ON ARTIN FORMALISM
FOR THE CONJECTURE OF
BLOCH AND KATO

DAVID BURNS

Abstract. We prove that the Tamagawa Number Conjecture of Bloch and Kato satisfies a natural 'Artin formalism' and then describe several unconditional consequences of this result.

1. Introduction

We fix a number field k, an algebraic closure k^c of k and a motive M that is defined over k. For each finite extension E of k in k^c we write M_E for the corresponding motive $h^0(\mathrm{Spec}(E)) \otimes_{h^0(\mathrm{Spec}(k))} M$ defined over E. We recall that if M is equal to the r-fold Tate twist $h^n(X)(r)$ of the motive that arises from the cohomology in degree n of a variety X defined over k, then M_E simply identifies with $h^n(X_{/E})(r)$ where $X_{/E}$ denotes X regarded as defined over E.

In this article we shall first prove (as Theorem 3.1) that as E varies all finite extensions of k in k^c the Tamagawa Number Conjecture of Bloch and Kato [3] for the motive M_E satisfies the same 'Artin formalism' that has played a key role in recent work of Dokchitser and Dokchitser [8, 9, 10] and Bartel [1]. The method that we use to prove this result can also be used in exactly the same way to prove the analogous (but finer) result for motives with coefficients in any given number field but, except for Remark 3.2, we prefer for clarity of exposition to omit any further discussion of motives with (non-trivial) coefficients. We then discuss several unconditional consequences of Theorem 3.1 and also describe a more conceptual approach to the theory of 'regulator constants' that was introduced in [10].

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2. The Bloch-Kato Conjecture

2.1. *Virtual objects* Throughout this article we shall use Deligne's formalism of virtual objects and so we now quickly introduce some of the necessary notation.

We fix an associative unital noetherian ring R. Modules over R are to be understood, unless explicitly stated otherwise, as left modules.

We write $V(R)$ for the Picard category of virtual objects over R that is defined by Deligne in [7]. For any finitely generated projective R-module P we write $[M]_R$ for the associated object of $V(R)$. We write $(X,Y) \mapsto X \cdot Y$ for the product in $V(R)$ and define $\mathbf{1}_R$ to be the unit object $[0]_R$ of $V(R)$. For each object X of $V(R)$ we fix an 'inverse object' X^{-1} and a 'contraction morphism' $\mathrm{ev}_X : X \cdot X^{-1} \to \mathbf{1}_R$ in $V(R)$. We also write $\mathcal{P}_0$ for the Picard category with unique object $\mathbf{1}_{\mathcal{P}_0}$ and the group $\mathrm{Aut}_{\mathcal{P}_0}(\mathbf{1}_{\mathcal{P}_0})$ trivial.

We let $D(R)$ denote the derived category of R-modules and $D^{\mathrm{p}}(R)$ the full triangulated subcategory of $D(R)$ comprising complexes that are quasi-isomorphic to a bounded complex of finitely generated projective R-modules. We often (and without explicit comment) identify an R-module M with the object $M[0]$ of $D(R)$ that is equal to M in degree 0 and is zero in all other degrees.

If R is a discrete valuation ring of characteristic 0 and F is a field that contains R, then for every object C of $D^{\mathrm{p}}(R)$ and every morphism $t : F \otimes_R [C]_R \to \mathbf{1}_F$ in $V(F)$ we write $\chi^{\mathrm{ref}}(C,t)$ for the element of the relative algebraic K-group $K_0(R,F)$ that corresponds to the isomorphism class of the pair $([C]_R,t)$ under the explicit isomorphism of abelian groups $\pi_0(V(R)\times_{V(F)}\mathcal{P}_0)\cong K_0(R,F)$ that is described in [5, Prop. 2.8]. (This element $\chi^{\mathrm{ref}}(C,t)$ is often referred to as the 'refined Euler characteristic' of the pair (C,t).) For the reader's convenience, we have recalled some of the basic properties of these constructions in an Appendix.

2.2. *Review of the Bloch-Kato Conjecture* We continue to use the general notation introduced in §2.1. For any Galois extension of fields F'/F we also set $G_{F'/F} := \mathrm{Gal}(F'/F)$.

We fix a finite Galois extension K of k in k^c and a finite set of places S of k containing the set S_∞ of all archimedean places, all which ramify in K/k and all at which M has bad reduction. For each intermediate field E of K/k we write S_E for the set of places of E above those in S and for each prime p we write S_p for the union of S and all places of k that divide p. We fix a full $G_{k^c/k}$-stable $\mathbb{Z}_p$-lattice T_p in the p-adic realisation of M and for each field E as above we set $T_{p,E} := T_p\otimes_{\mathbb{Z}_p}\prod_{E\to k^c}\mathbb{Z}_p$, endowed with the diagonal left action of $G_{k^c/k}$ and also, if E/k is Galois, with the obvious commuting left action of the group ring $\mathbb{Z}_p[G_{E/k}]$.

For any finite set of places Σ of E that contains $S_{\infty,E}$ we write $\mathcal{O}_{E,\Sigma}$ for the subring of E comprising elements that are integral at all places outside Σ and we also abbreviate $\mathcal{O}_{E,S_\infty}$ to $\mathcal{O}_E$. We now assume that E/k is Galois and recall that the compactly supported étale cohomology $R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S_p},T_{p,E})$ of $T_{p,E}$ on $\mathrm{Spec}(\mathcal{O}_{k,S_p})$ is an object of $D^{\mathrm{p}}(\mathbb{Z}_p[G_{E/k}])$ (by, for example, [12, Th. 5.1]). We further recall that, under certain standard conjectures (see Remark 2.1 below), the approach of [5, §3.4] constructs a canonical object $\Xi(M,E/k)$ of $V(\mathbb{Q}[G_{E/k}])$ together with a canonical morphism in $V(\mathbb{Q}_p[G_{E/k}])$

$$\vartheta_p(M,S,E/k) : \mathbb{Q}_p\otimes_{\mathbb{Q}}\Xi(M,E/k)\to\mathbb{Q}_p\otimes_{\mathbb{Z}_p}[R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S_p},T_{p,E})]_{\mathbb{Z}_p[G_{E/k}]}$$

and a canonical morphism in $V(\mathbb{R}[G_{E/k}])$

$$\vartheta_\infty(M,E/k) : \mathbb{R}\otimes_{\mathbb{Q}}\Xi(M,E/k)\to\mathbf{1}_{\mathbb{R}[G_{E/k}]}.$$

For each prime p we fix an isomorphism $j:\mathbb{C}\cong\mathbb{C}_p$ and then write

$$t_j^{\mathrm{BK}}(M,S,E/k) : \mathbb{C}_p\otimes_{\mathbb{Z}_p}[R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S_p},T_{p,E})]_{\mathbb{Z}_p[G_{E/k}]}\to\mathbf{1}_{\mathbb{C}_p[G_{E/k}]}$$

for the composite morphism $(\mathbb{C}_p\otimes_{\mathbb{R},j}\vartheta_\infty(M,E/k))\circ(\mathbb{C}_p\otimes_{\mathbb{Q}_p}\vartheta_p(M,S,E/k)^{-1})$.

In the case $E=k$ we abbreviate $\Xi(M,E/k)$, $\vartheta_p(M,S,E/k)$, $\vartheta_\infty(M,E/k)$ and $t_j^{\mathrm{BK}}(M,S,E/k)$ to $\Xi(M)$, $\vartheta_p(M,S)$, $\vartheta_\infty(M)$ and $t_j^{\mathrm{BK}}(M,S)$. In particular, for each intermediate field E of K/k we obtain an element

$$\chi_j^{\mathrm{BK}}(M_E) := \chi^{\mathrm{ref}}(R\Gamma_{c,\text{ét}}(\mathcal{O}_{E,S_{p,E}},T_p),t_j^{\mathrm{BK}}(M_E,S_E))\in K_0(\mathbb{Z}_p,\mathbb{C}_p).$$

This element $\chi_j^{\mathrm{BK}}(M_E)$ is independent of the choices of S and T_p (by, for example, [5, Lem. 5]) and can, and will, be regarded as a free rank one $\mathbb{Z}_p$-sublattice of

$\mathbb{C}_p$ via the natural identification $K_0(\mathbb{Z}_p,\mathbb{C}_p)\cong\mathbb{C}_p^\times/\mathbb{Z}_p^\times$. Further, if the 'Coherence hypothesis' of [5, §3.3] is valid for the motive M_E, then there exists an object $\Xi(M_E)_{\mathbb{Z}}$ of $V(\mathbb{Z})$ with $\mathbb{Q}\otimes_{\mathbb{Z}}\Xi(M_E)_{\mathbb{Z}}=\Xi(M_E)$ and such that for all primes p one has

$$\vartheta_p(M_E,S_E)(\mathbb{Z}_p\otimes_{\mathbb{Z}}\Xi(M_E)_{\mathbb{Z}})=[R\Gamma_{c,\text{ét}}(\mathcal{O}_{E,S_{p,E}},T_p)]_{\mathbb{Z}_p} \tag{1}$$

(cf. [5, Lem. 6]). Under this hypothesis we set

$$\chi^{\mathrm{BK}}(M_E):=\vartheta_\infty(M_E)(\Xi(M_E)_{\mathbb{Z}})\subset\mathbf{1}_{\mathbb{R}}.$$

Then, after identifying $\mathbf{1}_{\mathbb{R}}$ with the graded module $(\mathbb{R},0)$ (as in §A.1), the Tamagawa number conjecture of Bloch and Kato [3], as reformulated and extended by Fontaine and Perrin-Riou in [13], is equivalent to the following equality of sublattices of $\mathbb{R}$

$$L^*(M_E,0)\cdot\mathbb{Z}=\chi^{\mathrm{BK}}(M_E), \tag{2}$$

where $L^*(M_E,0)$ is the leading term at $z=0$ of the L-function of the motive M_E defined over E.

Remark 2.1. To define the object $\Xi(M,E/k)$ and morphism $t_j^{\mathrm{BK}}(M,S,E/k)$ used above one must assume both the validity of the equivariant Deligne-Beilinson Conjecture [5, Conj. 1] and the existence of equivariant Chern class maps [5, Conj. 2] for the motive $h^0(\mathrm{Spec}(E))\otimes_{h^0(\mathrm{Spec}(k))}M$, regarded as defined over k and with coefficients $\mathbb{Q}[G_{E/k}]$.

3. The main result

3.1. *Statement of the main result* For each subgroup Δ of a finite group Γ and each commutative ring R we write $R[\Gamma/\Delta]$ for the free R-module on the set $\{\Delta\gamma:\gamma\in\Gamma\}$ of right cosets of Δ in Γ. We regard $R[\Gamma/\Delta]$ as a (right) $R[\Gamma]$-module via the obvious right multiplication action of Γ on $\{\Delta\gamma:\gamma\in\Gamma\}$.

The following result is a natural generalisation of [10, Th. 2.3] (for more details of this connection see §4.1).

Theorem 3.1. *Let K/k be a finite Galois extension of number fields and set $G:=G_{K/k}$. Let M be a motive defined over k and for each Galois extension F/E with $k\subseteq E\subseteq F\subseteq K$ assume the validity of both of the conjectures discussed in Remark 2.1 for the motive $h^0(\mathrm{Spec}(F))\otimes_{h^0(\mathrm{Spec}(k))}M$, regarded as defined over E and with coefficients $\mathbb{Q}[G_{F/E}]$. Let $\{H_a:a\in A\}$ and $\{H'_b:b\in B\}$ be finite sets of subgroups of G such that the right $\mathbb{C}[G]$-modules $\bigoplus_{a\in A}\mathbb{C}[G/H_a]$ and $\bigoplus_{b\in B}\mathbb{C}[G/H'_b]$ are isomorphic. Then for every prime p and every isomorphism $j:\mathbb{C}\cong\mathbb{C}_p$ one has an equality*

$$\prod_{a\in A}\chi_j^{\mathrm{BK}}(M_{K^{H_a}})=\prod_{b\in B}\chi_j^{\mathrm{BK}}(M_{K^{H'_b}}) \tag{3}$$

in $K_0(\mathbb{Z}_p,\mathbb{C}_p)\cong\mathbb{C}_p^\times/\mathbb{Z}_p^\times$. In particular, if the Coherence hypothesis of [5, §3.3] *is valid for the pair $(M_K,\mathbb{Q}[G])$, then in $\mathbb{R}$ one has an equality of (free rank one) $\mathbb{Z}$-lattices*

$$\prod_{a\in A}\chi^{\mathrm{BK}}(M_{K^{H_a}})=\prod_{b\in B}\chi^{\mathrm{BK}}(M_{K^{H'_b}}). \tag{4}$$

Remark 3.2. Fix a number field C with ring of integers $\mathcal{O}$ and set $C_{\mathbb{R}} := \mathbb{R}\otimes_{\mathbb{Q}} C$. If M has coefficients in C, then each motive M_E has coefficients in C and the construction of [5] gives, modulo the relevant cases of the conjectures discussed in Remark 2.1 and of the Coherence hypothesis of [5, §3.3], an object $\Xi(M_E)_{\mathcal{O}}$ of $V(\mathcal{O})$ and a canonical morphism $\mathbb{R}\otimes_{\mathbb{Z}}\Xi(M_E)_{\mathcal{O}}\to \mathbf{1}_{C_{\mathbb{R}}}$ in $V(C_{\mathbb{R}})$. The leading term at $z=0$ of the C-equivariant L-function of M_E is a unit of $C_{\mathbb{R}}$ and the analogue of the conjectural equality (2) is an equality of invertible $\mathcal{O}$-submodules of $C_{\mathbb{R}}$. The result of Theorem 3.1 (and the proof presented below) extends directly to this setting but, for clarity of exposition, we prefer to leave all further details in this regard to the reader.

For a discussion of several (unconditional) consequences of Theorem 3.1 see §4.

3.2. *Proof of the main result* Before starting the proof of Theorem 3.1 we record a useful preliminary result.

Lemma 3.3. *As p runs over all primes and j over all field isomorphisms $\mathbb{C}\cong\mathbb{C}_p$ the natural diagonal homomorphism $\mathbb{R}^\times/\mathbb{Z}^\times\to\prod_{p,j}\mathbb{C}_p^\times/\mathbb{Z}_p^\times$ is injective.*

Proof. Fix $x\in\mathbb{R}^\times$ with $j(x)\in\mathbb{Z}_p^\times\subseteq\mathbb{C}_p^\times$ for all p and j. If x was transcendental over $\mathbb{Q}$, then there would exist an isomorphism j with $j(x)\notin\mathbb{Q}_p$ and so our assumptions imply that x is algebraic over $\mathbb{Q}$. The fact that $j(x)$ belongs to $\mathbb{Q}_p$ for all p and j then also implies that all primes are completely split in the number field $\mathbb{Q}(x)$ generated by x over $\mathbb{Q}$ so that $\mathbb{Q}(x)=\mathbb{Q}$ and hence x is rational. Since $j(x)$ belongs to $\mathbb{Z}_p^\times$ for all p and j this then implies that x belongs to $\{-1,1\}$, as required. □

To start the proof of Theorem 3.1 we note that if the Coherence hypothesis of [5, §3.3] is valid for the pair $(M_K,\mathbb{Q}[G])$, then it is clearly also valid for the pair $(M_E,\mathbb{Q})$ for every intermediate field E of K/k and so all of the lattices $\chi^{\mathrm{BK}}(M_E)$ are well-defined. In addition, the equality (1) combines with the definitions of $\chi^{\mathrm{BK}}(M_E)$ and $\chi_j^{\mathrm{BK}}(M_E)$ to imply $\mathbb{Z}_p\otimes_{\mathbb{Z}} j(\chi^{\mathrm{BK}}(M_E)) = \chi_j^{\mathrm{BK}}(M_E)$. The latter equality then combines with the result of Lemma 3.3 to imply that the $\mathbb{Z}$-lattice $\chi^{\mathrm{BK}}(M_E)$ is uniquely determined by the $\mathbb{Z}_p$-lattices $\chi_j^{\mathrm{BK}}(M_E)$ for all primes p and all isomorphisms $j:\mathbb{C}\cong\mathbb{C}_p$.

This observation shows, in particular, that the equality (4) is a consequence of the validity of (3) for all primes p and all isomorphisms $j:\mathbb{C}\cong\mathbb{C}_p$. In the sequel it thus suffices to fix a prime p and an isomorphism $j:\mathbb{C}\cong\mathbb{C}_p$ and then prove the equality (3). For any finitely generated module Y we shall henceforth write Y_p and $Y_{\mathbb{C}_p}$ for $\mathbb{Z}_p\otimes_{\mathbb{Z}} Y$ and $\mathbb{C}_p\otimes_{\mathbb{Z}} Y$ respectively.

We define (right) G-modules by setting

$$\Pi := \bigoplus_{a\in A}\mathbb{Z}[G/H_a] \quad\text{and}\quad \Pi' := \bigoplus_{b\in B}\mathbb{Z}[G/H'_b]. \tag{5}$$

Then the assumption that the $\mathbb{C}[G]$-modules $\mathbb{C}\otimes_{\mathbb{Z}}\Pi$ and $\mathbb{C}\otimes_{\mathbb{Z}}\Pi'$ are isomorphic combines with Deuring's Theorem to imply the existence of a short exact sequence of G-modules of the form

$$0\to\Pi\xrightarrow{\varphi}\Pi'\to\mathrm{cok}(\varphi)\to 0 \tag{6}$$

in which, as φ is injective, the module $\mathrm{cok}(\varphi)$ is finite.

For any object C of $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ the image under $\mathbb{Z}_p \otimes_{\mathbb{Z}} -$ of the exact sequence (6) induces an exact triangle in $D^{\mathrm{p}}(\mathbb{Z}_p)$ of the form

$$(7)\quad \Pi_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C \xrightarrow{\varphi_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} \mathrm{id}_C} \Pi'_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C \to \mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C \to (\Pi_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)[1].$$

To study such triangles we shall use the following technical result. To state this result we note that if F is any field and W any finitely generated right $F[G]$-module, then the assignment $P \mapsto P_W := W \otimes_{F[G]} P$ (for each finitely generated $F[G]$-module P) induces a functor $V(F[G]) \to V(F)$. We write t_W for the image of a morphism t in $V(F[G])$ under this functor.

Lemma 3.4. *Let C be any object of $D^{\mathrm{p}}(\mathbb{Z}_p[G])$. Then in each degree m the module $H^m(\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)$ is finite. Further, if t is any morphism in $V(\mathbb{C}_p[G])$ of the form*

$$[\mathbb{C}_p[G] \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C]_{\mathbb{C}_p[G]} \to \prod_{m\in\mathbb{Z}} [H^m(\mathbb{C}_p[G] \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)]^{(-1)^m}_{\mathbb{C}_p[G]} \to \mathbf{1}_{\mathbb{C}_p[G]}$$

where the first arrow is the canonical morphism and the second is induced by a set of exact sequences of $\mathbb{C}_p[G]$-modules (in the sense of §A.3), then in the group $K_0(\mathbb{Z}_p, \mathbb{C}_p) \cong \mathbb{C}_p^\times/\mathbb{Z}_p^\times$ one has

$$(8)\quad \chi^{\mathrm{ref}}(\Pi'_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C, t_{\Pi'_{\mathbb{C}_p}}) = \chi^{\mathrm{ref}}(\Pi_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C, t_{\Pi_{\mathbb{C}_p}}) \cdot \prod_{m\in\mathbb{Z}} |H^m(\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)|^{(-1)^m}.$$

Proof. Since C belongs to $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ and $\mathrm{cok}(\varphi)_p$ is finite it is clear that each $\mathbb{Z}_p$-module $H^m(\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)$ is both finitely generated and torsion, and hence finite.

We next apply Lemma A.1 with $\mathfrak{N} := \{n \in \mathbb{Z} : H^n(\mathbb{C}_p[G] \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C) \neq 0\}$, $M_n := H^n(\mathbb{C}_p[G] \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)$ for each $n \in \mathfrak{N}$ and $\phi := \mathbb{C}_p \otimes_{\mathbb{Z}} \varphi$ to obtain a commutative diagram in $V(\mathbb{C}_p)$ of the form

$$\begin{tikzcd}
{[\Pi'_{\mathbb{C}_p} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C]_{\mathbb{C}_p}} \arrow[r] \arrow[d, "t_{\Pi'_{\mathbb{C}_p}}"] & {[\Pi_{\mathbb{C}_p} \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C]_{\mathbb{C}_p} \cdot [\mathbb{C}_p \otimes_{\mathbb{Z}_p} (\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)]_{\mathbb{C}_p}} \arrow[d, "t_{\Pi_{\mathbb{C}_p}} \cdot \mathrm{can}"] \\
\mathbf{1}_{\mathbb{C}_p} \arrow[r] & \mathbf{1}_{\mathbb{C}_p} \cdot \mathbf{1}_{\mathbb{C}_p}.
\end{tikzcd}$$

Here the top arrow is the morphism induced by the image under $\mathbb{C}_p[G] \otimes_{\mathbb{Z}_p[G]} -$ of the exact triangle (7), the bottom arrow is the canonical morphism and 'can' denotes the morphism $[\mathbb{C}_p \otimes_{\mathbb{Z}_p} (\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)]_{\mathbb{C}_p} \to [0]_{\mathbb{C}_p} = \mathbf{1}_{\mathbb{C}_p}$ that is induced by the acyclicity of $\mathbb{C}_p \otimes_{\mathbb{Z}_p} (\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)$ (which itself follows from the fact that each module $H^m(\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)$ is finite).

The last diagram combines with the exact triangle (7) and the explicit product structure of the group $\pi_0(V(\mathbb{Z}_p) \times_{V(\mathbb{C}_p)} \mathcal{P}_0) \cong K_0(\mathbb{Z}_p, \mathbb{C}_p) \cong \mathbb{C}_p^\times/\mathbb{Z}_p^\times$ to give an equality

$$\chi^{\mathrm{ref}}(\Pi'_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C, t_{\Pi'_{\mathbb{C}_p}}) = \chi^{\mathrm{ref}}(\Pi_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C, t_{\Pi_{\mathbb{C}_p}}) \cdot \chi^{\mathrm{ref}}(\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C, \mathrm{can}).$$

To deduce (8) it now suffices to recall from §A.2(i) that $\chi^{\rm ref}(\mathrm{cok}(\varphi)_p\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C,\mathrm{can})$ is equal to the image of $\prod_{m\in\mathbb{Z}}|H^m(\mathrm{cok}(\varphi)_p\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C)|^{(-1)^m}$ in $\mathbb{C}_p^\times/\mathbb{Z}_p^\times$. ☐

Returning to the proof of Theorem 3.1 we now set $C := R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S},T_{p,K})$ and $t^{\rm BK} := t_j^{\rm BK}(M,S,K/k)$. We note that the definition of $t^{\rm BK}$ in [5] implies that it is induced by a finite set of exact sequences of $\mathbb{C}_p[G]$-modules in the sense of §A.3 (with, in terms of the notation and numbering of [5], the necessary exact sequences being derived from (16), (17), (19), (22), (23), the central column of (26), (27), (28) and the isomorphism on cohomology induced by the quasi-isomorphism AV_f.) By applying Lemma 3.4 with our current choice of C and with $t = t^{\rm BK}$ we therefore deduce that

$$\begin{aligned}(9)\quad \chi^{\rm ref}(\Pi'_p\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C,t^{\rm BK}_{\Pi'_{\mathbb{C}_p}}) &=\\ &\chi^{\rm ref}(\Pi_p\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C,t^{\rm BK}_{\Pi_{\mathbb{C}_p}})\cdot\prod_{m\in\mathbb{Z}}|H^m(\mathrm{cok}(\varphi)_p\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C)|^{(-1)^m}.\end{aligned}$$

We note next that the identification $\Pi_p\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C=\bigoplus_{a\in A}(\mathbb{Z}_p[G/H_a]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C)$ gives an equality in $K_0(\mathbb{Z}_p,\mathbb{C}_p)$

$$(10)\qquad \chi^{\rm ref}(\Pi_p\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C,t^{\rm BK}_{\Pi_{\mathbb{C}_p}})=\prod_{a\in A}\chi^{\rm ref}(\mathbb{Z}_p[G/H_a]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C,t^{\rm BK}_{\mathbb{C}_p[G/H_a]}),$$

and similarly with Π replaced by Π' and $\{H_a\}_{a\in A}$ by $\{H'_b\}_{b\in B}$. To compute the individual terms in these products we use the following result.

Lemma 3.5. *Let C and $t^{\rm BK}$ be as above. Then for any subgroup I of G one has*

$$\chi^{\rm ref}(\mathbb{Z}_p[G/I]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C,t^{\rm BK}_{\mathbb{C}_p[G/I]})=\chi_j^{\rm BK}(M_{K^I})\in K_0(\mathbb{Z}_p,\mathbb{C}_p).$$

Proof. For each subgroup I of G there is a natural composite isomorphism in $D^{\rm p}(\mathbb{Z}_p)$ of the form

$$\begin{aligned}\iota_I:\mathbb{Z}_p[G/I]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S_p},T_{p,K})&\cong\mathbb{Z}_p\otimes^{\mathbb{L}}_{\mathbb{Z}_p[I]}R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S_p},T_{p,K})\\ &\cong R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S_p},T_{p,K^I})\cong R\Gamma_{c,\text{ét}}(\mathcal{O}_{K^I,S_{p,K^I}},T_p).\end{aligned}$$

The second isomorphism here is the standard descent isomorphism, the third is induced by Shapiro's Lemma and the first arises in the following way: for each finitely generated projective $\mathbb{Z}_p[G]$-module P one has a composite isomorphism of $\mathbb{Z}_p$-modules $\mathbb{Z}_p[G/I]\otimes_{\mathbb{Z}_p[G]}P=\{I\}\cdot\mathbb{Z}_p[G]\otimes_{\mathbb{Z}_p[G]}P\cong\{I\}\cdot\mathbb{Z}_p\otimes_{\mathbb{Z}_p[I]}P'\cong\mathbb{Z}_p\otimes_{\mathbb{Z}_p[I]}P'$ where $\{I\}$ denotes I regarded as an element of $\mathbb{Z}_p[G/I]$, P' denotes P regarded, by restriction of scalars, as a $\mathbb{Z}_p[I]$-module and for each π in P' the last map sends $\{I\}\otimes_{\mathbb{Z}_p[I]}\pi$ to $1\otimes_{\mathbb{Z}_p[I]}\pi$.

Further, the argument of [5, Prop. 4.1] shows that the above isomorphism ι_I combines with the definitions of $t^{\rm BK}:=t_j^{\rm BK}(M,S,K/k)$ and $t_I^{\rm BK}:=t_j^{\rm BK}(M_{K^I},S_{K^I})$ to give a commutative diagram in $V(\mathbb{C}_p)$ of the form

$$\begin{tikzcd}[\mathbb{C}_p[G/I]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]}C]_{\mathbb{C}_p} \arrow[rr, "{[\mathbb{C}_p\otimes_{\mathbb{Z}_p}\iota_I]_{\mathbb{C}_p}}"] \arrow[dr, "t^{\rm BK}_{\mathbb{C}_p[G/I]}"'] & & {[\mathbb{C}_p\otimes^{\mathbb{L}}_{\mathbb{Z}_p}R\Gamma_{c,\text{ét}}(\mathcal{O}_{K^I,S_{p,K^I}},T_p)]_{\mathbb{C}_p}} \arrow[dl, "t_I^{\rm BK}"] \\ & \mathbf{1}_{\mathbb{C}_p} & \end{tikzcd}$$

The pair $(\iota_I, [\mathbb{C}_p \otimes_{\mathbb{Z}_p} \iota_I]_{\mathbb{C}_p})$ therefore constitutes an isomorphism in $V(\mathbb{Z}_p) \times_{V(\mathbb{C}_p)} \mathcal{P}_0$ between $([\mathbb{Z}_p[G/I] \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C]_{\mathbb{Z}_p}, t^{\mathrm{BK}}_{\mathbb{C}_p[G/I]})$ and $([R\Gamma_{c,\text{ét}}(\mathcal{O}_{K^I,S_{p,K^I}}, T_p)]_{\mathbb{Z}_p}, t^{\mathrm{BK}}_I)$ and hence implies that in $\pi_0(V(\mathbb{Z}_p) \times_{V(\mathbb{C}_p)} \mathcal{P}_0) \cong K_0(\mathbb{Z}_p, \mathbb{C}_p)$ one has an equality

$$\chi^{\mathrm{ref}}(\mathbb{Z}_p[G/I] \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C, t^{\mathrm{BK}}_{\mathbb{C}_p[G/I]}) = \chi^{\mathrm{ref}}(R\Gamma_{c,\text{ét}}(\mathcal{O}_{K^I,S_{p,K^I}}, T_p), t^{\mathrm{BK}}_I) =: \chi^{\mathrm{BK}}_j(M_{K^I}),$$

as required. □

Upon combining Lemma 3.5 with the equalities (9) and (10) (and the analogous equality for Π') one obtains an equality

$$\prod_{b\in B} \chi^{\mathrm{BK}}_j(M_{K^{H'_b}}) = \prod_{a\in A} \chi^{\mathrm{BK}}_j(M_{K^{H_a}}) \cdot \prod_{m\in\mathbb{Z}} |H^m(\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)|^{(-1)^m}.$$

The proof of Theorem 3.1 is therefore completed by the following result.

Lemma 3.6. *In* $K_0(\mathbb{Z}_p, \mathbb{C}_p) \cong \mathbb{C}_p^\times/\mathbb{Z}_p^\times$ *one has*

$$\prod_{m\in\mathbb{Z}} |H^m(\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)|^{(-1)^m} = 1.$$

Proof. For each $\mathbb{Z}_p$-module M and natural number n we write $M^{(n)}$ for the direct sum of n copies of M.

We first claim that there are non-negative integers $\{n_m : m \in \mathbb{Z}\}$, with $n_m = 0$ for almost all m, such that C is represented by a (bounded) complex of the form

$$P^\bullet = [\cdots \to \mathbb{Z}_p[G]^{(n_m)} \to \mathbb{Z}_p[G]^{(n_{m+1})} \to \cdots]$$

where the term $\mathbb{Z}_p[G]^{(n_m)}$ occurs in degree m and one has $\sum_{m\in\mathbb{Z}}(-1)^m n_m = 0$. Indeed, since C belongs to $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ and the Krull-Schimdt-Azumaya Theorem applies to the algebra $\mathbb{Z}_p[G]$, to prove that C is isomorphic in $D^{\mathrm{p}}(\mathbb{Z}_p[G])$ to a complex of the form $P^\bullet$ it suffices to show that the Euler characteristic of C in $K_0(\mathbb{Z}_p[G])$ vanishes. We set $\mathbb{F} = \mathbb{Z}/p\mathbb{Z}$. Then the natural reduction map $K_0(\mathbb{Z}_p[G]) \to K_0(\mathbb{F}[G])$ is bijective (cf. [2, Chap. IX, Prop. 1.3]) and so it is enough to show that the Euler characteristic of $\mathbb{F}[G] \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C$ in $K_0(\mathbb{F}[G])$ vanishes. But $\mathbb{F}[G] \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C$ is naturally isomorphic in $D^{\mathrm{p}}(\mathbb{F}[G])$ to $R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,S_p}, T_p \otimes_{\mathbb{Z}_p} \prod_{K\to k^c} \mathbb{F})$ and the Euler characteristic in $K_0(\mathbb{F}[G])$ of the latter complex vanishes by Flach's equivariant refinement of Tate's formula for the global Euler characteristic [12, Th. 5.1].

Replacing C by $P^\bullet$ shows that the derived tensor product $\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C$ is represented by the bounded complex

$$\mathrm{cok}(\varphi)_p \otimes_{\mathbb{Z}_p[G]} P^\bullet = [\cdots \to \mathrm{cok}(\varphi)_p^{(n_m)} \to \mathrm{cok}(\varphi)_p^{(n_{m+1})} \to \cdots]$$

where each module $\mathrm{cok}(\varphi)_p^{(n_m)}$ occurs in degree m. By breaking this complex into the associated short exact sequences of boundaries, cycles and cohomology, and noting that all such groups are finite, one then computes that

$$\begin{aligned}\prod_{m\in\mathbb{Z}} |H^m(\mathrm{cok}(\varphi)_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p[G]} C)|^{(-1)^m} &= \prod_{m\in\mathbb{Z}} |H^m(\mathrm{cok}(\varphi)_p \otimes_{\mathbb{Z}_p[G]} P^\bullet)|^{(-1)^m}\\ &= \prod_{m\in\mathbb{Z}} |\mathrm{cok}(\varphi)_p^{(n_m)}|^{(-1)^m}\\ &= \prod_{m\in\mathbb{Z}} |\mathrm{cok}(\varphi)_p|^{(-1)^m n_m}\\ &=|\mathrm{cok}(\varphi)_p|^{\sum_{m\in\mathbb{Z}}(-1)^m n_m}\\ &=|\mathrm{cok}(\varphi)_p|^0\end{aligned}$$

which is equal to 1, as required. □

4. Applications

In this section we discuss several unconditional consequences of Theorem 3.1 and also describe an alternative approach to the theory of regulator constants recently introduced by Dokchitser and Dokchitser.

We fix a finite Galois extension of number fields K/k and an intermediate field E of K/k.

4.1. *Abelian varieties* Let X be an abelian variety defined over k and set $M := h^1(X)(1)$. If the (classical) Tate-Shafarevic group of X over E is finite, then the lattice $\chi^{\mathrm{BK}}(M_E)$ is defined unconditionally and is equal to the $\mathbb{Z}$-submodule of $\mathbb{R}$ that is generated by the 'BSD quotient' $\mathrm{BSD}(X/E)$ defined by Dokchitser and Dokchitser in [10, §2.1] (for a proof of this fact see, for example, Venjakob [18, §3.1]). Theorem 3.1 therefore gives an alternative proof of [10, Th. 2.3]. Without assuming the finiteness of any Tate-Shafarevic group, this approach also leads to an alternative (unconditional) proof of [8, Th. 1.5].

4.2. *Units and class groups* If $M = h^0(\mathrm{Spec}(k))$, then each lattice $\chi^{\mathrm{BK}}(M_E)$ is defined unconditionally and can be computed explicitly by using Kummer theory and class field theory, the L-function $L(M_E, z)$ is equal to the Dedekind zeta function of E and it is well known that the equality (2) is equivalent to the analytic class number formula for E. A slight generalisation of this result is also useful in applications and to quickly describe this we fix a finite set of places S of k that contains S_∞, a prime p and an isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$. Then it is known that there exists a natural morphism $t^{\mathrm{BK}}_{S,j} : \mathbb{C}_p \otimes_{\mathbb{Z}_p} [R\Gamma_{c,\text{ét}}(\mathcal{O}_{E,S_{p,E}}, \mathbb{Z}_p)]_{\mathbb{Z}_p} \to \mathbf{1}_{\mathbb{C}_p}$ which agrees with $t^{\mathrm{BK}}_j(M_E, S_E)$ if $S = S_\infty$, is induced by a set of exact sequences of $\mathbb{C}_p$-modules (in the sense of §A.3) and is such that $\chi^{\mathrm{ref}}(R\Gamma_{c,\text{ét}}(\mathcal{O}_{E,S_{p,E}}, \mathbb{Z}_p), t^{\mathrm{BK}}_{S,j})$ corresponds to the sublattice $(j(R_{E,S})h_{E,S}/w_E)\cdot\mathbb{Z}_p$ of $\mathbb{C}_p$ where $R_{E,S}$ is the S_E-regulator of E, $h_{E,S}$ the cardinality of $\mathrm{Cl}(\mathcal{O}_{E,S_E})$ and w_E the cardinality of the torsion subgroup of $E^\times$ (the construction of $t^{\mathrm{BK}}_{S,j}$ and the stated formula for $\chi^{\mathrm{ref}}(R\Gamma_c(\mathcal{O}_{E,S_{p,E}}, \mathbb{Z}_p), t^{\mathrm{BK}}_{S,j})$ follows, for example, from the proof of [4, Prop. 4.2.2] and the remark in §A.2(ii)). In this case Theorem 3.1 therefore gives an algebraic proof of the equality of [1, (1)], thereby showing that the results of Bartel's paper (and earlier results of, amongst others, de Smit [6]) do not in fact depend upon the analytic class number formula.

4.3. *Higher algebraic K-groups* Fix a strictly negative integer r. For each odd prime p and integer $i = 1, 2$ write

$$\mathrm{ch}^i_{E,p,r} : \mathbb{Z}_p \otimes_{\mathbb{Z}} K_{2-i-2r}(\mathcal{O}_E) \to H^i_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_E[\frac{1}{p}]), \mathbb{Z}_p(r))$$

for the Chern class homomorphisms constructed by Soulé [16] and Dwyer and Friedlander [11]. Then the kernel of $\mathrm{ch}^i_{E,p,r}$ is known to be finite and is conjectured to vanish by Quillen and Lichtenbaum. We recall that if $r = -1$, then the injectivity of $\mathrm{ch}^i_{E,p,r}$ has been proved by work of Tate [17], Levine [14] and Merkuriev and Suslin [15]. We further recall that it is widely believed the recent work of Voevodsky, Suslin and Rost on the Milnor and Bloch-Kato conjectures should lead to a proof of the injectivity of $\mathrm{ch}^i_{E,p,r}$ for all strictly negative r.

We write $h_{r,E}$ for the cardinality of the quotient of the finite group $K_{-2r}(\mathcal{O}_E)[\frac{1}{2}] = \bigoplus_{p\neq 2}(\mathbb{Z}_p \otimes_{\mathbb{Z}} K_{-2r}(\mathcal{O}_E))$ by $\bigoplus_{p\neq 2}\ker(\mathrm{ch}^2_{E,p,r})$ and $w_{r,E}$ for the cardinality of the quotient of the finite group $K_{1-2r}(\mathcal{O}_E)_{\mathrm{tor}}[\frac{1}{2}] = \bigoplus_{p\neq 2}(\mathbb{Z}_p \otimes_{\mathbb{Z}} K_{1-2r}(\mathcal{O}_E))_{\mathrm{tor}}$ by $\bigoplus_{p\neq 2}\ker(\mathrm{ch}^1_{E,p,r})_{\mathrm{tor}}$.

Set $M_r = h^0(\mathrm{Spec}(k))(r)$. Then the $\mathbb{Z}$-sublattice $\chi^{\mathrm{BK}}(M_{r,E})$ of $\mathbb{R}$ is defined unconditionally and $\chi^{\mathrm{BK}}(M_{r,E}) \otimes \mathbb{Z}[\frac{1}{2}]$ is generated over $\mathbb{Z}[\frac{1}{2}]$ by $h_{r,E}R_{r,E}/w_{r,E}$ where $R_{r,E}$ is the Borel regulator for $K_{1-2r}(\mathcal{O}_E)$ (indeed, taking account of the remark in §A.2(ii), this follows from the proof of [4, Prop. 4.2.6]). In this case therefore, Theorem 3.1 proves an unconditional analogue of the Artin formalism equality of [1, (1)] relating to higher algebraic K-groups. It would surely be interesting to derive explicit consequences of this result for the structure of algebraic K-groups that are analogous to the rather striking results obtained (for unit groups) by Bartel in [1].

4.4. *Regulator constants* The techniques of the present article also give a different approach to the theory of 'regulator constants' that was introduced by Dokchitser and Dokchitser in [10] and has also played a key role in several subsequent articles including [8, 9] and [1].

To explain this we fix a field F of characteristic 0 and let C denote the complex $I \xrightarrow{0} J$ where I and J are finitely generated $F[G]$-modules and I is placed in degree 0. We fix a finitely generated right $F[G]$-module W and set $I_W := W \otimes_{F[G]} I$, $J_W := W \otimes_{F[G]} J$ and $C_W := W \otimes_{F[G]} C$. We identify $V(F)$ with the category of graded line bundles on F as in §A.1 and set $Y^* := \mathrm{Hom}_F(Y, F)$ for any F-module Y. Then any isomorphism of $F[G]$-modules $\iota : I \xrightarrow{\sim} J$ induces a morphism $t_\iota : [C]_{F[G]} \to \mathbf{1}_{F[G]}$ in $V(F[G])$ (see §A.2(ii)) and the induced morphism $t_{\iota,W} : [C_W]_F \to \mathbf{1}_F$ in $V(F)$ sends each element $(x \otimes f, 0)$ of $(\wedge^d_F I_W \otimes_F (\wedge^d_F J_W)^*, 0) = [C_W]_F$ to $(f(\wedge^d_F(\mathrm{id}_W \otimes_{F[G]} \iota)(x)), 0) \in (F, 0) = \mathbf{1}_F$ with $d := \dim_F(I_W) = \dim_F(J_W)$. In the following result we write ψ^{tr} for the transpose of a linear map ψ.

Lemma 4.1. *Let $\phi : W \to W'$ be an isomorphism of finitely generated right $F[G]$-modules and set $\phi_I := \phi \otimes_{F[G]} \mathrm{id}_I$ and $\phi_J := \phi \otimes_{F[G]} \mathrm{id}_J$. Then for any non-zero elements x, y, x' and y' of $\wedge^d_F I_W, (\wedge^d_F J_W)^*, \wedge^d_F I_{W'}$ and $(\wedge^d_F J_{W'})^*$ respectively one has*

$$t_{\iota,W}(x \otimes y)t_{\iota,W'}(x' \otimes y')^{-1} = c(\phi)_{x,x'}c(\phi^{\mathrm{tr}})^{-1}_{y',y} \tag{11}$$

where $c(\phi)_{x,x'}$ and $c(\phi^{\mathrm{tr}})_{y',y}$ are the elements of $F^\times$ defined by setting $\wedge^d_F\phi_I(x) = c(\phi)_{x,x'} \cdot x'$ and $(\wedge^d_F\phi^{\mathrm{tr}}_J)(y') = c(\phi^{\mathrm{tr}})_{y',y} \cdot y$. In particular, the expression in (11) is independent of both ι and ϕ.

Proof. Setting $\iota_W := \mathrm{id}_W \otimes_{F[G]} \iota$ and $\iota_{W'} := \mathrm{id}_{W'} \otimes_{F[G]} \iota$ it is clear that $\phi_J \circ \iota_W = \iota_{W'} \circ \phi_I$. From the commutative diagram at the end of the proof of Lemma A.1 below with $N = J$ it therefore follows that $t_{\iota,W}(x \otimes y)$ is equal to

$$\begin{aligned}
\mathrm{ev}_{J_W}(\wedge_F^d \iota_W(x) \otimes y) =& \mathrm{ev}_{J_{W'}}(\wedge_F^d \phi_J(\wedge_F^d \iota_W(x)) \otimes (\wedge_F \phi_J^{\mathrm{tr}})^{-1}(y))\\
=& \mathrm{ev}_{J_{W'}}(\wedge_F^d \iota_{W'}(\wedge_F^d \phi_I(x)) \otimes (\wedge_F \phi_J^{\mathrm{tr}})^{-1}(y))\\
=& t_{\iota,W'}(\wedge_F^d \phi_I(x) \otimes (\wedge_F \phi_J^{\mathrm{tr}})^{-1}(y))\\
=& t_{\iota,W'}(c(\phi)_{x,x'} \cdot x' \otimes c(\phi^{\mathrm{tr}})^{-1}_{y',y} \cdot y')\\
=& c(\phi)_{x,x'} c(\phi^{\mathrm{tr}})^{-1}_{y',y} \cdot t_{\iota,W'}(x' \otimes y'),
\end{aligned}$$

as required to prove the equality (11). The final assertion of the lemma is then true because the left and right hand sides of (11) are obviously independent of ϕ and ι respectively. □

To explain the connection of Lemma 4.1 to the theory of regulator constants we now consider the special case that $J = I^*$ and the $F[G]$-modules W and W' are self-dual. Then each choice of isomorphisms of $F[G]$-modules $W \cong W^*$ and $W' \cong W'^*$ induces identifications of F-modules $(J_W)^* \cong J^*_{W^*} \cong (I^{**})_W \cong I_W$ and similarly $(J_{W'})^* \cong I_{W'}$ and hence also $(\wedge_F^d J_W)^* \cong \wedge_F^d (J_W)^* \cong \wedge_F^d I_W$ and $(\wedge_F^d J_{W'})^* \cong \wedge_F^d I_{W'}$. Fixing such identifications allows one to choose $x = y$ and $x' = y'$ in Lemma 4.1 and in this case (11) implies immediately that the quotient $t_{\iota,W}(x \otimes x) t_{\iota,W'}(x' \otimes x')^{-1}$ is independent of the choices of x and x' when considered as an element of $F^\times/(F^\times)^2$. In particular, if we now suppose that F contains a Dedekind domain R, that $I = F \otimes_R M$ for a finitely generated $R[G]$-module M, that the isomorphism of $F[G]$-modules $\iota : I \to I^*$ is induced by a non-degenerate G-invariant pairing $M \times M \to F$ and that W and W' are the (canonically self-dual) $F[G]$-modules $F \otimes_{\mathbb{Z}} \Pi$ and $F \otimes_{\mathbb{Z}} \Pi'$ defined in (5), then for a suitable choice of x and x' the quotient $t_{\iota,W}(x \otimes x) t_{\iota,W'}(x' \otimes x')^{-1}$ is a 'regulator constant' of the form defined in [10]. In this way Lemma 4.1 gives different proofs of the results in both [10, §2.3] and [1, §3].

Appendix A. Virtual objects and relative algebraic K-theory

For the reader's convenience we quickly recall some standard facts concerning virtual objects. Categories of virtual objects were first introduced by Deligne in [7] and a fuller review of their properties than is given here (in particular, of the connection to relative algebraic K-theory) can be found in [5, §2].

A.1. We fix an associative unital noetherian ring R. Modules over R are to be understood, unless explicitly stated otherwise, as left modules. We use the notation $V(R)$, $[M]_R$, $X \cdot Y$, $\mathbf{1}_R$, X^{-1}, ev_X and $D^{\mathrm{p}}(R)$ introduced in §2.1.

Each object C of $D^{\mathrm{p}}(R)$ gives rise to a canonical object $[C]_R$ of $V(R)$. If R is regular, then for any such C there is a canonical morphism in $V(R)$ of the form

$$\iota(C) : [C]_R \to \prod_{m \in \mathbb{Z}} [H^m(C)]_R^{(-1)^m}.$$

If R is a field, then one can identify $V(R)$ with the category of graded line bundles on R in such a way that for any R-module M of rank r one has $[M]_R = (\wedge_R^r M, r)$, $[M]_R^{-1} = (\mathrm{Hom}_R(\wedge_R^r M, R), -r)$, $[M]_R \cdot [N]_R = (\wedge_R^r M \otimes_R \wedge_R^{r'} N, r + r')$ for any R-module N of rank r', and ev_M is induced by the natural evaluation pairing $\wedge_R^r M \otimes_R \mathrm{Hom}_R(\wedge_R^r M, R) \to R$.

A.2. We now let R be a discrete valuation ring of characteristic 0 and F a field that contains R. We fix an object C of $D^{\mathrm{p}}(R)$ and set $C_F := F \otimes_R^{\mathbb{L}} C$. For each morphism $t : [C_F]_F \to \mathbf{1}_F$ in $V(F)$ we recall the notation $\chi^{\mathrm{ref}}(C,t)$ introduced in §2.1. We also use the natural isomorphism $K_0(R,F) \cong F^\times/R^\times$ to identify $\chi^{\mathrm{ref}}(C,t)$ with a (free) rank one R-submodule of F. We recall the following special cases of this construction.
(i) If each cohomology module of C is finite, then C_F is acyclic and, with 'can' denoting the canonical morphism $F \otimes_R [C]_R \to [C_F]_F \to [0]_F = \mathbf{1}_F$, one has $\chi^{\mathrm{ref}}(C,\mathrm{can}) = R \cdot \prod_{m\in\mathbb{Z}} |H^m(C)|^{(-1)^m}$.
(ii) If C is acyclic outside degrees a and $a+1$ (for any given integer a), then any isomorphism of F-modules $\lambda : H^a(C_F) \cong H^{a+1}(C_F)$ induces a composite morphism

$$\begin{aligned} t_\lambda : F \otimes_R [C]_R \to [C_F]_F &\xrightarrow{\iota(C_F)} [H^a(C_F)]_F^{(-1)^a} \cdot [H^{a+1}(C_F)]_F^{(-1)^{a+1}} \\ &\xrightarrow{[\lambda]_F^{(-1)^a}\cdot \mathrm{id}} [H^{a+1}(C_F)]_F^{(-1)^a} \cdot [H^{a+1}(C_F)]_F^{(-1)^{a+1}} \xrightarrow{\mathrm{ev}_{[H^{a+1}(C_F)]_F^{(-1)^a}}} \mathbf{1}_F. \end{aligned}$$

One has $\chi^{\mathrm{ref}}(C,t_\lambda)^{(-1)^a} = R \cdot \det(\lambda)|H^a(C)_{\mathrm{tor}}|^{-1}|H^{a+1}(C)_{\mathrm{tor}}|$ where $\det(\lambda) \in F$ is computed with respect to any choice of R-bases of the lattices $H^a(C)/H^a(C)_{\mathrm{tor}}$ and $H^{a+1}(C)/H^{a+1}(C)_{\mathrm{tor}}$.

A.3. We require a variant of the well-known construction recalled in §A.2(ii). To describe this we fix a field F of characteristic 0, let Λ denote either $F[G]$ or F and suppose given a finite set $\mathfrak{M}$ and for each $\mathfrak{m}$ in $\mathfrak{M}$ an exact sequence $\mathcal{E}_{\mathfrak{m}}$ of finitely generated (projective) Λ-modules of the form $\cdots \to \mathcal{E}_{\mathfrak{m}}^{a-1} \to \mathcal{E}_{\mathfrak{m}}^{a} \to \mathcal{E}_{\mathfrak{m}}^{a+1} \to \cdots$ in which only finitely many modules $\mathcal{E}_{\mathfrak{m}}^a$ are non-zero. We regard each sequence $\mathcal{E}_{\mathfrak{m}}$ as an acyclic complex, with the module $\mathcal{E}_{\mathfrak{m}}^a$ placed in degree a, and write $\iota_{\mathcal{E}_{\mathfrak{m}}}$ for the natural morphism $\prod_{a\in\mathbb{Z}} [\mathcal{E}_{\mathfrak{m}}^a]_\Lambda^{(-1)^a} \to [\mathcal{E}_{\mathfrak{m}}]_\Lambda \to \mathbf{1}_\Lambda$ in $V(\Lambda)$.

Given a finite set of integers $\mathfrak{N}$ and for each n in $\mathfrak{N}$ a finitely generated Λ-module M_n we say that a morphism $t : \prod_{n\in\mathfrak{N}} [M_n]_\Lambda^{(-1)^n} \to \mathbf{1}_\Lambda$ in $V(\Lambda)$ is 'induced by $\{\mathcal{E}_{\mathfrak{m}}\}_{\mathfrak{m}\in\mathfrak{M}}$' if for each $n \in \mathfrak{N}$ one has $M_n = \mathcal{E}_{\mathfrak{m}_n}^{a_n}$ with $\mathfrak{m}_n \in \mathfrak{M}$ and $a_n \in \mathbb{Z}$ and if $n \neq n'$ then either $\mathfrak{m}_n \neq \mathfrak{m}_{n'}$ or $a_n \neq a_{n'}$, and the morphism t is equal to a composite of morphisms of the form $\iota_{\mathcal{E}_{\mathfrak{m}}}$ together with standard commutativity $(X \cdot Y \to Y \cdot X)$, associativity $(X \cdot (Y \cdot Z) \to (X \cdot Y) \cdot Z)$ and contraction (ev_W) morphisms for suitable Λ-modules X, Y, Z and W.

Lemma A.1. *Let $\mathfrak{N}$ be a finite set of integers, $\{M_n\}_{n\in\mathfrak{N}}$ a set of finitely generated $F[G]$-modules and $t : \prod_{n\in\mathfrak{N}} [M_n]_{F[G]}^{(-1)^n} \to \mathbf{1}_{F[G]}$ a morphism in $V(F[G])$ that is induced by a set of exact sequences of $F[G]$-modules. If $\phi : W \to W'$ is an isomorphism of finitely generated right $F[G]$-modules, then the following diagram commutes*

$$\begin{tikzcd}
\prod_{n\in\mathfrak{N}} [M_{n,W}]_F^{(-1)^n} \arrow[rr, "\prod_{n\in\mathfrak{N}} {[\phi\otimes_{F[G]}\mathrm{id}_{M_n}]}_F^{(-1)^n}"] \arrow[dr, "t_W"'] & & \prod_{n\in\mathfrak{N}} [M_{n,W'}]_F^{(-1)^n} \arrow[dl, "t_{W'}"] \\
& \mathbf{1}_F &
\end{tikzcd}$$

Proof. This is a straightforward consequence of the following fact: for any finitely generated $F[G]$-module P there is a commutative diagram in $V(F)$ of the form

$$\begin{tikzcd}
{[P_W]_F \cdot [P_W]_F^{-1}} \arrow[rr, "{[\phi_P]_F \cdot [\phi_P]_F^{-1}}"] \arrow[dr, "\mathrm{ev}_{P_W}"'] & & {[P_{W'}]_F \cdot [P_{W'}]_F^{-1}} \arrow[dl, "\mathrm{ev}_{P_{W'}}"] \\
& \mathbf{1}_F &
\end{tikzcd}$$

where $\phi_P := \phi \otimes_{F[G]} \mathrm{id}_P$. Indeed, if we identify $V(F)$ with the category of graded line bundles on F (as in §A.1), then this diagram is equivalent to the obvious commutative diagram

$$\begin{tikzcd}
(\wedge_F^d(P_W) \otimes_F (\wedge_F^d(P_W))^*, 0) \arrow[rr, "\wedge_F^d \phi_P \otimes_F (\wedge_F^d \phi_P)^{\mathrm{tr},-1}"] \arrow[dr, "\mathrm{ev}_{P_W}"'] & & (\wedge_F^d(P_{W'}) \otimes_F (\wedge_F^d(P_{W'}))^*, 0) \arrow[dl, "\mathrm{ev}_{P_{W'}}"] \\
& (F, 0) &
\end{tikzcd}$$

where we set $d := \dim_F(P_W) = \dim_F(P_{W'})$. ☐

References

[1] A. Bartel, *On Brauer-Kuroda type relations of S-class numbers in dihedral extensions*, submitted for publication.

[2] H. Bass, *Algebraic K-theory*, Benjamin, 1968.

[3] S. Bloch, K. Kato, *L-functions and Tamagawa numbers of motives*, in: *The Grothendieck Festschrift, vol. 1*, Prog. in Math. **86** (Birkhäuser, Boston, 1990), 333–400.

[4] D. Burns, *Leading terms and values of equivariant motivic L-functions*, Pure App. Math. Q. **6** (2010) (Special Issue in honor of John Tate, Part II) 83-172.

[5] D. Burns, M. Flach, *Tamagawa numbers for motives with (non-commutative) coefficients*, Doc. Math. **6** (2001), 501–570.

[6] B. de Smit, *Brauer-Kuroda relations for S-class numbers*, Acta Arith. **98** (2001) 133-146.

[7] P. Deligne, *Le déterminant de la cohomologie*, in: Current Trends in Arithmetical Algebraic Geometry, Contemp. Math. **67**, Amer. Math. Soc., 313-346, 1987.

[8] T. Dokchitser, V. Dokchitser, *Self-duality of Selmer groups*, Math. Proc. Cam. Phil. Soc. **146** (2009) 257-267.

[9] T. Dokchitser, V. Dokchitser, *Regulator constants and the parity conjecture*, Invent. Math. **178** (2009), 23-71.

[10] T. Dokchitser, V. Dokchitser, *On the Birch-Swinnerton-Dyer Quotients modulo squares*, Ann. Math. **172** (2010) 567-596.

[11] W. G. Dwyer, E. M. Friedlander, *Algebraic and étale K-theory*, Trans. Amer. Math. Soc. **292** (1985), 247-280.

[12] M. Flach, *Euler characteristics in relative K-groups*, Bull. London Math. Soc. **32** (2000) 272-284.

[13] J.-M. Fontaine, B. Perrin-Riou, *Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L*, in: *Motives (Seattle, 1991)*, Proc. Sympos. Pure Math. **55**, Part 1 (Amer. Math. Soc., Providence, 1994), 599–706.

[14] M. Levine, *The indecomposable K_3 of a field*, Ann Sci. École Norm. Sup. **22** (1989) 255-344.

[15] A. S. Merkurjev, A. A. Suslin, *The K_3-group of a field*, Izv. Nauk. SSSR **54** (1990) 339-356.

[16] C. Soulé, *K-théorie des anneaux d'entiers de corps de nombres et cohomologie étale*, Invent. Math. **55** (1979) 251-295.

[17] J. Tate, *Relations between K_2 and Galois Cohomology*, Invent. Math. **36** (1976) 257-274.

[18] O. Venjakob, *From the Birch and Swinnerton-Dyer Conjecture to non-commutative Iwasawa theory via the Equivariant Tamagawa Number Conjecture - a survey*, in *'L-functions and Galois Representations'*, Proceedings of the 2004 Durham Symposium, pp. 333-380, Cambridge Univ. Press, 2007.

E-mail address: `david.burns@kcl.ac.uk`

KING'S COLLEGE LONDON, DEPARTMENT OF MATHEMATICS, STRAND, LONDON WC2R 2LS, UNITED KINGDOM