WHAT IS MATHEMATICS ABOUT?

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This is a slightly expanded version of a popular lecture given at King's College London, based on ideas explained much more fully in my popular science book "Science in the Looking Glass", which is to be brought out in paperback by Oxford University Press in May 2007.

Mathematical Platonism

Plato declared that there existed an ideal realm of mathematical forms, and that philosophers could gain some access to this realm by intense thought, or could remember something about it from the period before they were born.

The realm of forms is supposed to be eternal, outside the limits of space and time.

Theorems are supposed to be true statements about entities that exist in the Platonic realm, irrespective of whether anyone ever knows their truth.

Gödel's Incompleteness Theorems

There exist statements in ordinary arithmetic that cannot be proved or disproved from within arithmetic.

The theorem makes no reference to 'truth' as distinguished from provability, but is often interpreted the following way.

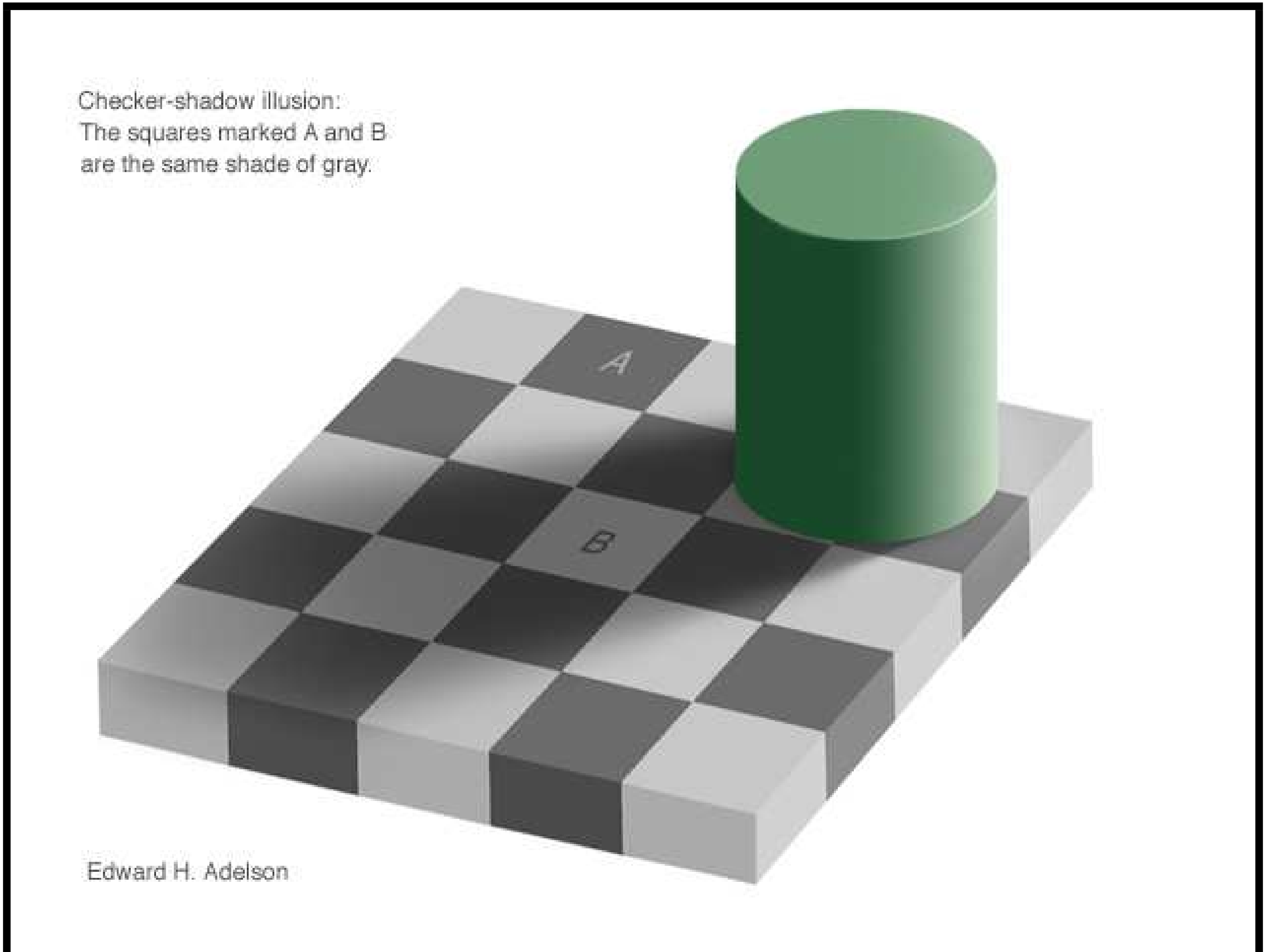
If a statement such as Goldbach's conjecture – every even number greater than 2 may be written as the sum of two primes – cannot be proved or disproved, then there cannot be a counterexample, so surely it must be true?

If you try to disprove Goldbach's conjecture by finding a counterexample, please note that it is true for all even numbers up to 4×10^{14}.

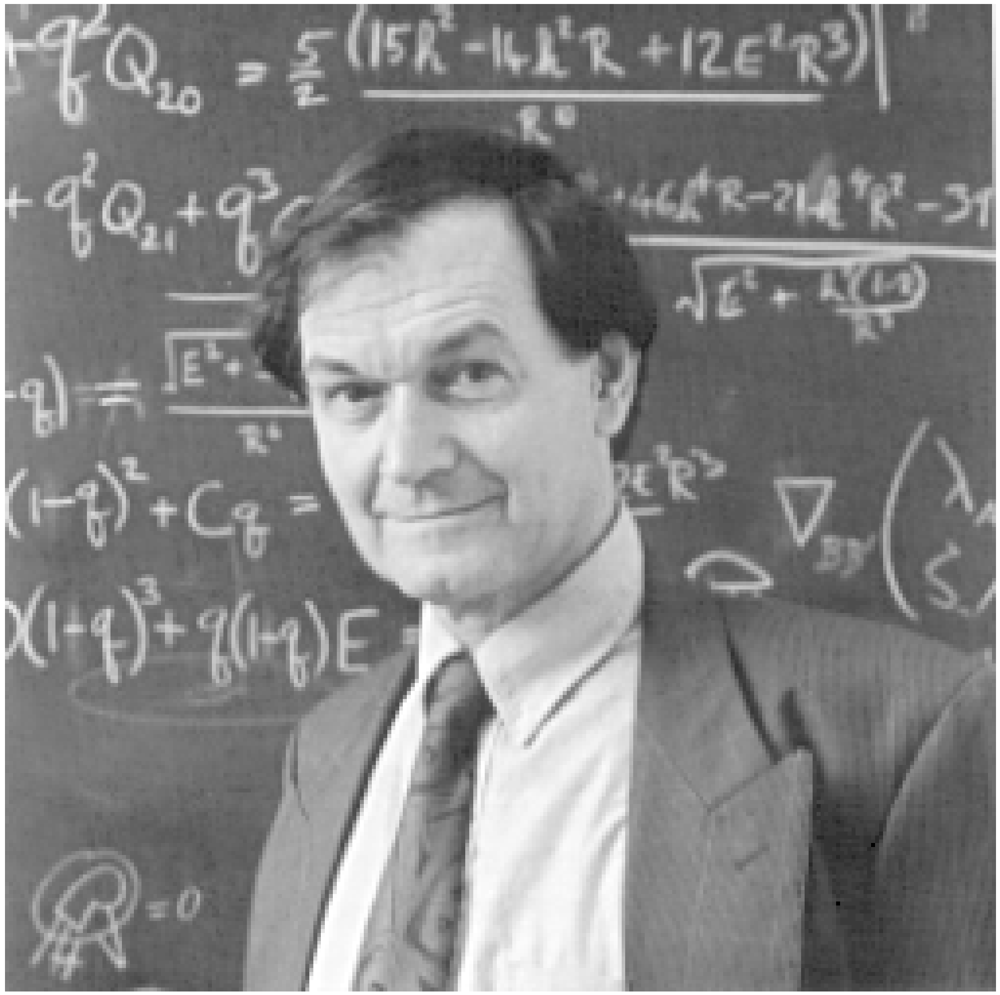
Kurt Gödel

Gödel was a Platonist and wrote:
"But, despite their remoteness from sense experience, we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true.
I don't see any reason why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception."

The following slide, in which the two squares labelled A and B have exactly the same luminosity, is intended to convince you that one should not have confidence in sense perception, and that one's intuitions are a very bad guide to how one's thought processes actually work.


Checker-shadow illusion:
The squares marked A and B
are the same shade of gray.
A
B
Edward H. Adelson

Roger Penrose



Quote from RP:

"When mathematicians communicate, this is made possible by each one having a *direct route to truth*, the consciousness of each being in a position to perceive mathematical truths directly, through this process of 'seeing'...

Since each can make contact with Plato's world directly, they can more readily communicate with each other than one might have expected."

“This is very much in accordance with Plato’s own idea that (say mathematical) discovery is just a form of remembering!

Indeed, I have often been struck by the similarity between just not being able to remember someone’s name, and just not being able to find the right mathematical concept.

In each case, the sought-for concept is, in a sense, *already present* in the mind, though this is a less usual form of words in the case of an undiscovered mathematical idea.”

Alain Connes

"Because this reality cannot be located in space and time, it affords – when one is fortunate enough to uncover the minutest part of it – a sensation of extraordinary pleasure through the feeling of timelessness that it produces. ...

If fact there will always be a property that holds for primordial reality, but which escapes the mode of exploration afforded by axiomatic, logico-deductive methods."

The problem with both the above is that they are simply expression of gut feelings. Either one agrees or one does not. There is no larger theory within which one can place such beliefs: they do not lead anywhere.

What problem does Platonism create?

One of the main problems, explained at length by Mark Balaguer, is that a being embodied in space and time can have no means of accessing an ideal realm.

If the realm did not exist, we could still pursue mathematics in the sense of writing down proofs and examining them for logical errors, so the existence of the realm seems to be unnecessary for the pursuit of mathematics.

What problem does Platonism solve? (1)

For: It helps one to feel that one is engaged in something objective.

Against: The case of chess shows that Platonism is not necessary for this. Chess is a human creation but the rules are precise and one cannot argue about who has won a game.

Reuben Hersh

By way of contrast:

"Mathematics is human. It's part of and fits into human culture. Mathematical knowledge isn't infallible. Like science, mathematics can advance by making mistakes, correcting and recorrecting them.

... Mathematical objects are a distinct variety of social-historic objects. They're a special part of culture. Literature, religion and banking are also special parts of culture."

Each is radically different from the others.

What problem does Platonism solve? (2)

For: Mathematics cannot have controlled the fundamental behaviour of the physical world long before humans evolved unless it is embedded in the objective structure of the world.

Against: Platonism does not help to understand how mathematics and the world might be related.

Mathematical Models

The following photo is meant to draw attention to the enormous gulf between any feasible mathematical model of the Earth and the Earth itself.

The Earth

The Problem of Regularity

We can only understand the world because it has a degree of regularity, both in space and time.

The fact that we describe that regularity mathematically is not the fundamental issue, since we must describe it somehow.

Mathematics was created by a process of selection to match the observed regularities.

Peano Arithmetic

Surely the existence and objective properties of the integers cannot be denied?

The issue here is whether Peano's axioms list some of the properties of an already existing infinite object, or axioms that we find particularly fruitful.

Where would this infinite set of all numbers exist? Even more simply in what sense does $10^{1000000}$ relate to counting? References to things being possible 'in principle' are simply admissions that they are not possible in fact.

A more fruitful question is to ask how the brain enables us to perform tasks in arithmetic, and how the subject developed in a historical context.

The Psychology of Counting

Brian Butterworth is one of the people using brain scanners and other experiments to understand how our sense of number is constructed inside the brain.

We use different parts of the brain to distinguish between numbers up to about 4 and to handle larger numbers.

Understanding number is not a function of general intelligence – it depends on particular structures in the brain functioning ‘normally’.

Dyscalculia

Students showing classical symptoms of dyscalculia may have normal abilities in other areas and appropriate or higher cognitive development.

Dyscalculic learners lack an intuitive grasp of numbers and have problems learning number facts and procedures by the usual methods of teaching.

Adults with identifiable damage in particular areas of the brain, often caused by strokes, may be able to converse intelligently on almost any topic, but be unable to say whether 7 is greater than or less than 9.

The Meaning of Existence

One can adopt a different meaning for the symbol $\exists$ from the usual one. Namely one uses the symbol to mean that there is an algorithm which enables one to construct the quantity whose existence is asserted.

In computational complexity studies one even needs to focus on polynomial algorithms. The following illustrates the issues in a simple example.

The Fibonacci Numbers

Consider the numbers $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...$ which form a sequence $F(n)$ satisfying

$$F(n) = F(n-1) + F(n-2).$$

What is the first digit of $F(n)$ for $n = 10^{1000000}$?

What is the last digit?

Which problem is more important?

The real answer is that neither is important. But the first digit gives one some information about the order of magnitude of the number.

‘In principle’ both questions are elementary. In fact they are not.

The last digit can be determined by using arithmetic mod 10.

As far as I know a method of finding the first digit does not exist.

The formula

$$F(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n$$

does not help. The second term on the right hand side is negligible, but to evaluate the leading digit of the first term one would have to evaluate $\sqrt{5}$ to $10^{1000000}$ decimal places.

Conclusion

Mathematicians are concerned with finding proofs, and very often with providing algorithms for constructing solutions.

Platonism does not contribute to this process, to understanding the historical development of the subject, to explaining how we come to have the mathematical abilities that we do, or finally to explaining why mathematics is useful for modelling some aspects of the world.