LEADING TERMS OF ARTIN L-FUNCTIONS AT $s = 0$ AND $s = 1$

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Abstract. We study the leading non-zero coefficients in the Taylor expansions at $s = 0$ and $s = 1$ of the equivariant Artin L-function that is associated to a Galois extension of number fields.

1. Introduction

In this manuscript we continue the study of the leading non-zero coefficients in the Taylor expansions at integer points of equivariant Artin L-functions that was initiated in [12].

To be more specific we fix a Galois extension of number fields L/K and a sufficiently large finite set S of places of K which in particular includes all archimedean places and all places which ramify in L/K. We set $G := \mathrm{Gal}(L/K)$ and for each rational integer m we write $L^*_{L/K,S}(m)$ for the invertible element of the centre $\zeta(\mathbb{R}[G])$ of the group ring $\mathbb{R}[G]$ that is equal to the leading non-zero coefficient in the Taylor expansion at $s = m$ of the S-truncated equivariant Artin L-function of L/K.

We shall formulate an explicit conjectural formula for the image of $L^*_{L/K,S}(1)$ under the canonical homomorphism ∂ from the unit group of $\zeta(\mathbb{R}[G])$ to the relative algebraic K-group $K_0(\mathbb{Z}[G], \mathbb{R})$. Our formula involves the Euler characteristic (in the sense of [8]) of a natural perfect complex of $\mathbb{Z}[G]$-modules that is constructed by using methods that are both explicit and comparatively elementary. The explicit nature of this formula allows us to prove rather easily that it simultaneously refines both the 'main conjecture' of Stark at $s = 1$ (as described by Tate in [45]) and the 'Ω_1-conjecture' formulated by Chinburg in [24]. More strikingly, after reinterpreting the conjectural formula in terms of étale cohomology, a delicate analysis of the passage to pro-p-completion (at each prime p) allows us to prove that our formula is also, under certain conditions, equivalent to the 'equivariant Tamagawa number conjecture' of [17], as applied to the pair $(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])$.

The latter comparison result is interesting for several reasons: it allows us to deduce the validity of our conjectural formula for $\partial(L^*_{L/K,S}(1))$ in the case that L is an abelian extension of $\mathbb{Q}$, it answers a question raised by Flach and the second named author in [15], and it also establishes the link between our explicit conjecture and the very general (and rather abstract) 'main conjecture of non-commutative Iwasawa theory' that was recently formulated by Fukaya and Kato in [31]. Indeed, this comparison result combines with the philosophy described by Huber and Kings in [35, §3.3] (and by Fukaya and Kato in [31, §2.3.5]) to suggest that the validity of our conjectural formula for $\partial(L^*_{L/K,S}(1))$ for all Galois extensions L/K could

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itself provide a pivotal step in proving the conjecture of Fukaya and Kato in full generality.

We shall also use the functional equation of the equivariant L-function to investigate the compatibility of our conjecture with the explicit conjectural formula for $\partial(L^*_{L/K,S}(0))$ that is studied in [12]. In particular, we prove that the mutual compatibility of these conjectures is equivalent to the validity of the 'epsilon constant conjecture' formulated in [2]. This result allows us to interpret results in [2], [4] and [5] as evidence for our conjecture. For example, by these means we deduce that the conjectural formulas for $\partial(L^*_{L/K,S}(0))$ and $\partial(L^*_{L/K,S}(1))$ are mutually compatible whenever L/K is tamely ramified or L is either an abelian extension of $\mathbb{Q}$ with odd conductor or a non-abelian extension of $\mathbb{Q}$ of degree 6.

In this regard, we also prove that the validity of the epsilon constant conjecture of [2] would imply that epsilon constants of symplectic characters are uniquely characterized by a natural algebraic invariant of L/K. This result shows that the epsilon constant conjecture implies an affirmative answer to a question that has been open ever since Cassou-Noguès and Taylor proved in [21] and [22] that symplectic epsilon constants of *tamely ramified* Galois extensions of number fields are characterized by algebraic invariants.

When taken together with the results of [12], [2] and [14] the present manuscript demonstrates that the use of equivariant L-functions and of the Euler characteristic formalism of [8] provides a universal framework of leading term conjectures which incorporates as consequences a wide variety of seemingly unrelated theorems and explicit conjectures ranging from Hilbert's Theorem 132 to the 'Ω-conjectures' of Chinburg, the explicit Galois structure results on units and ideal class groups proved by Fröhlich in [29] and [30], the refinement of Stark's conjecture formulated by Rubin and the 'refined class number formulas' conjectured by both Gross and Tate. In turn, such a universal approach gives new insight into various long-standing questions and conjectures. For example, in the setting of the present manuscript, the results we prove in §3 and §5 show that Chinburg's 'Ω_1-conjecture' and 'Ω_3-conjecture' are consequences of leading term conjectures at $s = 1$ and $s = 0$ respectively, whereas his 'Ω_2-conjecture' is most naturally interpreted as a consequence of the compatibility of these leading term conjectures with respect to the relevant functional equation. From a philosophical perspective, this qualitative distinction neatly accounts for the fact that the Ω_2-conjecture is much easier to study than either of the Ω_1-conjecture or Ω_3-conjecture and also provides a satisfactory answer to a problem emphasized by Fröhlich in both [29] and [20, §3]. Indeed, in [29, Introduction] Fröhlich writes of the 'amazing analogy' between the Galois structure theories of, on the one hand, unit groups and ideal class groups and, on the other hand, rings of algebraic integers and he stresses that providing a natural explanation of this analogy is 'an outstanding problem – possibly connected with a new interpretation of the functional equation'. The results we prove in §3 and §5 now provide just such an explanation of this analogy (see Remark 5.5 for further details in this regard).

In a little more detail, the basic contents of this manuscript is as follows. In §2 we review some basic algebraic K-theory and give a new (and more conceptual) description of the 'extended boundary homomorphism' ∂ that was introduced in [17]. We also review relevant facts concerning homological algebra and the Euler characteristic construction of [8] and recall the definition and basic properties of equivariant Artin L-functions. In §3 we formulate our conjectural description of $\partial(L^*_{L/K,S}(1))$, prove some of its basis properties and describe its relation to the conjectures of Stark and Chinburg. In §4 we review the conjectural description of $\partial(L^*_{L/K,S}(0))$ that is formulated in [12]. In §5 we use the functional equation of the equivariant L-function and the 'additivity criterion' proved in [8] to investigate the compatibility of the conjectures of §3 and §4 with the conjecture of [2]. We also prove that, if the central conjecture of [2] is valid, then epsilon constants of symplectic characters are uniquely characterized by natural algebraic invariants. In §6 we review relevant facts concerning étale cohomology and then establish the relation of the conjecture formulated in §3 with the relevant special case of the equivariant Tamagawa number conjecture of [17].

The present article incorporates updated versions of the unpublished manuscripts [11] and [7].

2. Preliminaries

In this section we summarise the necessary background from algebraic K-theory and homological algebra. Furthermore we recall the definition of equivariant Artin L-functions.

2.1. **Algebraic K-theory.**

We recall the definition of the relative K_0-group and describe the extended boundary homomorphism which takes values in such a group.

2.1.1. *Relative K_0-groups.* For any integral domain R of characteristic 0, any extension E of the field of fractions of R and any finite group G let $K_0(R[G],E)$ denote the relative algebraic K-group associated to the ring homomorphism $R[G]\to E[G]$; a description of $K_0(R[G],E)$ in terms of generators and relations is given in [42, p. 215]. Writing $K_0(R[G])$ for the Grothendieck group of the category of finitely generated projective $R[G]$-modules and $K_1(R[G])$ for the Whitehead group there is a long exact sequence of relative K-theory

$$K_1(R[G]) \longrightarrow K_1(E[G]) \xrightarrow{\partial^1_G} K_0(R[G],E) \xrightarrow{\partial^0_G} K_0(R[G]) \xrightarrow{\partial_G} K_0(E[G]) \tag{1}$$

(cf. [42, Chap. 15] or [13] for more details). The exact sequence (1) is functorial in the pair (R,E). In those cases where it is not clear from the context we will write $\partial^i_{R[G],E}$ for the map ∂^i_G. The projective class group $\mathrm{Cl}(R[G])$ of $R[G]$ is defined to be the kernel of ∂_G and is in fact independent of E.

Let $\zeta(E[G])^\times$ denote the multiplicative group of the centre of $E[G]$. The reduced norm map induces a homomorphism $\mathrm{nr}: K_1(E[G])\to\zeta(E[G])^\times$ and we denote its image by $\zeta(E[G])^{\times+}$. In this paper E will always be either $\mathbb{R}$ or an algebraically closed field or a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$ for some prime number p. In all these cases the homomorphism $\mathrm{nr}: K_1(E[G])\to\zeta(E[G])^\times$ is injective (cf. [26, Th. (45.3)]) and we will always identify $K_1(E[G])$ and $\zeta(E[G])^{\times+}$ via nr. In particular we will consider $\partial^1_G=\partial^1_{R[G],E}$ as a map $\zeta(E[G])^{\times+}\to K_0(R[G],E)$. If E is algebraically closed or a finite extension of $\mathbb{Q}_p$ then $\zeta(E[G])^{\times+}=\zeta(E[G])^\times$.

We recall that in the case $R=\mathbb{Z}$ and $E=\mathbb{Q}$ the canonical maps $K_0(\mathbb{Z}[G],\mathbb{Q})\to K_0(\mathbb{Z}_p[G],\mathbb{Q}_p)$ induce an isomorphism

$$K_0(\mathbb{Z}[G],\mathbb{Q})\cong\bigoplus_p K_0(\mathbb{Z}_p[G],\mathbb{Q}_p) \tag{2}$$

where the sum is over all prime numbers p.

Our main interest will be the case $R = \mathbb{Z}$ and $E = \mathbb{R}$. For every prime number p and every embedding $j : \mathbb{R} \to \mathbb{C}_p$ we obtain an induced map $j_* : K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$.

Lemma 2.1. *The map*

$$K_0(\mathbb{Z}[G], \mathbb{R}) \to \prod_{p,j} K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$$

is injective. Here the product runs over all prime numbers p and all embeddings $j : \mathbb{R} \to \mathbb{C}_p$.

Proof. We consider the exact sequence (1) for the pairs $(R, E) = (\mathbb{Z}, \mathbb{Q})$, $(\mathbb{Z}, \mathbb{R})$, $(\mathbb{Z}_p, \mathbb{Q}_p)$ and $(\mathbb{Z}_p, \mathbb{C}_p)$ and the maps between these sequences which are induced by the obvious inclusions and by an embedding $j : \mathbb{R} \to \mathbb{C}_p$. An easy diagram chase shows that there is a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0(\mathbb{Z}[G], \mathbb{Q}) & \longrightarrow & K_0(\mathbb{Z}[G], \mathbb{R}) & \longrightarrow & K_1(\mathbb{R}[G])/K_1(\mathbb{Q}[G]) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) & \longrightarrow & K_0(\mathbb{Z}_p[G], \mathbb{C}_p) & \longrightarrow & K_1(\mathbb{C}_p[G])/K_1(\mathbb{Q}_p[G]) & \longrightarrow & 0. \end{array}$$

Therefore it suffices to show that the maps

$$K_0(\mathbb{Z}[G], \mathbb{Q}) \to \prod_{p,j} K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \tag{3}$$

and

$$K_1(\mathbb{R}[G])/K_1(\mathbb{Q}[G]) \to \prod_{p,j} K_1(\mathbb{C}_p[G])/K_1(\mathbb{Q}_p[G]) \tag{4}$$

are injective. The injectivity of (3) follows immediately from (2).

Let $x \in K_1(\mathbb{R}[G])$ be such that $j_*(x) \in K_1(\mathbb{Q}_p[G]) \subseteq K_1(\mathbb{C}_p[G])$ for all p and j. Using the isomorphism $\mathrm{nr} : K_1(\mathbb{R}[G]) \xrightarrow{\cong} \zeta(\mathbb{R}[G])^{\times +}$ we have $x = \sum_g c_g g \in \zeta(\mathbb{R}[G])^{\times +}$ such that

$$j_*(x) = \sum_g j(c_g) g \in \zeta(\mathbb{Q}_p[G])^{\times}. \tag{5}$$

We claim that $\sum_g c_g g \in \mathbb{Q}[G]$. Let $g \in G$ and consider the coefficient c_g. If c_g was transcendental over $\mathbb{Q}$ then there would be an embedding $j : \mathbb{R} \to \mathbb{C}_p$ such that $j(c_g) \notin \mathbb{Q}_p$ contradicting (5). Therefore c_g is algebraic over $\mathbb{Q}$. Now $j(c_g) \in \mathbb{Q}_p$ for all p and embeddings j implies that all primes are completely split in the number field $\mathbb{Q}(c_g)$ and therefore $\mathbb{Q}(c_g) = \mathbb{Q}$. Hence $x \in \zeta(\mathbb{R}[G])^{\times +} \cap \mathbb{Q}[G] = \zeta(\mathbb{Q}[G])^{\times +} = K_1(\mathbb{Q}[G])$. This shows the injectivity of (4). □

2.1.2. *The extended boundary homomorphism.* We give a conceptual description of the 'extended boundary homomorphism' introduced in [17, Lem. 9].

Lemma 2.2. *There is a unique homomorphism $\hat{\partial}^1_G : \zeta(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G], \mathbb{R})$ such that for every embedding $\mathbb{R} \to \mathbb{C}_p$ the diagram*

$$\begin{array}{ccc}
\zeta(\mathbb{R}[G])^\times & \xrightarrow{\hat{\partial}^1_G} & K_0(\mathbb{Z}[G], \mathbb{R}) \\
\downarrow & & \downarrow \\
\zeta(\mathbb{C}_p[G])^\times & \xrightarrow{\partial^1_{\mathbb{Z}_p[G], \mathbb{C}_p}} & K_0(\mathbb{Z}_p[G], \mathbb{C}_p)
\end{array}$$

commutes. The restriction of $\hat{\partial}^1_G$ to $\zeta(\mathbb{R}[G])^{\times +}$ is the map $\partial^1_{\mathbb{Z}[G],\mathbb{R}}$.

Proof. To define $\hat{\partial}^1_G(x)$ for $x \in \zeta(\mathbb{R}[G])^\times$ we choose $\lambda \in \zeta(\mathbb{Q}[G])^\times$ such that $\lambda x \in \zeta(\mathbb{R}[G])^{\times +}$ and set

$$\hat{\partial}^1_G(x) := \partial^1_{\mathbb{Z}[G],\mathbb{R}}(\lambda x) - \sum_p \partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p}(\lambda).$$

Here the sum is over all prime numbers p and $\partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p}(\lambda)$ means the following: consider λ as an element of $\zeta(\mathbb{Q}_p[G])^\times$ via the inclusion $\zeta(\mathbb{Q}[G])^\times \subseteq \zeta(\mathbb{Q}_p[G])^\times$ then apply $\partial^1_{\mathbb{Z}_p[G],\mathbb{Q}_p}$ and consider the result as an element of $K_0(\mathbb{Z}[G], \mathbb{R})$ by the inclusion $K_0(\mathbb{Z}_p[G], \mathbb{Q}_p) \subseteq K_0(\mathbb{Z}[G], \mathbb{Q}) \subseteq K_0(\mathbb{Z}[G], \mathbb{R})$. One easily checks that $\hat{\partial}^1_G$ is well defined and a homomorphism. Obviously $\hat{\partial}^1_G(x) = \partial^1_{\mathbb{Z}[G],\mathbb{R}}(x)$ for $x \in \zeta(\mathbb{R}[G])^{\times +}$. The diagram commutes for every embedding $j : \mathbb{R} \to \mathbb{C}_p$ because for all prime numbers $q \neq p$ one has $j_*(\partial^1_{\mathbb{Z}_q[G],\mathbb{Q}_q}(\lambda)) = 0$ in $K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$. The uniqueness assertion is a consequence of Lemma 2.1. □

Remark 2.3. The construction of the map $\hat{\partial}^1_G$ in the proof of Lemma 2.2 shows that this map is the same as the homomorphism $\hat{\delta}^1_{\mathbb{Z}[G],\mathbb{R}}$ introduced in [17, Lem. 9].

2.1.3. *Change of group.* Let R, E and G be as in §2.1.1 and let H be a subgroup of G. Since $R[G]$ is free as $R[H]$-module, restriction of scalars is a functor from projective $R[G]$-modules to projective $R[H]$-modules and similarly from $E[G]$-modules to $E[H]$-modules. Therefore one obtains canonical restriction maps res^G_H for all K-groups in the exact sequence (1). Using the identification via the reduced norm we also obtain a restriction map $\mathrm{res}^G_H : \zeta(E[G])^{\times +} \to \zeta(E[H])^{\times +}$.

The functor $M \mapsto R[G] \otimes_{R[H]} M$ from projective $R[H]$-modules to projective $R[G]$-modules and the corresponding functor from $E[H]$-modules to $E[G]$-modules induce induction maps ind^G_H for all K-groups in the exact sequence (1). Again one also obtains an induction map $\mathrm{ind}^G_H : \zeta(E[H])^{\times +} \to \zeta(E[G])^{\times +}$.

If H is a normal subgroup of G then the functor $M \mapsto M^H$ from projective $R[G]$-modules to projective $R[G/H]$-modules and the corresponding functor from $E[G]$-modules to $E[G/H]$-modules induce quotient maps $\mathrm{q}^G_{G/H}$ for all K-groups in the exact sequence (1). Again one also obtains a quotient map $\mathrm{q}^G_{G/H} : \zeta(E[G])^{\times +} \to \zeta(E[G/H])^{\times +}$.

The extended boundary homomorphism is compatible with the restriction, induction and quotient maps. More precisely, the map $\mathrm{res}^G_H : \zeta(\mathbb{C}[G])^\times \to \zeta(\mathbb{C}[H])^\times$ restricts to a homomorphism $\mathrm{res}^G_H : \zeta(\mathbb{R}[G])^\times \to \zeta(\mathbb{R}[H])^\times$, and one has $\hat{\partial}^1_H \circ \mathrm{res}^G_H = \mathrm{res}^G_H \circ \hat{\partial}^1_G$ as maps $\zeta(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[H], \mathbb{R})$. Similarly $\hat{\partial}^1_G \circ \mathrm{ind}^G_H = \mathrm{ind}^G_H \circ \hat{\partial}^1_H : \zeta(\mathbb{R}[H])^\times \to K_0(\mathbb{Z}[G], \mathbb{R})$ and $\hat{\partial}^1_{G/H} \circ \mathrm{q}^G_{G/H} = \mathrm{q}^G_{G/H} \circ \hat{\partial}^1_G : \zeta(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G/H], \mathbb{R})$ if H is normal in G.

2.1.4. *Involutions.* We recall the definition of the involution ψ_G^* of $K_0(\mathbb{Z}[G],\mathbb{R})$ which is defined in [12, p. 217]. If P is a projective $\mathbb{Z}[G]$-module then $\mathrm{Hom}_{\mathbb{Z}}(P,\mathbb{Z})$ is a projective $\mathbb{Z}[G]$-module when endowed with the contragredient G-action. Every element in $K_0(\mathbb{Z}[G],\mathbb{R})$ is represented by a triple $[P_1,\varphi,P_2]$ where P_1, P_2 are finitely generated projective $\mathbb{Z}[G]$-modules and $\varphi : P_1\otimes_{\mathbb{Z}}\mathbb{R}\to P_2\otimes_{\mathbb{Z}}\mathbb{R}$ is an isomorphism of $\mathbb{R}[G]$-modules. The involution ψ_G^* is defined by

$$\psi_G^*([P_1,\varphi,P_2]) := [\mathrm{Hom}_{\mathbb{Z}}(P_1,\mathbb{Z}),\mathrm{Hom}_{\mathbb{R}}(\varphi,\mathbb{R})^{-1},\mathrm{Hom}_{\mathbb{Z}}(P_2,\mathbb{Z})].$$

One can show that ψ_G^* is compatible with the change of group homomorphisms defined in §2.1.3, i.e. for a subgroup H of G one has $\mathrm{res}_H^G\circ\psi_G^* = \psi_H^*\circ\mathrm{res}_H^G$, $\mathrm{ind}_H^G\circ\psi_H^* = \psi_G^*\circ\mathrm{ind}_H^G$ and $\mathrm{q}_{G/H}^G\circ\psi_G^* = \psi_{G/H}^*\circ\mathrm{q}_{G/H}^G$ if H is normal in G.

Let $\zeta(\mathbb{C}[G])$ denote the centre of $\mathbb{C}[G]$ and note that there is a canonical isomorphism $\zeta(\mathbb{C}[G]) = \prod_{\chi\in\mathrm{Irr}(G)}\mathbb{C}$ where we write $\mathrm{Irr}(G)$ for the set of irreducible $\mathbb{C}$-valued characters of G. On $\zeta(\mathbb{C}[G])$ there exists a natural involution $x\mapsto x^{\#}$ which is induced by the $\mathbb{C}$-linear anti-involution of $\mathbb{C}[G]$ which sends each element of G to its inverse. If $x=(x_\chi)_{\chi\in\mathrm{Irr}(G)}$ under the isomorphism $\zeta(\mathbb{C}[G]) = \prod_{\chi\in\mathrm{Irr}(G)}\mathbb{C}$ then $x^{\#} = (x_{\overline{\chi}})_{\chi\in\mathrm{Irr}(G)}$. This involution of $\zeta(\mathbb{C}[G])$ restricts to an involution of $\zeta(\mathbb{R}[G])^\times$ which is compatible with the change of group homomorphisms from §2.1.3.

Up to sign the involutions on $K_0(\mathbb{Z}[G],\mathbb{R})$ and $\zeta(\mathbb{R}[G])^\times$ are compatible with the extended boundary homomorphism. More precisely, if $x\in\zeta(\mathbb{R}[G])^\times$ then $\psi_G^*(\hat\partial_G^1(x)) = -\hat\partial_G^1(x^{\#})$ in $K_0(\mathbb{Z}[G],\mathbb{R})$.

2.2. **Homological algebra.** We fix the sign conventions used for the homological algebra constructions in this paper. Furthermore we prove an important lemma concerning extension classes and recall the notion of an Euler characteristic with values in a relative algebraic K-group.

2.2.1. *Complexes.* Let R be a ring. By a complex we mean a cocomplex of left R-modules. For any complex A we write $A[1]$ for the shifted complex that is $A[1]^i = A^{i+1}$ with differential $d_{A[1]}(a) = -d_A(a)$. The mapping cone $\mathrm{cone}(\omega)$ of a map of complexes $\omega : U\to V$ is the complex defined by $\mathrm{cone}(\omega)^i = V^i\oplus U^{i+1}$ with differential $d_{\mathrm{cone}(\omega)}(v,u) = (d_V(v)+\omega(u), -d_U(u))$. Let $\mathcal{D}(R)$ denote the derived category of the abelian category of R-modules. A triangle

$$A\xrightarrow{\alpha}B\xrightarrow{\beta}C\xrightarrow{\gamma}A[1]$$

in $\mathcal{D}(R)$ is called distinguished if it is isomorphic to a triangle of the form

$$U\xrightarrow{\omega}V\xrightarrow{\iota}\mathrm{cone}(\omega)\xrightarrow{\pi}U[1]$$

where ω is a map of complexes, $\iota : V\to\mathrm{cone}(\omega)$ is the canonical inclusion $\iota(v) = (v,0)$, and $\pi : \mathrm{cone}(\omega)\to U[1]$ is the negative of the canonical projection that is $\pi(v,u) = -u$. For typographic reasons we often write a distinguished triangle as $A\xrightarrow{\alpha}B\xrightarrow{\beta}C\xrightarrow{\gamma}$.

For every short exact sequence of complexes $0\to A\xrightarrow{\alpha}B\xrightarrow{\beta}C\to 0$ there exists a canonical map $\gamma : C\to A[1]$ in the derived category such that $A\xrightarrow{\alpha}B\xrightarrow{\beta}C\xrightarrow{\gamma}A[1]$ is a distinguished triangle. Our choice of triangulation guarantees that the cohomology sequences of the short exact sequence and of the distinguished triangle are the same, that is that the map on cohomology induced by γ coincides with the connecting homomorphism of the short exact sequence.

2.2.2. *Yoneda extensions.* We always use injective resolutions of the second variable to identify Yoneda Ext-groups (as defined in [34, IV.§9]) with derived functor Ext-groups. For a natural interpretation of the connecting homomorphism for derived functor Ext-groups in terms of Yoneda extensions see [16, Lem. 3].

We will frequently interpret certain complexes in the derived category in terms of Yoneda extension classes as in [16, p. 1353]. To a complex E which is acyclic outside degrees 0 and $n \geq 1$ one associates the class $e(E) \in \mathrm{Ext}_R^{n+1}(H^n(E), H^0(E))$ which is given by the truncated complex $E' := \tau^{\leq n}\tau^{\geq 0}E$ with the induced maps $H^0(E) \xrightarrow{\cong} H^0(E') \to (E')^0$ and $(E')^n \to H^n(E') \xrightarrow{\cong} H^n(E)$ considered as a Yoneda extension.

Lemma 2.4. *Let E and F be complexes which are acyclic outside degrees* 0 *and $n \geq 1$. Let $\alpha : H^0(E) \to H^0(F)$ and $\beta : H^n(E) \to H^n(F)$ be homomorphisms of R-modules inducing maps $\alpha_* : \mathrm{Ext}_R^{n+1}(H^n(E), H^0(E)) \to \mathrm{Ext}_R^{n+1}(H^n(E), H^0(F))$ and $\beta^* : \mathrm{Ext}_R^{n+1}(H^n(F), H^0(F)) \to \mathrm{Ext}_R^{n+1}(H^n(E), H^0(F))$. There exists a morphism $\varphi : E \to F$ in $\mathcal{D}(R)$ which induces α on H^0 and β on H^n if and only if $\alpha_*(e(E)) = \beta^*(e(F))$. If in addition $\mathrm{Ext}_R^n(H^n(E), H^0(F)) = 0$, then the morphism φ with this property is unique.*

Proof. Without loss of generality we can assume that $E^i = 0$ and $F^i = 0$ unless $0 \leq i \leq n$. By interpreting the maps α_* and β^* in terms of Yoneda extensions and by the definition of equivalence of Yoneda extensions it is easy to see that $\alpha_*(e(E)) = \beta^*(e(F))$ implies the existence of a morphism $E \to F$ in the derived category with the required property. Conversely, if $\varphi : E \to F$ is such a morphism in $\mathcal{D}(R)$ then there exists a complex G with $G^i = 0$ unless $0 \leq i \leq n$, a quasi-isomorphism $\lambda : G \to E$ and a map of complexes $\mu : G \to F$ such that $\varphi = \mu \circ \lambda^{-1}$. This easily implies $\alpha_*(e(E)) = \beta^*(e(F))$.

To show the uniqueness it suffices to prove that if $\varphi : E \to F$ induces the zero map on H^0 and H^n then $\varphi = 0$ in $\mathcal{D}(R)$. We first observe that there is a distinguished triangle

$$H^n(F)[-n-1] \longrightarrow H^0(F)[0] \longrightarrow F \longrightarrow H^n(F)[-n] \tag{6}$$

which on cohomology induces the canonical maps. Indeed, if $\widetilde{F}$ denotes the complex $F^0 \to F^1 \to \cdots \to F^n \to H^n(F)$ with F^0 in degree 0 then (6) arises from the short exact sequence of complexes

$$0 \longrightarrow H^n(F)[-n-1] \longrightarrow \widetilde{F} \longrightarrow F \longrightarrow 0$$

and the quasi-isomorphism $H^0(F)[0] \to \widetilde{F}$. From (6) we obtain an exact sequence of abelian groups

$$\mathrm{Hom}_{\mathcal{D}(R)}(E, H^0(F)[0]) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(E, F) \longrightarrow \mathrm{Hom}_{\mathcal{D}(R)}(E, H^n(F)[-n]).$$

The image of φ in $\mathrm{Hom}_{\mathcal{D}(R)}(E, H^n(F)[-n]) = \mathrm{Hom}_R(H^n(E), H^n(F))$ is trivial thus φ is the image of a map $\psi \in \mathrm{Hom}_{\mathcal{D}(R)}(E, H^0(F)[0])$. There is a distinguished triangle for E similar to (6) which gives an exact sequence

$$\mathrm{Hom}_{\mathcal{D}(R)}(H^n(E)[-n], H^0(F)[0]) \to \mathrm{Hom}_{\mathcal{D}(R)}(E, H^0(F)[0]) \to \mathrm{Hom}_{\mathcal{D}(R)}(H^0(E)[0], H^0(F)[0]).$$

The image of ψ in $\mathrm{Hom}_{\mathcal{D}(R)}(H^0(E)[0], H^0(F)[0]) = \mathrm{Hom}_R(H^0(E), H^0(F))$ is trivial and by assumption $\mathrm{Hom}_{\mathcal{D}(R)}(H^n(E)[-n], H^0(F)[0]) = \mathrm{Ext}^n_R(H^n(E), H^0(F)) = 0$. Thus $\psi = 0$ and hence also $\varphi = 0$. □

2.2.3. *Euler characteristics.* Let R, E and G be as in §2.1.1. For any object C of $\mathcal{D}(R[G])$ we write $H^{\mathrm{ev}}(C)$ and $H^{\mathrm{od}}(C)$ for the direct sums $\oplus_{i\,\mathrm{even}} H^i(C)$ and $\oplus_{i\,\mathrm{odd}} H^i(C)$ where i runs over all even and all odd integers respectively.

We write $\mathcal{D}^{\mathrm{perf}}(R[G])$ for the full triangulated subcategory of $\mathcal{D}(R[G])$ consisting of those complexes that are perfect (i.e. isomorphic in $\mathcal{D}(R[G])$ to a bounded complex of finitely generated projective $R[G]$-modules). Let C be an object in $\mathcal{D}^{\mathrm{perf}}(R[G])$ and t a trivialisation of R (over E), that is an isomorphism of $E[G]$-modules $t : H^{\mathrm{ev}}(C) \otimes_R E \xrightarrow{\cong} H^{\mathrm{od}}(C) \otimes_R E$. We write $\chi_{R[G],E}(C, t)$ for the Euler characteristic in $K_0(R[G], E)$ that is defined in [8, Def. 5.5] (where it is denoted by $\chi_{R[G],E[G]}(C, t)$).

To compute certain Euler characteristics and to compare our constructions to related results in the literature we will occasionally use the explicit approach described in [13] and [8, §6]. By this approach we obtain an element $\chi^{\mathrm{old}}_{R[G],E}(C, t^{-1}) \in K_0(R[G], E)$ for a complex C and trivialisation t as above. For the precise relation of $\chi_{R[G],E}(C, t)$ and $\chi^{\mathrm{old}}_{R[G],E}(C, t^{-1})$ see [8, Th. 6.2].

If R and E are clear from the context we also use the notation $\chi_G(C, t)$ and $\chi^{\mathrm{old}}_G(C, t^{-1})$.

2.3. **The equivariant Artin L-function.** We fix some basic notations for number fields which is used in the rest of this paper and recall the definition of the equivariant Artin L-function.

2.3.1. *Notation for number fields.* Let L be a number field. We write $\mathcal{O}_L$ for the ring of integers of L and $S(L)$ for the set of all places of L. For any place $w \in S(L)$ we denote the completion of L at w by L_w. For a non-archimedean place w we write $\mathcal{O}_w$ for the ring of integers of L_w, $\mathfrak{m}_w$ for the maximal ideal of $\mathcal{O}_w$, $\lambda(w) := \mathcal{O}_w/\mathfrak{m}_w$ for the residue field and $Nw := |\lambda(w)|$ for the cardinality of the residue field.

If L is an extension of K and $v \in S(K)$ then $S_v(L)$ is the set of all places of L above v. This applies in particular to $K = \mathbb{Q}$ where either $v = p$ is a prime number or $v = \infty$ is the archimedean place. We also use the notation $S_f(L)$ for the set of all non-archimedean places, $S_{\mathbb{R}}(L)$ for the set of real archimedean places and $S_{\mathbb{C}}(L)$ for the set of complex archimedean places.

From now on let L/K be a Galois extension of number fields with Galois group G. For $w \in S(L)$ we write G_w for the decomposition group of w. For a non-archimedean place w we denote the inertia group by I_w and we let $\sigma_w \in G_w$ be any lift of the (arithmetic) Frobenius in G_w/I_w. For any place $v \in S(K)$ we set $L_v := \prod_{w \in S_v(L)} L_w$ and (if $v \in S_f(L)$) $\mathcal{O}_{L,v} := \prod_{w \in S_v(L)} \mathcal{O}_w$ and $\mathfrak{m}_{L,v} := \prod_{w \in S_v(L)} \mathfrak{m}_w$. Note that L_v, $\mathcal{O}_{L,v}$ and $\mathfrak{m}_{L,v}$ are G-modules in an obvious way.

Let S be a finite subset of $S(K)$. The G-stable set of places of L which lie above a place in S will be denoted by the same letter S. This should not cause any confusion because places of K will be called v and places of L will be called w. For a finite subset S of $S(K)$ which contains all archimedean places we let $\mathcal{O}_{L,S}$ be the ring of S-integers in L. Note that $\mathcal{O}_{L,S}$ is again a G-module and that in the case $S = S_\infty(K)$ one has $\mathcal{O}_L = \mathcal{O}_{L,S}$.

2.3.2. *The L-function.* Let G be a finite group and denote the set of all irreducible $\mathbb{C}$-valued characters of G by $\mathrm{Irr}(G)$. We write $\zeta(\mathbb{C}[G])$ for the centre of $\mathbb{C}[G]$ and remark that there exists a canonical isomorphism $\zeta(\mathbb{C}[G]) = \prod_{\chi\in\mathrm{Irr}(G)}\mathbb{C}$. Recall that a meromorphic $\zeta(\mathbb{C}[G])$-valued function is a function of a complex variable s of the form $s \mapsto g(s) = (g(\chi, s))_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)}\mathbb{C} = \zeta(\mathbb{C}[G])$ where each function $s \mapsto g(\chi, s)$ is meromorphic.

From now on let L/K be a Galois extension of number fields with Galois group G. The equivariant Artin L-function of L/K is a meromorphic $\zeta(\mathbb{C}[G])$-valued function which is first defined on a right half plane as a product of equivariant local L-factors and then extended to the whole complex plane by meromorphic continuation.

Let $v \in S(K)$. Choose a place $w \in S_v(L)$ and denote the decomposition group of w by G_w. The local L-factor $L_{L_w/K_v}(s)$ is the meromorphic $\zeta(\mathbb{C}[G_w])$-valued function given by

$$L_{L_w/K_v}(s) := (L_{L_w/K_v}(\psi, s))_{\psi\in\mathrm{Irr}(G_w)}.$$

Here for each $\psi \in \mathrm{Irr}(G_w)$ (and more generally for each not necessarily irreducible character ψ) the meromorphic function $L_{L_w/K_v}(\psi, s)$ is defined as follows. First we choose a $\mathbb{C}[G_w]$-module V_ψ with character ψ. In the case $v \in S_f(K)$ we then define

$$L_{L_w/K_v}(\psi, s) := \det_{\mathbb{C}}(1 - \sigma_w(Nv)^{-s} | V_\psi^{I_w})^{-1}$$

where (as in §2.3.1) I_w denotes the inertia group of w, $\sigma_w \in G_w$ is a lift of the arithmetic Frobenius in G_w/I_w and Nv is the cardinality of the residue field of v. In the case $v \in S_\infty(K)$ we set $n_\psi := \dim_{\mathbb{C}}(V_\psi)$, $n_\psi^+ := \dim_{\mathbb{C}}(V_\psi^{G_w})$ and $n_\psi^- := n_\psi - n_\psi^+$, and define

$$L_{L_w/K_v}(\psi, s) := \begin{cases} \left[2(2\pi)^{-s}\Gamma(s)\right]^{n_\psi} & \text{if } v \in S_{\mathbb{C}}(K) \\ \left[\pi^{-s/2}\Gamma(s/2)\right]^{n_\psi^+}\left[\pi^{-(s+1)/2}\Gamma((s+1)/2)\right]^{n_\psi^-} & \text{if } v \in S_{\mathbb{R}}(K). \end{cases}$$

We note that $L_{L_w/K_v}(\psi + \psi', s) = L_{L_w/K_v}(\psi, s) \cdot L_{L_w/K_v}(\psi', s)$ for two characters ψ, ψ', thus one can in fact define $L_{L_w/K_v}(\psi, s)$ for any virtual character ψ of G_w.

Let $\mathrm{ind}_{G_w}^G : \zeta(\mathbb{C}[G_w]) \to \zeta(\mathbb{C}[G])$ be the map

$$(\alpha_\psi)_{\psi\in\mathrm{Irr}(G_w)} \mapsto \Big(\prod_{\psi\in\mathrm{Irr}(G_w)} \alpha_\psi^{\langle \mathrm{ind}_{G_w}^G \psi, \chi\rangle_G}\Big)_{\chi\in\mathrm{Irr}(G)}.$$

The restriction of this map to $\zeta(\mathbb{C}[G_w])^\times$ is the induction map $\mathrm{ind}_{G_w}^G : \zeta(\mathbb{C}[G_w])^\times \to \zeta(\mathbb{C}[G])^\times$ defined in §2.1.3 so that using the same name for these maps is justified. The meromorphic $\zeta(\mathbb{C}[G])$-valued function $s \mapsto \mathrm{ind}_{G_w}^G(L_{L_w/K_v}(s))$ depends only on the place v and not on the choice of $w \in S_v(L)$.

For a finite subset S of $S(K)$, the S-truncated equivariant Artin L-function $L_{L/K,S}(s)$ is the meromorphic $\zeta(\mathbb{C}[G])$-valued function which for $\mathrm{Re}(s) > 1$ is defined by the product

$$L_{L/K,S}(s) := \prod_{v\in S(K)\setminus S} \mathrm{ind}_{G_w}^G(L_{L_w/K_v}(s)). \tag{7}$$

For $S = \emptyset$ the empty set we also write $\Lambda_{L/K}(s) := L_{L/K,\emptyset}(s)$. It is not difficult to see that $\mathrm{ind}_{G_w}^G(L_{L_w/K_v}(s)) = \epsilon_v(s)$ where $\epsilon_v(s)$ is defined in [2, §3.1] and that therefore our definition of the equivariant L-function agrees with the definitions in [2] and [12].

2.3.3. *The functional equation.* One has $\Lambda_{L/K}(s) = (\Lambda_{L/K}(\chi, s))_{\chi\in\mathrm{Irr}(G)}$ where $\Lambda_{L/K}(\chi, s)$ is the completed Artin L-function of the character χ as defined in [28, Chap. I, §5]. This implies that the function $\Lambda_{L/K}(s)$ and more generally the functions $L_{L/K,S}(s)$ for any finite set S have meromorphic continuations to the whole complex plane.

We recall that $\Lambda_{L/K}(\chi, s)$ satisfies the functional equation

$$\Lambda_{L/K}(\overline{\chi}, s) = \varepsilon_{L/K}(\chi, s)\Lambda_{L/K}(\chi, 1-s)$$

where $\varepsilon_{L/K}(\chi, s) = W(\overline{\chi})\cdot\left(|d_{K/\mathbb{Q}}|^{\deg(\chi)} Nf(\chi)\right)^{1/2-s}$ with $W(\overline{\chi})$ denoting the Artin root number, $d_{K/\mathbb{Q}}$ the discriminant of the extension $K/\mathbb{Q}$, $\deg(\chi)$ the degree of the character χ and $f(\chi)$ the conductor of χ, cf. [28, Chap. I, p. 38]. We define a $\zeta(\mathbb{C}[G])$-valued epsilon function by $\varepsilon_{L/K}(s) := (\varepsilon_{L/K}(\chi, s))_{\chi\in\mathrm{Irr}(G)}$. From the functional equations for $\Lambda_{L/K}(\chi, s)$ one obtains the equivariant functional equation

$$\Lambda_{L/K}(s)^{\#} = \varepsilon_{L/K}(s)\Lambda_{L/K}(1-s) \tag{8}$$

where # denotes the involution from §2.1.4.

2.3.4. *The leading terms.* For a meromorphic $\mathbb{C}$-valued function $g(s)$ of a complex variable s which has algebraic order d at a point s_0 we set $g^*(s_0) := \lim_{s\to s_0}(s-s_0)^{-d}g(s) \in \mathbb{C}^\times$. For a meromorphic $\zeta(\mathbb{C}[G])$-valued function $g(s) = (g(\chi, s))_{\chi\in\mathrm{Irr}(G)}$ we set $g^*(s_0) := (g^*(\chi, s_0))_{\chi\in\mathrm{Irr}(G)} \in \zeta(\mathbb{C}[G])^\times$.

Lemma 2.5. *Let L/K be a Galois extension of number fields with Galois group G. Let $s_0 \in \mathbb{R}$.*

(i) *For each $v \in S(K)$ and $w \in S_v(L)$ one has $L^*_{L_w/K_v}(s_0) \in \zeta(\mathbb{R}[G_w])^{\times +}$.*

(ii) *Let S be a finite subset of $S(K)$. Then $L^*_{L/K,S}(s_0) \in \zeta(\mathbb{R}[G])^\times$ and moreover $L^*_{L/K,S}(s_0) \in \zeta(\mathbb{R}[G])^{\times +}$ if $s_0 \geq 1$.*

Proof. Recall that an element $x = (x_\chi)_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)} \mathbb{C}^\times = \zeta(\mathbb{C}[G])^\times$ belongs to $\zeta(\mathbb{R}[G])^\times$, respectively $\zeta(\mathbb{R}[G])^{\times +}$, if and only if $\overline{x_\chi} = x_{\overline{\chi}}$ for all χ, respectively $x \in \zeta(\mathbb{R}[G])^\times$ and x_χ is a strictly positive real number whenever χ is symplectic.

To prove claim (i) we first note that $\overline{L_{L_w/K_v}(\psi, s)} = L_{L_w/K_v}(\overline{\psi}, \overline{s})$ which implies $L^*_{L_w/K_v}(s_0) \in \zeta(\mathbb{R}[G_w])^\times$ for $s_0 \in \mathbb{R}$. In the case $v \in S_\infty(K)$ the group G_w has no irreducible symplectic characters. In the case $v \in S_f(K)$ we observe that

$$L_{L_w/K_v}(\psi, s) = (1-(Nv)^{-s})^{-n_\psi^+}\cdot \det_{\mathbb{C}}(1-\sigma_w(Nv)^{-s}|V_\psi^{I_w}/V_\psi^{G_w})^{-1} \tag{9}$$

where $n_\psi^+ = \dim_{\mathbb{C}}(V_\psi^{G_w})$ (compare the proof of [12, Lem. 2.0.1]). For a symplectic character $\psi \in \mathrm{Irr}(G_w)$ the inequality $L^*_{L_w/K_v}(\psi, s_0) > 0$ follows because n_ψ is even and the eigenvalues of σ_w on $V_\psi^{I_w}/V_\psi^{G_w}$ are either -1 or occur in complex conjugate pairs.

The statement $L^*_{L/K,S}(s_0) \in \zeta(\mathbb{R}[G])^\times$ for $s_0 \in \mathbb{R}$ in claim (ii) follows from $\overline{L_{L/K,S}(\chi, s)} = L_{L/K,S}(\overline{\chi}, \overline{s})$ as above. Now let $\chi \in \mathrm{Irr}(G)$ be a symplectic character. For $s_0 > 1$ one has $\mathrm{ind}_{G_w}^G(L_{L_w/K_v}(s_0)) = \mathrm{ind}_{G_w}^G(L^*_{L_w/K_v}(s_0)) \in \zeta(\mathbb{R}[G])^{\times +}$ by part (i), and since the product (7) converges for $s_0 > 1$ this implies $L^*_{L/K,S}(\chi, s_0) = L_{L/K,S}(\chi, s_0) > 0$. It is well known that $L_{L/K,S}(\chi, s)$ has no zero or pole at $s = 1$, hence $L^*_{L/K,S}(\chi, 1) = \lim_{s\to 1, s>1} L_{L/K,S}(\chi, s) > 0$. This shows the result for $s_0 = 1$. □

3. The leading term at $s = 1$

Let L/K be a Galois extension of number fields with Galois group G. Let S be a finite subset of $S(K)$ which contains all archimedean places and all places which ramify in L/K and is such that $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$. In this section we formulate an explicit conjectural description of $\hat{\partial}^1_G(L^*_{L/K,S}(1))$, and then describe some of its basic properties.

3.1. **Statement of the conjecture.** Recall that for a place $v \in S(K)$ we write $L_v := \prod_{w \in S_v(L)} L_w$ and for v non-archimedean $\mathcal{O}_{L,v} := \prod_{w \in S_v(L)} \mathcal{O}_w$ and $\mathfrak{m}_{L,v} := \prod_{w \in S_v(L)} \mathfrak{m}_w$. For each $v \in S_\infty(K)$ we let $\exp : L_v \to L_v^\times$ denote the product of the (real or complex) exponential maps $L_w \to L_w^\times$ for $w \in S_v(L)$. If $v \in S_f(K)$, then for sufficiently large i the exponential map $\exp : \mathfrak{m}^i_{L,v} \to L_v^\times$ is the product of the p-adic exponential maps $\mathfrak{m}^i_w \to L_w^\times$ for $w \in S_v(L)$.

To state our conjecture we need to choose certain lattices. For each $v \in S_f := S \cap S_f(K)$, with residue characteristic p, we choose a full projective $\mathbb{Z}_p[G]$-lattice $\mathcal{L}_v \subseteq \mathcal{O}_{L,v}$ which is contained in a sufficiently large power of $\mathfrak{m}_{L,v}$ to ensure that the exponential map is defined on $\mathcal{L}_v$. Let $\mathcal{L}$ be the full projective $\mathbb{Z}[G]$-sublattice of $\mathcal{O}_L$ which has p-adic completions

$$\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \Big(\prod_{v \in S_p(K) \setminus S} \mathcal{O}_{L,v} \Big) \times \Big(\prod_{v \in S_p(K) \cap S} \mathcal{L}_v \Big). \tag{10}$$

We set $L_S := \prod_{v \in S} L_v$ and $\mathcal{L}_S := \prod_{v \in S} \mathcal{L}_v$ (where $\mathcal{L}_v := L_v$ for each $v \in S_\infty(K)$) and we let $\exp_S$ denote the map $\mathcal{L}_S \to L_S^\times$ that is induced by the product of the respective exponential maps. We also write Δ_S for the natural diagonal embedding from $L^\times$ to $L_S^\times$.

Following the notation of [40, Chap. VIII] we write I_L for the group of idèles of L and regard $L^\times$ as embedded diagonally in I_L. The idèle class group is $C_L := I_L/L^\times$ and the S-idèle class group is $C_S(L) := I_L/(L^\times U_{L,S})$, where $U_{L,S} := \prod_{w \in S}\{1\} \times \prod_{w \notin S} \mathcal{O}_w^\times$. We remark that since $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$, the natural map $L_S^\times \to C_S(L)$ is surjective with kernel $\Delta_S(\mathcal{O}_{L,S}^\times)$.

There exists a canonical invariant isomorphism

$$\mathrm{inv}_{L/K,S} : H^2(G, C_S(L)) \xrightarrow{\cong} \frac{1}{|G|}\mathbb{Z}/\mathbb{Z}.$$

Indeed, the quotient map $C_L \to C_S(L)$ induces an isomorphism $H^2(G, C_L) \xrightarrow{\cong} H^2(G, C_S(L))$ and from class field theory one has a canonical invariant isomorphism $\mathrm{inv}_{L/K} : H^2(G, C_L) \xrightarrow{\cong} \frac{1}{|G|}\mathbb{Z}/\mathbb{Z}$ (as defined, for example, in [40, p. 379]). We let $e^{\mathrm{glob}}_{L/K,S}$ (or e^{glob}_S when L/K is clear from context) denote the global canonical class, i.e. the element of $\mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_S(L)) = H^2(G, C_S(L))$ that is sent by $\mathrm{inv}_{L/K,S}$ to $\frac{1}{|G|}$.

Let E_S be a complex in $\mathcal{D}(\mathbb{Z}[G])$ which corresponds (via §2.2.2) to e^{glob}_S. By Lemma 2.4 there exists a unique morphism $\alpha_S : \mathcal{L}_S[0] \oplus \mathcal{L}[-1] \to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$ for which $H^0(\alpha_S)$ is the composite $\mathcal{L}_S \xrightarrow{\exp_S} L_S^\times \to C_S(L)$ and $H^1(\alpha_S)$ is the restriction of the trace map $\mathrm{tr} : L \to \mathbb{Q}$ to $\mathcal{L}$. Let $E_S(\mathcal{L})$ be any complex which lies in a distinguished triangle in $\mathcal{D}(\mathbb{Z}[G])$ of the form

$$\mathcal{L}_S[0] \oplus \mathcal{L}[-1] \xrightarrow{\alpha_S} E_S \xrightarrow{\beta_S} E_S(\mathcal{L}) \xrightarrow{\gamma_S} . \tag{11}$$

To describe the cohomology of $E_S(\mathcal{L})$ we use the following notation. Let $L_\infty := \prod_{w\in S_\infty(L)} L_w$ and define $\mathrm{tr}_\infty : L_\infty \to \mathbb{R}$ by $(l_w)_{w\in S_\infty(L)} \mapsto \sum_{w\in S_\infty(L)} \mathrm{tr}_{L_w/\mathbb{R}}(l_w)$. We set $L^0_\infty := \ker(\mathrm{tr}_\infty)$ and $L^0 := \ker(\mathrm{tr} : L \to \mathbb{Q})$, and observe that one has a commutative diagram of short exact sequences of $\mathbb{R}[G]$-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^0 \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow{\subset} & L \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow{\mathrm{tr}\otimes_{\mathbb{Q}}\mathbb{R}} & \mathbb{R} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_L} & & \downarrow{\scriptstyle \mu'_L} & & \| & & \\
0 & \longrightarrow & L^0_\infty & \xrightarrow{\subset} & L_\infty & \xrightarrow{\mathrm{tr}_\infty} & \mathbb{R} & \longrightarrow & 0,
\end{array}$$

where μ'_L denotes the canonical isomorphism and μ_L its restriction to $L^0 \otimes_{\mathbb{Q}} \mathbb{R}$. We also use the notation $\exp_\infty : L_\infty \to L^\times_\infty$ for the product of the exponential maps and $\Delta_\infty : L^\times \to L^\times_\infty$ for the diagonal embedding. Finally we set $\log_\infty(\mathcal{O}^\times_L) := \{x \in L_\infty : \exp_\infty(x) \in \Delta_\infty(\mathcal{O}^\times_L)\}$ and remark that $\log_\infty(\mathcal{O}^\times_L)$ is a full lattice in L^0_∞.

Our conjectural formula for $\hat\partial^1_G(L^*_{L/K,S}(1))$ uses the Euler characteristic defined in part (iii) of the following lemma. In the sequel we shall abbreviate 'cohomologically-trivial' to 'c-t'.

Lemma 3.1. *The complex $E_S(\mathcal{L})$ defined by the distinguished triangle (11) has the following properties.*

(i) *$E_S(\mathcal{L})$ is a perfect complex of $\mathbb{Z}[G]$-modules.*

(ii) *$E_S(\mathcal{L}) \otimes \mathbb{Q}$ is acyclic outside degrees -1 and 0, and there exist canonical identifications of $\mathbb{Q}[G]$-modules $H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q} \cong \{x \in \mathcal{L}_S : \exp_S(x) \in \Delta_S(\mathcal{O}^\times_L)\} \otimes \mathbb{Q} \cong \log_\infty(\mathcal{O}^\times_L) \otimes \mathbb{Q}$ and $H^0(E_S(\mathcal{L})) \otimes \mathbb{Q} \cong L^0$.*

(iii) *The identifications from (ii) and the canonical isomorphism $\log_\infty(\mathcal{O}^\times_L) \otimes \mathbb{R} \cong L^0_\infty$ allow us to consider μ_L as a trivialisation of $E_S(\mathcal{L})$. The Euler characteristic $\chi_G(E_S(\mathcal{L}), \mu_L)$ in $K_0(\mathbb{Z}[G], \mathbb{R})$ depends only upon L/K and S.*

Proof. Since e^{glob}_S is a generator of $H^2(G, C_S(L))$, cup-product with e^{glob}_S induces isomorphisms between the (Tate) cohomology groups of $\mathbb{Z}$ and $C_S(L)$ (with a dimension shift of 2) and hence in $\mathcal{D}(\mathbb{Z}[G])$ the complex E_S is isomorphic to a bounded complex of G-modules each of which is c-t. Since this is also obviously true for $\mathcal{L}_S[0] \oplus \mathcal{L}[-1]$, the triangle (11) implies that $E_S(\mathcal{L})$ is isomorphic to a bounded complex of G-modules each of which is c-t. Claim (i) will therefore follow if we can show the cohomology of $E_S(\mathcal{L})$ to be finitely generated. But the exact cohomology sequence associated to (11) implies that $E_S(\mathcal{L})$ is acyclic outside degrees $-1, 0$ and 1 and that there is an exact sequence

$$0 \to H^{-1}(E_S(\mathcal{L})) \to \mathcal{L}_S \to C_S(L) \to H^0(E_S(\mathcal{L})) \to \mathcal{L} \xrightarrow{\mathrm{tr}} \mathbb{Z} \to H^1(E_S(\mathcal{L})) \to 0.$$

This immediately shows that $H^1(E_S(\mathcal{L}))$ is finite and $H^1(E_S(\mathcal{L})) \otimes \mathbb{Q} = 0$. The map $\mathcal{L}_S \to C_S(L)$ in the exact sequence is the composite $\mathcal{L}_S \xrightarrow{\exp_S} L^\times_S \to C_S(L)$ whose cokernel is easily seen to be finite, hence $H^0(E_S(\mathcal{L}))$ is finitely generated and there is an identification $H^0(E_S(\mathcal{L})) \otimes \mathbb{Q} \cong L^0$. Finally $H^{-1}(E_S(\mathcal{L})) \cong \{x \in \mathcal{L}_S : \exp_S(x) \in \Delta_S(\mathcal{O}^\times_L)\}$ and the projection map $\mathcal{L}_S \to L_\infty$ induces an isomorphism between this set and $\log_\infty(U) := \{x \in L_\infty : \exp_\infty(x) \in \Delta_\infty(U)\}$ where U is a subgroup of finite index in $\mathcal{O}_L^\times$. Since $\log_\infty(U)$ is a full lattice in L_∞^0 we see that $H^{-1}(E_S(\mathcal{L}))$ is finitely generated which completes the proof of (i). Moreover $\log_\infty(U) \otimes \mathbb{Q} = \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{Q}$ which completes the proof of (ii).

The element $\chi_G(E_S(\mathcal{L}), \mu_L)$ does not depend on the choice of E_S or of the distinguished triangle (11) because up to isomorphism in $\mathcal{D}(\mathbb{Z}[G])$ the complex $E_S(\mathcal{L})$ is independent of these choices. It remains to prove that $\chi_G(E_S(\mathcal{L}), \mu_L)$ is independent of the choice of $\mathcal{L}$. For each $v \in S_f$ we let $\mathcal{L}'_v \subseteq \mathcal{O}_{L,v}$ be lattices giving rise to a lattice $\mathcal{L}' \subseteq \mathcal{O}_L$ as above. We assume (as we may) that $\mathcal{L}'_v \subseteq \mathcal{L}_v$ for all $v \in S_f$ so $\mathcal{L}' \subseteq \mathcal{L}$. We set $\mathcal{L}'_S := \prod_{v\in S} \mathcal{L}'_v$ and consider the following commutative diagram of distinguished triangles in $\mathcal{D}(\mathbb{Z}[G])$ (the existence of such a diagram follows for example from [1, Prop. 1.1.11]).

$$\begin{array}{ccccccc}
\mathcal{L}'_S[0]\oplus\mathcal{L}'[-1] & \longrightarrow & E_S & \longrightarrow & E_S(\mathcal{L}') & \longrightarrow & \\
\downarrow{\scriptstyle\subseteq} & & \| & & \downarrow & & \\
\mathcal{L}_S[0]\oplus\mathcal{L}[-1] & \longrightarrow & E_S & \longrightarrow & E_S(\mathcal{L}) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{L}_S/\mathcal{L}'_S[0]\oplus\mathcal{L}/\mathcal{L}'[-1] & \longrightarrow & 0 & \longrightarrow & \mathcal{L}_S/\mathcal{L}'_S[1]\oplus\mathcal{L}/\mathcal{L}'[0] & = & \\
\downarrow & & \downarrow & & \downarrow & &
\end{array}$$

In this diagram the first two rows are distinguished triangles as in (11) and the first column is induced by the obvious short exact sequence. Now the G-modules $\mathcal{L}_S/\mathcal{L}'_S$ and $\mathcal{L}/\mathcal{L}'$ are c-t, finite and isomorphic. Hence the zero map is a trivialisation of $\mathcal{L}_S/\mathcal{L}'_S[1] \oplus \mathcal{L}/\mathcal{L}'[0]$ and the associated Euler characteristic is $\chi_G(\mathcal{L}_S/\mathcal{L}'_S[1] \oplus \mathcal{L}/\mathcal{L}'[0], 0) = 0 \in K_0(\mathbb{Z}[G], \mathbb{R})$. Upon applying [8, Th. 5.7] to the third column of the above diagram we thus deduce that $\chi_G(E_S(\mathcal{L}'), \mu_L) = \chi_G(E_S(\mathcal{L}), \mu_L)$, as required. □

Remark 3.2. It is occasionally convenient to give an explicit representative of $E_S(\mathcal{L})$ in the following way. Fix an extension $0 \to C_S(L) \xrightarrow{\subseteq} A \xrightarrow{d} B \to \mathbb{Z} \to 0$, which represents $e_S^{\mathrm{glob}} \in \mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_S(L))$ and in which the G-modules A and B are c-t. Then E_S can be taken to be the complex $A \xrightarrow{d} B$ in degrees 0 and 1, and the morphism $\mathcal{L}_S[0] \oplus \mathcal{L}[-1] \to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$ is represented by the map α of complexes which in degree 0 is $\mathcal{L}_S \xrightarrow{\exp_S} L_S^\times \to C_S(L) \subseteq A$ and in degree 1 is any lift $\mathcal{L} \xrightarrow{\mathrm{tr}'} B$ of $\mathcal{L} \xrightarrow{\mathrm{tr}} \mathbb{Z}$ through the given surjection $B \to \mathbb{Z}$.

The complex $E_S(\mathcal{L})$ can be taken to be the mapping cone of this map α of complexes, that is

$$\mathcal{L}_S \xrightarrow{(d^{-1},0)} A \oplus \mathcal{L} \xrightarrow{d^0} B$$

where $\mathcal{L}_S$ is placed in degree -1, $d^{-1} : \mathcal{L}_S \to A$ is as above and $d^0 = (d, \mathrm{tr}')$. Note that for these complexes the distinguished triangle (11) which gives rise to the identification of the cohomology of $E_S(\mathcal{L}) \otimes \mathbb{Q}$ in Lemma 3.1(ii) is

$$\mathcal{L}_S[0] \oplus \mathcal{L}[-1] \xrightarrow{\alpha} E_S \xrightarrow{\beta} E_S(\mathcal{L}) \xrightarrow{\gamma} \mathcal{L}_S[1] \oplus \mathcal{L}[0]$$

where α is as described above, β is the identity on A and B, and γ is minus the identity on $\mathcal{L}_S$ and $\mathcal{L}$.

We now formulate our conjectural description of $\hat{\partial}^1_G(L^*_{L/K,S}(1))$.

Conjecture 3.3. In $K_0(\mathbb{Z}[G],\mathbb{R})$ one has $\hat{\partial}^1_G(L^*_{L/K,S}(1)) = -\chi_G(E_S(\mathcal{L}),\mu_L)$.

3.2. **Basic properties.** To describe some basic properties of Conjecture 3.3 it is convenient to set

$$T\Omega(L/K,1) := \hat{\partial}^1_G(L^*_{L/K,S}(1)) + \chi_G(E_S(\mathcal{L}),\mu_L) \in K_0(\mathbb{Z}[G],\mathbb{R}).$$

Proposition 3.4. *The element $T\Omega(L/K,1)$ depends only upon L/K.*

Proof. Since $\chi_G(E_S(\mathcal{L}),\mu_L)$ depends only upon L/K and S (by Lemma 3.1(iii)) it suffices to prove that $T\Omega(L/K,1)$ is unchanged if one replaces S by $S' = S\cup\{v'\}$ where v' is any element of $S(K)\setminus S$. But $L^*_{L/K,S}(1) = L^*_{L/K,S'}(1)\cdot \mathrm{ind}^G_{G_w}(L^*_{L_w/K_{v'}}(1))$ where as in §2.3.2 we fix $w\in S_{v'}(L)$ and let G_w be the decomposition group of w. Hence we must show that

$$\chi_G(E_{S'}(\mathcal{L}'),\mu_L) - \chi_G(E_S(\mathcal{L}),\mu_L) = \hat{\partial}^1_G(\mathrm{ind}^G_{G_w}(L^*_{L_w/K_{v'}}(1))). \tag{12}$$

To simplify the notation we set $\mathcal{O}' := \mathcal{O}_{L,v'}$, $U' := \mathcal{O}^\times_{L,v'}$ and, for each integer $i\geq 1$, $U'^{(i)} := \prod_{w\in S_{v'}(L)}(1+\mathfrak{m}^i_w)$. Let π be a uniformising parameter for $\mathcal{O}_{v'}$. We choose a lattice $\mathcal{L}'\subseteq\mathcal{L}$ for S' such that $\mathcal{L}'_v = \mathcal{L}_v$ for $v\in S$ and $\mathcal{L}'_{v'} = \pi^m\mathcal{O}'$ where m is a sufficiently large integer. Note that $\pi^m\mathcal{O}'$ is a projective $\mathbb{Z}_p[G]$-lattice since v' is unramified in L/K and that $\exp(\mathcal{L}'_{v'}) = \exp(\pi^m\mathcal{O}') = U'^{(m)}$.

There is a canonical short exact sequence $0\to U'\to C_{S'}(L)\to C_S(L)\to 0$, and since the G-module U' is c-t one obtains an isomorphism $H^2(G,C_{S'}(L))\xrightarrow{\cong} H^2(G,C_S(L))$. This isomorphism maps $e^{\mathrm{glob}}_{S'}$ to e^{glob}_S and hence Lemma 2.4 implies the existence of a morphism $\kappa: E_{S'}\to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$ which induces the identity on H^1 and the map $C_{S'}(L)\to C_S(L)$ on H^0. We consider the following commutative diagram of distinguished triangles in $\mathcal{D}(\mathbb{Z}[G])$.

$$\begin{array}{ccccccc}
\mathcal{L}'_{v'}[0]\oplus(\mathcal{L}/\mathcal{L}')[-2] & \longrightarrow & \mathcal{L}'_{S'}[0]\oplus\mathcal{L}'[-1] & \longrightarrow & \mathcal{L}_S[0]\oplus\mathcal{L}[-1] & \longrightarrow & \\
\downarrow & & \downarrow{\scriptstyle \alpha'_S} & & \downarrow{\scriptstyle \alpha_S} & & \\
U'[0] & \longrightarrow & E_{S'} & \xrightarrow{\kappa} & E_S & \longrightarrow & \\
\downarrow & & \downarrow{\scriptstyle \beta_{S'}} & & \downarrow{\scriptstyle \beta_S} & & \\
(U'/U'^{(m)})[0]\oplus(\mathcal{L}/\mathcal{L}')[-1] & \longrightarrow & E_{S'}(\mathcal{L}') & \longrightarrow & E_S(\mathcal{L}) & \longrightarrow & \\
\downarrow & & \downarrow{\scriptstyle \gamma_{S'}} & & \downarrow{\scriptstyle \gamma_S} & &
\end{array}$$

In this diagram the upper row is induced by the obvious short exact sequences $0\to\mathcal{L}'_{v'}\to\mathcal{L}'_{S'}\to\mathcal{L}_S\to 0$ and $0\to\mathcal{L}'\to\mathcal{L}\to\mathcal{L}/\mathcal{L}'\to 0$; the first column is the distinguished triangle induced by the exact sequence $0\to\mathcal{L}'_{v'}\xrightarrow{\exp} U'\to U'/U'^{(m)}\to 0$; the second and third columns are the distinguished triangles defined as in (11). The existence of the diagram follows by applying [1, Prop. 1.1.11] to the upper right square, then rotating the resulting diagram and observing that the arrow $\mathcal{L}'_{v'}[0]\oplus(\mathcal{L}/\mathcal{L}')[-2]\to U'[0]$ is indeed as stated. Now the complex $(U'/U'^{(m)})[0]\oplus(\mathcal{L}/\mathcal{L}')[-1]$ has finite cohomology groups and so by applying [8, Th. 5.7] to the third row of the diagram we find that

$$\chi_G(E_{S'}(\mathcal{L}'), \mu_L) - \chi_G(E_S(\mathcal{L}), \mu_L) = \chi_G((U'/U'^{(m)})[0] \oplus (\mathcal{L}/\mathcal{L}')[-1], 0). \tag{13}$$

To compute further we recall that any finite c-t G-module M gives rise to a canonical element (M) of $K_0(\mathbb{Z}[G], \mathbb{Q}) \subseteq K_0(\mathbb{Z}[G], \mathbb{R})$ and that if we consider $M[-i]$ as an object of $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$, then $\chi_G(M[-i], 0) = (-1)^{i+1}(M)$. Hence one has

$$\begin{aligned}\chi_G((U'/U'^{(m)})[0] \oplus (\mathcal{L}/\mathcal{L}')[-1], 0) &= -(U'/U'^{(m)}) + (\mathcal{L}/\mathcal{L}') \\ &= -(U'/U'^{(1)}) - (U'^{(1)}/U'^{(m)}) + (\mathcal{O}'/\pi\mathcal{O}') + (\pi\mathcal{O}'/\pi^m\mathcal{O}').\end{aligned}$$

From the isomorphisms $U'^{(i)}/U'^{(i+1)} \cong \pi^i\mathcal{O}'/\pi^{i+1}\mathcal{O}'$ for all $i \geq 1$ one deduces $(U'^{(1)}/U'^{(m)}) = (\pi\mathcal{O}'/\pi^m\mathcal{O}')$. In addition, writing $\lambda(w)$ for the residue field of $\mathcal{O}_w$, one has $\mathcal{O}'/\pi\mathcal{O}' \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} \lambda(w)$ and $U'/U'^{(1)} = \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} \lambda(w)^\times$, and so the last displayed formula implies that

$$\chi_G((U'/U'^{(m)})[0] \oplus (\mathcal{L}/\mathcal{L}')[-1], 0) = \mathrm{ind}_{G_w}^G(-(\lambda(w)^\times) + (\lambda(w))) \tag{14}$$

where $(\lambda(w)^\times)$ and $(\lambda(w))$ are considered as elements of $K_0(\mathbb{Z}[G_w], \mathbb{R})$. Now the exact sequence $0 \to \mathbb{Z}[G_w] \xrightarrow{Nv' - \sigma_w} \mathbb{Z}[G_w] \to \lambda(w)^\times \to 0$ implies $(\lambda(w)^\times) = \hat{\partial}^1_{G_w}\left((\det_{\mathbb{C}}(Nv' - \sigma_w \mid V_\psi))_{\psi \in \mathrm{Irr}(G_w)}\right)$ where Nv', σ_w and V_ψ are as in §2.3.2. Also, if v' has residue characteristic ℓ and residue degree f (i.e. $Nv' = \ell^f$) then $\lambda(w)$ is a free $(\mathbb{Z}/\ell)[G_w]$-module of rank f and so

$$\begin{aligned}(\lambda(w)) &= [\mathbb{Z}[G_w]^f, \cdot\ell, \mathbb{Z}[G_w]^f] \\ &= [\mathbb{Z}[G_w], \cdot Nv', \mathbb{Z}[G_w]] = \hat{\partial}^1_{G_w}\left((\det_{\mathbb{C}}(Nv' \mid V_\psi))_{\psi \in \mathrm{Irr}(G_w)}\right).\end{aligned}$$

It follows that

$$\begin{aligned}-(\lambda(w)^\times) + (\lambda(w)) &= \hat{\partial}^1_{G_w}\left((\det_{\mathbb{C}}(1 - \sigma_w(Nv')^{-1} \mid V_\psi)^{-1})_{\psi \in \mathrm{Irr}(G_w)}\right) \\ &= \hat{\partial}^1_{G_w}(L^*_{L_w/K_{v'}}(1)).\end{aligned} \tag{15}$$

Equations (13), (14) and (15) imply (12) which completes the proof. □

We next describe the behaviour of $T\Omega(L/K, 1)$ under the maps discussed in §2.1.3.

Proposition 3.5. *Let M be an intermediate field of L/K and $H = \mathrm{Gal}(L/M)$. Then*

(i) $\mathrm{res}^G_H(T\Omega(L/K, 1)) = T\Omega(L/M, 1)$.
(ii) *If H is normal in G, then* $\mathrm{q}^G_{G/H}(T\Omega(L/K, 1)) = T\Omega(M/K, 1)$.

Proof. Let $S = S_K$ be a finite set of places of K satisfying all conditions necessary to formulate Conjecture 3.3 for L/K. Then the set S_M consisting of all places of M lying above a place in S satisfies the corresponding conditions with respect to L/M. Further, if $v \in S_f$ has residue characteristic p, then one can choose a full projective $\mathbb{Z}_p[H]$-lattice $\mathcal{L}_u \subset \mathcal{O}_{L,u}$ for every $u \in S_v(M)$ such that $\mathcal{L}_v := \prod_{u \in S_v(M)} \mathcal{L}_u \subseteq \prod_{u \in S_v(M)} \mathcal{O}_{L,u} = \mathcal{O}_{L,v}$ is a full projective $\mathbb{Z}_p[G]$-lattice. We thus obtain the same lattice $\mathcal{L} \subseteq \mathcal{O}_L$ for the extensions L/K and L/M, and also $\mathcal{L}_{S_K} = \mathcal{L}_{S_M}$. Now $e^{\mathrm{glob}}_{L/M,S_M}$ is the image of $e^{\mathrm{glob}}_{L/K,S_K}$ under the restriction map $H^2(G, C_S(L)) \to H^2(H, C_S(L))$ and so the complex $E_{L/M,S_M}$ can be taken to be equal to $E_{L/K,S_K}$ with the group action restricted from G to H. It follows that the trivialised complex $(E_{L/M,S_M}(\mathcal{L}), \mu_L)$ for the extension L/M can be taken to equal

$(E_{L/K,S_K}(\mathcal{L}),\mu_L)$ for the extension L/K (again with the group action restricted from G to H) and so

$$\mathrm{res}^G_H(\chi_G(E_{L/K,S_K}(\mathcal{L}),\mu_L)) = \chi_H(E_{L/M,S_M}(\mathcal{L}),\mu_L) \in K_0(\mathbb{Z}[H],\mathbb{R}).$$

Claim (i) now follows from the equality $\mathrm{res}^G_H(\hat{\partial}^1_G(L^*_{L/K,S_K}(1))) = \hat{\partial}^1_H(L^*_{L/M,S_M}(1)) \in K_0(\mathbb{Z}[H],\mathbb{R})$ (which is itself a consequence of the fact that $\mathrm{res}^G_H(L^*_{L/K,S_K}(1)) = L^*_{L/M,S_M}(1) \in \zeta(\mathbb{R}[H])^\times$).

To prove claim (ii) we fix a finite set S of places of K containing all archimedean places and all places which ramify in L/K (and hence all which ramify in M/K) and which is sufficiently large to ensure that both $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$ and $\mathrm{Pic}(\mathcal{O}_{M,S}) = 0$. Set $Q := G/H$. One has $\mathrm{q}^G_Q(L^*_{L/K,S}(1)) = L^*_{M/K,S}(1) \in \zeta(\mathbb{R}[Q])^\times$ and hence $\mathrm{q}^G_Q(\hat{\partial}^1_G(L^*_{L/K,S}(1))) = \hat{\partial}^1_Q(L^*_{M/K,S}(1)) \in K_0(\mathbb{Z}[Q],\mathbb{R})$. In addition, if we have chosen lattices $\mathcal{L}_v \subseteq \mathcal{O}_{L,v}$ and $\mathcal{L} \subseteq \mathcal{O}_L$ as in (10) with respect to $(L/K,S)$, then $\mathcal{L}^H_v \subseteq \mathcal{O}^H_{L,v} = \mathcal{O}_{M,v}$ and $\mathcal{L}^H \subseteq \mathcal{O}_M$ satisfy (10) with respect to $(M/K,S)$, and so we need only show that $\mathrm{q}^G_Q(\chi_G(E_{L/K,S}(\mathcal{L}),\mu_L)) = \chi_Q(E_{M/K,S}(\mathcal{L}^H),\mu_M) \in K_0(\mathbb{Z}[Q],\mathbb{R})$.

We first make the following general observation which follows easily from the description of Euler characteristics given in [8, §6]. If $C \in \mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ is a complex of c-t G-modules and $t : H^{\mathrm{ev}}(C\otimes\mathbb{R}) \to H^{\mathrm{od}}(C\otimes\mathbb{R})$ a trivialisation of C then $\mathrm{q}^G_Q(\chi_G(C,t)) = \chi_Q(C^H,t^H)$ in $K_0(\mathbb{Z}[Q],\mathbb{R})$ where $C^H \in \mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[Q])$ is the complex of H-invariants and t^H is the trivialisation $H^{\mathrm{ev}}(C^H\otimes\mathbb{R}) \cong (H^{\mathrm{ev}}(C\otimes\mathbb{R}))^H \xrightarrow{t} (H^{\mathrm{od}}(C\otimes\mathbb{R}))^H \cong H^{\mathrm{od}}(C^H\otimes\mathbb{R})$.

To apply this we represent $e^{\mathrm{glob}}_{L/K,S}$ by an extension of $\mathbb{Z}[G]$-modules $0 \to C_S(L) \xrightarrow{\subseteq} A \xrightarrow{d} B \xrightarrow{\kappa} \mathbb{Z} \to 0$ with A and B c-t, and take $E_{L/K,S}$ to be the complex $A \xrightarrow{d} B$ (in degrees 0 and 1). Since $|H|\cdot e^{\mathrm{glob}}_{L/K,S}$ is the image of $e^{\mathrm{glob}}_{M/K,S}$ under the inflation map $H^2(Q,C_S(M)) \to H^2(G,C_S(L))$, the extension of $\mathbb{Z}[Q]$-modules $0 \to C_S(M) \xrightarrow{\subseteq} A^H \xrightarrow{d} B^H \xrightarrow{\kappa'} \mathbb{Z} \to 0$ with $\kappa' := \frac{1}{|H|}\kappa$ represents $e^{\mathrm{glob}}_{M/K,S}$. Thus we can take $E_{M/K,S}$ to be the complex $A^H \xrightarrow{d} B^H$. When combined with the fact that $\mathrm{tr}_{M/\mathbb{Q}} = \frac{1}{|H|}\mathrm{tr}_{L/\mathbb{Q}}$ (on M) and the explicit construction of $E_{L/K,S}(\mathcal{L})$ and $E_{M/K,S}(\mathcal{L}^H)$ in Remark 3.2 we see that $E_{M/K,S}(\mathcal{L}^H) = (E_{L/K,S}(\mathcal{L}))^H$ and $\mu_M = (\mu_L)^H$. This gives the required equality $\mathrm{q}^G_Q(\chi_G(E_{L/K,S}(\mathcal{L}),\mu_L)) = \chi_Q((E_{L/K,S}(\mathcal{L}))^H,(\mu_L)^H) = \chi_Q(E_{M/K,S}(\mathcal{L}^H),\mu_M)$. □

3.3. **The conjectures of Stark and Chinburg.** We show that Conjecture 3.3 refines both Stark's Conjecture at $s = 1$ (as discussed by Tate in [45, Chap. I, Conj. 8.2]) and also Chinburg's 'Ω_1-Conjecture' (as formulated in [24, Question 3.2] and [20, §4.2, Conj. 3]).

Proposition 3.6. *Let $T\Omega(L/K,1)$ be the element of $K_0(\mathbb{Z}[G],\mathbb{R})$ defined in §3.2 (so Conjecture 3.3 is equivalent to an equality $T\Omega(L/K,1) = 0$). Then both of the following assertions are valid.*

(i) $T\Omega(L/K,1) \in K_0(\mathbb{Z}[G],\mathbb{Q})$ *if and only if Stark's conjecture at $s = 1$ is valid for L/K.*

(ii) $T\Omega(L/K,1) \in \ker(K_0(\mathbb{Z}[G],\mathbb{R}) \xrightarrow{\partial^0_G} K_0(\mathbb{Z}[G]))$ *if and only if Chinburg's 'Ω_1-Conjecture' is valid for L/K.*

Proof. Since the $\mathbb{R}[G]$-modules $L^0 \otimes_{\mathbb{Q}} \mathbb{R}$ and $\log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{R}$ are isomorphic we may choose an isomorphism $\kappa : L^0 \xrightarrow{\cong} \log_\infty(\mathcal{O}_L^\times) \otimes \mathbb{Q}$ of $\mathbb{Q}[G]$-modules. Then $\chi_G(E_S(\mathcal{L}), \kappa \otimes_{\mathbb{Q}} \mathbb{R}) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ and so $T\Omega(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if $\chi_G(E_S(\mathcal{L}), \kappa \otimes_{\mathbb{Q}} \mathbb{R}) - T\Omega(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$. But [8, Prop. 5.6.2] implies that

$$\chi_G(E_S(\mathcal{L}), \kappa \otimes_{\mathbb{Q}} \mathbb{R}) - \chi_G(E_S(\mathcal{L}), \mu_L) = \partial_G^1([L^0 \otimes_{\mathbb{Q}} \mathbb{R}, \lambda])$$

with $\lambda := \mu_L^{-1} \circ (\kappa \otimes_{\mathbb{Q}} \mathbb{R})$. Since $\zeta(\mathbb{Q}[G])^\times$ is the full pre-image of $K_0(\mathbb{Z}[G], \mathbb{Q})$ under the map $\hat{\partial}_G^1 : \zeta(\mathbb{R}[G])^\times \to K_0(\mathbb{Z}[G], \mathbb{R})$ and $\partial_G^1([L^0 \otimes_{\mathbb{Q}} \mathbb{R}, \lambda]) = \hat{\partial}_G^1(\mathrm{nr}(\lambda))$, it follows that $T\Omega(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if $L^*_{L/K,S}(1)^{-1} \cdot \mathrm{nr}(\lambda) \in \zeta(\mathbb{Q}[G])^\times$.

Now $\zeta(\mathbb{Q}[G])^\times$ is equal to the subgroup of elements $(z_\psi)_{\psi \in \mathrm{Irr}(G)}$ of $\zeta(\mathbb{C}[G])^\times = \prod_{\psi \in \mathrm{Irr}(G)} \mathbb{C}^\times$ with the property that $\omega(z_\psi) = z_{\omega \circ \psi}$ for all $\omega \in \mathrm{Aut}(\mathbb{C})$ and all $\psi \in \mathrm{Irr}(G)$. Since for each $\psi \in \mathrm{Irr}(G)$ one has $L^*_{L/K,S}(1)_\psi = L^*_{L/K,S}(\psi, 1)$ and $\mathrm{nr}(\lambda)_\psi = \det(\lambda \mid \mathrm{Hom}_{\mathbb{C}[G]}(V_\psi, L^0 \otimes_{\mathbb{Q}} \mathbb{C}))$ where we write λ for the $\mathbb{C}$-linear automorphism of $\mathrm{Hom}_{\mathbb{C}[G]}(V_\psi, L^0 \otimes_{\mathbb{Q}} \mathbb{C})$ which is induced by $\lambda \otimes_{\mathbb{R}} \mathbb{C}$, we deduce that $T\Omega(L/K, 1) \in K_0(\mathbb{Z}[G], \mathbb{Q})$ if and only if for each $\psi \in \mathrm{Irr}(G)$ and each $\omega \in \mathrm{Aut}(\mathbb{C})$ one has

$$\omega\left(\frac{\det(\lambda \mid \mathrm{Hom}_{\mathbb{C}[G]}(V_\psi, L^0 \otimes_{\mathbb{Q}} \mathbb{C}))}{L^*_{L/K,S}(\psi, 1)}\right) = \frac{\det(\lambda \mid \mathrm{Hom}_{\mathbb{C}[G]}(V_{\omega\circ\psi}, L^0 \otimes_{\mathbb{Q}} \mathbb{C}))}{L^*_{L/K,S}(\omega \circ \psi, 1)}.$$

To show that this condition is equivalent to the validity of [45, Chap. I, Conj. 8.2] for each $\psi \in \mathrm{Irr}(G)$ one need only mimic the proof of [45, Chap. I, Prop. 6.1]. Indeed, the last displayed equality is the variant of [45, Chap. I, Conj. 8.2] that is alluded to in [45, top of p. 35]. This proves claim (i).

Chinburg's 'Ω_1-Conjecture' asserts the vanishing of the element $\Omega(L/K, 1)$ of $\mathrm{Cl}(\mathbb{Z}[G]) \subseteq K_0(\mathbb{Z}[G])$ that is defined in [24, Def. 3.1]. Since Lemma 2.5(ii) implies that $\partial_G^0(\hat{\partial}_G^1(L^*_{L/K,S}(1))) = 0$, claim (ii) will follow if $\partial_G^0(\chi_G(E_S(\mathcal{L}), \mu_L)) = \Omega(L/K, 1)$. After enlarging S if necessary, we may assume that $\mathcal{L}$ is a free $\mathbb{Z}[G]$-module (indeed one can take $\mathcal{L}$ to be a suitable integer multiple of any free $\mathcal{O}_K[G]$-submodule of $\mathcal{O}_L$, and then define $\mathcal{L}_v$ by (10)). We choose an extension $0 \to C_S(L) \xrightarrow{\subseteq} A \xrightarrow{d} B \to \mathbb{Z} \to 0$ representing e_S^{glob} as in Remark 3.2 but with the additional condition that B is a finitely generated free $\mathbb{Z}[G]$-module. We set $\mathcal{L}_f := \prod_{v \in S_f} \mathcal{L}_v$, we write $\exp(\mathcal{L}_f)$ for the image of $\mathcal{L}_f \subset \mathcal{L}_S$ under the composite $\mathcal{L}_S \xrightarrow{\exp_S} L_S^\times \to C_S(L)$ and set $A_{\mathcal{L}} := A/\exp(\mathcal{L}_f)$. Then one has a short exact sequence of complexes (with vertical differentials) of the form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & B & = & B & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle d^0} & & \uparrow & & \\
0 & \longrightarrow & \exp(\mathcal{L}_f) & \longrightarrow & A \oplus \mathcal{L} & \longrightarrow & A_{\mathcal{L}} \oplus \mathcal{L} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \exp_S} & & \uparrow{\scriptstyle (d^{-1},0)} & & \uparrow{\scriptstyle (d',0)} & & \\
0 & \longrightarrow & \mathcal{L}_f & \xrightarrow{\subset} & \mathcal{L}_S & \longrightarrow & L_\infty & \longrightarrow & 0
\end{array}$$

where the central column is the representative of $E_S(\mathcal{L})$ described in Remark 3.2. Since the left complex is acyclic this sequence implies that $\chi_G(E_S(\mathcal{L}), \mu_L) = \chi_G(E_S(\mathcal{L})', \mu_L)$ where $E_S(\mathcal{L})'$ denotes the complex given by the right column of the diagram. From [8, Prop. 5.6.1] we may therefore deduce that $\partial_G^0(\chi_G(E_S(\mathcal{L}), \mu_L)) \equiv \chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{d'} A_{\mathcal{L}}) \bmod F(\mathbb{Z}[G])$ where we write $\chi_{\mathbb{Z}[G]}$ for the Euler characteristic with values in $K_0(\mathbb{Z}[G])$ and $F(\mathbb{Z}[G])$ for the subgroup of $K_0(\mathbb{Z}[G])$ which is generated by the class $[\mathbb{Z}[G]]$.

For each $v \in S_\infty(K)$ we now fix a place w of L above v and a finitely generated $\mathbb{Z}[G_w]$-submodule W_v of $L_w^\times$ which satisfies the conditions of [24, Lem. 2.1(i), (ii)]. We let W_∞ denote the $\mathbb{Z}[G]$-submodule $\prod_{v\in S_\infty(K)} \mathrm{Ind}_{G_w}^{G}(W_v)$ of $L_\infty^\times$. Both $\Delta_S(\mathcal{O}_{L,S}^\times)$ and $\exp_S(\mathcal{L}_f)$ lie in the subgroup $W_\infty \times \prod_{w\in S_f} L_w^\times$ of $L_S^\times$, and we set $C_{L,S}^f := (W_\infty \times \prod_{w\in S_f} L_w^\times)/(\Delta_S(\mathcal{O}_{L,S}^\times)\cdot \exp_S(\mathcal{L}_f))$. Then we may construct a commutative diagram of the form

$$
\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
0 & \longrightarrow & C_{L,S}^f & \longrightarrow & A' & \longrightarrow & B & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\subset} & & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & C_S(L)/\exp(\mathcal{L}_f) & \xrightarrow{e} & A'_{\mathcal{L}} & \longrightarrow & B & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & L_\infty^\times/W_\infty & = & L_\infty^\times/W_\infty & & & & & & \\
 & & \downarrow & & \downarrow & & & & & & \\
 & & 0 & & 0. & & & & & &
\end{array}
\tag{16}
$$

In this diagram the exactness of the left hand column is clear and hence, since (by definition of W_∞) the G-module $L_\infty^\times/W_\infty$ is c-t, this column induces an isomorphism $\iota : \mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_{L,S}^f) \cong \mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_S(L)/\exp(\mathcal{L}_f))$. The upper row is chosen to be a representative of the pre-image under ι of the element corresponding to the exact sequence

$$0 \to C_S(L)/\exp(\mathcal{L}_f) \to A_{\mathcal{L}} \xrightarrow{d} B \to \mathbb{Z} \to 0, \tag{17}$$

and the second row of (16) is then constructed via the push-out of the given maps $C_{L,S}^f \to A'$ and $C_{L,S}^f \to C_S(L)/\exp(\mathcal{L}_f)$. The second column of (16) is thus exact and hence, since $A_{\mathcal{L}}$ and $L_\infty^\times/W_\infty$ are both c-t, the G-module A', resp. $A'_{\mathcal{L}}$, is finitely generated and c-t, resp. c-t. Now the complexes $L_\infty \xrightarrow{d'} A_{\mathcal{L}}$ and $L_\infty \xrightarrow{d''} A'_{\mathcal{L}}$ (where d'' is the composite of $L_\infty \xrightarrow{\exp_\infty} L_\infty^\times \to C_S(L)/\exp(\mathcal{L}_f)$ and the map e from (16)) are isomorphic in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}[G])$ since (17) and the second row of (16) represent the same element of $\mathrm{Ext}^2_{\mathbb{Z}[G]}(\mathbb{Z}, C_S(L)/\exp(\mathcal{L}_f))$, and hence $\chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{d'} A_{\mathcal{L}}) = \chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{d''} A'_{\mathcal{L}})$. In addition, by its very definition, $\Omega(L/K,1) = \chi_{\mathbb{Z}[G]}(A' \to B) \equiv \chi_{\mathbb{Z}[G]}(A'[0]) \bmod F(\mathbb{Z}[G])$. From the exact sequence of perfect complexes (with vertical differentials)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A'_{\mathcal{L}} & \longrightarrow & L_\infty^\times/W_\infty & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow{\scriptstyle d''} & & \uparrow{\scriptstyle \exp_\infty} & & \\
0 & \longrightarrow & 0 & \longrightarrow & L_\infty & = & L_\infty & \longrightarrow & 0
\end{array}
$$

(where the upper row comes from (16)) we may therefore deduce that

$$\begin{aligned}\partial^0_G(\chi_G(E_S(\mathcal{L}),\mu_L)) &\equiv \chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{d'} A_{\mathcal{L}})\\ &= \chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{d''} A'_{\mathcal{L}})\\ &= \chi_{\mathbb{Z}[G]}(A'[0]) + \chi_{\mathbb{Z}[G]}(L_\infty \xrightarrow{\exp_\infty} L^\times_\infty/W_\infty)\\ &\equiv \Omega(L/K,1) + \sum_{v\in S_\infty(K)} \mathrm{Ind}^G_{G_w}(\chi_{\mathbb{Z}[G_w]}(L_w \xrightarrow{\exp} L^\times_w/W_v))\end{aligned}$$

where, for each $v \in S_\infty(K)$, $\mathrm{Ind}^G_{G_w}$ is the natural induction map $K_0(\mathbb{Z}[G_w]) \to K_0(\mathbb{Z}[G])$. But $\chi_{\mathbb{Z}[G_w]}(L_w \xrightarrow{\exp} L^\times_w/W_v) \equiv 0 \bmod F(\mathbb{Z}[G_w])$ for every $v \in S_\infty(K)$ because $\mathrm{Cl}(\mathbb{Z}[G_w]) = 0$ (since $|G_w| \leq 2$). So the last displayed formula implies that $\partial^0_G(\chi_G(E_S(\mathcal{L}),\mu_L)) \equiv \Omega(L/K,1) \bmod F(\mathbb{Z}[G])$, and since both sides lie in $\mathrm{Cl}(\mathbb{Z}[G])$ this shows $\partial^0_G(\chi_G(E_S(\mathcal{L}),\mu_L)) = \Omega(L/K,1)$ as required. □

4. The leading term at $s = 0$

As in §3 we consider a Galois extension L/K of number fields with Galois group G and a finite set S of places of K containing all archimedean places, all places ramified in L/K, and for which $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$. In this section we formulate a conjectural description of $\hat{\partial}^1_G(L^*_{L/K,S}(0))$ in terms of a natural Euler characteristic. An equivalent form of this conjecture has already been studied in [12], see Remark 4.3 below.

We will use the following standard notation. If T is any finite set of places of K then Y_T denotes the G-module $Y_T := \prod_{w\in T}\mathbb{Z}$ where the product is over all places w of L lying above a place in T. There is a natural augmentation map $\mathrm{aug} : Y_T \to \mathbb{Z}$ and we define X_T to be its kernel.

Let P_S be a complex in $\mathcal{D}(\mathbb{Z}[G])$ which corresponds to the canonical extension class in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_S, \mathcal{O}^\times_{L,S})$ that is defined in [43] (see also [24] and [45, Chap. II] for a discussion of this class). Then P_S is a perfect complex which is acyclic outside degrees 0 and 1, and there are isomorphisms $H^0(P_S) \cong \mathcal{O}^\times_{L,S}$ and $H^1(P_S) \cong X_S$. Let $\mathrm{Reg}_S : \mathcal{O}^\times_{L,S} \to X_S \otimes_{\mathbb{Z}} \mathbb{R}$ be the regulator map $\mathrm{Reg}_S(u) := (\log|u|_w)_{w\in S}$, where the absolute values $|\cdot|_w$ are normalised as in [45, Chap. 0, §0]. It induces an isomorphism $\mathcal{O}^\times_{L,S} \otimes_{\mathbb{Z}} \mathbb{R} \to X_S \otimes_{\mathbb{Z}} \mathbb{R}$ of $\mathbb{R}[G]$-modules which we again denote by Reg_S. We take the negative of this regulator as trivialisation of the complex P_S. Recall that in §2.1.4 we defined an involution ψ^*_G of $K_0(\mathbb{Z}[G],\mathbb{R})$.

Conjecture 4.1. One has $\hat{\partial}^1_G(L^*_{L/K,S}(0)) = -\psi^*_G(\chi_G(P_S, -\mathrm{Reg}_S))$ in $K_0(\mathbb{Z}[G],\mathbb{R})$.

To study Conjecture 4.1 it is convenient to set

$$T\Omega(L/K,0) := \psi^*_G\big(\hat{\partial}^1_G(L^*_{L/K,S}(0)^\#) - \chi_G(P_S, -\mathrm{Reg}_S)\big) \in K_0(\mathbb{Z}[G],\mathbb{R}).$$

Since $\psi^*_G(\hat{\partial}^1_G(L^*_{L/K,S}(0)^\#)) = -\hat{\partial}^1_G(L^*_{L/K,S}(0))$ (see §2.1.4), Conjecture 4.1 is equivalent to the equality $T\Omega(L/K,0) = 0$. The element $T\Omega(L/K,0)$ and therefore also Conjecture 4.1 depend only on the extension L/K, as can be seen by an argument similar to the proof of Proposition 3.4. Alternatively this follows from [12, Th. 2.1.2] and Remark 4.3 below.

Remark 4.2. The invariant $T\Omega(L/K,0)$ is functorial in the field extension, i.e. if M is an intermediate field of L/K and $H = \mathrm{Gal}(L/M)$ then

(i) $\mathrm{res}^G_H(T\Omega(L/K,0)) = T\Omega(L/M,0)$, and
(ii) $\mathrm{q}^G_{G/H}(T\Omega(L/K,0)) = T\Omega(M/K,0)$ if H is normal in G.

To show this one can apply an argument similar to the proof of Proposition 3.5, or alternatively use [12, Prop. 2.1.4] and the following remark.

Remark 4.3. In [12, Th. 2.1.2] an invariant $T\Omega(L/K,0) \in K_0(\mathbb{Z}[G],\mathbb{R})$ is defined by (in our notation) $T\Omega(L/K,0) := \psi^*_G\big(\hat{\partial}^1_G(L^*_{L/K,S}(0)^\#) + \chi^{\mathrm{old}}_G(\Psi_S,(-\mathrm{Reg}_S)^{-1})\big)$. The complex Ψ_S used here is defined in [16, Prop. 3.1] (see also [12, Prop. 2.1.1]). The extension class of Ψ_S in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_S,\mathcal{O}^\times_{L,S})$ with respect to an injective resolution of $\mathcal{O}^\times_{L,S}$ is the negative of Tate's canonical class used to define P_S (this is not clear in [12] and [16] because various sign conventions are not specified there; however one can check that $T\Omega(L/K,0)$ in [12] would not be independent of S if the extension class of Ψ_S was Tate's canonical class itself and not its negative). Therefore $\chi^{\mathrm{old}}_G(\Psi_S,(-\mathrm{Reg}_S)^{-1}) = -\chi_G(\Psi_S,-\mathrm{Reg}_S) + \partial^1_G[X_S\otimes\mathbb{R},-\mathrm{id}] = -\chi_G(P_S,-\mathrm{Reg}_S)$ which shows that our definition of $T\Omega(L/K,0)$ agrees with the definition in [12]. Thus from [12, §2.2, §2.3] we can deduce the following results.

Proposition 4.4. *Let L/K be any Galois extension of number fields of group G.*

(i) *$T\Omega(L/K,0)$ belongs to $K_0(\mathbb{Z}[G],\mathbb{Q})$ if and only if the main conjecture of Stark at $s=0$ (as interpreted by Tate in* [45, Chap. I, Conj. (5.1)]*) is valid for L/K.*
(ii) *$T\Omega(L/K,0)$ belongs to the torsion subgroup of $K_0(\mathbb{Z}[G],\mathbb{Q})$ if and only if the Strong-Stark conjecture (as formulated by Chinburg in* [23, Conj. 2.2]*) is valid for L/K.*
(iii) *$\partial^0_G(\psi^*_G(T\Omega(L/K,0))) = W_{L/K} - \Omega(L/K,3)$ where $W_{L/K}$ is the 'Cassou-Noguès-Fröhlich root number class' and $\Omega(L/K,3)$ is the element defined by Chinburg in* [24] *(and denoted by $\Omega^m(L/K)$ in* [23]*). In particular, the vanishing of $\partial^0_G(T\Omega(L/K,0))$ is equivalent to the 'Ω_3-conjecture' that is formulated in* [24].
(iv) *$T\Omega(L/K,0) = 0$ if and only if the 'Lifted Root Number Conjecture' of Gruenberg, Ritter and Weiss* [32] *is valid for L/K.*

5. Functional equation compatibility

In the previous two sections we formulated conjectures for the leading terms of the equivariant Artin L-function at $s=0$ and $s=1$. In this section we show that the compatibility of these conjectures with respect to the functional equation of the equivariant Artin L-function gives rise to a natural conjecture for the ε-constant.

5.1. Statement of the main result.

Let L/K be a Galois extension of number fields, G its Galois group and S a finite set of places of K as in §3 and §4. Before we can formulate the main result we must introduce an invariant encoding certain semilocal information about the extension L/K.

5.1.1. *Definition of the semilocal terms.* The definition of the following invariants is motivated by similar constructions in [2]; see Remark 5.4 for a detailed comparison.

Let v be a place in S_f and denote its residue characteristic by p. We choose $w \in S_v(L)$ and let M_w be a complex in $\mathcal{D}(\mathbb{Z}[G_w])$ which corresponds to the local canonical class in $\mathrm{Ext}^2_{\mathbb{Z}[G_w]}(\mathbb{Z},L^\times_w) = H^2(G_w,L^\times_w)$, i.e. the pre-image of $\frac{1}{|G_w|}$ under the local invariant isomorphism $H^2(G_w,L^\times_w) \xrightarrow{\mathrm{inv}_{L_w/K_v}} \frac{1}{|G_w|}\mathbb{Z}/\mathbb{Z}$. Furthermore we choose a full projective $\mathbb{Z}_p[G_w]$-sublattice $\mathcal{L}_w$ of $\mathcal{O}_w$ which is contained in a sufficiently large power of $\mathfrak{m}_w$. The exponential map $\exp : \mathcal{L}_w \to L_w^\times = H^0(M_w)$ induces a morphism $\exp : \mathcal{L}_w[0] \to M_w$ in $\mathcal{D}(\mathbb{Z}[G_w])$ and we define a complex $M_w(\mathcal{L}_w)$ by the distinguished triangle

$$\mathcal{L}_w[0] \xrightarrow{\exp} M_w \longrightarrow M_w(\mathcal{L}_w) \longrightarrow .$$

From the corresponding cohomology sequence we see that $M_w(\mathcal{L}_w)$ is acyclic outside degrees 0 and 1, and that there are identifications $H^0(M_w(\mathcal{L}_w)) \cong L_w^\times/\exp(\mathcal{L}_w)$ and $H^1(M_w(\mathcal{L}_w)) \cong \mathbb{Z}$. Moreover one easily sees that the complex $M_w(\mathcal{L}_w)$ is perfect. The (normalised) valuation $L_w^\times \to \mathbb{Z}$ induces a trivialisation $\nu_w : H^0(M_w(\mathcal{L}_w) \otimes \mathbb{R}) \xrightarrow{\cong} H^1(M_w(\mathcal{L}_w) \otimes \mathbb{R})$ and we can consider the Euler characteristic $\chi_{G_w}(M_w(\mathcal{L}_w), \nu_w) \in K_0(\mathbb{Z}[G_w], \mathbb{R})$. We also define

$$m_w := \frac{\alpha_w \cdot L^*_{L_w/K_v}(0)^\#}{L^*_{L_w/K_v}(1)} \in \zeta(\mathbb{R}[G_w])^{\times +} \tag{18}$$

where $\alpha_w = (\alpha_{w,\chi})_{\chi \in \mathrm{Irr}(G_w)} \in \prod_{\chi \in \mathrm{Irr}(G_w)} \mathbb{C}^\times = \zeta(\mathbb{C}[G_w])^\times$ is the element with $\alpha_{w,\chi} = \log(Nw)$ if χ is the trivial character and $\alpha_{w,\chi} = 1$ otherwise.

From the local lattices $\mathcal{L}_w \subseteq \mathcal{O}_w$ we obtain a global lattice $\mathcal{L} \subseteq \mathcal{O}_L$ as follows. For each $v \in S_f$, with residue characteristic p, we first define a $\mathbb{Z}_p[G]$-lattice $\mathcal{L}_v$ by $\mathcal{L}_v := \mathbb{Z}_p[G] \otimes_{\mathbb{Z}_p[G_w]} \mathcal{L}_w \subseteq \mathcal{O}_{L,v}$. We then define the $\mathbb{Z}[G]$-lattice $\mathcal{L} \subseteq \mathcal{O}_L$ by specifying its completions as in (10).

Let $\Sigma(L)$ denote the set of all embeddings $L \to \mathbb{C}$. Then $H_L := \prod_{\sigma \in \Sigma(L)} \mathbb{Z}$ is a $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module and we write $\rho_L : L \otimes_\mathbb{Q} \mathbb{C} \to H_L \otimes_\mathbb{Z} \mathbb{C}$ for the $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-equivariant isomorphism $l \otimes z \mapsto (\sigma(l)z)_{\sigma \in \Sigma(L)}$ (note that $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts only on the second factor of $L \otimes_\mathbb{Q} \mathbb{C}$ but on both factors of $H_L \otimes_\mathbb{Z} \mathbb{C}$). For any $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-module X we write X^+ and X^- for the submodules on which complex conjugation acts by $+1$ and -1 respectively. We define π_L to be the composite isomorphism of $\mathbb{R}[G]$-modules

$$\begin{aligned} L \otimes_\mathbb{Q} \mathbb{R} &= (L \otimes_\mathbb{Q} \mathbb{C})^+ \\ &\xrightarrow{\rho_L} (H_L \otimes_\mathbb{Z} \mathbb{C})^+ \\ &= (H_L^+ \otimes_\mathbb{Z} \mathbb{R}) \oplus (H_L^- \otimes_\mathbb{Z} (i\mathbb{R})) \\ &\xrightarrow{\mathrm{id} \oplus (\cdot i)} (H_L^+ \otimes_\mathbb{Z} \mathbb{R}) \oplus (H_L^- \otimes_\mathbb{Z} \mathbb{R}) \\ &= H_L \otimes_\mathbb{Z} \mathbb{R}. \end{aligned}$$

The map π_L depends on the choice of $i = \sqrt{-1} \in \mathbb{C}$, but one can check that the element $[\mathcal{L}, \pi_L, H_L] \in K_0(\mathbb{Z}[G], \mathbb{R})$ is independent of this choice.

We now set

$$R\Omega^{\mathrm{loc}}(L/K, 1) := [\mathcal{L}, \pi_L, H_L] + \sum_{v \in S_f} \mathrm{ind}_{G_w}^G \left(\chi_{G_w}(M_w(\mathcal{L}_w), \nu_w) + \partial^1_{G_w}(m_w) \right)$$

and

$$T\Omega^{\mathrm{loc}}(L/K, 1) := \hat{\partial}^1_G(\varepsilon_{L/K}(0)) - R\Omega^{\mathrm{loc}}(L/K, 1)$$

in $K_0(\mathbb{Z}[G], \mathbb{R})$. One can show that $R\Omega^{\mathrm{loc}}(L/K, 1)$ and $T\Omega^{\mathrm{loc}}(L/K, 1)$ depend only on the extension L/K.

Remark 5.1. The invariant $T\Omega^{\mathrm{loc}}(L/K, 1)$ is functorial in the field extension, i.e. if M is an intermediate field of L/K and $H = \mathrm{Gal}(L/M)$ then

(i) $\mathrm{res}^G_H(T\Omega^{\mathrm{loc}}(L/K,1)) = T\Omega^{\mathrm{loc}}(L/M,1)$, and
(ii) $\mathrm{q}^G_{G/H}(T\Omega^{\mathrm{loc}}(L/K,1)) = T\Omega^{\mathrm{loc}}(M/K,1)$ if H is normal in G.

5.1.2. *The comparison result.* We can now state the main result which describes the relation of the invariants $T\Omega(L/K,0)$ and $T\Omega(L/K,1)$, and therefore of the conjectures for the leading terms at $s=0$ and $s=1$.

Theorem 5.2. *One has*

$$\psi^*_G(T\Omega(L/K,0)) - T\Omega(L/K,1) = T\Omega^{\mathrm{loc}}(L/K,1)$$

in $K_0(\mathbb{Z}[G],\mathbb{R})$.

We will use the functional equation of the equivariant Artin L-function to prove Theorem 5.2 in §5.2. By this theorem the leading term Conjectures 3.3 and 4.1 force the following conjecture for the epsilon constant.

Conjecture 5.3. One has $T\Omega^{\mathrm{loc}}(L/K,1) = 0$ in $K_0(\mathbb{Z}[G],\mathbb{R})$. Equivalently, one has $\hat{\partial}^1_G(\varepsilon_{L/K}(0)) = R\Omega^{\mathrm{loc}}(L/K,1)$ in $K_0(\mathbb{Z}[G],\mathbb{R})$.

Remark 5.4. Conjecture 5.3 is equivalent to the conjecture formulated by Bley and the second named author in [2, Conj. 4.1]. To see this we first recall that, in the notation of that paper, [2, Conj. 4.1] is the conjectural equality $\mathcal{E}_{L/K} = \delta_{L/K}(\mathcal{L}) + \sum_{v\in S_f} I_G(v,\mathcal{L})$ in $K_0(\mathbb{Z}[G],\mathbb{R})$ for S and $\mathcal{L}$ as in §5.1.1. We claim that $T\Omega^{\mathrm{loc}}(L/K,1) = \mathcal{E}_{L/K} - \delta_{L/K}(\mathcal{L}) - \sum_{v\in S_f} I_G(v,\mathcal{L})$ which then immediately implies the equivalence of the conjectures. It is straightforward to verify that $\hat{\partial}^1_G(\varepsilon_{L/K}(0)) = \mathcal{E}_{L/K}$, $[\mathcal{L},\pi_L,H_L] = \delta_{L/K}(\mathcal{L})$ and that m_w as defined in (18) agrees with m_w defined in [2, p. 561]. Since by definition $I_G(v,\mathcal{L}) = \mathrm{ind}^G_{G_w}\big(\partial^1_{G_w}(m_w) - \chi^{\mathrm{old}}_{G_w}(K^\bullet_w(\exp(\mathcal{L}_w)),\nu_w^{-1})\big)$ with $K^\bullet_w(\exp(\mathcal{L}_w))$ as in [2, (18)], it remains to show that $\chi_{G_w}(M_w(\mathcal{L}_w),\nu_w) = -\chi^{\mathrm{old}}_{G_w}(K^\bullet_w(\exp(\mathcal{L}_w)),\nu_w^{-1})$. This holds because the extension class of $K^\bullet_w(\exp(\mathcal{L}_w))$ in $\mathrm{Ext}^2_{\mathbb{Z}[G_w]}(\mathbb{Z}, L^\times_w/\exp(\mathcal{L}_w))$ is the negative of the class of $M_w(\mathcal{L}_w)$.

Remark 5.5. By combining Remark 5.4 with [2, Rem. 4.2(iv)] we may deduce that $\partial^0_G(T\Omega^{\mathrm{loc}}(L/K,1)) = W_{L/K} - \Omega(L/K,2)$ where $\Omega(L/K,2)$ is the element defined by Chinburg in [24] and so Conjecture 5.3 is a refinement of the 'Ω_2-conjecture' that is formulated in [24, Question 3.1].

Now if L/K is tamely ramified, then $\mathcal{O}_L$ is a projective $\mathbb{Z}[G]$-module and one has $\Omega(L/K,2) = [\mathcal{O}_L] - [K:\mathbb{Q}]\cdot[\mathbb{Z}[G]] \in K_0(\mathbb{Z}[G])$ [24, Th. 3.2]. The study of $\Omega(L/K,2)$ can therefore be regarded as a natural generalisation of the Galois structure theory of rings of algebraic integers that is described by Fröhlich in [28] (indeed, a similarly explicit interpretation of $\Omega(L/K,2)$ is also valid for wildly ramified extensions [33, Th. 4.1]). On the other hand, the study of $\Omega(L/K,3)$ is a natural generalisation of the explicit study of the Galois structures of unit groups and ideal class groups that was undertaken by Fröhlich in [29] and [30] (in this regard see, for example, the formulas for $\Omega(L/K,3)$ that are obtained in [10, Th. 1.2, Th. 1.7]).

From this viewpoint, Proposition 4.4(iii) shows that the 'multiplicative' Galois structure results obtained by Fröhlich in [29] and [30] are explicit consequences (for special families of extensions) of the natural leading term conjecture for equivariant Artin L-functions at $s=0$, whilst Theorem 5.2 shows that the 'additive' Galois structure results that Fröhlich discusses in [28] and [29] reflect the compatibility of the leading term conjectures at $s=0$ and $s=1$ with respect to the functional equation of the equivariant Artin L-function. We thereby resolve the problem posed by Fröhlich in [29, Introduction] of using the functional equation to give a natural explanation of the 'amazing analogy' between the Galois structure theories of unit groups and ideal class groups and of rings of algebraic integers that he stresses in both [29] and [20, §3].

Remark 5.6. Conjecture 5.3 is essentially of a local nature. More precisely, in [5] a conjecture for the equivariant local epsilon constant of a Galois extension of p-adic fields is formulated. It is then shown that the validity of this local conjecture for all completions L_w/K_v of L/K implies the validity of Conjecture 5.3. Such a local approach lies (implicitly or explicitly) behind the proof of the known cases mentioned in the following proposition.

Proposition 5.7. *For every Galois extension L/K, the invariant $T\Omega^{\mathrm{loc}}(L/K, 1)$ lies in $K_0(\mathbb{Z}[G], \mathbb{Q})_{\mathrm{tors}}$, the torsion subgroup of $K_0(\mathbb{Z}[G], \mathbb{Q}) \subset K_0(\mathbb{Z}[G], \mathbb{R})$. Moreover $T\Omega^{\mathrm{loc}}(L/K, 1)$ is known to vanish in the following cases:*

(i) *all tamely ramified extensions L/K,*
(ii) *L is an abelian extension of $\mathbb{Q}$ with odd conductor,*
(iii) *L is a non-abelian extension of $\mathbb{Q}$ of degree six.*

Proof. The first statement is [2, Cor. 6.3(i)]. Case (i) is [2, Cor. 7.7]. Cases (ii) and (iii) follow by combining Remark 5.1 with [2, Cor. 5.4(ii)] and [4, Th. 1.1] respectively. □

Recall that if χ is a symplectic character of G then the Artin root number $W(\chi)$ is either 1 or -1. In the case where L/K is tamely ramified, this sign has been determined by Cassou-Noguès and Taylor in terms of a natural algebraic invariant (see [21], [22]). Assuming the validity of Conjecture 5.3, one has the following generalisation of this result to wildly ramified extensions.

Theorem 5.8. *If Conjecture 5.3 is valid, i.e. if the leading term conjectures at $s = 0$ and $s = 1$ are compatible, then for every symplectic character χ of G the Artin root number $W(\chi)$ is determined by the algebraic invariant $R\Omega^{\mathrm{loc}}(L/K, 1)$ in $K_0(\mathbb{Z}[G], \mathbb{R})$.*

Theorem 5.8 was first shown in [6, §7] but for easier reference we have included the proof in §5.3.

5.2. **Proof of the main result.** By the functorial properties of the invariants (see Proposition 3.5 and Remarks 4.2 and 5.1) it suffices to show Theorem 5.2 for K totally real and L totally complex. We will assume this for the rest of this section. We fix a finite set S of places of K containing $S_\infty(K)$, all places ramified in L/K and for which $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$. Furthermore for each $v \in S_f$ we fix a place $w \in S_v(L)$, a complex M_w and a lattice $\mathcal{L}_w \subseteq \mathcal{O}_w$ as in §5.1.1. These lattices give rise to $\mathcal{L}_v := \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} \mathcal{L}_w$ and $\mathcal{L}$ as above. In §5.2.1 we will use the functional equation of the equivariant L-function to compute the quotient of the leading terms at $s = 0$ and $s = 1$. Then in §5.2.2 – §5.2.5 we will apply the additivity of Euler characteristics in distinguished triangles and some explicit computations to express the sum $\chi_G(P_S, -\mathrm{Reg}_S) + \chi_G(E_S(\mathcal{L}), \mu_L)$ in terms of certain semilocal invariants. After these preliminary steps the proof of Theorem 5.2 will be given in §5.2.6.

5.2.1. *Functional equation of the equivariant L-function.* We now consider the behaviour of the leading terms of the S-truncated L-function with respect to the functional equation. To simplify the notation we use the following convention. If W is a finitely generated $\mathbb{Z}[G]$-module (resp. $\mathbb{R}[G]$-module) and $\alpha \in \mathbb{R}^\times$ then $[W, \alpha]$ denotes the element in $K_1(\mathbb{R}[G])$ which is represented by the $\mathbb{R}[G]$-module $W \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. W) with automorphism given by multiplication with α.

Lemma 5.9. *One has*

$$\hat{\partial}^1_G \left(\frac{L^*_{L/K,S}(0)^\#}{L^*_{L/K,S}(1)} \right) = \hat{\partial}^1_G(\varepsilon_{L/K}(0)) + \sum_{v \in S_f} \mathrm{ind}^G_{G_w} \hat{\partial}^1_{G_w} \left(\frac{L^*_{L_w/K_v}(1)}{L^*_{L_w/K_v}(0)^\#} \right) + \partial^1_G \big(-[H_L^-, \pi] - [H_L^+, 2] + [\mathbb{R}, -1] \big)$$

in $K_0(\mathbb{Z}[G], \mathbb{R})$.

Proof. Taking the leading term at $s_0 = 0$ of both sides of the functional equation (8) we obtain the equality

$$\Lambda^*_{L/K}(0)^\# = \varepsilon_{L/K}(0) \cdot \alpha \cdot \Lambda^*_{L/K}(1)$$

in $\zeta(\mathbb{R}[G])^\times$ where $\alpha = (\alpha_\chi)_{\chi \in \mathrm{Irr}(G)} \in \prod_{\chi \in \mathrm{Irr}(G)} \mathbb{C}^\times = \zeta(\mathbb{C}[G])^\times$ is the element with $\alpha_\chi = -1$ if χ is the trivial character and $\alpha_\chi = 1$ otherwise.

The relation to the leading term of the S-truncated L-function is given by

$$\Lambda^*_{L/K}(s_0) = L^*_{L/K,S}(s_0) \cdot \prod_{v \in S} \mathrm{ind}^G_{G_w}(L^*_{L_w/K_v}(s_0)).$$

Since the induction $\mathrm{ind}^G_{G_w}$ and the involution $x \mapsto x^\#$ commute we find

$$\frac{L^*_{L/K,S}(0)^\#}{L^*_{L/K,S}(1)} = \varepsilon_{L/K}(0) \cdot \alpha \cdot \prod_{v \in S} \mathrm{ind}^G_{G_w} \left(\frac{L^*_{L_w/K_v}(1)}{L^*_{L_w/K_v}(0)^\#} \right). \tag{19}$$

The product of the leading terms of the local archimedean factors can be written in the following more explicit form. Let $v \in S_\infty(K)$, $\psi \in \mathrm{Irr}(G_w)$ and n_ψ^+, n_ψ^- as in §2.3.2. The well-known properties of the Γ-function imply $L^*_{L_w/K_v}(\psi, 0) = 2^{n_\psi^+}$ and $L^*_{L_w/K_v}(\psi, 1) = \pi^{-n_\psi^-}$. From this one easily deduces

$$\prod_{v \in S_\infty(K)} \mathrm{ind}^G_{G_w} \left(\frac{L^*_{L_w/K_v}(1)}{L^*_{L_w/K_v}(0)^\#} \right) = -[H_L^-, \pi] - [H_L^+, 2]$$

in $\zeta(\mathbb{R}[G])^{\times +} \cong K_1(\mathbb{R}[G])$. Therefore the lemma follows by applying $\hat{\partial}^1_G$ to (19). $\square$

5.2.2. *The exact sequence.* As in the case of a non-archimedean place in §5.1.1, for each $v \in S_\infty(K)$ we fix $w \in S_v(L)$ and let M_w be a complex in $\mathcal{D}(\mathbb{Z}[G_w])$ which represents the canonical class in $\mathrm{Ext}^2_{\mathbb{Z}[G_w]}(\mathbb{Z}, L_w^\times) = H^2(G_w, L_w^\times) \cong \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Then for every $v \in S$ the complex $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} M_w$ in $\mathcal{D}(\mathbb{Z}[G])$ is acyclic outside degrees 0 and 1, and there are isomorphisms $H^0(\mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} M_w) \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} L_w^\times \cong \prod_{w \in S_v(L)} L_w^\times$ and $H^1(\mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} M_w) \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} \mathbb{Z} \cong \prod_{w \in S_v(L)} \mathbb{Z}$. We set $M_{S_f} := \bigoplus_{v \in S_f} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} M_w$ and $M_{S_\infty} := \bigoplus_{v \in S_\infty} \mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} M_w$.

The local and global invariant maps are compatible, that is for every $w \in S(L)$ the following diagram commutes

$$\begin{array}{ccc} H^2(G_w, L_w^\times) & \xrightarrow{\mathrm{inv}_{L_w/K_v}} & \frac{1}{|G_w|}\mathbb{Z}/\mathbb{Z} \\ \downarrow & & \| \\ H^2(G_w, C_S(L)) & \xrightarrow{\mathrm{inv}_{L/L^{G_w},S}} & \frac{1}{|G_w|}\mathbb{Z}/\mathbb{Z}. \end{array}$$

Here the left vertical arrow is induced by the map $L_w^\times \to C_S(L)$ which is the composite of the inclusion $L_w^\times \to I_L$ and the canonical map $I_L \to C_S(L)$. Therefore by Lemma 2.4 there exists a morphism $M_w \to E_S$ in $\mathcal{D}(\mathbb{Z}[G_w])$ which induces the map $L_w^\times \to C_S(L)$ on H^0 and the identity $\mathbb{Z} \xrightarrow{=} \mathbb{Z}$ on H^1. From this we obtain a morphism $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G_w]} M_w \to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$ and adding over all $v \in S$ gives a map $M_{S_f} \oplus M_{S_\infty} \to E_S$ in $\mathcal{D}(\mathbb{Z}[G])$. From the definition of the canonical class in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_S, \mathcal{O}_{L,S}^\times)$ one sees that the complex P_S lies in the distinguished triangle

$$P_S \longrightarrow M_{S_f} \oplus M_{S_\infty} \longrightarrow E_S \longrightarrow . \tag{20}$$

Note that the cohomology sequence of (20) identifies with the canonical sequence

$$0 \to \mathcal{O}_{L,S}^\times \to L_S^\times \to C_S(L) \xrightarrow{0} X_S \to Y_S \to \mathbb{Z} \to 0.$$

5.2.3. *Replacing the trace map by the zero map.* From now on we simply write $\exp : \mathcal{L}_S[0] \to E_S$ for the map in $\mathcal{D}(\mathbb{Z}[G])$ which induces $\mathcal{L}_S \xrightarrow{\exp_S} L_S^\times \to C_S(L) \cong H^0(E_S)$ on H^0. Similarly we write $\mathrm{tr} : \mathcal{L}[-1] \to E_S$ for the map which induces $\mathcal{L} \xrightarrow{\mathrm{tr}} \mathbb{Z} \cong H^1(E_S)$ on H^1. Recall that the complex $E_S(\mathcal{L})$ is defined by the distinguished triangle

$$\mathcal{L}_S[0] \oplus \mathcal{L}[-1] \xrightarrow{\exp \oplus \mathrm{tr}} E_S \longrightarrow E_S(\mathcal{L}) \longrightarrow \tag{21}$$

whose cohomology sequence induces the identifications

$$H^i(E_S(\mathcal{L}) \otimes \mathbb{R}) = \begin{cases} L_\infty^0 & \text{if } i = -1, \\ L^0 \otimes \mathbb{R} & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and that we consider the canonical isomorphism $\mu_L : L^0 \otimes \mathbb{R} \to L_\infty^0$ as trivialisation of the complex $E_S(\mathcal{L})$.

Instead of the trace map $\mathrm{tr} : \mathcal{L} \to \mathbb{Z} = H^1(E_S)$ we now consider the zero map $0 : \mathcal{L} \to H^1(E_S)$. We define a complex $F_S(\mathcal{L})$ by the distinguished triangle

$$\mathcal{L}_S[0] \oplus \mathcal{L}[-1] \xrightarrow{\exp \oplus 0} E_S \longrightarrow F_S(\mathcal{L}) \longrightarrow \tag{22}$$

whose cohomology sequence induces identifications

$$H^i(F_S(\mathcal{L}) \otimes \mathbb{R}) = \begin{cases} L_\infty^0 & \text{if } i = -1, \\ L \otimes \mathbb{R} & \text{if } i = 0, \\ \mathbb{R} & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We define a trivialisation $t_F : L \otimes \mathbb{R} \to L_\infty^0 \oplus \mathbb{R}$ of $F_S(\mathcal{L})$ by $L \otimes \mathbb{R} \xrightarrow{\mu_L'} L_\infty \cong L_\infty^0 \oplus \mathbb{R}$, where the last isomorphism is induced by the canonical splitting of the surjection $L_\infty \xrightarrow{\mathrm{tr}} \mathbb{R}$, i.e. by $\mathbb{R} \to L_\infty$, $x \mapsto (x/[L : \mathbb{Q}])_{w \in S_\infty}$.

Lemma 5.10. *One has*

$$\chi_G(E_S(\mathcal{L}), \mu_L) = \chi_G(F_S(\mathcal{L}), t_F)$$

in $K_0(\mathbb{Z}[G], \mathbb{R})$.

Proof. We will show below that there exists a distinguished triangle in $\mathcal{D}(\mathbb{Z}[G])$

$$F_S(\mathcal{L}) \longrightarrow E_S(\mathcal{L}) \longrightarrow \mathcal{L}[1] \oplus \mathcal{L}[0] \longrightarrow \tag{23}$$

whose cohomology sequence after tensoring with $\mathbb{R}$ identifies with (starting with $H^{-1}(F_S(\mathcal{L}) \otimes \mathbb{R})$)

$$L^0_\infty \xrightarrow{\mathrm{id}} L^0_\infty \xrightarrow{0} L \otimes \mathbb{R} \xrightarrow{\mathrm{id}} L \otimes \mathbb{R} \xrightarrow{0} L^0 \otimes \mathbb{R} \xrightarrow{\mathrm{incl}} L \otimes \mathbb{R} \xrightarrow{-\mathrm{tr}} \mathbb{R}.$$

On the complex $\mathcal{L}[1] \oplus \mathcal{L}[0]$ we take the trivialisation $-\mathrm{id} : L \otimes \mathbb{R} \to L \otimes \mathbb{R}$. One then easily verifies that the distinguished triangle (23) with the trivialisations t_F, μ_L and $-\mathrm{id}$ satisfies the additivity criterion of [8, Cor. 6.6]. Thus

$$\chi_G(E_S(\mathcal{L}), \mu_L) = \chi_G(F_S(\mathcal{L}), t_F) + \chi_G(\mathcal{L}[1] \oplus \mathcal{L}[0], -\mathrm{id})$$

and obviously $\chi_G(\mathcal{L}[1] \oplus \mathcal{L}[0], -\mathrm{id}) = 0$ in $K_0(\mathbb{Z}[G], \mathbb{R})$.

It remains to show the existence of the distinguished triangle (23) with the claimed cohomology sequence. One can construct the following commutative diagram of distinguished triangles

$$\begin{array}{ccccccc}
F_S(\mathcal{L}) & \longrightarrow & E_S(\mathcal{L}) & \longrightarrow & \mathcal{L}[1] \oplus \mathcal{L}[0] & \longrightarrow & \\
\downarrow & & \downarrow & & \| & & \\
\mathcal{L}_S[1] \oplus \mathcal{L}[0] & \xrightarrow{\mathrm{id} \oplus 0} & \mathcal{L}_S[1] \oplus \mathcal{L}[0] & \xrightarrow{0 \oplus \mathrm{id}} & \mathcal{L}[1] \oplus \mathcal{L}[0] & \xrightarrow{a} & \\
\downarrow{\scriptstyle \exp \oplus 0} & & \downarrow{\scriptstyle \exp \oplus \mathrm{tr}} & & \downarrow & & \\
E_S[1] & = & E_S[1] & \longrightarrow & 0 & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & &
\end{array} \tag{24}$$

where $a : \mathcal{L}[1] \oplus \mathcal{L}[0] \to \mathcal{L}_S[2] \oplus \mathcal{L}[1]$ is the map $(k, l) \mapsto (0, k)$, and the first two columns are the distinguished triangles obtained from (22) and (21) by rotation (without changing the signs of the maps; these still are distinguished triangles because we rotated twice). In fact we can construct (24) in such a way that it is isomorphic to a commutative diagram of short exact sequences. (To see this first replace the square

$$\begin{array}{ccc}
\mathcal{L}_S[1] \oplus \mathcal{L}[0] & \xrightarrow{\mathrm{id} \oplus 0} & \mathcal{L}_S[1] \oplus \mathcal{L}[0] \\
\downarrow{\scriptstyle \exp \oplus 0} & & \downarrow{\scriptstyle \exp \oplus \mathrm{tr}} \\
E_S[1] & = & E_S[1]
\end{array}$$

by an isomorphic square of complexes and maps of complexes with injective horizontal arrow and surjective vertical arrows. Then complete it to a commutative diagram of short exact sequences.) The first row of (24) is the distinguished triangle we want. Its cohomology sequence has the required form as can be checked by an easy diagram chase (using that (24) is isomorphic to a diagram of short exact sequences and therefore all computations can be done with cocycles). $\square$

5.2.4. *The semilocal complexes.* We now construct two complexes M^{finite} and M^{arch} in $\mathcal{D}(\mathbb{Z}[G])$ with trivialisations $t_{M^{\text{finite}}}$ and $t_{M^{\text{arch}}}$ respectively, and show that their Euler characteristics are closely related to the terms defined in §5.1.1.

Recall that M_{S_f} is the complex $\bigoplus_{v\in S_f}\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}M_w$. One has $H^0(M_{S_f}) = L_{S_f}^\times$ and we define M^{finite} by the distinguished triangle

$$\mathcal{L}_{S_f}[0] \xrightarrow{\exp} M_{S_f} \longrightarrow M^{\text{finite}} \longrightarrow .$$

From its cohomology sequence we see that for the non-zero cohomology groups there are identifications $H^0(M^{\text{finite}}) \cong \prod_{w\in S_f}\frac{L_w^\times}{\exp(\mathcal{L}_w)}$ and $H^1(M^{\text{finite}}) \cong Y_{S_f}$. On the complex M^{finite} we consider the trivialisation $t_{M^{\text{finite}}} : \prod_{w\in S_f}\frac{L_w^\times}{\exp(\mathcal{L}_w)}\otimes\mathbb{R} \to Y_{S_f}\otimes\mathbb{R}$, $(x_w)_{w\in S_f}\mapsto (v_w(x_w)\cdot\log Nw)_{w\in S_f}$ where v_w is the normalised valuation of $L_w^\times$ and Nw is the cardinality of the residue field of w.

Lemma 5.11. *One has*

$$\chi_G(M^{\text{finite}}, t_{M^{\text{finite}}}) = \sum_{v\in S_f}\operatorname{ind}_{G_w}^G\left(\chi_{G_w}(M_w(\mathcal{L}_w),\nu_w) + \partial^1_{G_w}[\mathbb{R},\log Nw]\right)$$

in $K_0(\mathbb{Z}[G],\mathbb{R})$.

Proof. This follows easily from [8, Prop. 5.6.2]. □

Before defining M^{arch} we must introduce some notation. For every $w\in S_\infty$ we denote by $\mathbf{R}(L_w)$ and $\mathbf{I}(L_w)$ the real and imaginary axis in L_w respectively, i.e. the $\mathbb{R}$-line generated by 1 in L_w and the $\mathbb{R}$-line generated by a square root of -1 in L_w. Furthermore we set $\mathbf{R}(L_\infty) := \prod_{w\in S_\infty}\mathbf{R}(L_w)$ and $\mathbf{I}(L_\infty) := \prod_{w\in S_\infty}\mathbf{I}(L_w)$. Then $\mathbf{R}(L_\infty)$ and $\mathbf{I}(L_\infty)$ are $\mathbb{R}[G]$-submodules of L_∞ and one has $L_\infty = \mathbf{I}(L_\infty)\oplus\mathbf{R}(L_\infty)$. Note that $\mathbf{I}(L_\infty)$ lies in the kernel of the trace map $\operatorname{tr}: L_\infty\to\mathbb{R}$. We let $\mathbf{R}^0(L_\infty)$ be the kernel of the restriction of tr to $\mathbf{R}(L_\infty)$, thus there is a short exact sequence $0\to\mathbf{R}^0(L_\infty)\to\mathbf{R}(L_\infty)\xrightarrow{\operatorname{tr}}\mathbb{R}\to 0$. Using the canonical splitting $\mathbb{R}\to\mathbf{R}(L_\infty)$, $x\mapsto (x/[L:\mathbb{Q}])_{w\in S_\infty}$ we obtain the direct sum decomposition $\mathbf{R}(L_\infty) = \mathbf{R}^0(L_\infty)\oplus\mathbb{R}$.

Let $\mathbf{R}(L_\infty)\xrightarrow{\cong} Y_{S_\infty}\otimes_{\mathbb{Z}}\mathbb{R}$ be the isomorphism of $\mathbb{R}[G]$-modules given by $(x_w)_{w\in S_\infty}\mapsto (2x_w)_{w\in S_\infty}$. Note that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{R}^0(L_\infty) & \longrightarrow & \mathbf{R}(L_\infty) & \xrightarrow{\operatorname{tr}} & \mathbb{R} & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \| & & \\ 0 & \longrightarrow & X_{S_\infty}\otimes\mathbb{R} & \longrightarrow & Y_{S_\infty}\otimes\mathbb{R} & \xrightarrow{\operatorname{aug}} & \mathbb{R} & \longrightarrow & 0 \end{array}$$

and that the above splitting of $\mathbf{R}(L_\infty)\xrightarrow{\operatorname{tr}}\mathbb{R}$ corresponds to the canonical splitting of $Y_{S_\infty}\otimes\mathbb{R}\xrightarrow{\operatorname{aug}}\mathbb{R}$.

Recall that M_{S_∞} is the complex $\bigoplus_{v\in S_\infty}\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}M_w$. One has $H^0(M_{S_\infty}) = L_\infty^\times$ and we define M^{arch} by the distinguished triangle

$$L_\infty[0]\oplus\mathcal{L}[-1] \xrightarrow{\exp\oplus 0} M_{S_\infty} \longrightarrow M^{\text{arch}} \longrightarrow .$$

From its cohomology sequence we see that for the non-zero cohomology groups there are identifications $H^{-1}(M^{\text{arch}}) \cong \prod_{w\in S_\infty}2\pi\sqrt{-1}\mathbb{Z}\subset L_\infty$, $H^0(M^{\text{arch}})\cong\mathcal{L}$ and $H^1(M^{\text{arch}})\cong Y_{S_\infty}$. Note that $\prod_{w\in S_\infty}2\pi\sqrt{-1}\mathbb{Z}$ is a full lattice in $\mathbf{I}(L_\infty)$ and therefore $(\prod_{w\in S_\infty}2\pi\sqrt{-1}\mathbb{Z})\otimes\mathbb{R}\cong\mathbf{I}(L_\infty)$. We define the trivialisation $t_{M^{\text{arch}}}$ by

$\mathcal{L}\otimes\mathbb{R}\xrightarrow{\mu'_L} L_\infty = \mathbf{I}(L_\infty)\oplus\mathbf{R}^0(L_\infty)\oplus\mathbb{R}\xrightarrow{\mathrm{id}\oplus(-\mathrm{id})\oplus\mathrm{id}}\mathbf{I}(L_\infty)\oplus\mathbf{R}^0(L_\infty)\oplus\mathbb{R} = \mathbf{I}(L_\infty)\oplus\mathbf{R}(L_\infty)\xrightarrow{\cong}\mathbf{I}(L_\infty)\oplus Y_{S_\infty}\otimes\mathbb{R}$.

Lemma 5.12. *One has*

$$\chi_G(M^{\mathrm{arch}}, t_{M^{\mathrm{arch}}}) = [\mathcal{L}, \pi_L, H_L] + \partial^1_G\big(-[H_L^-, -\pi] - [H_L^+, 2] + [\mathbb{R}, -1]\big)$$

in $K_0(\mathbb{Z}[G], \mathbb{R})$.

Proof. First we choose a special representative of the complex M^{arch} in $\mathcal{D}(\mathbb{Z}[G])$. We note that for each $v\in S_\infty$ the complex $M_w(L_w)$ in $\mathcal{D}(\mathbb{Z}[G_w])$ defined by the distinguished triangle $L_w[0]\xrightarrow{\exp} M_w\longrightarrow M_w(L_w)\longrightarrow$ has non-zero cohomology $H^{-1}(M_w(L_w)) = 2\pi\sqrt{-1}\mathbb{Z}\subset L_w$ and $H^1(M_w(L_w)) = \mathbb{Z}$, and that its extension class in $\mathrm{Ext}^3_{\mathbb{Z}[G_w]}(\mathbb{Z}, 2\pi\sqrt{-1}\mathbb{Z})\cong\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is non-trivial. On the other hand let $H_w := \prod_{\sigma\in\Sigma(L_w)}\mathbb{Z}$ where $\Sigma(L_w)$ is the set of continuous isomorphisms $L_w\to\mathbb{C}$, and consider the complex of G_w-modules $N_w : H_w\xrightarrow{1+\tau} H_w\xrightarrow{1-\tau} H_w$ with non-zero terms in degrees -1, 0 and 1, where $\tau\in\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ denotes complex conjugation. Then N_w is acyclic outside degrees -1 and 1, and one has $H^{-1}(N_w) = H_w^-$, $H^1(N_w) = H_w/H_w^-$. Moreover N_w has non-trivial extension class in $\mathrm{Ext}^3_{\mathbb{Z}[G_w]}(H_w/H_w^-, H_w^-)\cong\frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Thus by fixing isomorphisms $2\pi\sqrt{-1}\mathbb{Z}\cong H_w^-$ and $\mathbb{Z}\cong H_w/H_w^-$ we obtain an induced isomorphism $M_w(L_w)\cong N_w$ in $\mathcal{D}(\mathbb{Z}[G_w])$. Applying $\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}$ and summing over all $v\in S_\infty$ we see that M^{arch} is isomorphic to $N\oplus\mathcal{L}[0]$ in $\mathcal{D}(\mathbb{Z}[G])$ where N is the complex

$$H_L\xrightarrow{1+\tau} H_L\xrightarrow{1-\tau} H_L.$$

The trivialisation $t_{M^{\mathrm{arch}}}$ corresponds to the following trivialisation t on $N\oplus\mathcal{L}[0]$

$$\begin{aligned}\mathcal{L}\otimes\mathbb{R}&\xrightarrow{t_{M^{\mathrm{arch}}}}\mathbf{I}(L_\infty)\oplus Y_{S_\infty}\otimes\mathbb{R}\\ &\xrightarrow{\cong}\prod_{v\in S_\infty}(\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}(2\pi\sqrt{-1}\mathbb{Z}))\otimes\mathbb{R}\oplus\prod_{v\in S_\infty}(\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}\mathbb{Z})\otimes\mathbb{R}\\ &\xrightarrow{\cong}\prod_{v\in S_\infty}(\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}H_w^-)\otimes\mathbb{R}\oplus\prod_{v\in S_\infty}(\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_w]}H_w/H_w^-)\otimes\mathbb{R}\\ &= H_L^-\otimes\mathbb{R}\oplus H_L/H_L^-\otimes\mathbb{R}.\end{aligned}$$

We compute the Euler characteristic $\chi_G(M^{\mathrm{arch}}, t_{M^{\mathrm{arch}}}) = \chi_G(N\oplus\mathcal{L}[0], t)$ by using splittings, i.e. we use the refined Euler characteristic χ_G^{old} discussed in [8, §6]. The relation of χ_G and χ_G^{old} is given in [8, Th. 6.2]. With the canonical splittings (given by the direct sum decomposition $H_L\otimes\mathbb{R} = H_L^-\otimes\mathbb{R}\oplus H_L^+\otimes\mathbb{R}$) we find $\chi_G^{\mathrm{old}}(N\oplus\mathcal{L}[0], t^{-1}) = [H_L\oplus H_L, g, H_L\oplus\mathcal{L}]$ where $g : (H_L\oplus H_L)\otimes\mathbb{R}\to(H_L\oplus\mathcal{L})\otimes\mathbb{R}$ is the isomorphism

$$\begin{aligned}H_L\otimes\mathbb{R}\oplus H_L\otimes\mathbb{R} &= (H_L^-\otimes\mathbb{R}\oplus H_L^+\otimes\mathbb{R})\oplus(H_L^-\otimes\mathbb{R}\oplus H_L^+\otimes\mathbb{R})\\ &\xrightarrow{1\oplus2\oplus1/2\oplus1} H_L^-\otimes\mathbb{R}\oplus(H_L^+\otimes\mathbb{R}\oplus H_L^-\otimes\mathbb{R})\oplus H_L^+\otimes\mathbb{R}\\ &\cong H_L^-\otimes\mathbb{R}\oplus H_L\otimes\mathbb{R}\oplus(H_L/H_L^-)\otimes\mathbb{R}\\ &\xrightarrow{\mathrm{id}\oplus t^{-1}} H_L\otimes\mathbb{R}\oplus\mathcal{L}\otimes\mathbb{R}\end{aligned}$$

where the isomorphism $H_L^+\otimes\mathbb{R}\cong(H_L/H_L^-)\otimes\mathbb{R}$ is induced by the embedding $H_L^+\to H_L/H_L^-$ and for the arrow marked $\mathrm{id}\oplus t^{-1}$ we have first used the obvious permutation and then the identity on $H_L \otimes \mathbb{R}$ and the isomorphism t^{-1}. Let $h : \mathcal{L}\otimes\mathbb{R} \to H_L^- \otimes \mathbb{R} \oplus H_L/H_L^- \otimes \mathbb{R} \cong H_L \otimes \mathbb{R}$ be defined as t except that we omit $-\mathrm{id}$ on $\mathbf{R}^0(L_\infty)$. One then has

$$[H_L \oplus H_L, g, H_L \oplus \mathcal{L}] = -[\mathcal{L}, h, H_L] + \partial_G^1\big([X_{S_\infty}, -1] - [H_L^-, 2] + [H_L^+, -2]\big).$$

Finally we claim that

$$[\mathcal{L}, h, H_L] = [\mathcal{L}, \pi_L, H_L] - \partial_G^1[H_L^-, 2\pi]. \tag{25}$$

Indeed, the map $L_\infty \xrightarrow{(\mu_L')^{-1}} L\otimes\mathbb{R} \subset L\otimes\mathbb{C} \xrightarrow{\rho_L} H_L \otimes \mathbb{C}$ gives an identification of L_∞ with $(H_L\otimes\mathbb{C})^+$. Under this identification one has $\mathbf{R}(L_\infty) \cong H_L^+\otimes\mathbb{R}$ and $\mathbf{I}(L_\infty) \cong H_L^- \otimes i\mathbb{R}$. In the case $K=\mathbb{Q}$ it easily follows that for the correct choice of $i=\sqrt{-1}\in\mathbb{C}$ in the definition of π_L the map $h : \mathcal{L}\otimes\mathbb{R} \to H_L\otimes\mathbb{R}$ agrees with $\mathcal{L} \otimes \mathbb{R} \xrightarrow{\pi_L} H_L^+\otimes\mathbb{R}\oplus H_L^-\otimes\mathbb{R} \xrightarrow{\mathrm{id}\oplus 1/(2\pi)} H_L^+\otimes\mathbb{R}\oplus H_L^-\otimes\mathbb{R}$ which implies (25). In the case of a general totally real field K the two maps still agree if one chooses the correct i in the definition of π_L for each v-component of $H_L^- = \prod_{v\in S_\infty(K)} \big(\prod_{\sigma\in\Sigma(L),\sigma|_K=v}\mathbb{Z}\big)^-$ separately; one easily checks that $[\mathcal{L},\pi_L,H_L]$ is independent of all such choices.

Summarising we find

$$\begin{aligned}\chi_G(M^{\mathrm{arch}}, t_{M^{\mathrm{arch}}}) &= \chi_G(N\oplus\mathcal{L}[0], t)\\ &= -\chi_G^{\mathrm{old}}(N\oplus\mathcal{L}[0], t^{-1}) + \partial_G^1[H_L^-, -1]\\ &= [\mathcal{L}, h, H_L] + \partial_G^1\big(-[X_{S_\infty}, -1] + [H_L^-, 2] - [H_L^+, -2] + [H_L^-, -1]\big)\\ &= [\mathcal{L}, \pi_L, H_L] + \partial_G^1\big(-[H_L^-, -\pi] - [H_L^+, 2] + [\mathbb{R}, -1]\big).\end{aligned}$$

This completes the proof of Lemma 5.12. ☐

5.2.5. *Relation of Euler characteristics.*

Lemma 5.13. *One has*

$$\chi_G(M^{\mathrm{finite}}, t_{M^{\mathrm{finite}}}) + \chi_G(M^{\mathrm{arch}}, t_{M^{\mathrm{arch}}}) = \chi_G(P_S, -\mathrm{Reg}_S) + \chi_G(F_S(\mathcal{L}), t_F)$$

in $K_0(\mathbb{Z}[G],\mathbb{R})$.

For the proof of Lemma 5.13 we need the following result.

Lemma 5.14. *There exists a distinguished triangle*

$$P_S \longrightarrow M^{\mathrm{finite}} \oplus M^{\mathrm{arch}} \longrightarrow F_S(\mathcal{L}) \longrightarrow$$

in $\mathcal{D}(\mathbb{Z}[G])$ *whose cohomology sequence after tensoring with* $\mathbb{R}$ *identifies with (starting with* $H^{-1}((M^{\mathrm{finite}}\oplus M^{\mathrm{arch}})\otimes\mathbb{R})$*)*

$$\mathbf{I}(L_\infty) \xrightarrow{\mathrm{incl}} L_\infty^0 \xrightarrow{-\exp} \mathcal{O}_{L,S}^\times\otimes\mathbb{R} \xrightarrow{a} \Big(\prod_{w\in S_f}\frac{L_w^\times}{\exp(\mathcal{L}_w)}\Big)\otimes\mathbb{R}\oplus L\otimes\mathbb{R} \xrightarrow{b} L\otimes\mathbb{R} \xrightarrow{0}$$

$$X_S\otimes\mathbb{R} \xrightarrow{\mathrm{incl}} Y_S\otimes\mathbb{R} \to \mathbb{R}.$$

Here the map $-\exp : L_\infty^0 \to \mathcal{O}_{L,S}^\times\otimes\mathbb{R}$ *is the composite* $L_\infty^0 = \log_\infty(\mathcal{O}_L^\times)\otimes\mathbb{R} \xrightarrow{-\mathrm{id}} \log_\infty(\mathcal{O}_L^\times)\otimes\mathbb{R} \xrightarrow{\exp} \mathcal{O}_L^\times\otimes\mathbb{R} \xrightarrow{\mathrm{incl}} \mathcal{O}_{L,S}^\times\otimes\mathbb{R}$, *the map* a *is induced by the canonical maps* $\mathcal{O}_{L,S}^\times \to L_w^\times$, *and the map* b *is the identity on* $L\otimes\mathbb{R}$ *and zero on the first summand.*

Proof. We can construct the following commutative diagram of distinguished triangles

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{L}_{S_f}[0] \oplus \mathcal{L}_{S_\infty}[0] \oplus \mathcal{L}[-1] & = & \mathcal{L}_S[0] \oplus \mathcal{L}[-1] & \longrightarrow & \\
\downarrow & & \downarrow{\scriptstyle \exp\oplus\exp\oplus 0} & & \downarrow{\scriptstyle \exp\oplus 0} & & \\
P_S & \longrightarrow & M_{S_f} \oplus M_{S_\infty} & \longrightarrow & E_S & \longrightarrow & \\
\| & & \downarrow & & \downarrow & & \\
P_S & \longrightarrow & M^{\mathrm{finite}} \oplus M^{\mathrm{arch}} & \longrightarrow & F_S(\mathcal{L}) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & &
\end{array}$$

where the second row is (20). By an argument similar as in the proof of Lemma 5.10 we can assume that this diagram is isomorphic to a commutative diagram of short exact sequences. The third row is the distinguished triangle we want and a careful analysis shows that the associated cohomology sequence is of the form stated. □

Proof of Lemma 5.13. Consider the $\mathbb{R}[G]$-modules and isomorphisms in the following (in general non-commutative) diagram

$$\begin{array}{ccc}
\Big(\prod_{w\in S_f} \frac{L_w^\times}{\exp(\mathcal{L}_w)} \otimes \mathbb{R} \oplus L\otimes\mathbb{R}\Big) \oplus (\mathcal{O}_L^\times \otimes \mathbb{R}) & \xrightarrow{s^{\mathrm{ev}}} & (\mathcal{O}_{L,S}^\times \otimes \mathbb{R}) \oplus (L\otimes \mathbb{R}) \\
\downarrow{\scriptstyle t_{M^{\mathrm{finite}}}\oplus t_{M^{\mathrm{arch}}}\oplus \mathrm{id}} & & \downarrow{\scriptstyle (-\mathrm{Reg}_S)\oplus t_F} \\
(\mathbf{I}(L_\infty)\oplus Y_{S,\mathbb{R}}) \oplus (\mathcal{O}_L^\times\otimes\mathbb{R}) & \xrightarrow{s^{\mathrm{od}}} & (X_{S,\mathbb{R}}) \oplus (L_\infty^0 \oplus \mathbb{R}).
\end{array}$$

Here we write $Y_{S,\mathbb{R}} := Y_S \otimes \mathbb{R}$ etc., and for the left vertical map we have composed $t_{M^{\mathrm{finite}}} \oplus t_{M^{\mathrm{arch}}}$ with the canonical isomorphism $Y_{S_f,\mathbb{R}} \oplus \mathbf{I}(L_\infty) \oplus Y_{S_\infty,\mathbb{R}} \cong \mathbf{I}(L_\infty) \oplus Y_{S,\mathbb{R}}$. The maps s^{ev} and s^{od} are given by splittings of the exact sequences

$$0 \to \mathcal{O}_L^\times \otimes \mathbb{R} \to \mathcal{O}_{L,S}^\times \otimes \mathbb{R} \xrightarrow{a} \prod_{w\in S_f} \frac{L_w^\times}{\exp(\mathcal{L}_w)} \otimes \mathbb{R} \oplus L\otimes\mathbb{R} \xrightarrow{b} L\otimes\mathbb{R} \to 0$$

and

$$0 \to X_{S,\mathbb{R}} \to \mathbf{I}(L_\infty) \oplus Y_{S,\mathbb{R}} \to L_\infty^0 \oplus \mathbb{R} \xrightarrow{-\exp} \mathcal{O}_L^\times \otimes \mathbb{R} \to 0$$

respectively. Lemma 5.13 follows from the additivity criterion [8, Cor. 6.6] applied to the distinguished triangle in Lemma 5.14 once we have shown that the automorphism

$$(t^{-1}_{M^{\mathrm{finite}}} \oplus t^{-1}_{M^{\mathrm{arch}}} \oplus \mathrm{id}) \circ (s^{\mathrm{od}})^{-1} \circ ((-\mathrm{Reg}_S) \oplus t_F) \circ s^{\mathrm{ev}} \tag{26}$$

of $\big(\prod_{w\in S_f} \frac{L_w^\times}{\exp(\mathcal{L}_w)} \otimes \mathbb{R} \oplus L\otimes\mathbb{R}\big) \oplus (\mathcal{O}_L^\times \otimes \mathbb{R})$ has reduced norm 1.

To compute the reduced norm of (26) we use the following isomorphisms to replace various of the modules: $-\mathrm{Reg}_{S_\infty} : \mathcal{O}_L^\times \otimes \mathbb{R} \xrightarrow{\cong} X_{S_\infty,\mathbb{R}}$, $-\mathrm{Reg}_S : \mathcal{O}_{L,S}^\times \otimes \mathbb{R} \xrightarrow{\cong} X_{S,\mathbb{R}}$, $\prod_{w\in S_f}(v_w(\cdot)\cdot \log Nw) : \prod_{w\in S_f} \frac{L_w^\times}{\exp(\mathcal{L}_w)} \otimes \mathbb{R} \xrightarrow{\cong} Y_{S_f,\mathbb{R}}$, $Y_{S,\mathbb{R}} = Y_{S_f,\mathbb{R}} \oplus Y_{S_\infty,\mathbb{R}} = Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}$, $L_\infty^0 = \mathbf{I}(L_\infty) \oplus \mathbf{R}^0(L_\infty) \cong \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}}$ and $L\otimes\mathbb{R} \xrightarrow{\mu'_L} L_\infty = \mathbf{I}(L_\infty) \oplus \mathbf{R}^0(L_\infty) \oplus \mathbb{R} \cong \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}$.

The above diagram then becomes

$$\begin{array}{ccc}(Y_{S_f,\mathbb{R}} \oplus \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}) \oplus (X_{S_\infty,\mathbb{R}}) & \longrightarrow & (X_{S,\mathbb{R}}) \oplus (\mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}) \\ \downarrow c & & \downarrow \mathrm{id} \\ (\mathbf{I}(L_\infty) \oplus Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}) \oplus (X_{S_\infty,\mathbb{R}}) & \longrightarrow & (X_{S,\mathbb{R}}) \oplus (\mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R})\end{array}$$

with $c(\alpha,\beta,\gamma,\delta,\epsilon) = (\beta,\alpha,-\gamma,\delta,\epsilon)$, the upper horizontal arrow is induced by splittings of

$$0 \to X_{S_\infty,\mathbb{R}} \to X_{S,\mathbb{R}} \to Y_{S_f,\mathbb{R}} \oplus \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \to \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \to 0,$$

consisting of the canonical short exact sequence $X_{S_\infty,\mathbb{R}} \to X_{S,\mathbb{R}} \to Y_{S_f,\mathbb{R}}$ and the identity $\mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \xrightarrow{=} \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}$, and the lower horizontal arrow is induced by splittings of

$$0 \to X_{S,\mathbb{R}} \to \mathbf{I}(L_\infty) \oplus Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \to \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \to X_{S_\infty,\mathbb{R}} \to 0,$$

consisting of the canonical short exact sequences $X_{S,\mathbb{R}} \to Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R} \to \mathbb{R}$ (recall that $Y_{S,\mathbb{R}} = Y_{S_f,\mathbb{R}} \oplus X_{S_\infty,\mathbb{R}} \oplus \mathbb{R}$) and $\mathbf{I}(L_\infty) \to \mathbf{I}(L_\infty) \oplus X_{S_\infty,\mathbb{R}} \to X_{S_\infty,\mathbb{R}}$. It is easy to write down splittings of these sequences and to verify that the resulting automorphism has reduced norm 1. □

5.2.6. *Completion of the proof.* We now collect all the previous results to complete the proof of Theorem 5.2.

Proof of Theorem 5.2. By the definitions of $T\Omega(L/K,0)$ and $T\Omega(L/K,1)$, and by Lemmas 5.10 and 5.13 we have

$$\begin{aligned}&\psi_G^*(T\Omega(L/K,0)) - T\Omega(L/K,1)\\ &\quad= \hat{\partial}_G^1(L^*_{L/K,S}(0)^\#) - \chi_G(P_S, -\mathrm{Reg}_S) - \hat{\partial}_G^1(L^*_{L/K,S}(1)) - \chi_G(F_S(\mathcal{L}), t_F)\\ &\quad= \hat{\partial}_G^1\left(\frac{L^*_{L/K,S}(0)^\#}{L^*_{L/K,S}(1)}\right) - \chi_G(M^{\mathrm{finite}}, t_{M^{\mathrm{finite}}}) - \chi_G(M^{\mathrm{arch}}, t_{M^{\mathrm{arch}}}).\end{aligned}$$

Lemmas 5.9, 5.11 and 5.12 show that this is equal to

$$\begin{aligned}&\hat{\partial}_G^1(\varepsilon_{L/K}(0)) - [\mathcal{L}, \pi_L, H_L]\\ &\quad- \sum_{v\in S_f} \mathrm{ind}_{G_w}^G\left(-\hat{\partial}_{G_w}^1\left(\frac{L^*_{L_w/K_v}(1)}{L^*_{L_w/K_v}(0)^\#}\right) + \chi_{G_w}(M_w(\mathcal{L}_w), \nu_w) + \partial_{G_w}^1[\mathbb{R}, \log Nw]\right)\\ &\quad+ \partial_G^1\big(-[H_L^-, \pi] - [H_L^+, 2] + [\mathbb{R}, -1] + [H_L^-, -\pi] + [H_L^+, 2] - [\mathbb{R}, -1]\big)\end{aligned}$$

which is $T\Omega^{\mathrm{loc}}(L/K,1)$ since $\partial_G^1[H_L^-, -1] = 0$. □

5.3. **Proof of Theorem 5.8.** The following proof of Theorem 5.8 is taken from [6, §7]. As a preliminary step of independent interest we show that the invariant $R\Omega^{\mathrm{loc}}(L/K,1)$ in $K_0(\mathbb{Z}[G],\mathbb{R})$ allows one to determine the absolute norm of the Artin conductor of every character of G.

5.3.1. *Determining conductors.* In the following result $|\cdot|$ denotes the usual absolute value on the complex numbers $\mathbb{C}$.

Lemma 5.15. *Let $\alpha = (\alpha_\chi)_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)} \mathbb{C}^\times$ and assume that $|\omega(\alpha_\chi)| = |\alpha_\chi|$ for all $\chi \in \mathrm{Irr}(G)$ and all automorphisms ω of $\mathbb{C}$. Then for every $\chi \in \mathrm{Irr}(G)$ the absolute value $|\alpha_\chi|$ is determined by $\partial^1_{\mathbb{Z}[G],\mathbb{C}}(\alpha) \in K_0(\mathbb{Z}[G],\mathbb{C})$.*

Proof. The hypothesis implies that all α_χ are algebraic over $\mathbb{Q}$. Therefore there exists a finite extension E of $\mathbb{Q}$ in $\mathbb{C}$ such that $\partial^1_{\mathbb{Z}[G],\mathbb{C}}(\alpha) \in K_0(\mathbb{Z}[G],E)$ and which is big enough to ensure that every irreducible representation of G is realisable over E. This implies that $\alpha \in \zeta(E[G])^\times = \prod_{\chi\in\mathrm{Irr}(G)} E^\times$.

Using the hypothesis and the product formula for the field E we obtain

$$|\alpha_\chi|^{[E:\mathbb{Q}]} = \prod_{v\in S_\infty(E)} |\alpha_\chi|_v = \prod_{v\in S_f(E)} |\alpha_\chi|_v^{-1}$$

where the valuations $|\cdot|_v$ are normalised as usual. It therefore suffices to show that $\partial^1_{\mathbb{Z}[G],E}(\alpha)$ determines $|\alpha_\chi|_v$ for every non-archimedean place v of E.

Let v be a non-archimedean place of E and let p be the residue characteristic of v. Choose an embedding $j : E \to \overline{\mathbb{Q}_p}$ corresponding to the place v. Then j induces maps of the centre of the group rings and of the relative algebraic K-groups making the diagram

$$\begin{array}{ccc} \zeta(E[G])^\times & \xrightarrow{\partial^1_{\mathbb{Z}[G],E}} & K_0(\mathbb{Z}[G],E) \\ \downarrow j & & \downarrow j \\ \zeta(\overline{\mathbb{Q}_p}[G])^\times & \xrightarrow{\partial^1_{\mathbb{Z}_p[G],\overline{\mathbb{Q}_p}}} & K_0(\mathbb{Z}_p[G],\overline{\mathbb{Q}_p}) \end{array}$$

commutative. But it is well known that $\partial^1_{\mathbb{Z}_p[G],\overline{\mathbb{Q}_p}}(j(\alpha))$ determines $j(\alpha)_{j\circ\chi} = j(\alpha_\chi)$ up to a unit in $\overline{\mathbb{Q}_p}$ for every $\chi \in \mathrm{Irr}(G)$, hence it determines $|\alpha_\chi|_v$. □

Corollary 5.16. *For each character χ of G the absolute norm of the Artin conductor of χ is determined by $R\Omega^{\mathrm{loc}}(L/K,1) \in K_0(\mathbb{Z}[G],\mathbb{R})$.*

Proof. Let n be the order of the finite group $K_0(\mathbb{Z}[G],\mathbb{Q})_{\mathrm{tors}}$. Since the invariant $T\Omega^{\mathrm{loc}}(L/K,1) = \hat{\partial}^1_G(\varepsilon_{L/K}(0)) - R\Omega^{\mathrm{loc}}(L/K,1)$ lies in $K_0(\mathbb{Z}[G],\mathbb{Q})_{\mathrm{tors}}$ (cf. Proposition 5.7) and $\varepsilon_{L/K}(0)^{2n} \in \zeta(\mathbb{R}[G])^{\times+}$ one has $2n\cdot R\Omega^{\mathrm{loc}}(L/K,1) = 2n\cdot \hat{\partial}^1_G(\varepsilon_{L/K}(0)) = \partial^1_G(\varepsilon_{L/K}(0)^{2n})$. Furthermore

$$\varepsilon_{L/K}(0)^{2n} = (W(\overline{\chi})^{2n} Nf(\chi)^n |d_{K/\mathbb{Q}}|^{\deg(\chi)n})_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)} \mathbb{C}^\times$$

by definition, and since $Nf(\chi)$ and $|d_{K/\mathbb{Q}}|$ are both rational integers and $W(\overline{\chi})$ is an algebraic number with absolute value equal to 1 for every archimedean place (this follows for example from [44, Lem. on p. 98]), one sees that $\varepsilon_{L/K}(0)^{2n}$ satisfies the hypothesis of Lemma 5.15. Thus Lemma 5.15 shows that $2n\cdot R\Omega^{\mathrm{loc}}(L/K,1)$ determines $\left|W(\overline{\chi})^{2n} Nf(\chi)^n |d_{K/\mathbb{Q}}|^{\deg(\chi)n}\right| = Nf(\chi)^n |d_{K/\mathbb{Q}}|^{\deg(\chi)n}$ for every $\chi \in \mathrm{Irr}(G)$. This allows us to find $|d_{K/\mathbb{Q}}|$ because the conductor of the trivial character is equal to 1. We then get $Nf(\chi)$ for every $\chi \in \mathrm{Irr}(G)$ and finally $Nf(\chi)$ for arbitrary characters χ because the Artin conductor is multiplicative. □

5.3.2. *Determining symplectic epsilon constants.* We denote the group of all complex valued (virtual) characters of G by R_G and the subgroup of symplectic characters by R_G^s. Let $\alpha = (\alpha_\chi)_{\chi\in\mathrm{Irr}(G)} \in \prod_{\chi\in\mathrm{Irr}(G)} \mathbb{C}^\times = \zeta(\mathbb{C}[G])^\times$. The map $\mathrm{Irr}(G) \to \mathbb{C}^\times$ given by $\chi \mapsto \alpha_\chi$ has a unique extension to a homomorphism of abelian groups $R_G \to \mathbb{C}^\times$ and thus defines α_χ for every $\chi \in R_G$.

Lemma 5.17. *Let $\alpha \in \zeta(\mathbb{R}[G])^\times$ be such that $\alpha_\chi \in \{\pm 1\}$ for all $\chi \in R_G^s$. If $\hat\partial_G^1(\alpha) = 0 \in K_0(\mathbb{Z}[G],\mathbb{R})$ then $\alpha_\chi = 1$ for all $\chi \in R_G^s$.*

Proof. We reduce to [21, Prop. (6.1)] and use the same notation as there. In particular, J denotes the idèle group of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$. We write $+1_\infty$ for the idèle with component 1 at all places and -1_∞ for the idèle with component -1 at all non-archimedean places and component 1 at all archimedean places. Let $f \in \mathrm{Hom}(R_G^s, \pm 1_\infty)$ be the homomorphism given by

$$\chi \mapsto \begin{cases} +1_\infty & \text{if } \alpha_\chi = +1 \\ -1_\infty & \text{if } \alpha_\chi = -1 \end{cases}$$

for $\chi \in R_G^s$. To apply [21, Prop. (6.1)] we must show that $f \in \mathrm{Det}^s(\tilde{\mathbb{Z}}[G]^\times)$ where $\tilde{\mathbb{Z}}[G] = \mathbb{R}[G] \times \prod_p \mathbb{Z}_p[G]$ with p running through all rational prime numbers and Det^s denoting the restriction of the determinantal homomorphisms to R_G^s as discussed in [21, §3]. The archimedean component of f is obviously contained in $\mathrm{Det}^s(\mathbb{R}[G]^\times)$.

Let p be a prime number. Let $j : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ be any embedding and extend it to an embedding $j : \mathbb{C} \to \overline{\mathbb{Q}_p}$. Then j induces maps $j : \zeta(\mathbb{R}[G])^\times \to \zeta(\overline{\mathbb{Q}_p}[G])^\times$ and $K_0(\mathbb{Z}[G],\mathbb{R}) \to K_0(\mathbb{Z}_p[G],\overline{\mathbb{Q}_p})$ such that

$$\begin{array}{ccc} \zeta(\mathbb{R}[G])^\times & \xrightarrow{\hat\partial_G^1} & K_0(\mathbb{Z}[G],\mathbb{R}) \\ \downarrow j & & \downarrow \\ \zeta(\overline{\mathbb{Q}_p}[G])^\times & \xrightarrow{\partial^1_{\mathbb{Z}_p[G],\overline{\mathbb{Q}_p}}} & K_0(\mathbb{Z}_p[G],\overline{\mathbb{Q}_p}) \end{array}$$

commutes. From the hypothesis it follows that

$$j(\alpha) \in \mathrm{im}(K_1(\mathbb{Z}_p[G]) \to \zeta(\overline{\mathbb{Q}_p}[G])^\times). \tag{27}$$

In particular $j(\alpha) \in \zeta(\mathbb{Q}_p[G])^\times$ which implies $\alpha_{j^{-1}\circ\omega\circ j\circ\chi} = \alpha_\chi$ for all $\chi \in R_G^s$ and $\omega \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. Since this is true for all p and embeddings j we find that $\alpha_{\omega\circ\chi} = \alpha_\chi$ for all $\chi \in R_G^s$ and $\omega \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The p-component of f is the map $f_p : \chi \mapsto \alpha_\chi \in (\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times$ for $\chi \in R_G^s$. By the argument above f_p lies in $\mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(R_G^s, (\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times)$. Denote the group of symplectic $\overline{\mathbb{Q}_p}$-valued characters by $R_{G,p}^s$. An embedding $j : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_p}$ induces a homomorphism

$$j_* : \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}(R_G^s, (\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times) \to \mathrm{Hom}_{\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)}(R_{G,p}^s, \overline{\mathbb{Q}_p}^\times),$$

and to show that f_p lies in $\mathrm{Det}^s(\mathbb{Z}_p[G]^\times)$ it suffices to show that $j_*(f_p)$ lies in $\mathrm{Det}^s(\mathbb{Z}_p[G]^\times)$ (compare the diagram in [21, p. 254]). But $j_*(f_p)$ is the homomorphism which corresponds to $j(\alpha) \in \zeta(\overline{\mathbb{Q}_p}[G])^\times$ and therefore lies in $\mathrm{Det}^s(\mathbb{Z}_p[G]^\times)$ by (27).

We have shown that $f \in \mathrm{Hom}(R_G^s, \pm 1_\infty) \cap \mathrm{Det}^s(\tilde{\mathbb{Z}}[G]^\times)$. By [21, Prop. (6.1)] this intersection consists only of the trivial homomorphism, hence $\alpha_\chi = 1$ for all $\chi \in R_G^s$. □

Proof of Theorem 5.8. By the proof of Corollary 5.16 the element $\hat{\partial}_G^1(\varepsilon_{L/K}(0))$ determines $Nf(\chi)|d_{K/\mathbb{Q}}|^{\deg(\chi)}$ for every $\chi \in \mathrm{Irr}(G)$. We set $\delta_\chi := \sqrt{Nf(\chi)|d_{K/\mathbb{Q}}|^{\deg(\chi)}}$ (positive square root), $\delta := (\delta_\chi)_{\chi \in \mathrm{Irr}(G)}$ and $\alpha := \varepsilon_{L/K}(0)\delta^{-1}$. Then δ and α lie in $\zeta(\mathbb{R}[G])^\times$ and $\alpha = (W(\overline{\chi}))_{\chi \in \mathrm{Irr}(G)}$.

Since $W(\overline{\chi}) \in \{\pm 1\}$ for every $\chi \in R_G^s$, we can apply Lemma 5.17 to α and conclude that the root numbers $W(\overline{\chi})$ for $\chi \in R_G^s$ are determined by $\hat{\partial}_G^1(\alpha) = \hat{\partial}_G^1(\varepsilon_{L/K}(0)) - \hat{\partial}_G^1(\delta)$ and therefore also by $\hat{\partial}_G^1(\varepsilon_{L/K}(0))$. Thus assuming the validity of Conjecture 5.3, the symplectic root numbers are determined by $R\Omega^{\mathrm{loc}}(L/K, 1)$. □

6. The equivariant Tamagawa number conjecture

We fix L/K and S as in §3. In this section we shall prove that Conjecture 3.3 is equivalent under certain hypotheses to [17, Conj. 4(iv)] for the pair $(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])$. We recall that [17, Conj. 4(iv)] is itself a natural equivariant refinement of the 'Tamagawa number conjecture' originally formulated by Bloch and Kato in [3] and then extended by Fontaine and Perrin-Riou in [27] and by Kato in [36].

6.1. **Statement of the main results.** In order to state the main result of this section we recall that an element $T\Omega(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])$ of $K_0(\mathbb{Z}[G], \mathbb{R})$ is defined (unconditionally) in [17, Conj. 4(iii)] and that [loc. cit., Conj. 4(iv)] asserts the vanishing of $T\Omega(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])$.

Theorem 6.1. *Assume that both of the following hypotheses are satisfied.*

(i) $K = \mathbb{Q}$ *and* L *is a CM field.*

(ii) *The natural localisation map* $\lambda_p : \mathcal{O}_L^\times \otimes \mathbb{Z}_p \to \prod_{w \in S_p(L)} U_{L_w}^{(1)}$ *is injective.*

Then $T\Omega(L/K, 1)$ *and* $T\Omega(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])$ *have the same image under the map* $K_0(\mathbb{Z}[G], \mathbb{R}) \to K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ *that is induced by any choice of embedding* $\mathbb{R} \to \mathbb{C}_p$. *In particular, if* λ_p *is injective for all primes* p, *then*

$$T\Omega(L/K, 1) = T\Omega(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])$$

and the validity of Conjecture 3.3 is equivalent to the validity of [17, Conj. 4(iv)] *for the pair* $(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])$.

Remark 6.2. The functorial properties of Conjecture 3.3 that are described in Proposition 3.5 mean that it is enough to prove the conjecture in the case that $K = \mathbb{Q}$ and L is a totally imaginary Galois extension of $\mathbb{Q}$. But the assumption that L is a CM field is in general restrictive and the injectivity of λ_p is equivalent to the validity of Leopoldt's Conjecture for the field L and prime p. However, the latter hypotheses are introduced solely to simplify certain aspects of the proof and we expect the equality $T\Omega(L/K, 1) = T\Omega(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])$ to be valid without assuming either of them.

Remark 6.3. When combined with (the argument used to prove) Proposition 3.6(ii), the equality $T\Omega(L/K, 1) = T\Omega(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])$ of Theorem 6.1 implies that $\partial_G^0(T\Omega(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G]))$ is equal to the element $\Omega(L/K, 1)$ of $K_0(\mathbb{Z}[G])$

that is defined by Chinburg in [24]. This answers the second half of the question raised by Flach and the second named author in [15, Question 1.54].

Remark 6.4. Theorem 6.1 establishes connections between Conjecture 3.3 and other interesting conjectures. For example, [17, Conj. 4(iv)] is a consequence of the 'main conjecture of non-commutative Iwasawa theory' that is formulated by Fukaya and Kato in [31, Conj. 2.3.2]. One may therefore regard the study of Conjecture 3.3 as an attempt to provide supporting evidence for the conjecture of Fukaya and Kato. In another more concrete direction, it can be shown that the validity of [17, Conj. 4(iv)] for the pair $(h^0(\operatorname{Spec} L)(1), \mathbb{Z}[G])$ implies a refined version of an explicit conjecture concerning the values of certain 'twisted Zeta functions' that is formulated (for abelian G) by Solomon in [41] – see forthcoming work of Andrew Jones in this regard.

Corollary 6.5. *Let L/K be a Galois extension of number fields and assume that L is abelian over $\mathbb{Q}$.*

(i) *Conjecture 3.3 for the extension L/K is valid.*
(ii) *If the conductor of $L/\mathbb{Q}$ is odd then Conjecture 4.1 for the extension L/K is valid.*

Proof. Following Remark 6.2 we may assume that $K = \mathbb{Q}$ and L is a totally imaginary abelian extension of $\mathbb{Q}$ and hence automatically a CM field. In addition, in this case Brumer has proved that the hypothesis of Theorem 6.1(ii) is valid for all primes p [9] and the validity of [17, Conj. 4(iv)] for the pair $(h^0(\operatorname{Spec} L)(1), \mathbb{Z}[G])$ is a consequence of the main result of Flach and the second named author in [19]. The validity of Conjecture 3.3 therefore follows immediately from the second assertion of Theorem 6.1. Given this, claim (ii) follows directly from Proposition 5.7(ii) and Theorem 5.2. □

6.2. **Preliminaries concerning étale cohomology.** To relate Conjecture 3.3 to [17, Conj. 4(iv)] we shall make use of certain constructions in étale cohomology that are made in [16]. However, unfortunately the relevant parts of [16] contain some errors (cf. for example the proof of Lemma 6.9) and so we shall first present a corrected version of these constructions.

We fix L/K and S as in §3 and we continue to use the notation of §2.3.1. For each $w \in S(L)$ we denote the algebraic closure of L in L_w by L_w^h. Note that if $w \in S_f(L)$, then L_w^h is the field of fractions of the henselisation of (the localisation of) $\mathcal{O}_L$ at w (compare [38, Chap. I, Exam. 4.10(a)]). For $w \in S_f(L)$ we denote the ring of integers in L_w^h by $\mathcal{O}_w^h$, we write $\mathfrak{m}_w^h$ for the maximal ideal of $\mathcal{O}_w^h$ and set $\lambda(w) := \mathcal{O}_w^h/\mathfrak{m}_w^h = \mathcal{O}_w/\mathfrak{m}_w$. For any place $v \in S(K)$, resp. $v \in S_f(K)$, we define G-modules by setting $L_v^h := \prod_{w \in S_v(L)} L_w^h$, resp. $\mathcal{O}_{L,v}^h := \prod_{w \in S_v(L)} \mathcal{O}_w^h$ and $\mathfrak{m}_{L,v}^h := \prod_{w \in S_v(L)} \mathfrak{m}_w^h$.

We will need to consider the following canonical morphisms of schemes: the inclusion of the generic point $g : \operatorname{Spec} L \to \operatorname{Spec} \mathcal{O}_{L,S}$, for any $w \in S(L)$ the maps $g_w^h : \operatorname{Spec} L_w^h \to \operatorname{Spec} \mathcal{O}_{L,S}$, $f_w : \operatorname{Spec} L_w \to \operatorname{Spec} L_w^h$ and $g_w = g_w^h \circ f_w : \operatorname{Spec} L_w \to \operatorname{Spec} \mathcal{O}_{L,S}$, and for $w \notin S$ the inclusion of the closed point $i_w : \operatorname{Spec} \lambda(w) \to \operatorname{Spec} \mathcal{O}_{L,S}$.

6.2.1. *General conventions.* Let X be any scheme and $\mathcal{F}$ an étale sheaf on X, i.e. a sheaf on the étale site X_{et}. By $R\Gamma(X, \mathcal{F})$ we denote the complex in the derived category $\mathcal{D}(\mathbb{Z})$ which is obtained by applying the right derived functor of the global section functor $\Gamma(X,-)$ to the sheaf $\mathcal{F}$; thus $R\Gamma(X,\mathcal{F})$ is defined up to canonical isomorphism in $\mathcal{D}(\mathbb{Z})$. If $X = \operatorname{Spec} R$ for some commutative ring R, then we will write $R\Gamma(R,\mathcal{F})$ for $R\Gamma(\operatorname{Spec} R,\mathcal{F})$ and $H^i(R,\mathcal{F})$ for the cohomology groups $H^i(R\Gamma(R,\mathcal{F}))$.

Now let $v \in S(K)$, $w \in S_v(L)$ and let $\mathcal{F}$ be an étale sheaf on $\operatorname{Spec} K_v^h$. The G_w-action on $\operatorname{Spec} L_w^h$ induces a G_w-action on the sections $\Gamma(\operatorname{Spec} L_w^h,\mathcal{F})$ and hence the complex $R\Gamma(L_w^h,\mathcal{F})$ naturally lies in $\mathcal{D}(\mathbb{Z}[G_w])$. Similarly, if $\mathcal{F}$ is an étale sheaf on $\operatorname{Spec}\mathcal{O}_{K,S}$, then $R\Gamma(\mathcal{O}_{L,S},\mathcal{F})$ belongs to $\mathcal{D}(\mathbb{Z}[G])$. Finally for $v \in S(K)$ and $\mathcal{F}$ an étale sheaf on $\operatorname{Spec}\mathcal{O}_{K,S}$ we want to consider $\bigoplus_{w\in S_v(L)} R\Gamma(L_w^h,(g_w^h)^*\mathcal{F})$ as a complex in $\mathcal{D}(\mathbb{Z}[G])$. To do this we must choose the complexes $R\Gamma(L_w^h,(g_w^h)^*\mathcal{F})$ compatibly. This is possible because

$$\bigoplus_{w\in S_v(L)} R\Gamma(L_w^h,(g_w^h)^*\mathcal{F}) \cong R\Gamma(\coprod_{w\in S_v(L)} \operatorname{Spec} L_w^h,(g^h)^*\mathcal{F}),$$

where $g^h : \coprod_{w\in S_v(L)} \operatorname{Spec} L_w^h \to \operatorname{Spec}\mathcal{O}_{L,S}$ is the natural map, and $\coprod_{w\in S_v(L)} \operatorname{Spec} L_w^h$ is an étale cover of $\operatorname{Spec} K_v^h$ with group G. Alternatively we could fix $w' \in S_v(L)$ and use the decomposition $\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_{w'}]} R\Gamma(L_{w'}^h,(g_{w'}^h)^*\mathcal{F}) = \bigoplus_{w\in S_v(L)} C_w$ where C_w is naturally isomorphic to $R\Gamma(L_w^h,(g_w^h)^*\mathcal{F})$ in $\mathcal{D}(\mathbb{Z}[G_w])$. Of course the same is true with L_w^h and g_w^h replaced by L_w and g_w respectively.

6.2.2. *Local cohomology.* Let w be a place of L and recall that $f_w : \operatorname{Spec} L_w \to \operatorname{Spec} L_w^h$ corresponds to the inclusion $L_w^h \to L_w$. For any étale sheaf $\mathcal{F}$ on $\operatorname{Spec} L_w^h$ the canonical map $R\Gamma(L_w^h,\mathcal{F}) \longrightarrow R\Gamma(L_w, f_w^*\mathcal{F})$ is an isomorphism in $\mathcal{D}(\mathbb{Z}[G_w])$. Indeed, if $\overline{L_w}$ is an algebraic closure of L_w and $\overline{L_w^h}$ is the algebraic closure of L_w^h in $\overline{L_w}$, then the restriction map gives an isomorphism $\operatorname{Gal}(\overline{L_w}/L_w) \xrightarrow{\cong} \operatorname{Gal}(\overline{L_w^h}/L_w^h)$. Thus, upon identifying étale cohomology and Galois cohomology the claimed isomorphism follows.

If $\mathcal{F} = \mathbb{G}_m$ on $(\operatorname{Spec} L_w^h)_{\mathrm{et}}$, then $f_w^*\mathbb{G}_m$ is not isomorphic to the sheaf $\mathbb{G}_m$ on $(\operatorname{Spec} L_w)_{\mathrm{et}}$. However the complexes $R\Gamma(L_w^h,\mathbb{G}_m) \cong R\Gamma(L_w, f_w^*\mathbb{G}_m)$ and $R\Gamma(L_w,\mathbb{G}_m)$ are related as follows.

Lemma 6.6. *There is a distinguished triangle in* $\mathcal{D}(\mathbb{Z}[G_w])$

$$R\Gamma(L_w^h,\mathbb{G}_m) \longrightarrow R\Gamma(L_w,\mathbb{G}_m) \longrightarrow (L_w^\times/(L_w^h)^\times)[0] \longrightarrow,$$

whose cohomology sequence in degree 0 *identifies with the canonical short exact sequence*

$$0 \longrightarrow (L_w^h)^\times \longrightarrow L_w^\times \longrightarrow L_w^\times/(L_w^h)^\times \longrightarrow 0.$$

The G_w*-module* $L_w^\times/(L_w^h)^\times$ *is uniquely divisible and hence c-t.*

Proof. There is a canonical injection $f_w^*\mathbb{G}_m \to \mathbb{G}_m$ of sheaves on $(\operatorname{Spec} L_w)_{\mathrm{et}}$ such that the sequence

$$0 \longrightarrow f_w^*\mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m/f_w^*\mathbb{G}_m \longrightarrow 0$$

corresponds to the exact sequence $0 \to \overline{L_w^h}^\times \to \overline{L_w}^\times \to \overline{L_w}^\times/\overline{L_w^h}^\times \to 0$ of $\operatorname{Gal}(\overline{L_w}/L_w)$-modules. Now $\overline{L_w}^\times/\overline{L_w^h}^\times$ is uniquely divisible. Also, the isomorphism $\operatorname{Gal}(\overline{L_w}/L_w) \cong \operatorname{Gal}(\overline{L_w^h}/L_w^h)$ combines with Hilbert's Theorem 90 to imply $H^0(\operatorname{Gal}(\overline{L_w}/L_w),\overline{L_w}^\times/\overline{L_w^h}^\times) = L_w^\times/(L_w^h)^\times$. It follows that $L_w^\times/(L_w^h)^\times$ is uniquely divisible and hence c-t (as a G_w-module). In addition, by applying $R\Gamma(L_w,-)$ to the displayed exact sequence we obtain the claimed distinguished triangle. □

Lemma 6.7. *There are canonical isomorphisms of G_w-modules*

$$H^i(L_w, \mathbb{G}_m) \cong \begin{cases} L_w^\times & \text{if } i = 0, \\ 0 & \text{if } i = 1, \\ \mathrm{Br}(L_w) & \text{if } i = 2. \end{cases}$$

If w is non-archimedean then $H^i(L_w, \mathbb{G}_m) = 0$ for $i \geq 3$ and the local invariant isomorphism gives a canonical identification $\mathrm{Br}(L_w) \cong \mathbb{Q}/\mathbb{Z}$. With respect to this identification the class of $R\Gamma(L_w, \mathbb{G}_m)$ in $\mathrm{Ext}^3_{G_w}(\mathbb{Q}/\mathbb{Z}, L_w^\times) \cong H^2(G_w, L_w^\times)$ is the local canonical class.

Proof. This is [16, Prop. 3.5.(a)]. □

6.2.3. *Cohomology with compact support.* Recall that $g_w^h : \mathrm{Spec}\, L_w^h \to \mathrm{Spec}\, \mathcal{O}_{L,S}$ is the canonical map. For any étale sheaf $\mathcal{F}$ on $\mathrm{Spec}\, \mathcal{O}_{K,S}$ we define the complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F})$ in $\mathcal{D}(\mathbb{Z}[G])$ by

$$R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F}) := \mathrm{cone}\left(R\Gamma(\mathcal{O}_{L,S}, \mathcal{F}) \to \bigoplus_{w \in S} R\Gamma(L_w^h, (g_w^h)^* \mathcal{F}) \right)[-1], \tag{28}$$

thus this complex lies in a distinguished triangle

$$R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F}) \longrightarrow R\Gamma(\mathcal{O}_{L,S}, \mathcal{F}) \longrightarrow \bigoplus_{w \in S} R\Gamma(L_w^h, (g_w^h)^* \mathcal{F}) \longrightarrow . \tag{29}$$

In [16, (3)] a complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathcal{F})$ is defined just as in (28) but with L_w^h and g_w^h replaced by L_w and the map $g_w : \mathrm{Spec}\, L_w \to \mathrm{Spec}\, \mathcal{O}_{L,S}$ respectively. However, the observation made at the beginning of §6.2.2 ensures that this definition coincides with that given above.

6.2.4. *The cohomology of $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)$.* We define a G-module $C_S^h(L)$ in the same way as $C_S(L)$ is defined in §3.1 but with L_w replaced by L_w^h for each $w \in S(L)$ and $\mathcal{O}_w$ replaced by $\mathcal{O}_w^h$ for each $w \in S_f(L)$. Then, since we assume $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$, the natural map $\prod_{w \in S} (L_w^h)^\times \to C_S^h(L)$ is surjective with kernel $\mathcal{O}_{L,S}^\times$.

Lemma 6.8. *There are canonical isomorphisms of G-modules*

$$H^i(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) \cong \begin{cases} C_S^h(L) & \text{if } i = 1, \\ \mathbb{Q}/\mathbb{Z} & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first note that there are canonical isomorphisms of G-modules

$$H^i(\mathcal{O}_{L,S}, \mathbb{G}_m) \cong \begin{cases} \mathcal{O}_{L,S}^\times & \text{if } i = 0, \\ 0 & \text{if } i = 1, \\ \ker\left(\mathrm{Br}(L) \to \bigoplus_{w \notin S} \mathrm{Br}(L_w)\right) & \text{if } i = 2, \\ \bigoplus_{w \in S_{\mathbb{R}}(L)} H^i(L_w, \mathbb{G}_m) & \text{if } i \geq 3, \end{cases}$$

(cf. [39, Chap. II, Prop. 2.1, Rem. 2.2] and recall that we assume $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$). Now, for every $w \in S$ one has $(g_w^h)^* \mathbb{G}_m = \mathbb{G}_m$ on $(\mathrm{Spec}\, L_w^h)_{\mathrm{et}}$. The cohomology sequence of the distinguished triangle (29) with $\mathcal{F} = \mathbb{G}_m$ thus combines with Lemmas 6.6 and 6.7 and the above displayed isomorphisms to give exact sequences

$$0 \to H^0(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) \to \mathcal{O}_{L,S}^\times \to \bigoplus\nolimits_{w \in S} (L_w^h)^\times \to H^1(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{G}_m)) \to 0$$

and

$$0 \to H^2(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)) \to \ker\big(\mathrm{Br}(L) \to \bigoplus\nolimits_{w\notin S} \mathrm{Br}(L_w)\big) \to \bigoplus\nolimits_{w\in S} \mathrm{Br}(L_w) \to H^3(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)) \to 0$$

and an equality $H^i(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)) = 0$ for each $i \geq 4$. All maps here are the canonical ones, thus for $i=0$ and $i=1$ the claimed description follows immediately and for $i=2$ and $i=3$ it follows by using the canonical exact sequence $0 \to \mathrm{Br}(L) \to \bigoplus_{w\in S(L)} \mathrm{Br}(L_w) \to \mathbb{Q}/\mathbb{Z} \to 0$. □

6.2.5. *The complex* $\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)$. For every $w \in S$ there is a canonical map $g_w : \operatorname{Spec} L_w \to \operatorname{Spec} \mathcal{O}_{L,S}$ of schemes and an inclusion $g_w^*\mathbb{G}_m \to \mathbb{G}_m$ of étale sheaves on $\operatorname{Spec} L_w$. Thus we can consider the composite morphism

$$R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m) \longrightarrow \bigoplus_{w\in S} R\Gamma(L_w, g_w^*\mathbb{G}_m) \longrightarrow \bigoplus_{w\in S} R\Gamma(L_w,\mathbb{G}_m)$$

in $\mathcal{D}(\mathbb{Z}[G])$. We then define the complex $\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)$ by setting

$$\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m) := \mathrm{cone}\left(R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m) \longrightarrow \bigoplus_{w\in S} R\Gamma(L_w,\mathbb{G}_m)\right)[-1].$$

Lemma 6.9. *There are canonical isomorphisms of G-modules*

$$H^i(\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)) \cong \begin{cases} C_S(L) & \text{if } i=1,\\ \mathbb{Q}/\mathbb{Z} & \text{if } i=3,\\ 0 & \text{otherwise.}\end{cases}$$

The class of $\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1]$ in $\mathrm{Ext}^3_G(\mathbb{Q}/\mathbb{Z}, C_S(L)) \cong H^2(G, C_S(L))$ is the global canonical class.

Proof. The computation of the cohomology is similar to the proof of Lemma 6.8, except that the role of (29) is now played by the distinguished triangle

$$\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m) \longrightarrow R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m) \longrightarrow \bigoplus_{w\in S} R\Gamma(L_w,\mathbb{G}_m) \longrightarrow$$

that is induced by the definition of $\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)$. In degree 1 we also use the fact that, since $\mathrm{Pic}(\mathcal{O}_{L,S}) = 0$, $C_S(L)$ is canonically isomorphic to the cokernel of the diagonal embedding $\mathcal{O}_{L,S}^\times \to \prod_{w\in S} L_w^\times$. For the extension class see [16, Prop. 3.5(b)] (but note that the result and proof in [16] apply to $\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)$ rather than to $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)$ as incorrectly stated in loc. cit.). □

Lemma 6.10. *There is a distinguished triangle in $\mathcal{D}(\mathbb{Z}[G])$*

$$R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m) \longrightarrow \widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m) \longrightarrow \bigoplus_{w\in S}(L_w^\times/(L_w^h)^\times)[-1] \longrightarrow$$

which induces the canonical exact sequence $0 \to C_S^h(L) \to C_S(L) \to \prod_{w\in S} L_w^\times/(L_w^h)^\times \to 0$ on cohomology in degree 1 and the identity map $\mathbb{Q}/\mathbb{Z} \xrightarrow{=} \mathbb{Q}/\mathbb{Z}$ on cohomology in degree 3.

Proof. By using the distinguished triangle in Lemma 6.6 for each $w \in S$ we can construct a commutative diagram of distinguished triangles in $\mathcal{D}(\mathbb{Z}[G])$

$$\begin{array}{ccccccc}
R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m) & \longrightarrow & R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m) & \longrightarrow & \bigoplus_{w\in S} R\Gamma(L_w, g_w^*\mathbb{G}_m) & \longrightarrow \\
\downarrow & & \Vert & & \downarrow & \\
\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m) & \longrightarrow & R\Gamma(\mathcal{O}_{L,S},\mathbb{G}_m) & \longrightarrow & \bigoplus_{w\in S} R\Gamma(L_w,\mathbb{G}_m) & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\bigoplus_{w\in S}(L_w^\times/(L_w^h)^\times)[-1] & \longrightarrow & 0 & \longrightarrow & \bigoplus_{w\in S}(L_w^\times/(L_w^h)^\times)[0] & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow &
\end{array}$$

The first column of this diagram is the claimed triangle and the induced cohomology sequence is easily computed by means of the diagram. □

6.3. **The proof of Theorem 6.1.** The validity of the second assertion of Theorem 6.1 follows directly upon combining the first assertion of the theorem with Lemma 2.1. To prove Theorem 6.1 it therefore suffices to fix a prime p and an embedding $j:\mathbb{R}\to\mathbb{C}_p$ and to prove that, writing $j_*: K_0(\mathbb{Z}[G],\mathbb{R})\to K_0(\mathbb{Z}_p[G],\mathbb{C}_p)$ for the map induced by j, one has

$$j_*(T\Omega(L/K,1)) = j_*(T\Omega(h^0(\operatorname{Spec} L)(1),\mathbb{Z}[G])). \tag{30}$$

The proof of this equality will occupy the remainder of the manuscript.

6.3.1. *Pro-p-completion.* Taking advantage of Proposition 3.4 we assume henceforth that $p\in S$. As in §3.1 we choose lattices $\mathcal{L}_v$ for $v\in S_f$ and define $\mathcal{L}$ by (10). We fix an algebraic closure $\overline{K}$ of K and for each natural number n we write μ_{p^n} for the group of p^n-th roots of unity in $\overline{K}$ (regarded as an étale sheaf on $\operatorname{Spec}(\mathcal{O}_{K,S})$ in the natural way). We write K_S for the maximal extension of K inside $\overline{K}$ which is unramified outside S and let $\mathbb{Z}_p(1)$ denote the continuous $\operatorname{Gal}(K_S/K)$-module $\varprojlim_n \mu_{p^n}$ where the limit is taken with respect to p-th power maps. In this section we shall relate $E_S(\mathcal{L})\otimes\mathbb{Z}_p$ to the complex $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p(1))$ that is defined in [17, p. 522].

For any abelian group A and natural number m we write $A_{[m]}$ for the kernel of the endomorphism given by multiplication by m. For each natural number n we let $L(1)_{p,n}$ denote the $\mathbb{Z}/p^n[G]$-module $\prod_{w\in S_\infty(L)}(L_w^\times)_{[p^n]}\subset L_\infty^\times$. We then define a $\mathbb{Z}_p[G]$-module by setting $L(1)_p := \varprojlim_n L(1)_{p,n}$ where the transition morphisms are induced by raising to the power p. We also set $L_p := \prod_{w\in S_p(K)} L_w$ and recall that $\mathcal{L}_p$ is a full projective $\mathbb{Z}_p[G]$-sublattice of L_p.

Proposition 6.11. *There exists a distinguished triangle in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ of the form*

$$\mathcal{L}_p[0]\oplus\mathcal{L}_p[-1]\to R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p(1))[2]\to E_S(\mathcal{L})\otimes\mathbb{Z}_p\to . \tag{31}$$

If L is totally imaginary and λ_p is injective, then there are canonical isomorphisms

$$H^i(R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p(1)))\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \cong \begin{cases} (\varprojlim_n C_S(L)_{[p^n]}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, & \text{if } i=1 \\ (\varprojlim_n C_S(L)/p^n) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p, & \text{if } i=2 \\ \mathbb{Q}_p, & \text{if } i=3 \\ 0, & \text{otherwise.} \end{cases}$$

With respect to these isomorphisms and the description of the cohomology groups $H^i(E_S(\mathcal{L}))\otimes\mathbb{Q}$ given in Lemma 3.1(ii), the image under $-\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ of the cohomology sequence of (31) is equal to

$$(32)\quad 0 \to L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\theta_1} H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q}_p \xrightarrow{\theta_2} L_p \xrightarrow{\exp_p} \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{0}$$
$$H^0(E_S(\mathcal{L})) \otimes \mathbb{Q}_p \xrightarrow{\subset} L_p \xrightarrow{\mathrm{tr}_{L_p}} \mathbb{Q}_p \to 0$$

where θ_2 is induced by the projection $L_S \to L_p$ and θ_1 sends $(r_w\cdot\{\exp(2\pi\sqrt{-1}/p^n)\}_{n\geq 0})_{w\in S_\infty(L)}$ to the element $(r_w \cdot 2\pi\sqrt{-1})_{w\in S_\infty(L)}$ of $\ker(\exp_\infty) \otimes \mathbb{Q}_p \subset H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q}_p$.

The proof of Proposition 6.11 will occupy the rest of this section (but is itself of no further use in the sequel). As the first step in this proof we introduce a useful auxiliary complex.

Lemma 6.12. *There exists a complex Q in $\mathcal{D}(\mathbb{Z}[G])$ which corresponds (via §2.2.2) to the extension class e_S^{glob} and also possesses all of the following properties.*

- (i) *Q is a complex of $\mathbb{Z}$-torsion-free G-modules of the form $Q^{-1} \to Q^0 \to Q^1$ (where the first term is placed in degree -1).*
- (ii) *The morphism α_S used in the distinguished triangle (11) is represented by a morphism of complexes of G-modules $\alpha : \mathcal{L}_S[0] \oplus \mathcal{L}[-1] \to Q$.*
- (iii) *For each natural number n the complex Q/p^n consists of finite projective $\mathbb{Z}/p^n[G]$-modules.*

Proof. At the outset we fix a representative of e_S^{glob} of the form $A \xrightarrow{d} B$ as in Remark 3.2 with B a finitely generated projective $\mathbb{Z}[G]$-module. We write d^{-1} for the composite of $\exp_S : \mathcal{L}_S \to C_S(L)$ and the inclusion $C_S(L) \subset A$. Since $\mathrm{cok}(\exp_S)$ is finite we may choose a finitely generated free $\mathbb{Z}[G]$-module F and a homomorphism $\pi : F \to A$ such that the morphism $(d^{-1},\pi) : \mathcal{L}_S \oplus F \to A$ is surjective. We take Q to be the complex $\ker((d^{-1},\pi)) \xrightarrow{\subset} \mathcal{L}_S \oplus F \xrightarrow{d\circ(d^{-1},\pi)} B$ where the first term is placed in degree -1. Then (d^{-1},π) restricts to give a surjection $\ker(d\circ(d^{-1},\pi)) \to C_S(L)$ which in turn induces an identification of $H^0(Q)$ with $C_S(L)$. Via this identification, the morphism from Q to $A \to B$ that is equal to (d^{-1},π) in degree 0 and to the identity map in degree 1 induces the identity map on cohomology in each degree and so Q represents e_S^{glob}. Further, we obtain a morphism α as in claim (ii) by defining α^0 to be the inclusion $\mathcal{L}_S \subset \mathcal{L}_S\oplus F$ and α^1 to be the map tr' described in Remark 3.2.

Regarding claim (iii) we note $(\mathcal{L}_S \oplus F)/p^n$ and B/p^n are both finite and projective as $\mathbb{Z}/p^n[G]$-modules and that $\ker((d^{-1},\pi))/p^n$ is finite, and so it suffices to prove $\ker((d^{-1},\pi))/p^n$ is a projective $\mathbb{Z}/p^n[G]$-module. The exact sequence $0 \to \ker((d^{-1},\pi)) \to \mathcal{L}_S \oplus F \to A \to 0$ implies that the G-module $\ker((d^{-1},\pi))$ is c-t. Since $\ker((d^{-1},\pi))$ is also $\mathbb{Z}$-torsion-free this implies that there exists an exact sequence of finite $\mathbb{Z}/p^n[G]$-modules

$$0 \to P_{-1} \to P_0 \to \ker((d^{-1},\pi))/p^n \to 0$$

in which P_{-1} and P_0 are both projective. But any finite $\mathbb{Z}/p^n[G]$-module that is projective is also injective (because the functor $\mathrm{Hom}_{\mathbb{Z}/p^n}(-,\mathbb{Z}/p^n)$ is exact on the category of finite $\mathbb{Z}/p^n[G]$-modules and if P is any finite projective $\mathbb{Z}/p^n[G]$-module, then $\mathrm{Hom}_{\mathbb{Z}/p^n}(P,\mathbb{Z}/p^n)$ (endowed with contragredient G-action) is also projective). The displayed exact sequence therefore splits and so $\ker((d^{-1},\pi))/p^n$ is indeed projective. $\square$

We now fix a morphism α as in Lemma 6.12(ii). Then, for each natural number n one has a commutative diagram of morphisms of complexes of G-modules

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & \mathcal{L}_S[0]\oplus\mathcal{L}[-1] & \xrightarrow{\alpha} & Q & \xrightarrow{\beta} & \mathrm{cone}(\alpha) & \xrightarrow{\gamma} \\
 & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle p^n} & \\
(33) & & \mathcal{L}_S[0]\oplus\mathcal{L}[-1] & \xrightarrow{\alpha} & Q & \xrightarrow{\beta} & \mathrm{cone}(\alpha) & \xrightarrow{\gamma} \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & \mathcal{L}_S/p^n[0]\oplus\mathcal{L}/p^n[-1] & \xrightarrow{\alpha/p^n} & Q/p^n & \xrightarrow{\beta/p^n} & \mathrm{cone}(\alpha/p^n) & \xrightarrow{\gamma/p^n} \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & 0 & & 0 & & 0. &
\end{array}$$

In this diagram the maps β and γ come from the definition of $\mathrm{cone}(\alpha)$ and the conventions we fix in §2.2.1 and so the first (and second) row is an explicit representative of the triangle (11). Also, the columns are the short exact sequences which result from the fact that $\mathcal{L}_S$, $\mathcal{L}$ and each term of Q (and hence also of $\mathrm{cone}(\alpha)$) is $\mathbb{Z}$-torsion-free. Now $\mathcal{L}_p$ is canonically isomorphic to both $\varprojlim_n \mathcal{L}_S/p^n$ and $\varprojlim_n \mathcal{L}/p^n$ and, as $\mathrm{cone}(\alpha)$ is perfect, there are natural isomorphisms $\mathrm{cone}(\alpha)\otimes\mathbb{Z}_p \cong \varprojlim_n \mathrm{cone}(\alpha/p^n) \cong \mathrm{cone}(\varprojlim_n \alpha/p^n)$ in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ (where in all cases the limits are taken with respect to the natural transition morphisms). Hence, upon passing to the inverse limit of the lower row of (33) and setting $Q_{\mathrm{lim}} := \varprojlim_n Q/p^n$ (with respect to the natural transition morphisms), we obtain a distinguished triangle in $\mathcal{D}^{\mathrm{perf}}(\mathbb{Z}_p[G])$ of the form

$$\mathcal{L}_p[0]\oplus\mathcal{L}_p[-1] \xrightarrow{\varprojlim_n \alpha/p^n} Q_{\mathrm{lim}} \xrightarrow{\varprojlim_n \beta/p^n} \mathrm{cone}(\alpha)\otimes\mathbb{Z}_p \xrightarrow{\varprojlim_n \gamma/p^n} . \tag{34}$$

We must now relate Q_{lim} to the complex $R\Gamma_c(\mathcal{O}_{L,S},\mathbb{Z}_p(1))[2]$. To do this we let $\widehat{Q}$ denote the complex

$$Q^{-1} \to Q^0 \to Q^1 \to \mathbb{Q}$$

where Q^{-1} is placed in degree -1, the first two arrows are the differentials of Q and the third is the natural map $Q^1 \to H^1(Q) = \mathbb{Z} \subset \mathbb{Q}$. Then the second assertion of Lemma 6.9 combines with the fact that Q corresponds to e_S^{glob} (and our convention that Yoneda and derived functor Ext-groups are identified by means of an injective resolution of the second variable) to imply the existence of an isomorphism $\xi : \widehat{Q} \cong$

$\widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1]$ in $\mathcal{D}(\mathbb{Z}[G])$ which induces the identity map on each degree of cohomology.

We now consider the following diagram in $\mathcal{D}(\mathbb{Z}[G])$

$$\begin{array}{ccccccc} \widehat{Q} & \xrightarrow{p^n} & \widehat{Q} & \longrightarrow & Q/p^n & \longrightarrow & \\ \downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\xi} & & & & \\ \widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1] & \xrightarrow{p^n} & \widehat{R\Gamma}_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1] & \longrightarrow & R\Gamma_c(\mathcal{O}_{L,S},\mu_{p^n})[2] & \longrightarrow & , \end{array} \tag{35}$$

where the upper row is the distinguished triangle associated to the natural short exact sequence $0 \to \widehat{Q} \xrightarrow{p^n} \widehat{Q} \to Q/p^n \to 0$ and the lower row is the triangle that is obtained in the following way. By rotating the distinguished triangle

$$R\Gamma_c(\mathcal{O}_{L,S},\mu_{p^n}) \xrightarrow{\theta} R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m) \xrightarrow{p^n} R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m) \xrightarrow{\kappa} \tag{36}$$

that is induced by the exact sequence $0 \to \mu_{p^n} \to \mathbb{G}_m \xrightarrow{p^n} \mathbb{G}_m \to 0$ of sheaves one finds that there is a distinguished triangle of the form

$$R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1] \xrightarrow{p^n} R\Gamma_c(\mathcal{O}_{L,S},\mathbb{G}_m)[1] \xrightarrow{\kappa[1]} R\Gamma_c(\mathcal{O}_{L,S},\mu_{p^n})[2] \xrightarrow{\theta[2]} .$$

Upon combining the latter triangle with that of Lemma 6.10 and the fact that each module $L_w^\times/(L_w^h)^\times$ is uniquely divisible (by Lemma 6.6) one obtains the lower row of (35). Since the left hand square of (35) commutes there exists an isomorphism

$$\xi_n : Q/p^n \to R\Gamma_c(\mathcal{O}_{L,S},\mu_{p^n})[2]$$

in $\mathcal{D}(\mathbb{Z}[G])$ which makes the diagram into an isomorphism of distinguished triangles. The isomorphisms ξ_n can be chosen to be compatible with the inverse systems (over n).

Before stating the next result we note that, since μ_{p^n} is a continuous $\mathrm{Gal}(K_S/K)$-module, we can identify $R\Gamma_c(\mathcal{O}_{L,S},\mu_{p^n})$ with a concrete complex of $\mathbb{Z}/p^n[G]$-modules that is constructed by using continuous cochains as in [17, p. 522].

Lemma 6.13. *For each natural number n there exists a morphism of complexes of $\mathbb{Z}/p^n[G]$-modules $\tilde{\xi}_n : Q/p^n \to R\Gamma_c(\mathcal{O}_{L,S},\mu_{p^n})[2]$ which satisfies both of the following conditions.*

(i) *One has a commutative diagram of morphisms of complexes of G-modules*

$$\begin{array}{ccc} Q/p^n & \longrightarrow & Q/p^{n-1} \\ \downarrow{\scriptstyle\tilde{\xi}_n} & & \downarrow{\scriptstyle\tilde{\xi}_{n-1}} \\ R\Gamma_c(\mathcal{O}_{L,S},\mu_{p^n})[2] & \longrightarrow & R\Gamma_c(\mathcal{O}_{L,S},\mu_{p^{n-1}})[2], \end{array}$$

where the upper arrow is the canonical projection and the lower arrow is the map induced by composing cochains with the p-th power map $\mu_{p^n} \to \mu_{p^{n-1}}$.

(ii) *For each integer i one has $\varprojlim_n H^i(\tilde{\xi}_n) = \varprojlim_n H^i(\xi_n)$, where the first limit is taken with respect to the maps induced on cohomology by the morphisms described in claim (i).*

Proof. The argument we use here is modelled closely on that of [6, Lem. 4.21]. We set $R\Gamma_c(\mu_{p^n}) := R\Gamma_c(\mathcal{O}_{L,S},\mu_{p^n})$ and $\mathcal{D}_p := \mathcal{D}(\mathbb{Z}_p[G])$.

Then, since both Q/p^n and $R\Gamma_c(\mu_{p^n})[2]$ are cohomologically bounded complexes of $\mathbb{Z}/p^n[G]$-modules, the natural restriction of scalars homomorphism

$$\mathrm{Hom}_{\mathcal{D}_p}(Q/p^n, R\Gamma_c(\mu_{p^n})[2]) \to \mathrm{Hom}_{\mathcal{D}(\mathbb{Z}[G])}(Q/p^n, R\Gamma_c(\mu_{p^n})[2]) \tag{37}$$

is bijective (cf. [16, Lem. 17]). There exists a natural distinguished triangle in $\mathcal{D}_p$

$$Q_{\lim} \xrightarrow{p^n} Q_{\lim} \xrightarrow{\pi_n} Q/p^n \to Q_{\lim}[1] \tag{38}$$

and we use (37) to regard the composite $\xi_n \circ \pi_n$ as an element of $\mathrm{Hom}_{\mathcal{D}_p}(Q_{\lim}, R\Gamma_c(\mu_{p^n})[2])$. Now Lemma 6.12(iii) combines with [25, Prop. (6.17)] (with $A = \mathbb{Z}_p[G], R := \mathbb{Z}_p$ and $N = p^n \cdot \mathbb{Z}_p[G]$ for varying n) to imply that each term of $Q_{\lim}$ is a projective $\mathbb{Z}_p[G]$-module. Hence in $\mathcal{D}_p$ the morphism $\xi_n \circ \pi_n$ can be represented by an actual map of complexes $Q_{\lim} \to R\Gamma_c(\mu_{p^n})[2]$ which is itself unique up to homotopy. Since $R\Gamma_c(\mu_{p^n})[2]$ consists of p^n-torsion modules, this map factors through π_n and so we obtain a map of complexes $\tilde{\xi}_n : Q/p^n \to R\Gamma_c(\mu_{p^n})[2]$ such that $\xi_n \circ \pi_n = \tilde{\xi}_n \circ \pi_n$ in $\mathcal{D}_p$.

Further, it is straightforward to check that the diagram in claim (i) commutes up to homotopy. Hence, since each term of Q/p^n is a projective $\mathbb{Z}/p^n[G]$-module, the same argument as in [16, p. 1367] shows that one can always modify the maps $\tilde{\xi}_n$ by homotopies (if necessary) in order to ensure that the given diagram is commutative.

To prove claim (ii) we note that (38) induces an exact sequence of Hom-groups

$$\mathrm{Hom}_{\mathcal{D}_p}(Q_{\lim}[1], R\Gamma_c(\mu_{p^n})[2]) \to \mathrm{Hom}_{\mathcal{D}_p}(Q/p^n, R\Gamma_c(\mu_{p^n})[2]) \to \mathrm{Hom}_{\mathcal{D}_p}(Q_{\lim}, R\Gamma_c(\mu_{p^n})[2]).$$

We set $f_n := \xi_n - \tilde{\xi}_n \in \mathrm{Hom}_{\mathcal{D}_p}(Q/p^n, R\Gamma_c(\mu_{p^n})[2])$. Then, since $f_n \circ \pi_n = 0$, the displayed sequence implies that f_n factors through the morphism $Q/p^n \to Q_{\lim}[1]$. It follows that, for each integer i, the morphism $H^i(f_n)$ factors through the map $H^i(Q/p^n) \to H^{i+1}(Q_{\lim})_{[p^n]}$. But each module $H^{i+1}(Q_{\lim})$ is finitely generated over $\mathbb{Z}_p$ and hence its $\mathbb{Z}_p$-torsion subgroup is finite. Thus one has $\varprojlim_n H^i(f_n) = 0$, as is required to complete the proof of claim (ii). □

Now the complex $R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))$ defined in [17, p. 522] is equal to $\varprojlim_n R\Gamma_c(\mathcal{O}_{L,S}, \mu_{p^n})$, where the transition morphisms are as described in Lemma 6.13(i). Since each ξ_n is an isomorphism in $\mathcal{D}(\mathbb{Z}[G])$, from Lemma 6.13(ii) we may therefore deduce that $\varprojlim_n \tilde{\xi}_n$ is a quasi-isomorphism of complexes of $\mathbb{Z}_p[G]$-modules of the form

$$Q_{\lim} \to R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))[2]. \tag{39}$$

This isomorphism combines with (34) to give a triangle of the form (31).

To compute the cohomology sequence of (31) we assume henceforth that L is totally imaginary and λ_p is injective.

To compute the modules $H^i(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)))$ explicitly one can combine the cohomology sequence of (36) with the identifications of Lemma 6.8 and then pass to the inverse limit over n (or, equivalently, combine the cohomology sequence of the lower row of (35) with the descriptions of Lemma 6.9 and then pass to the inverse limit over n). In particular, by these means one obtains the explicit descriptions of the spaces $H^i(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ given in Proposition 6.11. In this regard we note only that in the cases $i = 1$ and $i = 2$, the second displayed isomorphisms in the descriptions of $H^i(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ in Proposition 6.11 rely on the assumption that L is totally imaginary and the fact that the diagonal map

$\mathcal{O}_{L,S}^{\times} \otimes \mathbb{Q}_p \to \prod_{w \in S_f}(\varprojlim_n L_w^{\times}/(L_w^{\times})^{p^n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is injective (as follows from the assumed injectivity of λ_p) and has cokernel naturally isomorphic to $\mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

On the other hand, the cohomology of Q_{lim} can be computed explicitly by combining the cohomology sequence of the second column of (33) with the description of the cohomology of Q that is implied by the fact that Q corresponds to the extension class e_S^{glob} and then passing to the inverse limit over n. In this way one finds that there is a natural identification $H^i(Q_{\mathrm{lim}}) = H^{i+2}(R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)))$ for each integer i and indeed that, with respect to these identifications, the isomorphism (39) induces the identity map in each degree of cohomology.

By combining the observations made in the previous two paragraphs with an explicit computation of all of the morphisms on cohomology that occur in (34) one finds that the image under $- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ of the cohomology sequence of (31) is equal to the sequence (32). This therefore completes the proof of Proposition 6.11.

6.3.2. *The element* $j_*(T\Omega(L/K,1))$. In the sequel we shall always assume that the hypotheses (i) and (ii) of Theorem 6.1 are satisfied.

We write L^+ for the maximal real subfield of L and we observe that the extension $L^+/\mathbb{Q}$ is Galois. For each place $w_0 \in S_\infty(L)$ we write σ_{w_0} for the complex embedding $L = L \otimes_{\mathbb{Q}} \mathbb{Q} \subset L \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{w \in S_\infty(L)} L_w \to L_{w_0} = \mathbb{C}$ (where the arrow denotes the natural projection map). For any complex number z and complex embedding $\sigma : L \to \mathbb{C}$ we let $\overline{z}$ and $\overline{\sigma}$ denote the complex conjugate of z and the composite of σ and complex conjugation respectively. We shall also use the following abbreviations: for any subring Λ of $\mathbb{C}_p$ which contains $\mathbb{Z}_p$ we set $R\Gamma_c(\Lambda(1)) := R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \Lambda$ and $H_c^i(\Lambda(1)) := H^i(R\Gamma_c(\Lambda(1)))$ for each integer i.

To state the next result we must define isomorphisms of $\mathbb{C}_p[G]$-modules $\tau^j : H_c^{\mathrm{ev}}(\mathbb{C}_p(1)) \cong H_c^{\mathrm{od}}(\mathbb{C}_p(1))$ and $\xi^j : L_p \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong L_p \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. To this end we write $\mathrm{Irr}_p(G)$ for the set of irreducible $\mathbb{C}_p$-valued characters of G. We let $\mathbf{1}_G$ denote the trivial character of G and we refer to an element of $\mathrm{Irr}_p(G)$ as 'even', resp. 'odd', if it is inflated from an element of $\mathrm{Irr}_p(\mathrm{Gal}(L^+/\mathbb{Q}))$, resp. otherwise. For each $\chi \in \mathrm{Irr}_p(G)$ we fix a $\mathbb{C}_p[G]$-module V_χ which realises χ. Given a $\mathbb{C}_p[G]$-module M, resp. a homomorphism of $\mathbb{C}_p[G]$-modules $\phi : M \to N$, and any $\chi \in \mathrm{Irr}_p(G)$ we set $M_\chi := \mathrm{Hom}_{\mathbb{C}_p[G]}(V_\chi, M)$ and write $\phi_\chi : M_\chi \to N_\chi$ for the induced map of $\mathbb{C}_p$-modules. For any finitely generated $\mathbb{Q}_p[G]$-module P, resp. homomorphism of finitely generated $\mathbb{Q}_p[G]$-modules $\psi : P \to Q$, we also set $P_\chi := (P \otimes_{\mathbb{Q}_p} \mathbb{C}_p)_\chi$ and $\psi_\chi := (\psi \otimes_{\mathbb{Q}_p} \mathrm{id}_{\mathbb{C}_p})_\chi$ for each $\chi \in \mathrm{Irr}_p(G)$.

Since any homomorphism of $\mathbb{C}_p[G]$-modules ϕ is uniquely specified by its 'χ-components' ϕ_χ for each $\chi \in \mathrm{Irr}_p(G)$ we may define τ^j and ξ^j as follows.

$$\tau_\chi^j := \begin{cases} H_c^2(\mathbb{Q}_p(1))_\chi = \mathbb{Z}_p^{(1)} \otimes_{\mathbb{Z}_p} \mathbb{C}_p \xrightarrow{|G|\cdot \log_p} \mathbb{C}_p = H_c^3(\mathbb{Q}_p(1))_\chi, & \text{if } \chi = \mathbf{1}_G \\ H_c^2(\mathbb{Q}_p(1))_\chi \xrightarrow{\log_{L_{p,\chi}}} L_{p,\chi} \xrightarrow{\iota_1} L(1)_{p,\chi} = H_c^1(\mathbb{Q}_p(1))_\chi, & \text{if } \chi \text{ is odd} \\ \mathrm{id}_0, & \text{otherwise} \end{cases}$$

$$\xi_\chi^j := \begin{cases} \mathrm{id}_{L_{p,\chi}}, & \text{if } \chi = \mathbf{1}_G \text{ or } \chi \text{ is odd} \\ L_{p,\chi} \xrightarrow{(\log_{L_{p,\chi}} \circ \lambda_{p,\chi})^{-1}} (\mathcal{O}_L^{\times} \otimes \mathbb{Q}_p)_\chi \xrightarrow{\iota_2} L_{p,\chi}, & \text{otherwise.} \end{cases}$$

Here we have used the following notation: id_W is the identity map on each space W; $\log_{L_p}$ is the homomorphism $\prod_{w\in S_p(L)} U^{(1)}_{L_w} \to \prod_{w\in S_p(L)} L_w = L_p$ induced by the p-adic logarithm maps $U^{(1)}_{L_w} \to L_w$; ι_1 is the inverse of the χ-component of the composite

$$L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p \to (L \otimes_{\mathbb{Q}} \mathbb{R}) \otimes_{\mathbb{R},j} \mathbb{C}_p \cong L_p \otimes_{\mathbb{Q}_p} \mathbb{C}_p$$

where the arrow is induced by the map

$$L(1)_p \to \{(z_\sigma) \in \bigoplus_{\sigma: L\to\mathbb{C}} 2\pi i \cdot \mathbb{Z} : z_{\overline{\sigma}} = -z_\sigma\} \otimes \mathbb{Z}_p$$

which sends $(n_w \cdot \{\exp(2\pi\sqrt{-1}/p^n)\}_{n\geq 0})_{w\in S_\infty(L)}$ to $\sum_{w\in S_\infty(L)}((2\pi\sqrt{-1}\cdot n_w)_{\sigma_w} - (2\pi\sqrt{-1}\cdot n_w)_{\overline{\sigma_w}})$ together with the canonical isomorphism

$$L \otimes_{\mathbb{Q}} \mathbb{R} \cong \{(z_\sigma) \in \bigoplus_{\sigma: L\to\mathbb{C}} \mathbb{C} : z_{\overline{\sigma}} = \overline{z_\sigma}\}; \tag{40}$$

ι_2 is induced by combining the equality $\mathcal{O}_L^\times \otimes \mathbb{R} = \mathcal{O}_{L^+}^\times \otimes \mathbb{R}$ with the isomorphism $\mathrm{Reg}_{S_\infty(L^+)} \otimes_{\mathbb{R},j} \mathbb{C}_p$ and the identification $(X_{L^+,S_\infty(L^+)} \otimes \mathbb{C}_p)_\chi = (\prod_{S_\infty(L^+)} \mathbb{C}_p)_\chi \cong (L^+ \otimes_{\mathbb{Q}} \mathbb{C}_p)_\chi \cong L_{p,\chi}$.

We can now state the main result of this section.

Lemma 6.14. *In $K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ one has*

$$j_*(T\Omega(L/K,1)) = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}(j(L^*_{L/K,S}(1))) + \chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(R\Gamma_c(\mathbb{Z}_p(1)), \tau^j) - \chi_{\mathbb{Z}_p[G],\mathbb{C}_p}(\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1], \xi^j).$$

Proof. We abbreviate '$\chi_{\mathbb{Z}[G],\mathbb{R}}$' and '$\chi_{\mathbb{Z}_p[G],\mathbb{C}_p}$' to '$\chi$' and '$\chi_p$' respectively.

Then it is clear that $j_*(\chi(E_S(\mathcal{L}), \mu_L)) = \chi_p(E_S(\mathcal{L}) \otimes \mathbb{Z}_p, \mu_L \otimes_{\mathbb{R},j} \mathbb{C}_p)$ and that Lemma 2.2 implies $j_*(L^*_{L/K,S}(1)) = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}(j(L^*_{L/K,S}(1)))$. It is therefore enough to prove that in $K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ one has

$$\chi_p(E_S(\mathcal{L}) \otimes \mathbb{Z}_p, \mu_L \otimes_{\mathbb{R},j} \mathbb{C}_p) = \chi_p(R\Gamma_c(\mathbb{Z}_p(1)), \tau^j) - \chi_p(\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1], \xi^j).$$

To do this we apply [8, Cor. 6.6] to the triangle (31) in Proposition 6.11. Thus, in terms of the notation of [8, Cor. 6.6], we now set $\Lambda := \mathbb{Z}_p[G], \Sigma := \mathbb{C}_p[G], P := \mathcal{L}_p[0] \oplus \mathcal{L}_p[-1], Q := R\Gamma_c(\mathcal{O}_{L,S}, \mathbb{Z}_p(1))[2]$ and $R := E_S(\mathcal{L}) \otimes \mathbb{Z}_p$ and we let a denote the morphism from P to Q that occurs in (31). We must verify that for each $\chi \in \mathrm{Irr}_p(G)$ the determinant of the automorphism κ_χ of $(H^{\mathrm{ev}}Q_\Sigma \oplus \ker(H^{\mathrm{ev}}a_\Sigma) \oplus \ker(H^{\mathrm{od}}a_\Sigma))_\chi$ that is defined in [8, Cor. 6.6] is equal to 1.

If $\chi = \mathbf{1}_G$, then $(E_S(\mathcal{L}) \otimes \mathbb{Z}_p)_\chi$ is acyclic and the χ-component of the exact sequence (32) gives

$$0 \to 0 \to 0 \to \mathbb{C}_p \xrightarrow{\exp_p} \mathbb{Z}_p^{(1)} \otimes_{\mathbb{Z}_p} \mathbb{C}_p \to 0 \to \mathbb{C}_p \xrightarrow{|G|} \mathbb{C}_p \to 0.$$

It follows that $(H^{\mathrm{ev}}Q_\Sigma \oplus \ker(H^{\mathrm{ev}}a_\Sigma) \oplus \ker(H^{\mathrm{od}}a_\Sigma))_\chi = (\mathbb{Z}_p^{(1)} \otimes_{\mathbb{Z}_p} \mathbb{C}_p) \oplus 0 \oplus 0$ and, given the definitions of τ^j_χ and ξ^j_χ, an explicit computation shows that κ_χ is the identity map on this space.

If χ is even and $\chi \neq \mathbf{1}_G$, then the χ-component of (32) gives

$$0 \to 0 \to (H^{-1}(E_S(\mathcal{L}))\otimes\mathbb{Q}_p)_\chi \xrightarrow{\theta_{2,\chi}} L_{p,\chi} \to 0 \to (H^0(E_S(\mathcal{L}))\otimes\mathbb{Q}_p)_\chi \xrightarrow{=} L_{p,\chi} \to 0 \to 0.$$

Hence $(H^{\rm ev}Q_\Sigma \oplus \ker(H^{\rm ev}a_\Sigma) \oplus \ker(H^{\rm od}a_\Sigma))_\chi = 0 \oplus L_{p,\chi} \oplus L_{p,\chi}$ and the commutative diagram

$$\begin{array}{ccc} (H^0(E_S(\mathcal{L})) \otimes \mathbb{Q}_p)_\chi & \xrightarrow{=} & L_{p,\chi} \\ \big\downarrow (\mu_L \otimes_{\mathbb{R},j} \mathbb{C}_p)_\chi & & \big\uparrow \xi^j_\chi \\ (H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q}_p)_\chi & \xrightarrow{\theta_{2,\chi}} & L_{p,\chi} \end{array}$$

implies that κ_χ has the form

$$\begin{pmatrix} 0 & -\xi^j_\chi \\ (\xi^j_\chi)^{-1} & 0 \end{pmatrix},$$

and so $\det(\kappa_\chi) = 1$. (The commutativity of the given diagram relies on the explicit definition of ξ^j_χ and the fact that, with respect to the description of $H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q}_p$ used in Proposition 6.11, θ_2 is induced by the projection $L_S \to L_p$ whilst μ_L uses the projection $L_S \to L_\infty$.)

Finally, if χ is odd, then the χ-component of (32) gives

$$0 \to L(1)_{p,\chi} \xrightarrow{\theta_{1,\chi}} (H^{-1}(E_S(\mathcal{L})) \otimes \mathbb{Q}_p)_\chi \xrightarrow{0} L_{p,\chi} \xrightarrow{\exp_{p,\chi}} H^2_c(\mathbb{Q}_p(1))_\chi \xrightarrow{0} (H^0(E_S(\mathcal{L})) \otimes \mathbb{Q}_p)_\chi \xrightarrow{=} L_{p,\chi} \to 0 \to 0$$

and hence $(H^{\rm ev}Q_\Sigma \oplus \ker(H^{\rm ev}a_\Sigma) \oplus \ker(H^{\rm od}a_\Sigma))_\chi = H^2_c(\mathbb{Q}_p(1))_\chi \oplus 0 \oplus L_{p,\chi}$. By an explicit computation (using the definitions of ξ^j_χ and τ^j_χ) one finds that κ_χ is of the form

$$\begin{pmatrix} 0 & -\log_{L_{p,\chi}} \\ \exp_{p,\chi} & 0 \end{pmatrix}$$

and hence that $\det(\kappa_\chi) = 1$, as required. □

6.3.3. *Virtual objects.* For comparison to the constructions of [17] it is convenient to reinterpret the result of Lemma 6.14 in terms of the language of virtual objects that is used in loc. cit.

To this end we consider the Picard categories $\mathcal{V}(\mathbb{C}_p)$, $\mathcal{V}(\mathbb{Z}_p[G])$, $\mathcal{V}(\mathbb{C}_p[G])$ and $\mathcal{V}(\mathbb{Z}_p[G], \mathbb{C}_p[G])$ discussed in [8, §5.1]. If R denotes either $\mathbb{C}_p$, $\mathbb{Z}_p[G]$ or $\mathbb{C}_p[G]$, then we fix a unit object $\mathbf{1}_{\mathcal{V}(R)}$ of $\mathcal{V}(R)$ and for each object X of $\mathcal{V}(R)$ we fix an object X^{-1} of $\mathcal{V}(R)$ and an isomorphism $X \otimes X^{-1} \cong \mathbf{1}_{\mathcal{V}(R)}$ in $\mathcal{V}(R)$. We also write $\iota : \pi_0\mathcal{V}(\mathbb{Z}_p[G], \mathbb{C}_p[G]) \cong K_0(\mathbb{Z}_p[G], \mathbb{C}_p)$ for the isomorphism of abelian groups that is described in [8, Lem. 5.1].

Then the definition of the Euler characteristic χ_p in [8, Def. 5.3] ensures that $\chi_p(R\Gamma_c(\mathbb{Z}_p(1)), \tau^j) = \iota(([R\Gamma_c(\mathbb{Z}_p(1))], \hat{\tau}^j))$ with $\hat{\tau}^j$ equal to the following isomorphism in $\mathcal{V}(\mathbb{C}_p[G])$

$$\begin{aligned} [R\Gamma_c(\mathbb{C}_p(1))] &\cong [H^1_c(\mathbb{C}_p(1))]^{-1} \otimes [H^2_c(\mathbb{C}_p(1))] \otimes [H^3_c(\mathbb{C}_p(1))]^{-1} \\ &\cong [H^2_c(\mathbb{C}_p(1))] \otimes [H^1_c(\mathbb{C}_p(1)) \oplus H^3_c(\mathbb{C}_p(1))]^{-1} \\ &\cong [H^1_c(\mathbb{C}_p(1)) \oplus H^3_c(\mathbb{C}_p(1))] \otimes [H^1_c(\mathbb{C}_p(1)) \oplus H^3_c(\mathbb{C}_p(1))]^{-1} \\ &\cong \mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])} \end{aligned} \tag{41}$$

where the first map is the canonical isomorphism induced by [8, Prop. 3.1], the second and fourth maps are clear and the third is induced by τ^j. One also has

$$-\chi_p(\mathcal{L}_p[0] \oplus \mathcal{L}_p[-1], \xi^j) = \iota((\mathbf{1}_{\mathcal{V}(\mathbb{Z}_p[G])}, (\hat{\xi}^j)^{-1}))$$

where $\hat{\xi}^j$ denotes ξ^j considered as an element of $\mathrm{Aut}_{\mathcal{V}(\mathbb{C}_p[G])}([L_p \otimes_{\mathbb{Q}_p} \mathbb{C}_p]) \cong \mathrm{Aut}_{\mathcal{V}(\mathbb{C}_p[G])}(\mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])})$. The result of Lemma 6.14 is therefore equivalent to an equality

$$j_*(T\Omega(L/K,1)) = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}(j(L^*_{L/K,S}(1))) + \iota(([R\Gamma_c(\mathbb{Z}_p(1))], (\hat{\xi}^j)^{-1} \circ \hat{\tau}^j)). \tag{42}$$

6.3.4. *The element* $j_*(T\Omega(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G]))$. We let H_B denote the $G \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-submodule $\bigoplus_{L \to \mathbb{C}} 2\pi i \cdot \mathbb{Z}$ of $\bigoplus_{L \to \mathbb{C}} \mathbb{C}$ upon which $G \times \{1\}$ acts via L and $\{1\} \times \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts diagonally. We write H_B^+ for the G-module of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant elements in H_B and we let H^1_f denote the $\mathbb{Q}_p[G]$-module $\mathrm{im}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

For the motive $h^0(\mathrm{Spec}\, L)(1)$, considered as defined over K and with coefficients $\mathbb{Q}[G]$, the isomorphism of virtual objects ϑ_∞ that is constructed in [17, §3.2, just prior to Lem. 7] can be explicitly described by using the observations of [15, §1.1, §1.3]. With this in mind, and with the role of graded determinants replaced by virtual objects, the argument used in [18] to prove that [18, (11)] is a special case of [17, Conj. 4.1(iv)] shows that

$$j_*(T\Omega(h^0(\mathrm{Spec}\, L)(1), \mathbb{Z}[G])) = \partial^1_{\mathbb{Z}_p[G],\mathbb{C}_p}(j(L^*_{L/K,S}(1))) + \iota(([R\Gamma_c(\mathbb{Z}_p(1))], \omega^j)) \tag{43}$$

with ω^j equal to the morphism in $\mathcal{V}(\mathbb{C}_p[G])$ defined by

$$\begin{aligned}
[R\Gamma_c(\mathbb{C}_p(1))] &\cong [H^1_c(\mathbb{C}_p(1))]^{-1} \otimes [H^2_c(\mathbb{C}_p(1))] \otimes [H^3_c(\mathbb{C}_p(1))]^{-1} \\
&\cong [H^1_c(\mathbb{C}_p(1))]^{-1} \otimes ([H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p] \otimes [H^2_c(\mathbb{C}_p(1))]) \\
&\quad \otimes [H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p]^{-1} \otimes [H^3_c(\mathbb{C}_p(1))]^{-1} \\
&\cong ([H^+_B \otimes \mathbb{C}_p]^{-1} \otimes [L \otimes_{\mathbb{Q}} \mathbb{C}_p]) \otimes ([\mathcal{O}_L^\times \otimes \mathbb{C}_p]^{-1} \otimes [\mathbb{C}_p]^{-1}) \\
&\cong [\prod_{S_\infty(L)} \mathbb{C}_p] \otimes [\prod_{S_\infty(L)} \mathbb{C}_p]^{-1} \\
&\cong \mathbf{1}_{\mathcal{V}(\mathbb{C}_p[G])}.
\end{aligned} \tag{44}$$

In this displayed formula we have used the following notation: the first map coincides with the first map in (41); the second and fifth maps are clear; the third map is induced by the exact sequence

$$\begin{aligned}
0 \to L(1)_p \otimes_{\mathbb{Z}_p} \mathbb{C}_p \xrightarrow{=} H^1_c(\mathbb{C}_p(1)) \xrightarrow{0} H^1_f \otimes_{\mathbb{Q}_p} \mathbb{C}_p \xrightarrow{\subseteq} \prod_{w \in S_p(L)} U^{(1)}_{L_w} \otimes_{\mathbb{Z}_p} \mathbb{C}_p \\
\xrightarrow{\pi} H^2_c(\mathbb{C}_p(1)) \to 0 \to 0 \to H^3_c(\mathbb{C}_p(1)) \xrightarrow{=} \mathbb{C}_p \to 0,
\end{aligned}$$

where π is induced by the identification $H^2_c(\mathbb{Q}_p(1)) \cong \mathrm{cok}(\lambda_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ used in (32) (note that the displayed sequence is the cohomology sequence of the distinguished triangle of [18, (3)] with $M = h^0(\mathrm{Spec}\, L)(1)$ and $A = \mathbb{Q}[G]$), together with the isomorphism $L(1)_p \to H^+_B \otimes \mathbb{Z}_p$ which sends each element $(n_w \cdot \{\exp(2\pi\sqrt{-1}/p^n)\}_{n \geq 0})_{w \in S_\infty(L)}$ to $\sum_{w \in S_\infty(L)}((2\pi\sqrt{-1} \cdot n_w)_{\sigma_w} - (2\pi\sqrt{-1} \cdot n_w)_{\overline{\sigma_w}})$ and the isomorphism $\prod_{w \in S_p(L)} U^{(1)}_{L_w} \otimes_{\mathbb{Z}_p} \mathbb{C}_p \cong L \otimes_{\mathbb{Q}} \mathbb{C}_p$ that is induced by $\log_{L_p}$; the fourth map is induced by (the images under $- \otimes_{\mathbb{R},j} \mathbb{C}_p$ of) the standard exact sequence

$$0 \to \mathcal{O}_L^\times \otimes \mathbb{R} \xrightarrow{\mathrm{Reg}_{S_\infty(L)}} \prod_{S_\infty(L)} \mathbb{R} \to \mathbb{R} \to 0, \tag{45}$$

the identification (40) and the exact sequence

$$0 \to H_B^+ \otimes \mathbb{R} \xrightarrow{\subseteq} \{(z_\sigma) \in \bigoplus_{\sigma: L \to \mathbb{C}} \mathbb{C} : z_{\overline{\sigma}} = \overline{z_\sigma}\} \to \prod_{S_\infty(L)} \mathbb{R} \to 0 \tag{46}$$

in which the third arrow sends each element (z_σ) to $(z_{\sigma_w} + z_{\overline{\sigma_w}})_{w \in S_\infty(L)}$.

6.3.5. *Completion of the proof.* In view of (42) and (43), the required equality (30) will follow if we can prove that $\omega^j = (\hat{\xi}^j)^{-1} \circ \hat{\tau}^j$ (as morphisms in $\mathcal{V}(\mathbb{C}_p[G])$), or equivalently that, in the obvious notation, $\omega^j_\chi = (\hat{\xi}^j_\chi)^{-1} \circ \hat{\tau}^j_\chi$ (as morphisms in $\mathcal{V}(\mathbb{C}_p)$) for all $\chi \in \mathrm{Irr}_p(G)$. For each such χ one can verify the required equality by an explicit comparison of the definitions of ω^j_χ, $\hat{\xi}^j_\chi$ and $\hat{\tau}^j_\chi$.

For example, if $\chi = \mathbf{1}_G$, then $\hat{\xi}^j_\chi$ is the identity element of $\mathrm{Aut}_{\mathcal{V}(\mathbb{C}_p)}(\mathbf{1}_{\mathcal{V}(\mathbb{C}_p)})$ and the required equality $\omega^j_\chi = \hat{\tau}^j_\chi$ follows directly upon comparing the χ-components of (44) and (41) and then using the following observation: the factor $|G|$ in the definition of τ^j_χ is accounted for by the equality $|G| = 2|S_\infty(L)|$ and the fact that the χ-components of the images under $- \otimes_{\mathbb{R},j} \mathbb{C}_p$ of the exact sequences (45) and (46) identify with $0 \to 0 \to \mathbb{C}_p \xrightarrow{|S_\infty(L)|} \mathbb{C}_p \to 0$ and $0 \to 0 \to \mathbb{C}_p \xrightarrow{2} \mathbb{C}_p \to 0$ respectively.

In a similar way, if χ is odd, then $\hat{\xi}^j_\chi$ is the identity element of $\mathrm{Aut}_{\mathcal{V}(\mathbb{C}_p)}(\mathbf{1}_{\mathcal{V}(\mathbb{C}_p)})$ and the equality $\omega^j_\chi = \hat{\tau}^j_\chi$ can be verified by means of an explicit comparison of the χ-components of the isomorphisms (44) and (41). When making this comparison it is useful to note that in this case the χ-components of $H^3_c(\mathbb{C}_p(1))$ and $H^1_f \otimes_{\mathbb{Z}_p} \mathbb{C}_p$ and of the image under $- \otimes_{\mathbb{R},j} \mathbb{C}_p$ of all of the spaces which occur in (45) are zero.

Finally, if χ is even and $\chi \neq \mathbf{1}_G$, then $[R\Gamma_c(\mathbb{C}_p(1))_\chi] = \mathbf{1}_{\mathcal{V}(\mathbb{C}_p)}$ and τ^j_χ is the identity element of $\mathrm{Aut}_{\mathcal{V}(\mathbb{C}_p)}(\mathbf{1}_{\mathcal{V}(\mathbb{C}_p)})$, whilst an explicit check shows that $(\hat{\xi}^j_\chi)^{-1}$ coincides with the element of $\mathrm{Aut}_{\mathcal{V}(\mathbb{C}_p)}(\mathbf{1}_{\mathcal{V}(\mathbb{C}_p)})$ that is induced by the χ-component of (44). In this regard we note that the occurrences of $\mathrm{Reg}_{S_\infty(L^+)}$ in the definition of ξ^j_χ and of $\mathrm{Reg}_{S_\infty(L)}$ in (44) (via (45)) are compatible since $\mathrm{Reg}_{S_\infty(L)} = 2 \cdot \mathrm{Reg}_{S_\infty(L^+)}$ on $\mathcal{O}^\times_{L^+} \otimes \mathbb{R}$ whilst the χ-component of the image under $- \otimes_{\mathbb{R},j} \mathbb{C}_p$ of (46) identifies with $0 \to 0 \to (\prod_{S_\infty(L)} \mathbb{C}_p)_\chi \xrightarrow{2} (\prod_{S_\infty(L)} \mathbb{C}_p)_\chi \to 0$.

By these means one completes the proof of Theorem 6.1.

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