\begin{aligned}
\|f^{-1}\| &\leq \sum_{n=0}^{\infty}(\delta\sigma)^n c_g(n+1)^k c^{n+1}(1-\delta)^{-n-1}\sigma^{-n-1} \\
&= \frac{c_g}{\delta\sigma}\sum_{n=0}^{\infty}(n+1)^k\left(\frac{\delta c}{1-\delta}\right)^{n+1} \\
&= \frac{c_g}{\delta\sigma}\sum_{n=0}^{\infty}(n+1)^k\left(\frac{c}{1+2c}\right)^{n+1} \\
&\leq \frac{c_g}{\delta\sigma}\sum_{n=0}^{\infty}(n+1)^k 2^{-n-1}.
\end{aligned}$$

The norm of $1/f$ is effectively computable in the following sense. If there is a constructive proof that $\mathcal{D}$ is dense in $\mathcal{A}$ then we can approximate a given function f by an element g of $\mathcal{D}$ explicitly and this should provide a bound for c_g. In all of the applications below it is evident that these comments are justified; we discuss the approximation procedure in some detail in Example 4.

Example 2 Let $\mathcal{A}$ be the algebra of all continuous periodic functions $f(x) = \sum_{n\in\mathbf{Z}} a_n \mathrm{e}^{-inx}$ on $X = [-\pi,\pi]$ for which the Wiener norm

$$\|f\| = \sum_{n\in\mathbf{Z}} |a_n|$$

is finite. We have to verify (1) under the stated hypotheses on g, where we take $\mathcal{D}$ to be the set of all C^∞ periodic functions on $[-\pi,\pi]$. Various other choices of $\mathcal{D}$ are possible.

We have

$$\begin{aligned}
\|g^{-n}\| &\leq \|g^{-n}\|_\infty + 2\|(g^{-n})'\|_\infty \\
&= \|g^{-n}\|_\infty + 2n\|g'g^{-n-1}\|_\infty \\
&\leq \sigma^{-n} + 2n\|g'\|_\infty\sigma^{-n-1}
\end{aligned}$$

which is of the stated form.

Example 3 A similar proof holds if $\mathcal{A}$ is the algebra of all continuous periodic functions on $[-\pi,\pi]^N$ for which the Wiener norm

$$\|f\| = \sum_{n\in\mathbf{Z}^N} |a_n|$$

is finite. We take $\mathcal{D}$ to be the set of all C^∞ periodic functions on $[-\pi,\pi]^N$ and put $M=[N/2]+1$. If $n\in\mathbf{Z}^N$ and α is a multi-index we adopt standard conventions such as

$$n^\alpha = n_1^{\alpha_1}...n_N^{\alpha_N}, \qquad |\alpha| = \alpha_1 + ...\alpha_N.$$

Let $\{a_n\}_{n\in\mathbf{Z}^N}$ be the sequence of Fourier coefficents of $g\in\mathcal{D}$. An application of the Schwarz inequality and standard results about Fourier series yield

$$\begin{aligned}\|a\|_1^2 &\leq c_1 \sum_{n\in\mathbf{Z}^N} (1+n^2)^M |a_n|^2 \\ &\leq c_2 \sum_{|\alpha|\leq M} \sum_{n\in\mathbf{Z}^N} |n^\alpha a_n|^2 \\ &\leq c_3 \sum_{|\alpha|\leq M} \|D^\alpha g\|_2^2 \\ &\leq c_4 \sum_{|\alpha|\leq M} \|D^\alpha g\|_\infty^2.\end{aligned}$$

Therefore

$$\|g\| \leq c_5 \sum_{|\alpha|\leq M} \|D^\alpha g\|_\infty$$

where the LHS is the Wiener norm of g. We finally use

$$D_i(g^{-n}) = -n(D_i g)g^{-n-1},$$

where $D_i=\partial/\partial x_i$, repeatedly to estimate $\|D^\alpha(g^{-n})\|_\infty$. The constant c_g obtained is a polynomial in the L^∞ norms of a finite number of derivatives of g.

Example 4 Let $\overline{\mathbf{R}}$ be the one-point compactification of $\mathbf{R}$ and let $\mathcal{A}$ be the algebra of all continuous functions on $\overline{\mathbf{R}}$ of the form $f=\alpha+f_0$, where $\alpha\in\mathbf{C}$,

$$f_0(x) = \int_{\mathbf{R}} \tilde{f}(\xi)\mathrm{e}^{-ix\xi}\,\mathrm{d}\xi,$$

and $\tilde{f}\in L^1(\mathbf{R})$. We make $\mathcal{A}$ into a Banach algebra with identity under pointwise multiplication by putting

$$\|f\| = |\alpha| + \|\tilde{f}\|_1.$$

This is consistent with the multiplication within $\mathbf{C}\oplus L^1(\mathbf{R})$ defined by

$$(\alpha,\tilde{f}).(\beta,\tilde{g}) = (\alpha\beta, \alpha\tilde{g}+\beta\tilde{f}+\tilde{f}*\tilde{g})$$

where $*$ denotes convolution. Note that $\mathcal{A}\subseteq C(\overline{\mathbf{R}})$, that $\|f\|_\infty\leq\|f\|$ for all $f\in\mathcal{A}$, and that $\alpha=f(\infty)$.

In order to apply Theorem 1 to this example we have to chose a dense subset $\mathcal{D}$ of $\mathcal{A}$. Out of many equally good possibilities we choose $\mathcal{D} := \mathbf{C} + W^{1,2}(\mathbf{R})$. If $f \in \mathcal{D}$ then

$$\begin{aligned} \|\tilde{f}\|_1 &\leq \left\{ \int_{\mathbf{R}} (1+|\xi|^2)^{-1} \,\mathrm{d}\xi \right\}^{1/2} \left\{ \int_{\mathbf{R}} (1+|\xi|^2) |f(\xi)|^2 \,\mathrm{d}\xi \right\}^{1/2} \\ &\leq \pi^{1/2} \left\{ \|\tilde{f}\|_2 + \|\xi\tilde{f}\|_2 \right\} \\ &\leq \|f_0\|_2 + \|f_0'\|_2. \end{aligned}$$

Therefore $f \in \mathcal{A}$ and

$$\|f\| \leq |\alpha| + \|f_0\|_2 + \|f_0'\|_2.$$

The density of $\mathcal{D}$ in $\mathcal{A}$ is proved as follows. Given $f \in \mathcal{A}$ and any positive integer n we may put $f_n = \alpha + (f_n)_0$ where

$$\tilde{f}_n(\xi) := \begin{cases} \tilde{f}(\xi) & \text{if } |\tilde{f}(\xi)| \leq n \text{ and } |\xi| \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We see immediately that $\|f_n - f\| \to 0$ in $\mathcal{A}$ and that $f_n \in W^{1,2}(\mathbf{R})$.

We have to prove the bound (1) of Theorem 1 on the assumption that $g \in \mathcal{D}$ satisfies $|g(x)| \geq \sigma > 0$ for all $x \in \mathbf{R}$. We first observe that $h = 1/g$ satisfies $h = \beta + h_0$ where $\beta = 1/\alpha$ and

$$h_0 = \frac{1}{\alpha + g_0} - \frac{1}{\alpha} = \frac{-g_0}{\alpha(\alpha + g_0)}.$$

The proof that $h \in \mathcal{D}$ is elementary and is a special case of the calculations below.

For every positive integer r we have

$$h_0^r = \frac{-g_0}{\alpha(\alpha + g_0)} \left\{ \frac{1}{\alpha + g_0} - \frac{1}{\alpha} \right\}^{r-1}.$$

This implies

$$|h_0^r| \leq |g_0| \sigma^{-2} \{2\sigma^{-1}\}^{r-1}$$

and hence

$$\|h_0^r\|_2 \leq \|g_0\|_2 \, 2^{r-1} \sigma^{-r-1}.$$

We also have

$$\begin{aligned} (h_0^r)' &= r h_0' h_0^{r-1} \\ &= \frac{-r g_0'}{(\alpha + g_0)^2} \left\{ \frac{1}{\alpha + g_0} - \frac{1}{\alpha} \right\}^{r-1}, \end{aligned}$$

from which we deduce that

$$\|(h_0^r)'\|_2 \leq r\|g_0'\|_2\, 2^{r-1}\sigma^{-r-1}.$$

Therefore

$$\begin{aligned}\|h_0^r\| &\leq \frac{\|g_0\|_2 + \|g_0'\|_2}{2\sigma} r 2^r \sigma^{-r} \\ &= c_g r 2^r \sigma^{-r}.\end{aligned}$$

It follows that for every positive integer n we have

$$\begin{aligned}\|h^n\| &= |\beta^n| + \left\| \sum_{r=1}^{n} \binom{n}{r} \beta^{n-r} h_0^r \right\| \\ &\leq \sigma^{-n} + \sum_{r=1}^{n} \binom{n}{r} \sigma^{r-n} \|h_0^r\| \\ &\leq \sigma^{-n} + \sum_{r=1}^{n} \binom{n}{r} \sigma^{r-n} c_g r 2^r \sigma^{-r} \\ &\leq \sigma^{-n} + n c_g \sigma^{-n} \sum_{r=1}^{n} \binom{n}{r} 2^r \\ &\leq (1+c_g) n 3^n \sigma^{-n}.\end{aligned}$$

This is of the form (1), as needed for the proof of Theorem 1.

We finally discuss the sense in which we have obtained an explicit estimate for $\|f^{-1}\|$. We assume that $f \in \mathcal{A}$ and $\sigma > 0$ are given. We assume that this means that $\tilde{f}$ is known, to the extent that effective estimates for $\|\tilde{f}\|_1$ and for $\|\tilde{g} - \tilde{f}\|_1$ are available for every n, where $\tilde{g} = \tilde{f}_n$ is defined by (2). Having chosen n, we have to provide a bound for c_g, i.e. for $\|g_0\|_2$ and $\|g_0'\|_2$. These are

$$\|g_0\|_2^2 = 2\pi\|\tilde{g}\|_2^2 \leq 2\pi n\|\tilde{g}\|_1 \leq 2\pi n\|\tilde{f}\|_1$$

and

$$\|g_0'\|_2^2 = 2\pi\|\xi\tilde{g}\|_2^2 \leq 2\pi n^3\|\tilde{g}\|_1 \leq 2\pi n^3\|\tilde{f}\|_1.$$

Example 5 Let $\mathcal{A}$ be the algebra of analytic functions f on $D := \{z : |z| < 1\}$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where

$$\|f\| := \sum_{n=0}^{\infty} |a_n| < \infty.$$

Every $f \in \mathcal{A}$ has a unique continuous extension to $\overline{D}$.

If $f \in \mathcal{A}$ and $|f(z)| \geq \sigma > 0$ for all $z \in D$, then $1/f$ is continuous on $\overline{D}$ and analytic on D. Note that by applying the maximum principle to $1/f$ we obtain

$$\min\{|f(z)| : z \in D\} = \min\{|f(z)| : |z| = 1\}$$

provided the LHS is non-zero.

Given $f \in \mathcal{A}$ we put

$$\hat{f}(\theta) := f(\mathrm{e}^{i\theta}) = \sum_{n=0}^{\infty} a_n \mathrm{e}^{in\theta}.$$

Wiener's theorem implies that if $|f(z)| \geq \sigma > 0$ for all $z \in D$ then

$$1/\hat{f}(\theta) = \sum_{n=-\infty}^{\infty} b_n \mathrm{e}^{in\theta}$$

where $\sum_{n=-\infty}^{\infty} |b_n| < \infty$. If $n > 0$ then

$$\begin{aligned} b_{-n} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{in\theta}}{\hat{f}(\theta)} \, \mathrm{d}\theta \\ &= \frac{1}{2\pi i} \int_{|z|=1} \frac{z^{n-1}}{f(z)} \, \mathrm{d}z \\ &= 0 \end{aligned}$$

by Cauchy's theorem. It follows that $1/f \in \mathcal{A}$, and that one has the same bound on $\|1/f\|$ as in Example 2.

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References

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Department of Mathematics
King's College
Strand
London WC2R 2LS

E.Brian.Davies@kcl.ac.uk