ON SPECIAL ELEMENTS IN HIGHER ALGEBRAIC K-THEORY AND THE LICHTENBAUM-GROSS CONJECTURE

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Abstract. We conjecture the existence of special elements in odd degree higher algebraic K-groups of number fields that are related in a precise way to the values at strictly negative integers of the derivatives of Artin L-functions of finite dimensional irreducible complex representations. We prove this conjecture for an important family of examples and also provide other evidence (both theoretical and numerical) in its support.

Introduction

An important conjecture of Gross asserts, roughly speaking, that the leading term at any strictly negative integer of the Artin L-function of a finite dimensional complex representation should be equal, to within an undetermined algebraic factor, to a regulator constructed from elements of the appropriate odd degree higher algebraic K-group. (Gross's Conjecture was first formulated in the late 1970s as a natural analogue of the seminal conjecture of Stark concerning the leading terms at zero of Artin L-functions but was only recently published in [22] and can by now be seen as a special case of the natural equivariant refinement of Beilinson's general conjectures on the leading terms of L-functions.) It is well known that providing an explicit upper bound on the (absolute norm of the) denominator of the undetermined algebraic factor in Gross's Conjecture would make it much easier to obtain numerical evidence for the conjecture - see, for example, the discussion of Dummit in [18, §14] regarding problems that arise when conducting numerical investigations of Stark's Conjecture. Perhaps more importantly, it is also likely that such a bound could be used to give some much needed insight into arithmetic properties of any possible non-Abelian analogues of the cyclotomic and elliptic elements in higher algebraic K-theory that have been constructed by Beilinson, Deligne and Soulé. However, apart from an important, but rather inexplicit, modification of the related Lichtenbaum-Gross Conjecture that was formulated by Chinburg, Kolster, Pappas and Snaith in [14], and a similarly inexplicit refinement of the Lichtenbaum-Gross Conjecture that we recently learnt has been formulated by Nickel in [28], the present authors are not aware of any other predictions, let alone results, concerning this problem.

In the current article we make a first step in this direction by investigating the possible existence of elements in odd degree higher algebraic K-groups of number fields that are related in a very explicit way to the values at strictly negative integers of the first derivatives of Artin L-functions. By developing a suggestion of the first

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author in [8, Rem. 5.1.5], which was itself partly motivated by the results of Stark, Tate and Chinburg that are discussed by Tate in [35, Chap. III], we formulate a precise conjecture in this regard (see Conjecture 1.2 and Propositions 2.2(i) and 2.6). For the L-functions of representations that factor through Abelian extensions of $\mathbb{Q}$ the theory of cyclotomic elements in higher algebraic K-theory gives an explicit construction of the elements that are predicted to exist by our conjecture (see Theorem 3.1). More generally, it is straightforward to see that our conjecture refines Gross's Conjecture and in Theorem 3.3 we prove in addition that, in certain cases, it is implied by the modified Lichtenbaum-Gross Conjecture of [14, Conj. 6.12] and also that the elements it predicts should encode explicit information about the structure of even degree higher algebraic K-groups in a way that is strikingly parallel to the way in which cyclotomic units are known to encode information about the structure of ideal class groups (see Remark 3.5). For some cases of our conjecture that are not related to the Lichtenbaum-Gross Conjecture we provide (in §5) supporting numerical evidence for various types of representations, including certain dihedral and tetrahedral representations, the standard representation of S_5, and also those studied by Tate and Buhler [7] and Chinburg [13]. As a preliminary step to describing this evidence, which we believe may itself be of some independent interest, in Theorem 4.1 we make precise the relation between a version of the (second) Bloch group and K_3 of a field, and, if the field is a number field, the Beilinson regulator map.

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1. Statement of the conjecture

1.1. Throughout this article we use the following general notation.

We fix an algebraic closure $\mathbb{Q}^c$ of $\mathbb{Q}$ and for any Galois extension of fields F/E we set $G_{F/E} := \mathrm{Gal}(F/E)$. For each non-zero integer a and number field F we write $\mu_a(F)$ for the finite module $H^0(G_{\mathbb{Q}^c/F}, \mathbb{Q}/\mathbb{Z}(a))$ where $\mathbb{Q}/\mathbb{Z}(a)$ denotes the group $\mathbb{Q}/\mathbb{Z}$ regarded as a $G_{\mathbb{Q}^c/\mathbb{Q}}$-module by setting $g(x) := \chi_{\mathrm{cyc}}(g)^a x$ for all $g \in G_{\mathbb{Q}^c/\mathbb{Q}}$ and $x \in \mathbb{Q}/\mathbb{Z}$ where χ_{cyc} is the cyclotomic character. We also set $w_a(F) := |\mu_a(F)|$ and $\mathbb{R}(a) := (2\pi i)^a\mathbb{R}$.

For any Abelian group A we write A_{tor} for its torsion subgroup. Unadorned tensor products are to be regarded as taken in the category of Abelian groups.

1.2. To state our conjecture we fix a finite Galois extension of number fields F/k and a finite set of places S of k containing the set S_∞ of all archimedean places. We also fix an irreducible finite dimensional complex character χ of $G := G_{F/k}$ and a subfield E of $\mathbb{C}$ that is both Galois and of finite degree over $\mathbb{Q}$ and over which χ can be realised. We write $\check{\chi}$ for the contragredient of χ and e_χ for the primitive central idempotent $\chi(1)|G|^{-1}\sum_{g\in G}\check{\chi}(g)g$ of $E[G]$. We set $\Gamma := G_{E/\mathbb{Q}}$ and write $\mathcal{O}$ for the ring of integers in E. For any G-module M we write the natural semi-linear action of Γ on $E \otimes M$ as $(\gamma, x) \mapsto x^\gamma$ (so $x^\gamma = \gamma(e) \otimes m$ if $x = e \otimes m$ with $e \in E$ and $m \in M$).

We fix a strictly negative integer r and for any field L we write the higher algebraic K-group $K_{1-2r}(L)$ additively. We consider the composite homomorphism

$$\mathrm{reg}_{1-r} : K_{1-2r}(\mathbb{C}) \to H^1_{\mathcal{D}}(\mathrm{Spec}(\mathbb{C}), \mathbb{R}(1-r)) = \mathbb{C}/\mathbb{R}(1-r) \xrightarrow{\sim} \mathbb{R}(-r) \quad (1.1)$$

where the first arrow is the Beilinson regulator map and the second is the isomorphism induced by the decomposition $\mathbb{C} = \mathbb{R}(1-r) \oplus \mathbb{R}(-r)$. We let τ be the non-trivial element in $G_{\mathbb{C}/\mathbb{R}}$ and use the same symbol to denote the induced involution on $K_{1-2r}(\mathbb{C})$. We recall that $\mathrm{reg}_{1-r} \circ \tau = (-1)^r \mathrm{reg}_{1-r}$. We note that the index $1-r$ corresponds to the only Adams weight in algebraic K-theory on which reg_{1-r} is non-trivial. In particular reg_{1-r} is trivial on the image in $K_{1-2r}(\mathbb{C})$ of the Milnor K-group $K^M_{1-2r}(\mathbb{C})$ since this has weight $1-2r$.

We regard the set Σ_F of embeddings $F \to \mathbb{C}$ as a left $G \times G_{\mathbb{C}/\mathbb{R}}$-module by setting $(g \times \omega)(\sigma) = \omega \circ \sigma \circ g^{-1}$ for each $g \in G, \omega \in G_{\mathbb{C}/\mathbb{R}}$ and $\sigma \in \Sigma_F$. For each $\sigma \in \Sigma_F$ we write $\mathrm{reg}_{1-r,\sigma} : \mathcal{O} \otimes K_{1-2r}(F) \to \mathbb{C}$ for the $\mathcal{O}$-linear map sending each $e \otimes x$ with $e \in \mathcal{O}$ and $x \in K_{1-2r}(F)$ to $e \cdot \mathrm{reg}_{1-r}(\sigma(x))$, where σ also denotes the induced homomorphism $K_{1-2r}(F) \to K_{1-2r}(\mathbb{C})$. We write τ_σ for the generator of the decomposition subgroup D_σ in G of the place of F that corresponds to σ.

For each γ in Γ we write $L'_S(r, \chi^\gamma)$ for the value at $s = r$ of the first derivative of the S-truncated Artin L-function $L_S(s, \chi^\gamma)$ of χ^γ. In the case $S = S_\infty$ we often abbreviate $L_S(s, \chi^\gamma)$ and $L'_S(r, \chi^\gamma)$ to $L(s, \chi^\gamma)$ and $L'(r, \chi^\gamma)$ respectively. Finally we set $w_r(\chi) := w_{1-r}(F^{\ker(\chi^\gamma)})^{\chi^\gamma(1)}$ for any $\gamma \in \Gamma$, and we note that this number is indeed independent of the choice of γ.

We can now state the central conjecture of this article.

Conjecture 1.2. *Assume that $L(r, \check{\chi}) = 0$. For each σ in Σ_F there exists an element $\epsilon_\sigma(\chi, S)$ of $\mathcal{O} \otimes K_{1-2r}(F)$ with the following property: for all $\sigma' \in \Sigma_F$ and all $\gamma \in \Gamma$ one has*

$$(2\pi i)^r \mathrm{reg}_{1-r,\sigma'}(\epsilon_\sigma(\chi, S)^\gamma) = w_r(\chi)\gamma(c^\chi_{\sigma',\sigma}) L'_S(r, \check{\chi}^\gamma) \quad (1.3)$$

with

$$c^\chi_{\sigma',\sigma} := \begin{cases} \check{\chi}(g) + (-1)^r \check{\chi}(g\tau_\sigma) & \text{if } \sigma(k) \subset \mathbb{R} \text{ and } \sigma' = g(\sigma) \text{ for some } g \in G, \\ \check{\chi}(g) & \text{if } \sigma(k) \not\subseteq \mathbb{R} \text{ and } \sigma' = g(\sigma) \text{ for some } g \in G, \\ (-1)^r \check{\chi}(g) & \text{if } \sigma(k) \not\subseteq \mathbb{R} \text{ and } \sigma' = \tau \circ g(\sigma) \text{ for some } g \in G, \\ 0 & \text{otherwise.} \end{cases}$$

1.3. We make several straightforward remarks about Conjecture 1.2.

Remark 1.4.

(i) If Conjecture 1.2 holds for σ and χ, then it holds for σ and χ^δ for any δ in Γ with $\epsilon_\sigma(\chi^\delta, S) = \epsilon_\sigma(\chi, S)^\delta$.

(ii) If Conjecture 1.2 holds for σ and χ, then it holds for $\tau \circ \sigma$ and χ with $\epsilon_{\tau\circ\sigma}(\chi, S) = (-1)^r \epsilon_\sigma(\chi, S)$.

(iii) If Conjecture 1.2 holds for σ and χ, and h is in G, then it holds for $h(\sigma)$ and χ with $\epsilon_{h(\sigma)}(\chi, S) = h(\epsilon_\sigma(\chi, S))$. This is true because both $\mathrm{reg}_{1-r,\sigma'}(h(\epsilon_\sigma(\chi, S))) = \mathrm{reg}_{1-r,h^{-1}(\sigma')}(\epsilon_\sigma(\chi, S))$ and $c^\chi_{h^{-1}(\sigma'),\sigma} = c^\chi_{\sigma',h(\sigma)}$ (since $\tau_{h(\sigma)} = h\tau_\sigma h^{-1}$ if $\sigma(k) \subset \mathbb{R}$ and $\check{\chi}$ is a class function on G).

(iv) Any element $\epsilon_\sigma(\chi, S)$ that satisfies (1.3) for all σ' and any fixed γ is necessarily unique modulo $\mathcal{O} \otimes K_{1-2r}(F)_{\mathrm{tor}}$ (see the proof of Proposition 2.1(iv)(a) below).

Remark 1.5. If $L'(r,\check{\chi})=0$, then $L'(r,\check{\chi}^\gamma)=0$ for all $\gamma\in\Gamma$ and so Conjecture 1.2 is interesting only if $L(r,\check{\chi})=0\neq L'(r,\check{\chi})$. By computing orders of vanishing via an explicit analysis of the Gamma factors and functional equation of $L(s,\check{\chi})$ (as in [27, Chap. VII, §12]), one finds that this is the case in precisely the following situations:

(i) k has exactly one complex place, $\chi(1)=1$ and $\chi(1)+(-1)^r\chi(\tau_{\sigma'})=0$ for all $\sigma'\in\Sigma_F$ with $\sigma'(k)\subset\mathbb{R}$;
(ii) k is totally real, $\chi(1)+(-1)^r\chi(\tau_\sigma)=2$ for some $\sigma\in\Sigma_F$ and $\chi(1)+(-1)^r\chi(\tau_{\sigma'})=0$ for all $\sigma'\in\Sigma_F\setminus\{g(\sigma):g\in G\}$.

In both of these situations χ is also realisable over its character field $\mathbb{Q}(\chi)$. Indeed, this is obvious in case (i) and in case (ii) can be shown as follows. For any σ in Σ_F and any χ, we let $e_{\sigma,\chi}$ in $\mathbb{Q}(\chi)[G]$ denote the idempotent $\frac{1}{2}(1+(-1)^r\tau_\sigma)e_\chi$ if $\sigma(k)\subset\mathbb{R}$ and the central idempotent e_χ otherwise. In case (ii) above we consider $e_{\sigma,\chi}$ for the given σ. Then the character of the $\mathbb{Q}(\chi)[G]$-module $V:=\mathbb{Q}(\chi)[G]e_{\sigma,\chi}$ is equal to $m\chi$ for some natural number m and we must show $m=1$. But in $\mathbb{C}[G]e_{\sigma,\chi}\cong M_{\chi(1)}(\mathbb{C})$ the idempotent $e_{\sigma,\chi}$ identifies with a matrix of rank one (since $\chi(1)+(-1)^r\chi(\tau_\sigma)=2$) and so $\chi(1)=\dim_{\mathbb{C}}(\mathbb{C}[G]e_{\sigma,\chi})=\dim_{\mathbb{C}}(\mathbb{C}\otimes_{\mathbb{Q}(\chi)}V)=m\chi(1)$, as required.

Remark 1.6. With H the image of $G_{F/k}$ under a representation underlying χ, H is Abelian in case (i) of Remark 1.5. In case (ii) with r odd, the image of τ_σ has determinant -1, hence $|H:[H,H]|$ is even and H cannot be perfect (consequently, H cannot be both simple and non-Abelian). But in case (ii) with r even, we can have H simple and non-Abelian. For example, for the splitting field F of $f(x)=x^5+2x^3-4x^2-2x+4$ over $\mathbb{Q}$ one has $G_{F/\mathbb{Q}}\simeq A_5$, τ_σ corresponds to a $2-2$-cycle in A_5 as $f(x)$ has exactly one real root, hence $\chi(1)+\chi(\tau_\sigma)=2$ for χ the character of either 3-dimensional irreducible representation of A_5.

There are a small number of cases in which the elements predicted by Conjecture 1.2 can be explained theoretically.

If $k=\mathbb{Q}$ then Remark 1.5(ii) implies that $L(r,\check{\chi})=0\neq L'(r,\check{\chi})$ if and only if $\chi(\tau_\sigma)=(-1)^r(2-\chi(1))$ for all $\sigma\in\Sigma_F$, which certainly holds if $\chi(1)=1$ and $\chi(\tau_\sigma)=(-1)^r$ for all σ. In this case Proposition 2.1(ii) below also allows us to assume that F is an Abelian extension of $\mathbb{Q}$ and so one can explicitly construct elements $\epsilon_\sigma(\chi,S)$ as in Conjecture 1.2 by using the cyclotomic elements in algebraic K-theory of Deligne et al. (see the proof of Proposition 3.1 below).

If k is imaginary quadratic, then Remark 1.5(i) implies that $L(r,\check{\chi})=0\neq L'(r,\check{\chi})$ if and only if $\chi(1)=1$ and in any such case Deninger has proved Gross's Conjecture (in [17, Th. 3.1]). It seems likely that Deninger's methods can be used to directly construct the corresponding elements in Conjecture 1.2 both for any such χ and, when F is Galois over $\mathbb{Q}$, also for the character $\mathrm{Ind}_G^{G_{F/\mathbb{Q}}}(\chi)$ if this is irreducible. For example, interesting work in this direction (which does not however bound the denominators of the elements that arise) is described by Levin in [26].

In general, however, we are not aware of any theoretical constructions that can account for the existence of the elements $\epsilon_\sigma(\chi,S)$ in Conjecture 1.2 for other classes of characters χ. In §3 we will also see that, for certain sets S, these mysterious elements should encode information about the explicit module structure of even degree algebraic K-groups in a manner that is strikingly parallel to the way in which cyclotomic units encode structural information about ideal class groups (see Remark 3.5).

2. Reductions and reformulations

2.1. Conjecture 1.2 admits several useful reformulations that we shall describe below. We begin by recording some of its properties. Here we write $\overline{\epsilon}_\sigma(\chi, S)$ for the image in $E \otimes K_{1-2r}(F)$ of the conjectural element $\epsilon_\sigma(\chi, S)$.

Proposition 2.1.

(i) *It is enough to verify Conjecture 1.2 with* $S = S_\infty$.
(ii) *It is enough to verify Conjecture 1.2 after replacing* F *by* $F^{\ker(\chi)}$.
(iii) *It is enough to verify Conjecture 1.2 when* $L'(r, \check{\chi}) \neq 0$ *and with* $E = \mathbb{Q}(\chi)$.
(iv) *Assume that Conjecture 1.2 is valid for* σ *in* Σ_F.
  (a) $\overline{\epsilon}_\sigma(\chi, S)$ *is uniquely specified by the formulas in Conjecture 1.2.*
  (b) $\overline{\epsilon}_\sigma(\chi, S) = e_\chi \overline{\epsilon}_\sigma(\chi, S)$*; if* $\sigma(k) \subset \mathbb{R}$*, then* $\tau_\sigma(\overline{\epsilon}_\sigma(\chi, S)) = (-1)^r \overline{\epsilon}_\sigma(\chi, S)$.

Proof. If Conjecture 1.2 is valid as stated, then for any $v \notin S$ it is validated with S replaced by $S' = S \cup \{v\}$ by setting $\epsilon_\sigma(\chi, S') := \det_{\mathbb{C}}(1 - \sigma_w \mathrm{N}v^{-r} \mid V_\chi^{I_w})\epsilon_\sigma(\chi, S)$ where w is any place of F above v, G_w and I_w are the decomposition and inertia subgroups of w in G, σ_w the Frobenius automorphism in G_w/I_w and V_χ a $\mathbb{C}[G]$-module of character χ. (Here we use the fact that, since $r < 0$, the element $\det_{\mathbb{C}}(1 - \sigma_w \mathrm{N}v^{-r} \mid V_\chi^{I_w})$ belongs to $\mathcal{O}$.) This proves claim (i).

Claim (ii) is easily checked and the first assertion of (iii) is obvious since if $L'(r, \check{\chi}) = 0$, then $L'(r, \check{\chi}^\gamma) = 0$ for all $\gamma \in \Gamma$ and so Conjecture 1.2 is validated by setting $\epsilon_\sigma(\chi, S) := 0$. Finally, it is clear that if Conjecture 1.2 is valid for a given field E, then it is valid for any larger such field and so the second claim of (iii) is true because $L'(r, \check{\chi}) \neq 0$ implies that χ can be realised over $\mathbb{Q}(\chi)$ by Remark 1.5.

To prove claim (iv) we set $\overline{\epsilon}_\sigma := \overline{\epsilon}_\sigma(\chi, S)$. We extend $\mathrm{reg}_{1-r,\sigma'}$ to a homomorphism of E-modules $E \otimes K_{1-2r}(F) \to \mathbb{C}$ in the natural way, write $\widetilde{\mathrm{reg}}_{\sigma'}$ for $(2\pi i)^r \mathrm{reg}_{1-r,\sigma'}$ and recall that $\bigcap_{\sigma' \in \Sigma_F} \ker(\widetilde{\mathrm{reg}}_{\sigma'})$ vanishes (by Borel [5]). It is thus clear that $\overline{\epsilon}_\sigma$ is uniquely specified by (1.3) for all σ' and with γ equal to the trivial element, proving (iv)(a). For the same reason, to prove (iv)(b) it suffices to show that $\widetilde{\mathrm{reg}}_{\sigma'}(e_\chi \overline{\epsilon}_\sigma) = \widetilde{\mathrm{reg}}_{\sigma'}(\overline{\epsilon}_\sigma)$ and, if $\sigma(k) \subset \mathbb{R}$, that $\widetilde{\mathrm{reg}}_{\sigma'}(\tau_\sigma(\overline{\epsilon}_\sigma)) = (-1)^r \widetilde{\mathrm{reg}}_{\sigma'}(\overline{\epsilon}_\sigma)$, for all σ'. But $\widetilde{\mathrm{reg}}_{\sigma'}(e_\chi \overline{\epsilon}_\sigma) = \chi(1)|G|^{-1} \sum_{h \in G} \check{\chi}(h) \widetilde{\mathrm{reg}}_{h^{-1}(\sigma')}(\overline{\epsilon}_\sigma) = \widetilde{\mathrm{reg}}_{\sigma'}(\overline{\epsilon}_\sigma)$ where the last equality follows by combining the conjectural equality (1.3) (with γ trivial) with the fact that $\chi(1)|G|^{-1} \sum_{h \in G} \check{\chi}(h)\check{\chi}(h^{-1}g) = \check{\chi}(g)$ for all $g \in G$, so that $\chi(1)|G|^{-1} \sum_{h \in G} \check{\chi}(h) c^\chi_{h^{-1}(\sigma'),\sigma} = c^\chi_{\sigma',\sigma}$. Moreover, if $\sigma(k) \subset \mathbb{R}$ then one easily checks that $c^\chi_{\tau_\sigma^{-1}(\sigma'),\sigma} = (-1)^r c^\chi_{\sigma',\sigma}$, and combining this with (1.3) (with γ trivial) we find that $\widetilde{\mathrm{reg}}_{\sigma'}(\tau_\sigma(\overline{\epsilon}_\sigma)) = \widetilde{\mathrm{reg}}_{\tau_\sigma^{-1}(\sigma')}(\overline{\epsilon}_\sigma) = (-1)^r \widetilde{\mathrm{reg}}_{\sigma'}(\overline{\epsilon}_\sigma)$. □

In the next result we reformulate Conjecture 1.2 in a style reminiscent of the refinement of (a special case of) Stark's conjecture that is stated by Chinburg in [13]. Here we write $\mathcal{D}_E$ for the different of $E/\mathbb{Q}$.

Proposition 2.2. *Let* σ *be in* Σ_F.

(i) *If Conjecture 1.2 holds for* σ*, then for each* d *in* $\mathcal{D}_E^{-1}$ *the element*

$$\mathrm{Tr}_d(\epsilon_\sigma(\chi, S)) := \sum_{\gamma \in \Gamma} \gamma(d) \epsilon_\sigma(\chi, S)^\gamma$$

belongs to $K_{1-2r}(F)$ *and for all* $\sigma' \in \Sigma_F$ *satisfies*

$$(2\pi i)^r \mathrm{reg}_{1-r,\sigma'}(\mathrm{Tr}_d(\epsilon_\sigma(\chi,S))) = w_r(\chi)\sum_{\gamma\in\Gamma}\gamma(d)\gamma(c^{\chi}_{\sigma',\sigma})L'_S(r,\check{\chi}^{\gamma})\,. \tag{2.3}$$

(ii) *If for each* d *in* $\mathcal{D}_E^{-1}$ *there exists an element* $\alpha_{d,\sigma}(\chi,S)$ *of* $K_{1-2r}(F)$ *such that* (2.3) *holds with* $\alpha_{d,\sigma}(\chi,S)$ *in place of* $\mathrm{Tr}_d(\epsilon_\sigma(\chi,S))$*, then Conjecture 1.2 holds for* σ*.*

Proof. To prove (i) we note Conjecture 1.2 implies that for every γ in Γ one has $\gamma(d)\epsilon_\sigma(\chi,S)^\gamma \in \mathcal{D}_E^{-1}\otimes K_{1-2r}(F)$ and also $(2\pi i)^r\mathrm{reg}_{1-r,\sigma'}(\gamma(d)\epsilon_\sigma(\chi,S)^\gamma) = w_r(\chi)\gamma(d)\gamma(c^{\chi}_{\sigma',\sigma})L'_S(r,\check{\chi}^\gamma)$. It is thus clear that $\mathrm{Tr}_d(\epsilon_\sigma(\chi,S))$ satisfies (2.3). In addition one has $\mathrm{Tr}_{E/\mathbb{Q}}(\mathcal{D}_E^{-1})\subseteq\mathbb{Z}$ and so

$$\mathrm{Tr}_d(\epsilon_\sigma(\chi,S)) = \sum_{\gamma\in\Gamma}(d\epsilon_\sigma(\chi,S))^\gamma \in \mathrm{Tr}_{E/\mathbb{Q}}(\mathcal{D}_E^{-1})\otimes K_{1-2r}(F)\subseteq K_{1-2r}(F),$$

as required. For claim (ii), let $\{d_j\}_j$ be a $\mathbb{Z}$-basis of $\mathcal{D}_E^{-1}$, $\{e_j\}_j$ the dual $\mathbb{Z}$-basis of $\mathcal{O}$ with respect to the trace pairing, and set $\epsilon_\sigma(\chi,S) := \sum_j e_j\otimes\alpha_{d_j,\sigma}(\chi,S)$. If γ is in Γ, then

$$\begin{aligned}(2\pi i)^r\mathrm{reg}_{1-r,\sigma'}(\epsilon_\sigma(\chi,S)^\gamma) &= w_r(\chi)\sum_{\gamma',j}\gamma(e_j)\gamma'(d_j)\gamma'(c^{\chi}_{\sigma',\sigma})L'_S(r,\check{\chi}^{\gamma'})\\ &= w_r(\chi)\gamma(c^{\chi}_{\sigma',\sigma})L'_S(r,\check{\chi}^{\gamma})\end{aligned}$$

because if $\Gamma=\{\gamma_j\}_j$, then $(\gamma_j(d_i))(\gamma_i(e_j))$ is the identity matrix, and the same holds for $(\gamma_i(e_j))(\gamma_j(d_i))$. ☐

2.2. In this subsection we give (in Proposition 2.6(ii)) a more concise reformulation of Conjecture 1.2 that requires some preliminaries. This reformulation will be particularly useful in Section 5 when we numerically verify Conjecture 1.2 in the case that $S=S_\infty$ and $L'_S(r,\chi)\neq 0$.

Claim (iii) of Proposition 2.1 implies that one only needs to consider the case that $L_S(r,\chi)=0\neq L'_S(r,\chi)$. Remark 1.5 specifies when this happens, and we let $\Sigma_F^{r,\chi}$ denote the subset of Σ_F comprising the $2|G|$ elements σ with $\sigma(k)\not\subseteq\mathbb{R}$ in case (i) of that remark, and comprising the $|G|$ elements σ with $\chi(1)+(-1)^r\chi(\tau_\sigma)=2$ in case (ii). (Equivalently, one has $\sigma\notin\Sigma_F^{r,\chi}$ if and only if both $\sigma(k)\subset\mathbb{R}$ and τ_σ acts as multiplication by $(-1)^{r+1}$ in the representation underlying χ.) Note that $\Sigma_F^{r,\chi}=\Sigma_F^{r,\chi^\gamma}$ for γ in Γ. We recall that the idempotent $e_{\sigma,\chi}$ was defined in Remark 1.5 for any σ in Σ_F.

Lemma 2.4. *Assume* $L'(r,\check{\chi})=0$ *and* $L'(r,\check{\chi})\neq 0$*.*

(i) *If either* σ *or* σ' *is not in* $\Sigma_F^{r,\chi}$*, then* $c^{\chi}_{\sigma',\sigma}=0$*.*
(ii) *For* σ *in* Σ_F*, one has* $e_{\sigma,\chi}=0$ *if and only if* $\sigma\notin\Sigma_F^{r,\chi}$*.*
(iii) *If* g *is in* G *and* σ *in* Σ_F*, then* $c^{\chi}_{\sigma,\sigma}e_{\sigma,\chi}g^{-1}e_{\sigma,\chi}=c^{\chi}_{g(\sigma),\sigma}e_{\sigma,\chi}$*.*

Proof. For claim (i) we note that $c^{\chi}_{\sigma',\sigma}\neq 0$ implies that both σ and σ' are in $\Sigma_F^{r,\chi}$ or neither are. If $\sigma\notin\Sigma_F^{r,\chi}$ then $\sigma(k)\subset\mathbb{R}$, and $c^{\chi}_{\sigma',\sigma}=0$ because of the definitions. For claim (ii) we note that $e_{\sigma,\chi}=0$ from the definitions if $\sigma\notin\Sigma_F^{r,\chi}$, and that $e_{\sigma,\chi}\neq 0$ if $\sigma\in\Sigma_F^{r,\chi}$: when $\sigma(k)\not\subseteq\mathbb{R}$ this is obvious, and if $\sigma(k)\subset\mathbb{R}$ then it follows by choosing an isomorphism $\mathbb{Q}(\chi)[G]e_\chi\simeq M_{\chi(1)}(\mathbb{Q}(\chi))$ such that $1+(-1)^r\tau_\sigma$ corresponds to the diagonal matrix with entries $2,0,\ldots,0$ on the diagonal. Claim (iii) for σ in $\Sigma_F^{r,\chi}$

follows from similar considerations using that $c^{\chi}_{\sigma,\sigma}$ is equal to either 1 or 2, and for other σ is obvious from (ii). □

Remark 2.5. Lemma 2.4(i) refines Proposition 2.1 as it implies that Conjecture 1.2 holds (with $\epsilon_\sigma(\chi,S)=0$) when $L'_S(r,\chi)\neq 0$ and $\sigma\notin\Sigma_F^{r;\chi}$. Considering also the first three parts of Remark 1.4, we see that verifying Conjecture 1.2 for a given χ and any given embedding σ in $\Sigma_F^{r;\chi}$ does so for all characters in the Γ-orbit $\{\chi^\gamma:\gamma\in\Gamma\}$ and for all embeddings σ in Σ_F.

In the next result let $K_{1-2r}(F)_{\rm tf}$ denote the image of $K_{1-2r}(F)$ in $\mathbb{Q}\otimes K_{1-2r}(F)$. Then $\mathrm{reg}_{1-r,\sigma'}$ factorizes through the map $\mathcal{O}\otimes K_{1-2r}(F)\to\mathcal{O}\otimes K_{1-2r}(F)_{\rm tf}$ and we use the same notation for the resulting map.

Proposition 2.6. *Let σ be in Σ_F. Then the following statements are equivalent.*

(i) *Conjecture 1.2 holds for σ.*
(ii) *There exists $\beta_\sigma(\chi,S)$ in $\mathcal{O}\otimes K_{1-2r}(F)_{\rm tf}\subseteq E\otimes K_{1-2r}(F)$ that satisfies $e_{\sigma,\chi}\beta_\sigma(\chi,S)=\beta_\sigma(\chi,S)$ and, for all γ in Γ, also*

$$(2\pi i)^r\mathrm{reg}_{1-r,\sigma}(\beta_\sigma(\chi,S)^\gamma)=w_r(\chi)\gamma(c^{\chi}_{\sigma,\sigma})L'_S(r,\check\chi^\gamma)\,.$$

Proof. That (i) implies (ii) can be seen from Proposition 2.1(iv)(b): we may simply set $\beta_\sigma(\chi,S):=\bar\epsilon_\sigma(\chi,S)$. For the converse, we note that Conjecture 1.2 always holds if $L'_S(r,\check\chi)=0$, or if $L'_S(r,\check\chi)\neq 0$ and $\sigma\notin\Sigma_F^{r;\chi}$, so we assume that both $L'_S(r,\check\chi)\neq0$ and $\sigma\in\Sigma_F^{r;\chi}$. Then $c^{\chi}_{\sigma,\sigma}$ is equal to either 1 or 2 and so it will suffice to show that

$$\gamma(c^{\chi}_{\sigma,\sigma})\mathrm{reg}_{1-r,\sigma'}(\beta_\sigma(\chi,S)^\gamma)=\gamma(c^{\chi}_{\sigma',\sigma})\mathrm{reg}_{1-r,\sigma}(\beta_\sigma(\chi,S)^\gamma) \tag{2.7}$$

for all $\sigma'\in\Sigma_F$ and all $\gamma\in\Gamma$, because then (1.3) holds for any lift $\epsilon_\sigma(\chi,S)$ of $\beta_\sigma(\chi,S)$ to $\mathcal{O}\otimes K_{1-2r}(F)$.

If $\sigma'\notin\Sigma_F^{r;\chi}$, then $c^{\chi}_{\sigma',\sigma}=0$ by Lemma 2.4(i). On the other hand, for $\gamma\in\Gamma$ we have $\beta_\sigma(\chi,S)^\gamma=e_{\chi^\gamma}\beta_\sigma(\chi,S)^\gamma$ because $\beta_\sigma(\chi,S)=e_\chi\beta_\sigma(\chi,S)$, and

$$\mathrm{reg}_{1-r,\sigma'}(\beta_\sigma(\chi,S)^\gamma)=\mathrm{reg}_{1-r,\sigma'}(\tfrac{1+(-1)^r\tau_{\sigma'}}{2}\beta_\sigma(\chi,S)^\gamma)=\mathrm{reg}_{1-r,\sigma'}(e_{\sigma',\chi^\gamma}\beta_\sigma(\chi,S)^\gamma)\,,$$

which is trivial by Lemma 2.4(ii) because $\Sigma_F^{r;\chi}=\Sigma_F^{r;\chi^\gamma}$, thus establishing (2.7) for such σ'.

If now $\sigma'\in\Sigma_F^{r;\chi}$, then we distinguish three cases: (a) $\sigma(k)\not\subseteq\mathbb{R}$ and $\sigma'=g(\sigma)$ for some g in G; (b) $\sigma(k)\not\subseteq\mathbb{R}$ and $\sigma'=\tau\circ g(\sigma)$ for some g in G; (c) $\sigma(k)\subset\mathbb{R}$ and $\sigma'=g(\sigma)$ for some g in G. In case (a) $e_{\sigma,\chi}$ is equal to the central idempotent e_χ, $c^{\chi}_{\sigma,\sigma}=\chi(1)=1$, and for $\gamma\in\Gamma$ we have $g^{-1}\beta_\sigma(\chi,S)^\gamma=e_{\chi^\gamma}g^{-1}e_{\chi^\gamma}\beta_\sigma(\chi,S)^\gamma=\gamma(c^{\chi}_{g(\sigma),\sigma})\beta_\sigma(\chi,S)^\gamma$ by Lemma 2.4(iii). Now (2.7) follows because $\mathrm{reg}_{1-r,g(\sigma)}(\beta_\sigma(\chi,S)^\gamma)=\mathrm{reg}_{1-r,\sigma}(g^{-1}\beta_\sigma(\chi,S)^\gamma)$. Case (b) is dealt with similarly using the equalities $\mathrm{reg}_{1-r,\sigma'}=(-1)^r\mathrm{reg}_{1-r,g(\sigma)}$ and $c^{\chi}_{\sigma',\sigma}=(-1)^rc^{\chi}_{g(\sigma),\sigma}$. In case (c) one has $c^{\chi}_{\sigma,\sigma}=2$ and the left hand side of (2.7) is equal to

$$2\mathrm{reg}_{1-r,\sigma}(g^{-1}\beta_\sigma(\chi,S)^\gamma)=\mathrm{reg}_{1-r,\sigma}((1+(-1)^r\tau_\sigma)g^{-1}\beta_\sigma(\chi,S)^\gamma)\,.$$

Using that $g^{-1}\beta_\sigma(\chi,S)^\gamma=g^{-1}e_{\sigma,\chi^\gamma}\beta_\sigma(\chi,S)^\gamma=e_{\chi^\gamma}g^{-1}e_{\sigma,\chi^\gamma}\beta_\sigma(\chi,S)^\gamma$ for the central idempotent e_{χ^γ} we see from Lemma 2.4(iii) that (2.7) again holds. □

2.3. We conclude this section with some more results of independent interest, which will also be used in the sequel.

Lemma 2.8. *If the irreducible character χ is realisable over E, then the multiplicity of the corresponding representation in $E \otimes K_{1-2r}(F)$ is equal to the order of vanishing of $L(s, \check{\chi})$ at $s = r$.*

Proof. For each integer m we define a $(G \times G_{\mathbb{C}/\mathbb{R}})$-module $B_m := \bigoplus_{\Sigma_F} (2\pi i)^{-m}\mathbb{Z}$, where G acts on Σ_F in the way specified just before Conjecture 1.2 and $G_{\mathbb{C}/\mathbb{R}}$ acts diagonally (on both Σ_F and $(2\pi i)^{-m}\mathbb{Z}$). Then, according to Borel's theorem [5], the G-invariant pairing

$$\mathbb{Q} \otimes B_{-r}^{G_{\mathbb{C}/\mathbb{R}}} \times \mathbb{Q} \otimes K_{1-2r}(F) \to \mathbb{R} \tag{2.9}$$

that is induced by mapping each element $((2\pi i)^r \sigma, \alpha)$ to $(2\pi i)^r \mathrm{reg}_{1-r}(\sigma(\alpha))$ is non-degenerate.

If we now extend coefficients to E, then the non-degeneracy of (2.9) implies that for any idempotent π in $E[G]$ one has $\dim_E(\pi(E \otimes K_{1-2r}(F))) = \dim_E(\check{\pi}(E \otimes B_r))$, where $\check{\pi}$ is the image of π under the E-linear anti-involution of $E[G]$ that inverts elements of G. Finally we note that if $\pi = e_\chi$, so $\check{\pi} = e_{\check{\chi}}$, then the description of the Gamma factors and functional equation of $L(s, \check{\chi})$ (as in [27, Chap. VII, §12]) implies that $\chi(1)^{-1}\dim_E(\check{\pi}(E \otimes B_r))$ is equal to the vanishing order of $L(s, \check{\chi})$ at $s = r$. □

Remark 2.10. There are several ways in which the observations made above are useful when making numerical investigations of Conjecture 1.2.

(i) If $L_S(r, \check{\chi}) = 0 \neq L'_S(r, \check{\chi})$, then by Proposition 2.1(iv)(b), $\overline{\epsilon}_\sigma(\chi, S)$ belongs to the $\mathcal{O}$-sublattice $e_{\sigma,\chi}(\mathcal{O} \otimes K_{1-2r}(F)_{\mathrm{tf}})$ of $E \otimes K_{1-2r}(F)$. If σ is in $\Sigma_F^{r,\chi}$ then this $\mathcal{O}$-lattice has rank one by Lemma 2.8 and the proof of Lemma 2.4(ii), and this provides a very strong restriction on where one searches to find $\overline{\epsilon}_\sigma(\chi, S)$. This also applies to $\beta_\sigma(\chi, S)$ in Proposition 2.6, and implies that $\beta_\sigma(\chi, S)$ is unique in this case.

(ii) If r is even, then Proposition 2.1(ii) and (the second part of) (iv)(b) combine to imply that the element $e|K_{1-2r}(F^{\ker(\chi)})_{\mathrm{tor}}|\epsilon_\sigma(\chi, S, d)$ belongs to the image of $K_{1-2r}(F^{\ker(\chi)D_\sigma})$ in $K_{1-2r}(F)$, where $e \in \{1, 2\}$ is the exponent of the Tate cohomology group $\hat{H}^0(\ker(\chi)D_\sigma/\ker(\chi), K_{1-2r}(F^{\ker(\chi)}))$. This observation is useful because it can be computationally much easier to search for elements in $K_{1-2r}(F')$ for proper subfields F' of F rather than in $K_{1-2r}(F)$ itself.

(iii) The first observation made in the proof of Lemma 2.8 implies that if F has signature $[r_1, r_2]$, then the rank of $K_{1-2r}(F)$ is equal to $r_1 + r_2$ if r is even and to r_2 if r is odd. This explicit (and well-known) formula will be useful in the sequel.

Remark 2.11. It is clear that any commutator in $G_{\mathbb{Q}^c/\mathbb{Q}}$ acts trivially on $\mu_a(F) = H^0(G_{\mathbb{Q}^c/F}, \mathbb{Q}/\mathbb{Z}(a))$. Hence one has $\mu_a(F) = \mu_a(F \cap \mathbb{Q}^{\mathrm{ab}})$, where $\mathbb{Q}^{\mathrm{ab}}$ is the maximal Abelian extension of $\mathbb{Q}$. If a is even, then one sees similarly that $\mathbb{Q}^{\mathrm{ab}}$ may be replaced by its maximal totally real subfield.

3. Theoretical evidence

In this section we prove Conjecture 1.2 for an important family of examples. We also describe the connection between Conjecture 1.2 and the modified Lichtenbaum-Gross Conjecture formulated by Chinburg et al. in [14] and show that the elements predicted by Conjecture 1.2 should encode information about the structure (as Galois modules) of certain even degree higher algebraic K-groups.

3.1. In the proof of the following result we use cyclotomic elements in higher algebraic K-theory to give an explicit construction of the elements that are predicted to exist by (the relevant special case of) Conjecture 1.2.

Theorem 3.1. *If $k = \mathbb{Q}$ and $\chi(1) = 1$, then Conjecture 1.2 is valid.*

Proof. By Proposition 2.1 it suffices to prove Conjecture 1.2 for $S = S_\infty$, and we may also replace F by $F^{\ker(\chi)}$. Moreover, by Proposition 2.6 it suffices to construct, for each σ in Σ_F, an element $\beta_\sigma(\chi, S_\infty)$ in $\mathcal{O} \otimes K_{1-2r}(F)_{\mathrm{tf}}$ satisfying both $e_{\sigma,\chi}\beta_\sigma(\chi, S_\infty) = \beta_\sigma(\chi, S_\infty)$ and

$$(2\pi i)^r \mathrm{reg}_{1-r,\sigma}(\beta_\sigma(\chi, S_\infty)^\gamma) = w_r(\chi)\gamma(c^\chi_{\sigma,\sigma})L'(r, \check{\chi}^\gamma)$$

for all γ in Γ. By the proof of that proposition, Conjecture 1.2 is then satisfied by any lift $\epsilon_\sigma(\chi, S_\infty)$ of $\beta_\sigma(\chi, S_\infty)$ to $\mathcal{O} \otimes K_{1-2r}(F)$.

We may view χ as a primitive Dirichlet character with conductor N, so that $F \subseteq \mathbb{Q}(\mu_N)$. Since the Dirichlet and Artin L-functions coincide for such Dirichlet characters, we have $L(1-r, \chi^\gamma) = \sum_{n\geq 1} \chi^\gamma(n)n^{r-1}$ for every γ in Γ, where $\chi^\gamma(n) = 0$ if $\gcd(n, N) \neq 1$. For any integer n and any primitive Dirichlet character ψ modulo N we set $\tau(\psi, n) := \sum_{j=0}^{N-1} \psi(j)e^{2\pi inj/N}$, and we abbreviate $\tau(\psi, 1)$ to $\tau(\psi)$. With $a = 0, 1$ such that $\chi(-1) = (-1)^a$ we obtain from Theorem 2.8 in Chapter VII (or from p. 541) in [27] that

$$L(1 - s, \check{\chi}^\gamma) = \frac{2i^a}{\tau(\chi^\gamma)} \left(\frac{N}{2\pi}\right)^s \cos(\pi(s - a)/2)\Gamma(s)L(s, \chi^\gamma)\,.$$

Therefore Conjecture 1.2 applies exactly when $a \equiv r$ modulo 2, so $\chi(-1) = (-1)^r$ (cf. Remark 1.5), and

$$L'(r, \check{\chi}^\gamma) = (-1)^{\frac{r+a}{2}} i^a N^{1-r}(2\pi)^r \frac{(-r)!}{2\tau(\chi^\gamma)} L(1 - r, \chi^\gamma)\,.$$

If $M \geq 2$ then by [23, Cor. 9.8], for every primitive Mth root of unity ω there is an element $\varphi_{r,M}(\omega)$ in $K_{1-2r}(\mathbb{Q}(\mu_M))_{\mathrm{tf}}$ with

$$\mathrm{reg}_{1-r,\sigma}(\varphi_{r,M}(\omega)) = -\frac{(-r)!\,M^{-r}}{2}(\mathrm{Li}_{1-r}(\sigma(\omega)) + (-1)^r\mathrm{Li}_{1-r}(\sigma(\omega^{-1})))$$

for every σ in $\Sigma_{\mathbb{Q}(\mu_M)}$, where $\mathrm{Li}_{1-r}(z) = \sum_{n\geq 1} \frac{z^n}{n^{1-r}}$ for z in $\mathbb{C}$ with $|z| \leq 1$. It follows that $\varphi_{r,M}(g(\omega)) = g\varphi_{r,M}(\omega)$ for g in $G_{\mathbb{Q}(\mu_N)/\mathbb{Q}}$ because they have the same value under $\mathrm{reg}_{1-r,\sigma}$ for all σ in $\Sigma_{\mathbb{Q}(\mu_N)}$ (cf. the proof of Proposition 2.1).

We assume first that χ is not trivial. Then $N \geq 3$ and we extend σ in Σ_F to an element in $\Sigma_{\mathbb{Q}(\mu_N)}$, also denoted by σ. Fixing a primitive Nth root of unity ω and an integer ℓ with $\sigma(\omega) = e^{2\pi i\ell/N}$ (and hence $\gcd(\ell, N) = 1$), we set

$$\beta_\sigma(\chi, S_\infty) := -w_r(\chi) \sum_{\substack{j=0,\dots,N-1\\ \gcd(j,N)=1}} \check{\chi}(j\ell) \otimes \varphi_{r,N}(\omega^j)$$

in $\mathcal{O}\otimes K_{1-2r}(\mathbb{Q}(\mu_N))_{\mathrm{tf}}$. Now if α is any element of $K_{1-2r}(\mathbb{Q}(\mu_N))_{\mathrm{tf}}$ then, by the formalism of pushforward and pullback, the sum $\sum_{g\in\ker(\chi)} g(\alpha)$ lies in $K_{1-2r}(F)_{\mathrm{tf}} \subseteq K_{1-2r}(\mathbb{Q}(\mu_N))_{\mathrm{tf}}$ because $F = \mathbb{Q}(\mu_N)^{\ker(\chi)}$. In particular, therefore, the element

$\beta_\sigma(\chi, S_\infty)$ belongs to $\mathcal{O} \otimes K_{1-2r}(F)_{\mathrm{tf}} \subseteq \mathcal{O} \otimes K_{1-2r}(\mathbb{Q}(\mu_N))_{\mathrm{tf}}$. It clearly also satisfies $e_{\sigma,\chi}\beta_\sigma(\chi, S_\infty) = \beta_\sigma(\chi, S_\infty)$, and we have

$$\begin{aligned}
&(2\pi i)^r \mathrm{reg}_{1-r,\sigma}(\beta_\sigma(\chi, S_\infty)^\gamma) \\
&= w_r(\chi)(2\pi i)^r(-r)!\, N^{-r} \sum_{j=0}^{N-1} \check{\chi}(j\ell)^\gamma \sum_{n\geq 1} \frac{\sigma(\omega)^{jn} + (-1)^r \sigma(\omega)^{-jn}}{2n^{1-r}} \\
&= w_r(\chi)(2\pi i)^r(-r)!\, N^{-r} \sum_{n\geq 1} \frac{\tau(\check{\chi}^\gamma, n) + (-1)^r \tau(\check{\chi}^\gamma, -n)}{2n^{1-r}} \\
&= w_r(\chi)(2\pi i)^r(-r)!\, N^{-r} \tau(\check{\chi}^\gamma) \sum_{n\geq 1} \frac{\chi^\gamma(n)}{n^{1-r}} \\
&= (-1)^{\frac{r-a}{2}} w_r(\chi) i^a (2\pi)^r(-r)!\, N^{-r} \frac{(-1)^r N}{\tau(\chi^\gamma)} L(1-r, \chi^\gamma) \\
&= w_r(\chi)\gamma(c^\chi_{\sigma,\sigma}) L'(r, \chi^\gamma)
\end{aligned}$$

because by [27, Chap. VII, Prop. 2.6], one has $\tau(\check{\chi}^\gamma, m) = \chi(m)\tau(\check{\chi}^\gamma)$, $\tau(\check{\chi}^\gamma) = \chi^\gamma(-1)\overline{\tau(\chi^\gamma)}$, $\tau(\chi^\gamma)\overline{\tau(\chi^\gamma)} = N$, $\chi(-1) = (-1)^r$, and $c^\chi_{\sigma,\sigma} = 2$.

We assume now that χ is trivial. Then r is even and we can take $F = \mathbb{Q}$. For every $N \geq 2$ we see as above that the element

$$\beta_N := -\sum_{1<d\,|N} \left(\frac{N}{d}\right)^{-r} \sum_{\substack{j=0,\dots,N-1 \\ \gcd(j,N)=N/d}} \varphi_{r,d}(\omega^j)$$

belongs to $K_{1-2r}(\mathbb{Q})_{\mathrm{tf}} \subseteq K_{1-2r}(\mathbb{Q}(\mu_N))_{\mathrm{tf}}$. For any σ in $\Sigma_{\mathbb{Q}(\mu_N)}$ we also find that

$$\mathrm{reg}_{1-r,\sigma}(\beta_N) = (-r)!N^{-r} \sum_{j=1}^{N-1} \sum_{n\geq 1} \frac{\sigma(\omega)^{jn} + (-1)^r \sigma(\omega)^{-jn}}{2n^{1-r}} = (-r)!(1-N^{-r})\zeta(1-r)$$

because $\sum_{j=0}^{N-1} \omega^{jn}$ is equal to N if N divides n and to 0 otherwise. Restricting σ to $\mathbb{Q}$, we therefore have

$$(2\pi i)^r \mathrm{reg}_{1-r,\sigma}(\beta_\sigma(\chi, S_\infty)) = (-1)^{\frac{r}{2}} w_r(\chi)(2\pi)^r(-r)!\, \zeta(1-r) = c^\chi_{\sigma,\sigma} w_r(\chi)\zeta'(r)$$

for a suitable $\mathbb{Z}$-linear combination $\beta_\sigma(\chi, S_\infty)$ of the elements β_N because $1 - N^{-r}$ is coprime to N. □

Remark 3.2. The special argument given for the trivial character in the proof of Theorem 3.1 is necessary because the element $\varphi_{r,N}(\omega)$ is not defined for $\omega = 1$. However, for each integer $r \leq -1$ and each Nth root of unity ω an element $[\omega]_{1-r}$ of $\mathbb{Q} \otimes K_{1-2r}(\mathbb{Q}(\mu_N))$ is constructed in [16] (see Theorem 3.15 or the beginning of § 5 of loc. cit.). If ω is a primitive Nth root of unity with $N \geq 2$ then comparing the formula for $\mathrm{reg}_{1-r,\sigma}([\omega]_{1-r})$ in [16, Prop. 4.1] with that for $\mathrm{reg}_{1-r,\sigma}(\varphi_{r,N}(\omega))$, we see that $N^{-r}[\omega]_{1-r}$ and $\varphi_{r,N}(\omega)$ coincide in $K_{1-2r}(\mathbb{Q}(\mu_N))_{\mathrm{tf}} \subseteq \mathbb{Q} \otimes K_{1-2r}(\mathbb{Q}(\mu_N))$ up to a universal choice of sign depending only on r. (The formula in loc. cit. is normalized, and due to a typographical error the factor $(n-1)$ should be $(n-1)!$.) Therefore $N^{-r}[\omega]_{1-r}$ is in $K_{1-2r}(\mathbb{Q}(\mu_N))_{\mathrm{tf}}$ for any Nth root of unity $\omega \neq 1$. It follows from Proposition 6.1 of loc. cit. and the action of $G_{\mathbb{Q}(\mu_N)/\mathbb{Q}}$ (cf. Remark 3.17 of loc. cit.), that $(1 - N^{-r})[1]_{1-r} = N^{-r} \sum_{j=1}^{N-1} [\omega^j]_{1-r} = \pm\beta_N$ is in $K_{1-2r}(\mathbb{Q})_{\mathrm{tf}}$.

The same then holds for $[1]_{1-r}$. If r is even then $\mathrm{reg}_{1-r,\sigma}([1]_{1-r}) = \pm(-r)!\,\zeta(1-r)$, and we can take $\beta_\sigma(\chi, S_\infty) = \pm w_r(\chi)[1]_{1-r}$.

3.2. In comparison with Theorem 3.1 the following result makes no assumptions on either k or $\chi(1)$ but deals only with sufficiently large sets of places S. This result will be proved in §3.6.

We write $\mathcal{O}_{F,S}$ for the subring of F comprising elements that are integral at all places that do not lie above a place in S.

Theorem 3.3. *We assume that the modified Lichtenbaum-Gross Conjecture (see Conjecture 3.10 below) is valid for χ.*

(i) *Then Conjecture 1.2 is valid for any (finite) set S that contains S_∞ and all places that ramify in F/k.*

(ii) *Further, for any set S as in (i) the element $\mathrm{Tr}_1(\epsilon_\sigma(\chi, S)) := \sum_{\gamma\in\Gamma} \epsilon_\sigma(\chi, S)^\gamma$ belongs to $K_{1-2r}(F)$ and for all ϕ in $\mathrm{Hom}_G(K_{1-2r}(F), \mathbb{Z}[G])$ one has*

$$\frac{|G|^2}{\chi(1)}\phi(\mathrm{Tr}_1(\epsilon_\sigma(\chi, S))) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\bigoplus_{p\neq 2} H^2_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{F,S}[\frac{1}{p}]), \mathbb{Z}_p(1-r))).$$

Corollary 3.4. *If $k = \mathbb{Q}$ and $\chi(1) = 1$, then Theorem 3.3 is valid unconditionally.*

Proof. In this case, and for any set S containing S_∞, the validity of Conjecture 1.2 is proved in Theorem 3.1, and the explicit description of $\epsilon_\sigma(\chi, S)$ given in the proof of the latter result even shows that $\mathrm{Tr}_1(\epsilon_\sigma(\chi, S))$ belongs to $w_r(\chi)\cdot K_{1-2r}(F)$. Moreover, in view of Theorem 3.3, Proposition 2.1(ii) and Remark 3.13(ii) below, to prove the second assertion of Theorem 3.3(ii) it is enough to recall that if F is an Abelian extension of $\mathbb{Q}$, then the equivariant Tamagawa number conjecture of [11, Conj. 4(iv)] is valid for the pair $(h^0(\mathrm{Spec}(F))(r), \mathbb{Z}[G])$. Indeed, this case of the latter conjecture is proved by Flach and the first author in [12, Cor. 1.2] (with corrections to the 2-primary part of the argument of loc. cit. recently provided by Flach in [20]). □

Remark 3.5. It is conjectured by Quillen and Lichtenbaum that for all odd primes p the natural Chern class homomorphism

$$\mathbb{Z}_p \otimes K_{-2r}(\mathcal{O}_{F,S}) \to H^2_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{F,S}[\frac{1}{p}]), \mathbb{Z}_p(1-r))$$

is bijective. In the case $r = -1$ this conjecture has been proved by Tate in [34]. (It is also known, by work of Suslin, that the Quillen-Lichtenbaum Conjecture is a consequence of a conjecture of Bloch and Kato relating Milnor K-theory to étale cohomology and it is widely believed that recent fundamental work of Voevodsky and Rost has led to a proof of this conjecture of Bloch and Kato.) Whenever the conjecture of Quillen and Lichtenbaum is valid the module $\bigoplus_{p\neq 2} H^2_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{F,S}[\frac{1}{p}]), \mathbb{Z}_p(1-r))$ can be replaced by $\mathbb{Z}[\frac{1}{2}] \otimes K_{-2r}(\mathcal{O}_{F,S})$ in the statement of Theorem 3.3(ii). In the setting of Corollary 3.1 we thereby obtain a rather striking analogue of the result of Rubin in [31, Th. (2.2) and the following Remark] concerning the annihilation of ideal class groups in absolutely Abelian fields.

Remark 3.6. We recently learnt Nickel has shown that for each strictly negative integer r the equivariant Tamagawa number conjecture for $(h^0(\mathrm{Spec}(F))(r), \mathbb{Z}[G])$ implies (via the reinterpretation given in [8, Prop. 4.2.6]) that certain elements constructed from the leading term at $s = r$ of truncated Artin L-functions should belong to $\mathbb{Z}[G]$ and annihilate $\bigoplus_{p\neq 2} H^2_{\text{ét}}(\text{Spec}(\mathcal{O}_{F,S}[\frac{1}{p}]), \mathbb{Z}_p(1-r))$ (cf. [28, Th. 4.1]). His prediction is however much less explicit than that given in Theorem 3.3(ii) above and it would be interesting to know if there is any direct link between them.

3.3. In this subsection we discuss some algebraic preliminaries that will be used in the proof of Theorem 3.3.

We fix G and χ as in §1.2 and, following Proposition 2.1(iii), we set $E := \mathbb{Q}(\chi)$. We write $\sum_{j=1}^{j=\chi(1)} f^j_\chi$ for a decomposition of e_χ as a sum of mutually orthogonal indecomposable idempotents in $E[G]$. We write $\mathcal{O}$ for the ring of algebraic integers in E and for each j we choose a maximal $\mathcal{O}$-order $\mathfrak{M}^j$ in $E[G]$ that contains f^j_χ. For any character $\psi = \chi^\gamma$, with $\gamma \in \Gamma$, we set $e_\psi := \psi(1)|G|^{-1}\sum_{g\in G}\check{\psi}(g)g = (e_\chi)^\gamma$ and $f^j_\psi := (f^j_\chi)^\gamma$ and define an $\mathcal{O}$-torsion-free right $\mathcal{O}[G]$-module $T^j_\psi := f^j_\psi(\mathfrak{M}^j)^\gamma = (f^j_\chi\mathfrak{M}^j)^\gamma$. The associated right $E[G]$-module $V^j_\psi := E\otimes_{\mathcal{O}} T^j_\psi$ has character ψ. For any (left) G-module M we define a module $M^j[\psi] := T^j_\psi \otimes M$ upon which each g in G acts on the left by $t\otimes m \mapsto tg^{-1}\otimes g(m)$ for each t in T^j_ψ and m in M. Then there is a natural isomorphism of (left) $\mathcal{O}[G]$-modules

$$M^j[\psi] \cong \text{Hom}_{\mathcal{O}}(T^{j,*}_\psi, \mathcal{O}\otimes M) \tag{3.7}$$

where G acts in the usual (diagonal) manner on the Hom-set and the module $T^{j,*}_\psi := \text{Hom}_{\mathcal{O}}(T^j_\psi, \mathcal{O})$ is endowed with the natural left G-action and hence spans a left $E[G]$-module of character ψ.

For any subgroup J of G we write M^J, resp. M_J, for the maximal submodule, resp. quotient, of M upon which J acts trivially. In particular we obtain a left, resp. right, exact functor $M\mapsto M^{j,\psi}$, resp. $M\mapsto M^j_\psi$, from the category of left G-modules to the category of $\mathcal{O}$-modules by setting $M^{j,\psi} := M^j[\psi]^G$ and $M^j_\psi := M^j[\psi]_G \cong T^j_\psi \otimes_{\mathbb{Z}[G]} M$. By replacing T^j_ψ by $\mathbb{Z}_p\otimes T^j_\psi$ we extend the notation $M^{j,\psi}$ and M^j_ψ to $\mathbb{Z}_p[G]$-modules M in the obvious way. We will often use the fact that the action of $\sum_{g\in G} g$ on $M^j[\psi]$ induces a homomorphism of $\mathcal{O}$-modules $t^j(M,\psi) : M^j_\psi \to M^{j,\psi}$ that has finite kernel and finite cokernel.

The module $E[G]\otimes M$ has two commuting left actions of G: the first via left multiplication on $E[G]$ and the second such that each g in G sends $x\otimes m$ to $xg^{-1}\otimes g(m)$ for x in $E[G]$ and m in M. We write $(E[G]\otimes M)^{G,2}$ for the subset of $E[G]\otimes M$ comprising elements that are invariant under the second action of G and use the first action of G on $E[G]\otimes M$ to regard $(E[G]\otimes M)^{G,2}$ as an $E[G]$-module. If M is finitely generated and torsion-free, then we always regard $M^{j,\psi}$ as an $\mathcal{O}$-submodule of $\mathcal{O}\otimes M$ by means of the identification described in the following result.

Lemma 3.8.

(i) *The E-linear map $E[G]\otimes M \to E\otimes M$ that sends $g\otimes m$ to $g(m)$ for each g in G and m in M restricts to give an isomorphism of $E[G]$-modules $\iota : (E[G]\otimes M)^{G,2} \cong E\otimes M$.*

(ii) *Assume now that M is finitely generated and torsion-free and for each character $\psi = \chi^\gamma$, with $\gamma\in\Gamma$, set $\text{pr}_\psi := \sum_{g\in G}\check{\psi}(g)g \in \mathcal{O}[G]$. Then one has $\text{pr}_\psi(\mathcal{O}\otimes M) \subseteq \psi(1)^{-1}\sum_{j=1}^{j=\psi(1)}\iota(M^{j,\psi}) \subseteq \mathcal{O}\otimes M$.*

Proof. Since $E[G]\otimes M$ is uniquely divisible it is cohomologically trivial with respect to any action of G. The submodule $(E[G]\otimes M)^{G,2}$ is thus equal to the E-linear span of elements of the form $(\sum_{g\in G} g)(x\otimes m)=\sum_{g\in G} xg^{-1}\otimes g(m)$ with x in $E[G]$ and m in M. Using this fact it is straightforward to check that the map ι is an isomorphism of $E[G]$-modules, as required to prove claim (i).

We next set $n:=\psi(1)$ and note that $\mathrm{pr}_\psi = n^{-1}|G|e_\psi$. Then the first inclusion in claim (ii) is true because for each m in M one has

$$\begin{aligned} n\cdot \mathrm{pr}_\psi(1\otimes m) &= |G|e_\psi(1\otimes m) = |G|\sum_{j=1}^{j=n} f_\psi^j(1\otimes m)\\ &= \sum_{j=1}^{j=n}\iota(\sum_{g\in G} f_\psi^j(g^{-1}\otimes g(m)) \in \sum_{j=1}^{j=n}\iota(M^{j,\psi}).\end{aligned}$$

To prove the second claimed inclusion it suffices to show $\iota(M^{j,\psi})\subseteq n(\mathcal{O}\otimes M)$ for each index j. To do this we set $N:=\mathcal{O}\otimes M$ and $N^*:=\mathrm{Hom}_{\mathcal{O}[G]}(N,\mathcal{O}[G])$. Then, as N is finitely generated and torsion-free, one has $N=\{x\in E\otimes_{\mathcal{O}} N : \theta(x)\in\mathcal{O}[G] \text{ for all } \theta\in N^*\}$. Thus it suffices to show that $\theta(M^{j,\psi})\subseteq n\mathcal{O}[G]$ for all $\theta\in N^*$. But $\theta(M^{j,\psi})\subseteq \mathbb{Z}[G]^{j,\psi}$ and, since $\mathbb{Z}[G]$ is cohomologically trivial and $\mathbb{Z}[G]^j_\psi = T^j_\psi$, one has $\mathbb{Z}[G]^{j,\psi}=\mathrm{im}(t^j(\mathbb{Z}[G],\psi)) = |G|T^j_\psi$. Thus it suffices to note that $|G|T^j_\psi = |G|f^j_\psi\mathfrak{M}^j\subseteq |G|e_\psi\mathfrak{M}^j\subseteq n\mathcal{O}[G]$, where the latter inclusion follows from Jacobinski's description in [24] of the central conductor of $\mathfrak{M}^j$ in $\mathcal{O}[G]$ (see also [15, Th. (27.13)]). □

For any G-module, resp. $\mathcal{O}$-module, M we write M_{tf} for the image of M in $\mathbb{Q}\otimes M$, resp. $E\otimes_{\mathcal{O}} M$. We often identify M_{tf} with the quotient M/M_{tor} in the natural way. If M is a G-module, then for each m in M we write $f^j_\psi(m)$ for the image of $f^j_\psi\otimes_{\mathbb{Z}[G]} m\in M^j_\psi$ in $M^j_{\psi,\mathrm{tf}}$.

3.4. In this subsection we recall the modified Lichtenbaum-Gross Conjecture of Chinburg, Kolster, Pappas and Snaith. Since we regard F/k as fixed we set $C_r := K_{1-2r}(F)$ and also recall the $G\times G_{\mathbb{C}/\mathbb{R}}$-modules B_m that are defined in the proof of Lemma 2.8. We note, in particular, that the non-degeneracy of the pairing (2.9) combines with the natural isomorphism of $G\times G_{\mathbb{C}/\mathbb{R}}$-modules $\mathrm{Hom}_{\mathbb{Z}}(B_{-r},\mathbb{Z})\cong B_r$ to imply that the map $\bigoplus_{\sigma\in\Sigma_F}\mathrm{reg}_{1-r,\sigma}$ induces a canonical isomorphism of $\mathbb{R}[G]$-modules

$$\mathrm{Reg}_r : \mathbb{R}\otimes C_r \xrightarrow{\sim} \mathbb{R}\otimes B_r^{G_{\mathbb{C}/\mathbb{R}}}.$$

The bijectivity of Reg_r then combines with Deuring's Theorem (cf. [15, §6, Exer. 6]) to imply the existence of an (in general non-canonical) isomorphism of $\mathbb{Q}[G]$-modules $\phi:\mathbb{Q}\otimes B_r^{G_{\mathbb{C}/\mathbb{R}}}\xrightarrow{\sim}\mathbb{Q}\otimes C_r$. For any such ϕ, any index j and any character $\psi=\chi^\gamma$, with $\gamma\in\Gamma$, we set

$$R_r^\phi(\psi) := \det_{\mathbb{C}}[(\mathbb{C}\otimes_{\mathbb{R}}\mathrm{Reg}_r)\circ(\mathbb{C}\otimes_{\mathbb{Q}}\phi) \mid (\mathbb{C}\otimes_E V^j_\psi)\otimes_{\mathbb{Z}[G]} B_r^{G_{\mathbb{C}/\mathbb{R}}}]\in\mathbb{C}^\times.$$

(This element doesn't depend on j because each $E[G]$-module V^j_ψ has the same character ψ.)

We now fix a finite set S of places of k that contains S_∞ and all places that ramify in F/k. Then the 'modified Lichtenbaum-Gross Conjecture' predicts an explicit Euler characteristic formula for the $\mathcal{O}$-module generated by $L^*_S(r,\check{\psi})/R^\phi_r(\psi)$

where $L_S^*(r,\check{\psi})$ is the leading non-zero coefficient in the Taylor expansion at $s=r$ of the function $L_S(s,\check{\psi})$. Before stating this conjecture we must recall a useful auxiliary result. In this result we set $\mathbb{Z}' := \mathbb{Z}[\frac{1}{2}]$ and for each odd prime ℓ we write $\mathrm{ch}^1_{\ell,1-r} : \mathbb{Z}_\ell \otimes K_{1-2r}(\mathcal{O}_{F,S}) \to H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{F,S}[\frac{1}{\ell}]), \mathbb{Z}_\ell(1-r))$ for the Chern class homomorphism constructed by Soulé [32] and Dwyer and Friedlander [19]. We also set $\mathrm{Tr}_{\mathbb{C}/\mathbb{R}} := 1+\tau \in \mathbb{Z}[G_{\mathbb{C}/\mathbb{R}}]$.

Lemma 3.9. *There exist finitely generated G-modules X_r and Y_r which possess all of the following properties.*

(i) $X_{r,\mathrm{tf}} \subseteq C_{r,\mathrm{tf}}$ *and* $Y_{r,\mathrm{tf}} = \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(B_r) \subseteq B_r^{G_{\mathbb{C}/\mathbb{R}}}$ *and both* $\mathbb{Z}'\otimes X_{r,\mathrm{tf}} = \mathbb{Z}'\otimes C_{r,\mathrm{tf}}$ *and* $\mathbb{Z}'\otimes Y_{r,\mathrm{tf}} = \mathbb{Z}'\otimes B_r^{G_{\mathbb{C}/\mathbb{R}}}$.

(ii) *For each prime ℓ there is an isomorphism of $\mathbb{Z}_\ell[G]$-modules of the form* $\mathbb{Z}_\ell\otimes X_r \cong H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{F,S}[\frac{1}{\ell}]), \mathbb{Z}_\ell(1-r))$. *If ℓ is odd the induced isomorphism* $\mathbb{Z}_\ell\otimes X_{r,\mathrm{tf}} = \mathbb{Z}_\ell\otimes K_{1-2r}(\mathcal{O}_{F,S})_{\mathrm{tf}} \to H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{F,S}[\frac{1}{\ell}]), \mathbb{Z}_\ell(1-r))_{\mathrm{tf}}$ *coincides with that induced by* $\mathrm{ch}^1_{\ell,1-r}$.

(iii) $X_{r,\mathrm{tor}} = \mu_{1-r}(F)$ *and* $Y_{r,\mathrm{tor}} = \bigoplus_\ell H^2_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{F,S}[\frac{1}{\ell}]), \mathbb{Z}_\ell(1-r))$ *where ℓ runs over all primes.*

(iv) *If the Quillen-Lichtenbaum Conjecture is valid for F and r, then one has* $\mathbb{Z}'\otimes X_r = \mathbb{Z}'\otimes C_r$ *and* $\mathbb{Z}'\otimes Y_{r,\mathrm{tor}} \cong \mathbb{Z}'\otimes K_{-2r}(\mathcal{O}_{F,S})$.

Proof. All claims in Lemma 3.9 except for the equality $Y_{r,\mathrm{tf}} = \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(B_r)$ follow directly from the constructions of [8, §11.1] (where X_r and Y_r correspond to the modules $N'_{r,0}$ and $N'_{r,1}$ respectively). Since for every odd prime p one has $\mathbb{Z}_p\otimes Y_{r,\mathrm{tf}} = \mathbb{Z}_p\otimes_{\mathbb{Z}'}(\mathbb{Z}'\otimes Y_{r,\mathrm{tf}}) = \mathbb{Z}_p\otimes_{\mathbb{Z}'}(\mathbb{Z}'\otimes B_r^{G_{\mathbb{C}/\mathbb{R}}}) = \mathbb{Z}_p\otimes B_r^{G_{\mathbb{C}/\mathbb{R}}} = \mathbb{Z}_p\otimes \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(B_r)$ it therefore suffices to prove that $\mathbb{Z}_2\otimes Y_{r,\mathrm{tf}} = \mathbb{Z}_2\otimes\mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(B_r)$. To show this we recall the construction of Y_r uses an isomorphism of $\mathbb{Z}_2[G]$-modules $\mathbb{Z}_2\otimes Y_r \cong H^1(C^\bullet_{2,r})$ with $C^\bullet_{2,r} := R\mathrm{Hom}_{\mathbb{Z}_2}(R\Gamma_{c,\text{ét}}(\mathcal{O}_{F,S}[\frac{1}{2}], \mathbb{Z}_2(r)), \mathbb{Z}_2[-2])$ where the subscript 'c' denotes cohomology with compact support, whilst from the long exact cohomology sequences associated to the commutative diagram of exact triangles given in [10, (114)] (with $p=2$ and r replaced by $1-r$) one obtains an exact commutative diagram of $\mathbb{Z}_2[G]$-modules of the form

$$\begin{array}{ccccccc}
0 \to H^1(C^\bullet_{2,r})_{\mathrm{tf}} \to & \bigoplus_{w|\infty} H^0_{\text{ét}}(F_w,\mathbb{Z}_2(-r)) & \to & H^3_{\text{ét}}(\mathcal{O}_{F,S}[\frac{1}{2}],\mathbb{Z}_2(1-r)) & \to 0 \\
& \theta\downarrow & & \cong\downarrow & \\
& \bigoplus_{w|\infty} H^3_{\text{ét}}(F_w,\mathbb{Z}_2(1-r)) & = & \bigoplus_{w|\infty} H^3_{\text{ét}}(F_w,\mathbb{Z}_2(1-r)) & \\
& \downarrow & & & \\
& 0. & & &
\end{array}$$

Here w runs over all archimedean places of F and each term $H^0_{\text{ét}}(F_w,\mathbb{Z}_2(-r))$ arises from the explicit description of the complex $R\Gamma_\Delta(F_w,\mathbb{Z}_2(r))^*[-3]$ that is given in [10, p. 1391]. Now $H^3_{\text{ét}}(F_w,\mathbb{Z}_2(1-r))$ is isomorphic to $\mathbb{Z}_2/2\mathbb{Z}_2$ if w is real and r is even and vanishes in all other cases. Thus, since the homomorphism θ in the above diagram is known to respect the direct sum decompositions of its source and target (see the proof of [10, Lem. 18]), the diagram induces an isomorphism

$$H^1(C^\bullet_{2,r})_{\mathrm{tf}} \cong \bigoplus_{w'} 2H^0_{\text{ét}}(F_{w'},\mathbb{Z}_2(-r)) \oplus \bigoplus_{w} H^0_{\text{ét}}(F_w,\mathbb{Z}_2(-r))$$

where w', resp. w, runs over all real, resp. complex, places of F. Finally we note that each choice of a topological generator of $\mathbb{Z}_2(-r)$ gives a natural isomorphism of the last displayed direct sum module with $\mathbb{Z}_2 \otimes \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(B_r)$. □

The isomorphism Reg_r combines with the two equalities $Y_{r,\mathrm{tf}} = \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(B_r)$ and $\mathbb{Q} \otimes X_r = \mathbb{Q} \otimes C_r$ coming from Lemma 3.9(i) to imply the existence of homomorphisms of G-modules

$$\lambda : Y_r \to X_r$$

that have both finite kernel and finite cokernel. In particular, for each such λ, the induced map $\mathbb{Q} \otimes \lambda : \mathbb{Q} \otimes B_r^{G_{\mathbb{C}/\mathbb{R}}} = \mathbb{Q} \otimes Y_r \to \mathbb{Q} \otimes X_r = \mathbb{Q} \otimes C_r$ is bijective and so for each character $\psi = \chi^\gamma$, with $\gamma \in \Gamma$, we may set $R_r^\lambda(\psi) := R_r^{\mathbb{Q}\otimes\lambda}(\psi)$. For each integer j with $1 \le j \le \chi(1)$ we then consider the composite homomorphism of $\mathcal{O}$-modules

$$t^j_{\lambda,\psi} : Y^j_{r,\psi} \xrightarrow{\lambda^j_\psi} X^j_{r,\psi} \xrightarrow{t^j(X_r,\psi)} X_r^{j,\psi}.$$

We write $\mathrm{Fit}_{\mathcal{O}}(M)$ for the Fitting ideal of a finitely generated $\mathcal{O}$-module M (so in particular $\mathrm{Fit}_{\mathcal{O}}(M) \subseteq \mathcal{O}$) and if $f : M \to M'$ is a homomorphism of finitely generated $\mathcal{O}$-modules that has both finite kernel and finite cokernel we define a fractional $\mathcal{O}$-ideal by setting $q(f) := \mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(f))\mathrm{Fit}_{\mathcal{O}}(\ker(f))^{-1}$.

Conjecture 3.10. *(The 'modified Lichtenbaum-Gross Conjecture') Let S be any finite set of places of k that contains S_∞ and all places that ramify in F/k. Then for every homomorphism λ and index j as above one has*

$$\frac{L_S^*(r,\check{\chi}^\alpha)}{R_r^\lambda(\chi^\alpha)} = \frac{L_S^*(r,\check{\chi})^\alpha}{R_r^\lambda(\chi)^\alpha} \tag{3.11}$$

for all $\alpha \in \mathrm{Aut}(\mathbb{C})$ and also

$$\frac{L_S^*(r,\check{\chi}^\gamma)}{R_r^\lambda(\chi^\gamma)}\mathcal{O} = q(t^j_{\lambda,\chi^\gamma})^{-1} \tag{3.12}$$

for all $\gamma \in \Gamma$.

Remark 3.13.

(i) After taking account of the isomorphism (3.7), the equality (3.11) coincides with the central conjecture of Gross in [22] (which is often referred to as the 'Gross-Stark Conjecture'). The equality (3.12) was first explicitly formulated by Chinburg, Kolster, Pappas and Snaith in [14, Conj. 6.12] (where it is referred to as a 'modified Lichtenbaum-Gross Conjecture'). The notation of loc. cit. is however different from that used here and the necessary translation is described in [8, §11.3].

(ii) The 'equivariant Tamagawa number' $T\Omega(h^0(\mathrm{Spec}(F))(r), \mathbb{Z}[G])$ that is defined (unconditionally) by Flach and the first author in [11, Conj. 4(iii)] is an element of the relative algebraic K-group $K_0(\mathbb{Z}[G], \mathbb{R}[G])$ and the relevant case of the equivariant Tamagawa number conjecture predicts that $T\Omega(h^0(\mathrm{Spec}(F))(r), \mathbb{Z}[G])$ vanishes. In [8, §11.1] it is shown that Conjecture 3.10 is valid if and only if $T\Omega(h^0(\mathrm{Spec}(F))(r), \mathbb{Z}[G])$ belongs to the kernel of a natural homomorphism $\rho_*^\chi : K_0(\mathbb{Z}[G], \mathbb{R}[G]) \to K_0(\mathcal{O}, \mathbb{C})$. When combined with the good functorial properties of $T\Omega(h^0(\mathrm{Spec}(F))(r), \mathbb{Z}[G])$ under change of extension F/k (that are proved in [11, Prop. 4.1b)]), this observation implies that the validity of Conjecture 3.10 is unchanged if one replaces F by any subfield F' of F that is Galois over k and such that χ factors through the projection $G \to G_{F'/k}$.

3.5. We now reinterpret the conjectural equality (3.12). To do this we set $\psi := \chi^\gamma$ for some fixed $\gamma \in \Gamma$ and $n := \psi(1) = \chi(1)$. We choose an integer j with $1 \le j \le n$ and set $\rho_r := \dim_E(V_\psi^j \otimes_{\mathbb{Z}[G]} B_r^{G_{\mathbb{C}/\mathbb{R}}}) = \dim_E(V_\psi^j \otimes_{\mathbb{Z}[G]} Y_r) = \dim_E(E \otimes_{\mathcal{O}} Y_{r,\psi}^j)$. Then ρ_r is independent of both j and γ and the argument of Lemma 2.8 shows that the function $L_S(s, \check{\psi})$ vanishes to order ρ_r at $s = r$. Now (3.11) implies that the quotient $L_S^{(\rho_r)}(r, \check{\psi})/R_r^\lambda(\psi)$ belongs to E and, after unwinding the definition of $R_r^\lambda(\psi)$, this implies that there is an equality of E-spaces

$$L_S^{(\rho_r)}(r, \check{\psi}) \cdot \wedge_E^{\rho_r}(V_\psi^j \otimes_{\mathbb{Z}[G]} Y_r) = \mathrm{Reg}_r^{(\psi)}(\wedge_E^{\rho_r}(V_\psi^j \otimes_{\mathbb{Z}[G]} C_r)),$$

where $\mathrm{Reg}_r^{(\psi)} : \mathbb{C} \otimes_E \wedge_E^{\rho_r}(V_\psi^j \otimes_{\mathbb{Z}[G]} C_r) \xrightarrow{\sim} \mathbb{C} \otimes_E \wedge_E^{\rho_r}(V_\psi^j \otimes_{\mathbb{Z}[G]} Y_r)$ is the isomorphism of $\mathbb{C}$-spaces induced in the obvious way by Reg_r. In the next result we show that (3.12) implies a natural integral refinement of this equality.

Proposition 3.14. *Assume that (3.12) is valid for χ. Set $K := F^{\ker(\chi)}$ and $\psi := \chi^\gamma$ with $\gamma \in \Gamma$.*

(i) *Then in $\mathbb{C} \otimes_E \wedge_E^{\rho_r}(V_\psi^j \otimes_{\mathbb{Z}[G]} Y_r) = \mathbb{C} \otimes_{\mathcal{O}} \wedge_{\mathcal{O}}^{\rho_r} Y_{r,\psi}^j$ one has*

$$|G|^{\rho_r} L_S^{(\rho_r)}(r, \check{\psi}) \mathrm{Fit}_{\mathcal{O}}(\mu_{1-r}(K)^{j,\psi}) \wedge_{\mathcal{O}}^{\rho_r}(Y_{r,\psi}^j)_{\mathrm{tf}} = \mathrm{Fit}_{\mathcal{O}}((Y_{r,\psi}^j)_{\mathrm{tor}}) \mathrm{Reg}_r^{(\psi)}(\wedge_{\mathcal{O}}^{\rho_r}(X_r^{j,\psi})_{\mathrm{tf}}).$$

(ii) *For each integer a with $1 \le a \le \rho_r$ fix σ_a in Σ_F and define $\tilde{\sigma}_a := \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}((2\pi i)^{-r}\sigma_a) \in Y_{r,\mathrm{tf}}$. Also set $\Sigma' := \{\sigma_a : 1 \le a \le \rho_r\} \subseteq \Sigma_F$. Then for every d' in $\mathrm{Fit}_{\mathcal{O}}(\mu_{1-r}(K)^{j,\psi})$ there exists a unique element $u_{\Sigma',\psi}^j(d')$ of $\mathrm{Fit}_{\mathcal{O}}((Y_{r,\psi}^j)_{\mathrm{tor}}) \wedge_{\mathcal{O}}^{\rho_r} (n^{-1}(C_r^{j,\psi})_{\mathrm{tf}})$ for which one has*

$$\mathrm{Reg}_r^{(\psi)}(u_{\Sigma',\psi}^j(d')) = d'(n^{-1}|G|)^{\rho_r} L_S^{(\rho_r)}(r, \check{\psi}) \wedge_{a=1}^{a=\rho_r} f_\psi^j(\tilde{\sigma}_a).$$

Proof. We set $\rho := \rho_r$. To derive the equality of claim (i) from the equality (3.12) in Conjecture 3.10 we note that

$$\begin{aligned}
L_S^{(\rho_r)}(r, \check{\psi}) \cdot \wedge_{\mathcal{O}}^{\rho}(Y_{r,\psi}^j)_{\mathrm{tf}} &= R_r^\lambda(\psi) q(t_{\lambda,\psi}^j)^{-1} \wedge_{\mathcal{O}}^{\rho}(Y_{r,\psi}^j)_{\mathrm{tf}} \\
&= q(t_{\lambda,\psi}^j)^{-1} (\mathrm{Reg}_r^{(\psi)} \circ \wedge_{\mathbb{C}}^{\rho}(\mathbb{C} \otimes_{\mathcal{O}} \lambda_\psi^j))(\wedge_{\mathcal{O}}^{\rho}(Y_{r,\psi}^j)_{\mathrm{tf}}) \\
&= \frac{q(\lambda_\psi^j)}{q(t_{\lambda,\psi}^j)} \frac{\mathrm{Fit}_{\mathcal{O}}((Y_{r,\psi}^j)_{\mathrm{tor}})}{\mathrm{Fit}_{\mathcal{O}}((X_{r,\psi}^j)_{\mathrm{tor}})} \mathrm{Reg}_r^{(\psi)}(\wedge_{\mathcal{O}}^{\rho}(X_{r,\psi}^j)_{\mathrm{tf}}) \\
&= q(t^j(X_r, \psi))^{-1} \frac{\mathrm{Fit}_{\mathcal{O}}((Y_{r,\psi}^j)_{\mathrm{tor}})}{\mathrm{Fit}_{\mathcal{O}}((X_{r,\psi}^j)_{\mathrm{tor}})} \mathrm{Reg}_r^{(\psi)}(\wedge_{\mathcal{O}}^{\rho}(X_{r,\psi}^j)_{\mathrm{tf}}) \\
&= |G|^{-\rho} \frac{\mathrm{Fit}_{\mathcal{O}}((Y_{r,\psi}^j)_{\mathrm{tor}})}{\mathrm{Fit}_{\mathcal{O}}((X_r^{j,\psi})_{\mathrm{tor}})} \mathrm{Reg}_r^{(\psi)}(\wedge_{\mathcal{O}}^{\rho}(X_r^{j,\psi})_{\mathrm{tf}}).
\end{aligned}$$

The first equality here follows immediately from (3.12), the second from the definition of $R_r^\lambda(\psi)$ and the fourth from the equality $q(t_{\lambda,\psi}^j) = q(t^j(X_r, \psi)) q(\lambda_\psi^j)$ which is a consequence of the kernel-cokernel sequence of the composite $t_{\lambda,\psi}^j = t^j(X_r, \psi) \circ \lambda_\psi^j$. In addition, the third and fifth displayed equalities follow by applying Lemma 3.15 below with f equal to $\lambda_\psi^j : Y_{r,\psi}^j \to X_{r,\psi}^j$ and $t^j(X_r, \psi) : X_{r,\psi}^j \to X_r^{j,\psi}$ respectively.

To deduce the equality of claim (i) from the above displayed formula it only remains to show that $(X_r^{j,\psi})_{\mathrm{tor}} = \mu_{1-r}(K)^{j,\psi}$. But $(X_r^{j,\psi})_{\mathrm{tor}} := (T_\psi^j \otimes X_r)_{\mathrm{tor}}^G = (T_\psi^j \otimes X_{r,\mathrm{tor}})^G = (T_\psi^j \otimes X_{r,\mathrm{tor}}^{G_{F/K}})^G =: (X_{r,\mathrm{tor}}^{G_{F/K}})^{j,\psi}$ where the second and third equalities follow from the fact that T_ψ^j is torsion-free and, in the latter case, that $G_{F/K} = \ker(\psi)$ acts trivially on T_ψ^j. The required equality is therefore true because Lemma 3.9(iii) implies $X_{r,\mathrm{tor}}^{G_{F/K}} = \mu_{1-r}(F)^{G_{F/K}} := H^0(G_{\mathbb{Q}^c/F}, \mathbb{Q}/\mathbb{Z}(1-r))^{G_{F/K}} = H^0(G_{\mathbb{Q}^c/K}, \mathbb{Q}/\mathbb{Z}(1-r)) =: \mu_{1-r}(K)$.

To prove claim (ii) we note that the element $\wedge_{a=1}^{a=\rho} f_\psi^j(\tilde\sigma_a)$ belongs to $\wedge_{\mathcal{O}}^{\rho}(Y_{r,\psi}^j)_{\mathrm{tf}}$. The existence of an element $u_{\Sigma',\psi}^j(d')$ of $\mathrm{Fit}_{\mathcal{O}}((Y_{r,\psi}^j)_{\mathrm{tor}}) \wedge_{\mathcal{O}}^{\rho} (n^{-1}(X_r^{j,\psi})_{\mathrm{tf}})$ which satisfies the displayed equality in claim (ii) therefore follows directly from claim (i). It thus suffices to note that Lemma 3.9(i) implies $(X_r^{j,\psi})_{\mathrm{tf}} \subseteq (C_r^{j,\psi})_{\mathrm{tf}}$ and that the uniqueness of $u_{\Sigma',\psi}^j(d')$ follows from the injectivity of $\mathrm{Reg}_r^{(\psi)}$. □

Lemma 3.15. *If $f : M \to N$ is any homomorphism of finitely generated $\mathcal{O}$-modules that has both finite kernel and finite cokernel, then there is an equality of $\mathcal{O}$-lattices*

$$\wedge_E^d(E\otimes_{\mathcal{O}} f)(\wedge_{\mathcal{O}}^d M_{\mathrm{tf}}) = q(f)\frac{\mathrm{Fit}_{\mathcal{O}}(M_{\mathrm{tor}})}{\mathrm{Fit}_{\mathcal{O}}(N_{\mathrm{tor}})}\wedge_{\mathcal{O}}^d N_{\mathrm{tf}}$$

with $d := \dim_E(E\otimes_{\mathcal{O}} M)$.

Proof. We consider the following exact commutative diagram

$$\begin{array}{ccccccccc}
 & & \ker(f_{\mathrm{tor}}) & = & \ker(f) & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_{\mathrm{tor}} & \longrightarrow & M & \longrightarrow & M_{\mathrm{tf}} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle f_{\mathrm{tor}}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_{\mathrm{tf}}} & & \\
0 & \longrightarrow & N_{\mathrm{tor}} & \longrightarrow & N & \longrightarrow & N_{\mathrm{tf}} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{cok}(f_{\mathrm{tor}}) & \longrightarrow & \mathrm{cok}(f) & \longrightarrow & \mathrm{cok}(f_{\mathrm{tf}}) & \longrightarrow & 0.
\end{array}$$

Here f_{tor} and f_{tf} denote the homomorphisms that are induced by the given map f; the equality $\ker(f_{\mathrm{tor}}) = \ker(f)$ and the injectivity of f_{tf} both follow from the assumption that $\ker(f)$ is finite and the exactness of the bottom row then follows from the Snake lemma. Now M_{tf}, and hence also N_{tf} since $\mathrm{cok}(f_{\mathrm{tf}})$ is finite, is a projective $\mathcal{O}$-module of rank d and so the definition of $\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(f_{\mathrm{tf}}))$ implies that $\wedge_E^d(E\otimes_{\mathcal{O}} f)(\wedge_{\mathcal{O}}^d M_{\mathrm{tf}}) = \mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(f_{\mathrm{tf}}))\cdot\wedge_{\mathcal{O}}^d N_{\mathrm{tf}}$. On the other hand, the exactness of the bottom row and left hand column of the above diagram combines with the multiplicativity of Fitting ideals on exact sequences of finite $\mathcal{O}$-modules and the definition of $q(f)$ to imply that $\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(f_{\mathrm{tf}})) = \mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(f))\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(f_{\mathrm{tor}}))^{-1} = q(f)\mathrm{Fit}_{\mathcal{O}}(\ker(f))\mathrm{Fit}_{\mathcal{O}}(\mathrm{cok}(f_{\mathrm{tor}}))^{-1} = q(f)\mathrm{Fit}_{\mathcal{O}}(M_{\mathrm{tor}})\mathrm{Fit}_{\mathcal{O}}(N_{\mathrm{tor}})^{-1}$. The claimed equality is thus clear. □

3.6. We now prove Theorem 3.3. To do this we set $n := \chi(1)$, $\psi := \chi^\gamma$ with $\gamma \in \Gamma$ and $\mathrm{pr}_\psi := n^{-1}|G|e_\psi = \sum_{g\in G}\check\psi(g)g \in \mathcal{O}[G]$ and note that $\sum_{j=1}^{j=n} f_\psi^j$ is a decomposition of e_ψ as a sum of indecomposable idempotents in $E[G]$. We set $K := F^{\ker(\chi)} = F^{\ker(\psi)}$ and $d' := w_{1-r}(K)^n$. For the given embedding σ we also set $\tilde\sigma := \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}((2\pi i)^{-r}\sigma)$ (so $\tilde\sigma \in Y_{r,\mathrm{tf}}$ by Lemma 3.9(i)) and then for each index j

as above we define

$$\epsilon_\sigma(\psi, j) := \frac{d'|G|}{n}\frac{L'_S(r,\check{\psi})}{R_r^\lambda(\psi)}\lambda_\psi^j(f_\psi^j(\tilde{\sigma})) \in \mathbb{C}\otimes_{\mathcal{O}} C_r^{j,\psi}\,. \tag{3.16}$$

We may assume that $\rho_r = 1$ since if $\rho_r > 1$, then $L'_S(r,\check{\chi}^\delta) = 0$ for all $\delta \in \Gamma$ and so Conjecture 1.2 is obviously valid. Then, since $\rho_r = 1$ one has $L^*_S(r,\check{\psi}) = L'_S(r,\check{\psi})$ and so, as $f_\psi^j(\tilde{\sigma})$ is a non-zero element of the dimension one E-space $E\otimes_{\mathcal{O}} Y^j_{r,\psi}$, the definition of $R_r^\lambda(\psi)$ implies that

$$\mathrm{Reg}_r^{(\psi)}(\epsilon_\sigma(\psi,j)) = d'n^{-1}|G|L'_S(r,\check{\psi})\cdot f_\psi^j(\tilde{\sigma}).$$

Now the finite group $\mu_{1-r}(K)$ is cyclic and so the (finite) $\mathcal{O}$-module $\mu_{1-r}(K)^{j,\psi} \subseteq T_\psi^j \otimes \mu_{1-r}(K)$ is both annihilated by $w_{1-r}(K)$ and generated by (at most) $\mathrm{rk}_{\mathcal{O}}(T_\psi^j) = n$ elements. It follows in particular that d' belongs to $\mathrm{Fit}_{\mathcal{O}}(\mu_{1-r}(K)^{j,\psi})$ and so the last displayed formula implies that $\epsilon_\sigma(\psi,j)$ is equal to the element $u^j_{\{\sigma\},\psi}(d')$ of $n^{-1}\mathrm{Fit}_{\mathcal{O}}((Y^j_{r,\psi})_{\mathrm{tor}})(C_r^{j,\psi})_{\mathrm{tf}}$ that occurs in Proposition 3.14(ii). Lemma 3.8(ii) thus implies that

$$\epsilon_\sigma(\psi,j) \in \mathrm{Fit}_{\mathcal{O}}((Y^j_{r,\psi})_{\mathrm{tor}})\otimes C_{r,\mathrm{tf}} \subseteq \mathcal{O}\otimes C_{r,\mathrm{tf}}. \tag{3.17}$$

The element $\overline{\epsilon}_\sigma(\psi,S) := \sum_{j=1}^{j=n}\epsilon_\sigma(\psi,j)$ therefore belongs to $\mathcal{O}\otimes C_{r,\mathrm{tf}}$ and satisfies the equality

$$\begin{aligned}
&(2\pi i)^r\mathrm{Reg}_r(\overline{\epsilon}_\sigma(\psi,S))\\
&= (2\pi i)^r\sum_{j=1}^{j=n} d'n^{-1}|G|L'_S(r,\check{\psi})\cdot f_\psi^j(\tilde{\sigma})\\
&= (2\pi i)^r d' L'_S(r,\check{\psi})\cdot \mathrm{pr}_\psi(\tilde{\sigma})\\
&= d'\sum_{g\in G} L'_S(r,\check{\psi})\check{\psi}(g)g(\sigma) + d'\sum_{g\in G}L'_S(r,\check{\psi})(-1)^r\check{\psi}(g)g(\tau\circ\sigma)\,.
\end{aligned} \tag{3.18}$$

Now the assumed validity of (3.11) combines with (3.16) to imply that for all $\alpha\in\mathrm{Aut}(\mathbb{C})$ and all j one has

$$\begin{aligned}
\epsilon_\sigma(\chi^\alpha, j) &:= \frac{d'|G|}{n}\frac{L'_S(r,\check{\chi}^\alpha)}{R_r^\lambda(\chi^\alpha)}\lambda^j_{\chi^\alpha}(f^j_{\chi^\alpha}(\tilde{\sigma}))\\
&= \left(\frac{d'|G|}{n}\frac{L'_S(r,\check{\chi})}{R_r^\lambda(\chi)}\lambda^j_\chi(f^j_\chi(\tilde{\sigma}))\right)^\alpha =: \epsilon_\sigma(\chi,j)^\alpha
\end{aligned} \tag{3.19}$$

(where in the last two terms we use the natural semi-linear action of $\mathrm{Aut}(\mathbb{C})$ on the space $\mathbb{C}\otimes_{\mathcal{O}} C_r^{j,\psi} \subset f_\psi^j\mathbb{C}[G]\otimes C_r$) and hence also $\overline{\epsilon}_\sigma(\psi,S) = \overline{\epsilon}_\sigma(\chi,S)^\gamma$. Given this equality, it is now straightforward to check, by unwinding the definition of Reg_r, that (3.18) implies that for all σ' in Σ_F the image of $\overline{\epsilon}_\sigma(\chi,S)^\gamma$ under $(2\pi i)^r\mathrm{reg}_{1-r,\sigma'}$ is equal to the right hand side of (1.3). To verify Conjecture 1.2, and hence complete the proof of Theorem 3.3(i), it thus suffices to define $\epsilon_\sigma(\chi,S)$ to be any pre-image of $\overline{\epsilon}_\sigma(\chi,S)$ under the natural map $\mathcal{O}\otimes C_r \to \mathcal{O}\otimes C_{r,\mathrm{tf}}$.

The containment $\mathrm{Tr}_1(\epsilon_\sigma(\chi,S)) \in K_{1-2r}(F)$ in Theorem 3.3(ii) follows directly from claim (i) and the result of Proposition 2.2(i) with $d=1$. To prove the rest of claim (ii) we set $\overline{\epsilon}(\chi) := \overline{\epsilon}_\sigma(\chi,S)$. We also note that for each prime ideal $\mathfrak{p}$ of $\mathcal{O}$ the isomorphism class of the $\mathcal{O}_\mathfrak{p}$-module $\mathcal{O}_\mathfrak{p}\otimes_{\mathcal{O}} Y^j_{r,\chi^\gamma}$ is independent of the choice of index j and hence that $\mathrm{Fit}_{\mathcal{O}}((Y^j_{r,\chi^\gamma})_{\mathrm{tor}}) = \mathrm{Fit}_{\mathcal{O}}((Y^1_{r,\chi^\gamma})_{\mathrm{tor}})$ for all j. From the containment (3.17) and equalities (3.19) we therefore know that both $\overline{\epsilon}(\chi)^\gamma$ belongs to $\mathrm{Fit}_{\mathcal{O}}((Y^1_{r,\chi^\gamma})_{\mathrm{tor}}) \otimes C_{r,\mathrm{tf}}$ and that $\overline{\epsilon}(\chi^\gamma) = \overline{\epsilon}(\chi)^\gamma e_{\chi^\gamma}$ and so for every ϕ in $\mathrm{Hom}_G(C_r, \mathbb{Z}[G]) \subseteq \mathrm{Hom}_{E[G]}(E \otimes C_{r,\mathrm{tf}}, E[G])$ the element $\phi(\overline{\epsilon}(\chi)^\gamma)$ belongs to $\mathrm{Fit}_{\mathcal{O}}((Y^1_{r,\chi^\gamma})_{\mathrm{tor}}) \cdot \mathcal{O}[G]e_{\chi^\gamma}$. Now Lemma 3.9(i) implies that $\mathbb{Z}' \otimes Y_{r,\mathrm{tf}} = \mathbb{Z}' \otimes B_r^{G_{\mathbb{C}/\mathbb{R}}}$ is a projective $\mathbb{Z}'[G]$-module so that $\mathbb{Z}' \otimes (Y^1_{r,\chi^\gamma})_{\mathrm{tor}}$ identifies with $\mathbb{Z}' \otimes (Y_{r,\mathrm{tor}})^1_{\chi^\gamma}$ and hence that $\phi(\overline{\epsilon}(\chi)^\gamma) \in \mathbb{Z}' \otimes \mathrm{Fit}_{\mathcal{O}}((Y_{r,\mathrm{tor}})^1_{\chi^\gamma}) \cdot \mathcal{O}[G]e_{\chi^\gamma}$. By applying [9, Lem. 11.1.2(i)] we can therefore deduce that the element $\chi(1)^{-1}|G|^2\phi(\overline{\epsilon}(\chi)^\gamma)$ belongs to $\mathbb{Z}' \otimes \mathrm{Ann}_{\mathbb{Z}[G]}(Y_{r,\mathrm{tor}})$. The displayed containment in Theorem 3.3(ii) thus follows directly from the equality $\phi(\mathrm{Tr}_1(\epsilon_\sigma(\chi, S))) = \sum_{\gamma\in\Gamma} \phi(\overline{\epsilon}(\chi)^\gamma)$ and the description of $Y_{r,\mathrm{tor}}$ given in Lemma 3.9(iii). This completes our proof of Theorem 3.3.

4. On K_3 and the regulator

In preparation for describing some numerical evidence for Conjecture 1.2 we now make precise the relation between a version of the (second) Bloch group and K_3 of a field, and, if the field is a number field, the Beilinson regulator map. This result may itself be of some independent interest and so, in order to keep open the possibility of extending it to higher Bloch groups, we have used the approach of [16] rather than the potentially more precise result of [33, Th. 5.2].

Let F be a field, and let $K_3(F)^{\mathrm{ind}}$ be the quotient of $K_3(F)$ by the image of the Milnor K-group $K_3^M(F)$. We recall that if F is a number field of signature $[r_1, r_2]$ then $K_3^M(F) \cong (\mathbb{Z}/2\mathbb{Z})^{r_1}$ by [1, Th. 2.1(3)] so $K_3(F)^{\mathrm{ind}}_{\mathrm{tf}} = K_3(F)_{\mathrm{tf}}$ naturally, and this is a free Abelian group of rank r_2 by Remark 2.10(iii). Finally, we set

$$\tilde{\wedge}^2 F^\times := F^\times \otimes F^\times / \langle (-x) \otimes x : x \in F^\times \rangle ,$$

(this does not coincide with the usual exterior power $\wedge^2 F^\times$ because of the negative sign in the denominator) and then write $\delta_{2,F}$ for the homomorphism from the free Abelian group $\mathbb{Z}[F^\times]$ on $F^\times$ to $\tilde{\wedge}^2 F^\times$ that is given by mapping a generator $\{x\}$ with x in $F^\times$ to $(1-x)\tilde{\wedge}\, x$ if $x \neq 1$ and to 0 if $x = 1$.

In the next proposition we refine various results in [16, §2-§5]. Here we write $D(z)$ for the Bloch-Wigner dilogarithm $\mathbb{C} \setminus \{0\} \to \mathbb{R}(1)$ that is defined in [3] by integrating the function $\log|w| d i\arg(1-w) - \log|1-w| d i\arg(w)$ along any path from a point $z_0 \in \mathbb{R} \setminus \{0,1\}$ to z (for $z = 1$ one uses a limit).

Theorem 4.1.

(i) *With notation as above, there is a homomorphism*

$$\varphi_F : \ker(\delta_{2,F}) \to K_3(F)^{\mathrm{ind}}_{\mathrm{tf}} ,$$

which is natural up to sign.

(ii) *If F is a number field, then φ_F has finite cokernel.*

(iii) *Moreover, there is a universal choice of sign, such that if F is any number field, and σ any embedding $F \to \mathbb{C}$, then the composition*

$$\ker(\delta_{2,F}) \overset{\varphi_F}{\to} K_3(F)^{\mathrm{ind}}_{\mathrm{tf}} = K_3(F)_{\mathrm{tf}} \overset{\sigma}{\to} K_3(\mathbb{C})_{\mathrm{tf}} \overset{\mathrm{reg}_2}{\to} \mathbb{R}(1)$$

is induced by mapping $\{x\}$ to $D(\sigma(x))$.

Proof. At the outset we note that our set-up is essentially that of [3] with the various modifications and improvements contained in [4] and [16] (but which are sometimes carried out only after tensoring with $\mathbb{Q}$). In particular, the reader may observe that the construction of the element $[x]_2'$ and the map to $K_3(F)_{\mathrm{tf}}^{\mathrm{ind}}$ described below become those in [16, §3] after tensoring with $\mathbb{Q}$ and decomposing according to the Adams eigenspaces.

For any Noetherian regular ring R we let $X_R = \mathbb{P}^1_R \setminus \{t = 1\}$, where t is the standard affine coordinate on $\mathbb{P}^1_R$. Then there is a long exact sequence of relative K-groups (terminating with $K_0(\square)$),

$$\cdots \to K_n(X_R;\square) \to K_n(X_R) \to K_n(\square) \to K_{n-1}(X_R;\square) \to \cdots$$

where $\square$ consists of the subset of X_R where $t = 0, \infty$, i.e., two copies of $\mathrm{Spec}(R)$. Since the pullback $K_n(R) \to K_n(X_R)$ along the natural map is an isomorphism (by Quillen [29, p.122]) and therefore the composition $K_n(R) \to K_n(X_R) \to K_n(\square) \cong K_n(R) \oplus K_n(R)$ is the diagonal map, we find an isomorphism

$$K_n(X_R;\square) \cong K_{n+1}(R) \tag{4.2}$$

for $n \geq 0$, with a map that is natural up to a universal sign.

We can combine localization with relativity under suitable assumptions (see [16, §2.2]). In particular, if we take $R = \mathbb{Z}[S, S^{-1}]$, then we have the exact sequence (terminating with a map $K_0(X_R;\square) \to 0$)

$$\cdots \to K_2(X_R;\square) \to K_2(X_R \setminus \{t = S\};\square) \to K_1(R') \to K_1(X_R;\square) \to \cdots$$

with $R' = \mathbb{Z}[S, S^{-1}, (1-S)^{-1}]$. Then $K_1(R') \cong \langle -1, S, 1-S\rangle$, and using [16, Lem. 3.14] one sees that the map $K_1(R') \to K_1(X_R;\square) \cong K_2(R)$ maps $(1-S)$ to $\pm\{1-S, S\} = 0$, so there exists an element $[S]_2^\sim$ in $K_2(X_R \setminus \{t = S\};\square)$ with image $(1-S)^{-1}$ in $K_1(\mathbb{Z}[S, S^{-1}, (1-S)^{-1}])$. Because $K_2(X_R;\square) \cong K_3(R) \cong K_3(\mathbb{Z}) \oplus K_2(\mathbb{Z})$ is torsion of exponent 48 (by Lee and Szczarba [25]) the element $[S]_2^\sim$ is unique up to such torsion. We fix a choice of $[S]_2^\sim$ in what follows.

For the field F, write $F^\flat = F\setminus\{0,1\}$ as well as $X_{F,\mathrm{loc}} = X_F\setminus\{t = u \text{ with } u \text{ in } F^\flat\}$. By comparing the exact localization sequence

$$\cdots \to K_1(X_F;\square) \to K_1(X_{F,\mathrm{loc}};\square) \to \coprod_{u\in F^\flat} \mathbb{Z} \to K_0(X_F;\square) \to \cdots \tag{4.3}$$

with the same one without relativity one sees as on [16, p.222] that

$$(1+I)^\times := K_1(X_{F,\mathrm{loc}};\square) = \left\{ f(t) = \prod_{i=1}^n \left(\frac{t-a_i}{t-1}\right)^{n_i} \text{ with } f(0) = f(\infty) = 1 \right\}$$

where $n \geq 0$, all n_i are in $\mathbb{Z}$, and all a_i are in $F^\flat$. Tensoring the short exact sequence

$$0 \to (1+I)^\times \to \coprod_{u\in F^\flat} \mathbb{Z} \to F^\times \to 0\,,$$

which is part of (4.3), with $F^\times$ over $\mathbb{Z}$, we obtain the top row of the diagram

$$\begin{array}{ccccccccc}
0 \to & \mathrm{Tor}_1^{\mathbb{Z}}(F^\times, F^\times) & \to & (1+I)^\times \otimes F^\times & \to & \coprod_{u\in F^\flat} F^\times & \to & F^\times \otimes F^\times & \to 0 \\
& \downarrow & & \downarrow & & \| & & \downarrow & \\
0 \to & \frac{K_2(X_F;\square)}{\mathrm{im}(\coprod_{u\in F^\flat} K_2(F))} & \to & K_2(X_{F,\mathrm{loc}};\square) & \xrightarrow{d^\sim} & \coprod_{u\in F^\flat} F^\times & \to & K_1(X_F;\square) & \to 0.
\end{array}$$

The bottom row is also obtained from (4.3), using [16, Lem. 3.14] and the fact that $K_2(F)$ is generated by symbols $\{a, b\} = a \cup b$. The second and last vertical maps are induced by the cup product, thus giving rise to the first vertical map which makes the diagram commutes.

Using [16, Lem. 3.14] yet again one sees that the image of $\coprod_{u\in F^\flat} K_2(F) \to K_2(X_F;\square)$ under the isomorphism $K_2(X_F;\square)\cong K_3(F)$ in (4.2) equals the image of $K_3^M(F)$ under the natural map to $K_3(F)$. So with $K_3(F)^{\mathrm{ind}} = K_3(F)/\mathrm{im}(K_3^M(F))$ and A the image of $\mathrm{Tor}_1^{\mathbb{Z}}(F^\times, F^\times)$ we obtain an exact sequence

$$0 \to K_3(F)^{\mathrm{ind}}/A \to \frac{K_2(X_{F,\mathrm{loc}};\square)}{\mathrm{im}((1+I)^\times \otimes F^\times)} \xrightarrow{d} F^\times \otimes F^\times \to K_2(F) \to 0\,,$$

which gives an isomorphism $\ker(d) \cong K_3(F)^{\mathrm{ind}}/A$.

For x in $F^\times$, let $[x]_2^\sim$ be the image of $x^*([S]_2^\sim) \in K_2(X_F \setminus \{t=x\};\square)$ under the localization $K_2(X_F\setminus\{t=x\};\square) \to K_2(X_{F,\mathrm{loc}};\square)$. Note that the boundary of $[x]_2^\sim$ under $d^\sim$ is $(1-x)^{-1}_{|t=x}$ if $x\neq 1$ and is trivial if $x=1$. Write $[x]_2$ for the image of $[x]_2^\sim$ in $K_2(X_{F,\mathrm{loc}};\square)/\mathrm{im}((1+I)^\times\otimes F^\times)$, let $B_2(F)$ be the subgroup of $K_2(X_{F,\mathrm{loc}};\square)/\mathrm{im}((1+I)^\times\otimes F^\times)$ generated by the $[x]_2$ with x in $F^\times$, and let d_2 be the restriction of d to $B_2(F)$. We then have an inclusion

$$\ker(d_2) \to K_3(F)^{\mathrm{ind}}/A\,. \tag{4.4}$$

Now let $\mathcal{N}$ be the subgroup of $B_2(F)$ generated by the classes of $[x]_2 + [1/x]_2$ with x in $F^\times$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{d_{2|\mathcal{N}}} & d(\mathcal{N}) \\ \downarrow & & \downarrow \\ B_2(F) & \xrightarrow{d_2} & F^\times \otimes F^\times\,. \end{array}$$

Next we note that $d_2(\mathcal{N}) = \langle (-x)\otimes x : x\in F^\times\rangle \subseteq F^\times\otimes F^\times$ so $F^\times\otimes F^\times/d_2(\mathcal{N}) = \tilde{\wedge}^2 F^\times$, and hence by taking quotients we get a short exact sequence

$$0 \to \ker(d_{2|\mathcal{N}}) \to \ker(d_2) \to \ker(d_2') \to 0 \tag{4.5}$$

where, setting $B_2'(F) := B_2(F)/\mathcal{N}$, we write

$$d_2' : B_2'(F) \to \tilde{\wedge}^2 F^\times$$

for the induced map. Writing $[x]_2'$ for the image of $[x]_2$ in $B_2'(F)$ we have that d_2' maps $[x]_2'$ to $(1-x)^{-1}\tilde{\wedge}\, x$ if $x\neq 1$ and to 0 if $x=1$.

We know from [16, Lem. 3.7] that $\ker(d_{2|\mathcal{N}})$ is torsion, so combining (4.5) with the map (4.4) and using that A is torsion, we obtain a map

$$\ker(d_2') \to K_3(F)^{\mathrm{ind}}_{\mathrm{tf}}$$

with torsion kernel. We note that this map is independent of the choice of $[S]_2^\sim$, which was unique up to torsion. The map φ_F in Theorem 4.1(i) is now induced by mapping $\{x\}$ in $\ker(\delta_{2,F}) \subseteq \mathbb{Z}[F^\times]$ to $[x]_2'$ in $\ker(d_2') \subseteq B_2'(F)$.

If $F\subseteq \mathbb{C}$ then [16, Prop. 4.1] describes the composition of the maps in $\ker(d_2') \to K_3(F)^{\mathrm{ind}}_{\mathrm{tf}} \to K_3(\mathbb{C})^{\mathrm{ind}}_{\mathrm{tf}}$ with those in (1.1) for $r=-1$, namely

$$K_3(\mathbb{C})^{\mathrm{ind}}_{\mathrm{tf}} \to H^1_{\mathcal{D}}(\mathrm{Spec}(\mathbb{C}), \mathbb{R}(2)) \cong \mathbb{R}(1)\,.$$

That proposition states that the total map is induced by a map $B_2'(F) \to \mathbb{R}(1)$ mapping $[z]_2'$ to $\pm D(z)$, with the (universal) sign depending on the choice of the sign in the isomorphism (4.2). (Note that in loc. cit. this was expressed using the function $P_2(z) = D(z)/i$.) Claim (iii) of the proposition then follows because the construction of $B_2'(F)$ and the map φ_F is natural, so that

$$\begin{array}{ccccc} \ker(\delta_{2,F}) & \longrightarrow & B_2'(F) & \longrightarrow & K_3(F)^{\mathrm{ind}}_{\mathrm{tf}} \\ \downarrow & & \downarrow & & \downarrow \\ \ker(\delta_{2,\mathbb{C}}) & \longrightarrow & B_2'(\mathbb{C}) & \longrightarrow & K_3(\mathbb{C})^{\mathrm{ind}}_{\mathrm{tf}} \end{array}$$

commutes, where the vertical maps are induced by the embedding $\sigma : F \to \mathbb{C}$. Finally, claim (ii) of the proposition follows from the facts that, as F is a number field, after tensoring with $\mathbb{Q}$, φ_F induces an isomorphism (by [16, Th. 5.3]) and $K_3(F)_{\mathrm{tf}}$ is finitely generated (by Quillen [30]). □

5. Numerical evidence

In this section we provide corroborating numerical evidence for Conjecture 1.2 in the case that $r = -1$, $k = \mathbb{Q}$ and $S = S_\infty$ (so that Theorem 3.3 does not apply), taking into account the restrictions on the choice of such examples as mentioned in Remark 1.6. For other interesting numerical work that is related to Conjecture 1.2 but uses elements in the K-group tensored with the rationals see the articles of Besser, Buckingham, Roblot and the second author [2, §7] and of Zagier and the third author [37, §5].

In the sequel we will use the symbol '$\doteq$' to indicate a numerical identity that we have checked to hold to many (and in all cases at least one hundred) decimal places.

We shall give examples of Galois extensions $F/\mathbb{Q}$ with an embedding $\sigma : F \to \mathbb{C}$, and an irreducible complex character χ of $G_{F/\mathbb{Q}}$ with $\chi(1) - \chi(\tau_\sigma) = 2$, so that $L(s, \check{\chi})$ vanishes to order one at $s = -1$ by Remark 1.5, $c^{\chi}_{\sigma,\sigma} = 2$ and $\Sigma_F^{-1,\chi} = \Sigma_F$. Let $\mathbb{Q}(\chi) \subset \mathbb{C}$ be the character field of χ with ring of integers $\mathcal{O}$. In each case, we found elements ξ in $\ker(\delta_{2,F})$ and used the resulting $\varphi_F(\xi)$ in $K_3(F)_{\mathrm{tf}}$ to construct an element β of $\mathcal{O} \otimes K_3(F)_{\mathrm{tf}}$ that satisfies $e_{\sigma,\chi}\beta = \beta$, as well as, for all γ in $G_{\mathbb{Q}(\chi)/\mathbb{Q}}$,

$$(2\pi i)^{-1} \mathrm{reg}_{2,\sigma}(\beta^\gamma) \doteq \gamma(e) L'(-1, \check{\chi}^\gamma)$$

for some e in $E^\times$. Using Remark 2.11 it is easy to check that $w_2(F^{\ker(\chi)}) = 24$ in all of our examples. In each case we can write $ef = c^{\chi}_{\sigma,\sigma} w_{-1}(\chi) = 2 \cdot 24^{\chi(1)}$ for some f in $\mathcal{O}$, so that $\beta_\sigma(\chi, S_\infty) = f\beta$ satisfies the requirements of Proposition 2.6(ii), thus verifying Conjecture 1.2 for χ and σ. In fact, in view of Lemma 2.4, this verifies Conjecture 1.2 for all characters in $\{\chi^\gamma : \gamma \in \Gamma\}$ and for all σ in Σ_F.

We used Theorem 4.1(iii) and GP-PARI [36] in order to compute $\mathrm{reg}_{2,\sigma}(\beta^\gamma)$, and the latest version of MAGMA [6] for the computation of $L'(-1, \check{\chi}^\gamma)$ (but for our original experiments some of these values were provided by A. Booker and X.-F. Roblot).

The elements in $\ker(\delta_{2,F})$ were found, using GP-PARI, roughly as follows. Let T be a finite set of places of $\mathbb{Q}$ including the archimedean place. Then, in a first step, we produce exceptional T-units $x_j \in \mathcal{O}_{F,T}^\times$ (recall that x_j is said to be 'exceptional' if also $1 - x_j \in \mathcal{O}_{F,T}^\times$), and we decompose both x_j and $1 - x_j$ with respect to a chosen set $\mathcal{B}$ of fundamental T-units (as provided by GP-PARI). The second step is then to look for linear relations among the elements $(1 - x_j) \tilde{\wedge} x_j$ of $\tilde{\bigwedge}^2 \langle \mathcal{B} \rangle$, the latter constituting a finite rank submodule of $\tilde{\bigwedge}^2 F^\times$. (For more details on the underlying algorithm and implementation, we refer to work in preparation [21] and to the third author's personal web page.) A theorem of Bloch and Suslin, combined with the explicit formula for the rank of K-groups given in Remark 2.10(iii), guarantees that, for sufficiently large T, there are $\mathbb{Z}$-linear combinations ξ of elements $\{x_j\}$ as above with $\xi \in \ker(\delta_{2,F})$ such that the $\varphi_F(\xi)$ generate a subgroup of finite index in $K_3(F)_{\mathrm{tf}}$ (cf. Theorem 4.1).

Depending on T, the calculation of the resulting elements in $\ker(\delta_{2,F})$, based on the LLL-algorithm, can be unfeasibly large, or the resulting image under φ_F does not have the required rank. For our most complicated examples, it could take several days of experimenting before a T was found for which this calculation took at most a few minutes and resulted in an image under φ_F of the right rank. The numerical verification of our identities up to high precision then took also at most a few minutes.

For each natural number m we now set $\zeta_m := \exp(2\pi i/m)$ and $\eta_m := \zeta_m + \zeta_m^{-1}$.

5.1. Dihedral representations. Let F be a Galois extension of $\mathbb{Q}$ with $G_{F/\mathbb{Q}}$ isomorphic to the dihedral group D_n for an odd integer n. If F is not totally real, then the element τ_σ in Conjecture 1.2 has order 2 and so for any irreducible 2-dimensional character χ of G one has $\chi(1) - \chi(\tau_\sigma) = 2$.

5.1.1. Let $F = \mathbb{Q}(a)$ denote the totally complex field of discriminant -47^5 that is the splitting field over $\mathbb{Q}$ of the irreducible polynomial $x^{10} - 5x^9 + 12x^8 - 18x^7 + 20x^6 - 18x^5 + 10x^4 - x^3 - 2x^2 + x + 1$ in $\mathbb{Q}[x]$, of which a is a root. This field is such that $G := G_{F/\mathbb{Q}} \cong D_5$ is generated by an element s of order 5 with $s(a) = \frac{1}{5}(3a^9 - 12a^8 + 25a^7 - 34a^6 + 38a^5 - 33a^4 + 16a^3 - 7a^2 + 2a + 3)$ and an element t of order 2 with $t(a) = 1 - a$. The irreducible 2-dimensional characters of $G_{F/\mathbb{Q}}$ take values in $\mathbb{Q}(\eta_5) = \mathbb{Q}(\sqrt{5})$, and form an orbit $\{\chi_1, \chi_2\}$ under the action of $\Gamma := G_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}$, where $\chi_j(s) = \zeta_5^j + \zeta_5^{-j}$ for $j \in \{1, 2\}$. MAGMA gives

$$L'(-1, \check{\chi}_1) \doteq -1.2094\ldots, \qquad L'(-1, \check{\chi}_2) \doteq -0.91109\ldots.$$

The elements $u := \frac{1}{5}(-a^8 + 3a^7 - 5a^6 + 6a^5 - 9a^4 + 7a^3 - 2a^2 + 4a + 3)$ and $u' = -(1-u)^2/u$ are exceptional units in $\mathcal{O}_F$ and we found that $\xi := 2(\{u\} + \{u'\})$ lies in $\ker(\delta_{2,F})$. We fix σ with $\sigma(a) = 1.367\ldots + 0.197\ldots \cdot i$, and let $\beta = 10e_{\sigma,\chi_1}\varphi_F(\xi) = \sum_{g \in G} \check{\chi}_1(g)(1 - \tau_\sigma)g\varphi_F(\xi)$ in $\mathcal{O} \otimes K_3(F)_{\mathrm{tf}}$, so that $e_{\sigma,\chi_1}\beta = \beta$. We then find, for both γ in Γ, that

$$(2\pi i)^{-1}\mathrm{reg}_{2,\sigma}(\beta^\gamma) \doteq \gamma(e)L'(-1, \check{\chi}_1^\gamma) \tag{5.1}$$

for $e = \sqrt{5}$ in $E^\times$, so that e does not divide $c_{\sigma,\sigma}^{\chi} w_{-1}(\chi_1) = 2 \cdot 24^2$ in $\mathcal{O}$. We find a different β as follows. If g in G has $\check{\chi}_1(g) \neq 0$, then $(\check{\chi}_1(g) - 2)/(\eta_5 - 2)$ is in $\mathcal{O}$, so we can write $5e_{\chi_1} = (\eta_5 - 2)z + 2z'$ with z and $z' = \sum_{i=0}^{i=4} s^i$ in $\mathcal{O}[G]$. If β' is in $K_3(F)_{\mathrm{tf}}$, then $(1+t)z\beta'$ is in $\mathcal{O} \otimes K_3(F)_{\mathrm{tf}}$ and $(\eta_5 - 2)(1+t)z\beta' = 5e_{\chi_1}(1+t)\beta'$ because $(1+t)z'\beta'$ lies in $K_3(F)_{\mathrm{tf}}^G = \{0\}$. Hence $\beta = (1 - \tau_\sigma)(1+t)z\varphi_F(\xi)$ in $\mathcal{O} \otimes K_3(F)_{\mathrm{tf}}$ also satisfies $e_{\sigma,\chi_1}\beta = \beta$. For this choice of β we find (5.1) is satisfied for both γ with $e = -2 - \sqrt{5}$, so we may take $\beta_\sigma(\chi_1, S_\infty) = f\beta$ with $f = (4 - 2\sqrt{5})24^2$ in $\mathcal{O}$.

5.1.2. Now let $F = \mathbb{Q}(a)$ denote the totally complex field of discriminant -71^7 that is the splitting field over $\mathbb{Q}$ of the irreducible polynomial $x^{14} - 4x^{12} - x^{11} + 5x^{10} + 6x^9 + 16x^8 + 25x^7 + 16x^6 + 6x^5 + 5x^4 - x^3 - 4x^2 + 1$ in $\mathbb{Q}[x]$, of which a is a root. This field is such that $G := G_{F/\mathbb{Q}} \cong D_7$ is generated by an element s of order 7 with $s(a) = \frac{1}{1139}(1685a^{13} - 343a^{12} - 6831a^{11} - 12a^{10} + 8661a^9 + 7815a^8 + 25608a^7 + 36171a^6 + 17365a^5 + 6516a^4 + 6531a^3 - 4219a^2 - 5336a + 1683)$ and an element t of order 2 with $t(a) = a^{-1}$. The irreducible 2-dimensional characters of $G_{F/\mathbb{Q}}$ take values in $\mathbb{Q}(\eta_7)$, where $\eta_7^3 + \eta_7^2 - 2\eta_7 - 1 = 0$, and form an orbit $\{\chi_1, \chi_2, \chi_3\}$ under the action of $\Gamma := G_{\mathbb{Q}(\eta_7)/\mathbb{Q}}$, where $\chi_j(s) = \zeta_7^j + \zeta_7^{-j}$ for $j \in \{1, 2, 3\}$. MAGMA gives

$$L'(-1, \check{\chi}_1) \doteq -2.6049\ldots, \quad L'(-1, \check{\chi}_2) \doteq -2.1887\ldots, \quad L'(-1, \check{\chi}_3) \doteq -1.5689\ldots.$$

Just as in §5.1.1 we found an element $\xi = \sum_{j \in J} n_j\{x_j\}$ in $\ker(\delta_{2,F})$ (with $|J| = 14$, all x_j exceptional units in $\mathcal{O}_F$, one coefficient n_j equal to -4 and all thirteen others equal to ± 2). We fix σ with $\sigma(a) = 0.450\ldots + 0.163\ldots \cdot i$ and let $\beta = 14 e_{\sigma,\chi_1}\varphi_F(\xi) = \sum_{g \in G} \check{\chi}_1(g)(1 - \tau_\sigma) g \varphi_F(\xi)$ in $\mathcal{O} \otimes K_3(F)_{\mathrm{tf}}$, so that $e_{\sigma,\chi_1}\beta = \beta$. We then find, for all three γ in Γ, that

$$(2\pi i)^{-1}\mathrm{reg}_{2,\sigma}(\beta^\gamma) \doteq \gamma(e) L'(-1, \check{\chi}_1^\gamma) \tag{5.2}$$

for $e = 2\eta_7^2 - \eta_7 + 1$ in $E^\times$, which has norm 49, so that again e does not divide $c^{\chi}_{\sigma,\sigma} w_{-1}(\chi_1) = 2 \cdot 24^2$ in $\mathcal{O}$. To find a more suitable element we write $7e_{\chi_1} = (\eta_7 - 2)z + 2z'$ with z and $z' = \sum_{i=0}^{i=6} s^i$ in $\mathcal{O}[G]$, and one sees as before that $\beta = (1 - \tau_\sigma)(1 + t) z \varphi_F(\xi)$ in $\mathcal{O} \otimes K_3(F)_{\mathrm{tf}}$ is fixed by e_{σ,χ_1}. For this β we find that (5.2) is satisfied for all three γ with $e = -3\eta_7^2 - 3\eta_7 - 1$, which has norm -13. We may then take

$$\beta = \left((-3\eta_7 - 3\eta_7^2) 14 e_{\sigma,\chi_1} + (2 - 3\eta_7^2)(1 - \tau_\sigma)(1 + t) z\right) \varphi_F(\xi)\,,$$

which also satisfies $e_{\sigma,\chi_1}\beta = \beta$, and for which (5.2) holds for all three γ with $e = 1$, and $\beta_\sigma(\chi_1, S_\infty) = 2 \cdot 24^2 \beta$.

5.2. A tetrahedral representation. Let F denote the Galois closure of the field $F' = \mathbb{Q}(\theta)$ with $\theta^4 = 1 - \theta$. Then F has discriminant 283^{12} and $G_{F/\mathbb{Q}}$ is isomorphic to the symmetric group S_4. In fact, since the polynomial $x^4 + x - 1$ has precisely two complex roots, under the natural identification of $G_{F/\mathbb{Q}}$ with S_4 the element τ_σ corresponds to a transposition. For the (rational valued, 3-dimensional) tetrahedral character χ_3 of $G_{F/\mathbb{Q}}$ one therefore has $\chi_3(1) - \chi_3(\tau_\sigma) = 2$. In this case MAGMA gives $L'(-1, \check{\chi}_3) \doteq 0.62475\ldots$. Note that the tetrahedral representation occurs with multiplicity one inside $\mathbb{Q} \otimes K_3(F)$ by Lemma 2.8 and that it has a unique one-dimensional $G_{F/F'}$-invariant subspace V. Because $K_3(F')_{\mathrm{tf}}$ has rank 1 and is of finite index in $K_3(F)_{\mathrm{tf}}^{G_{F/F'}}$, both groups lie in V and e_{χ_3} acts on them as the identity. This applies, in particular, to $\varphi_F(\{\theta\}) = \varphi_{F'}(\{\theta\})$. If σ is in Σ_F then we let $\beta = (1 - \tau_\sigma)\varphi_F(\{\theta\})$ in $K_3(F)_{\mathrm{tf}}$, so that $e_{\sigma,\chi_3}\beta = \beta$ and $\mathrm{reg}_{2,\sigma}(\beta) = 2\mathrm{reg}_{2,\sigma}(\varphi_F(\{\theta\}))$. For any σ with $\sigma(\theta) = 0.248\ldots + 1.033\ldots \cdot i$ we find

$$(2\pi i)^{-1}\mathrm{reg}_{2,\sigma}(\beta) \doteq \frac{1}{2} L'(-1, \check{\chi}_3)\,,$$

so we can take $\beta_\sigma(\chi, S_\infty) = 4 \cdot 24^3 \beta$.

Remark 5.3. In this case there is another rational valued, 3-dimensional character χ_3' of $G_{F/\mathbb{Q}}$ that is obtained by multiplying χ_3 by the alternating character. However, the function $L(s, \check{\chi}_3')$ vanishes to order 2 at $s = -1$ and so Conjecture 1.2 is trivially satisfied. In the spirit of Proposition 3.14, we investigated the value of the second derivative of $L(s, \check{\chi}_3')$ at $s = -1$. One has $L''(-1, \check{\chi}_3') \doteq -10541.7335\ldots$. Further, with F'' denoting the fixed field of F under some element of $G_{F/\mathbb{Q}}$ of order

4, we found two elements ξ_1, ξ_2 of $\ker(\delta_{2,F''})$ (each being a linear combination of about one hundred terms), together with two embeddings σ_1, σ_2 in Σ_F, such that

$$(2\pi i)^{-2}\det\big(\big(\mathrm{reg}_{2,\sigma_a}(\varphi_F(\xi_b)\big)_{1\le a,b\le 2}\big) \doteq \frac{1}{4}L''(-1,\check{\chi}_3').$$

5.3. An example for the standard representation of S_5. Let F denote the Galois closure of the field $F' = \mathbb{Q}(\theta)$ with θ a root of the irreducible polynomial $x^5 - x^3 - 2x^2 + 1$ in $\mathbb{Q}[x]$. Since this polynomial has precisely two complex roots, $G_{F/\mathbb{Q}}$ is isomorphic to the symmetric group S_5 and the element τ_σ for any σ corresponds to a transposition. In particular, the maximal Abelian extension of $\mathbb{Q}$ in F is imaginary quadratic, so $w_2(F) = w_2(\mathbb{Q}) = 24$ by Remark 2.11. The (rational valued, 4-dimensional) character χ_4 of $G_{F/\mathbb{Q}}$ of the standard representation of S_5, acting on $\{(x_1,\dots,x_5)$ in $\mathbb{C}^5$ with $x_1+\cdots+x_5 = 0\}$ by permuting the coordinates, satisfies $\chi_4(1) - \chi_4(\tau_\sigma) = 2$. In this case MAGMA gives $L'(-1,\check{\chi}_4) \doteq -1.9653\dots$. Note that the corresponding representation occurs with multiplicity one inside $\mathbb{Q}\otimes K_3(F)$ by Lemma 2.8 and that it has a unique one-dimensional $G_{F/F'}$-invariant subspace V, i.e., its intersection with $K_3(F')_{\mathrm{tf}}$ is an Abelian group of rank 1. Because $K_3(F')_{\mathrm{tf}}$ has rank 1 and is of finite index in $K_3(F)_{\mathrm{tf}}^{G_{F/F'}}$, both groups lie in V and e_{χ_4} acts on them as the identity. This applies, in particular, to $\varphi_F(\xi) = \varphi_{F'}(\xi)$ with $\xi = \{\theta^2\} + 2\{\theta^3\} - 2\{\theta - \theta^4\}$ in $\ker(\delta_{2,F'})$. If σ is in Σ_F then we let $\beta = (1-\tau_\sigma)\varphi_F(\xi)$ in $K_3(F)_{\mathrm{tf}}$, so that $e_{\sigma,\chi_4}\beta = \beta$ and $\mathrm{reg}_{2,\sigma}(\beta) = 2\mathrm{reg}_{2,\sigma}(\varphi_F(\xi))$. For any σ with $\sigma(\theta) = -0.656\ldots + 0.982\ldots\cdot i$ we find

$$(2\pi i)^{-1}\mathrm{reg}_{2,\sigma}(\beta) \doteq -\frac{1}{4}L'(-1,\check{\chi}_3),$$

so we can take $\beta_\sigma(\chi, S_\infty) = 8\cdot 24^4\beta$.

5.4. The Tate-Buhler-Chinburg representations. As a final example we considered one of the fields F of degree 48 over $\mathbb{Q}$ studied by Tate and Buhler [7] and Chinburg [13, §III.A.]. This field has discriminant $7^{24}19^{32}$ and $G := G_{F/\mathbb{Q}}$ is isomorphic to the amalgamated product $(SL_2(\mathbb{F}_3)\times\mathbb{Z}/4\mathbb{Z})/\langle((\begin{smallmatrix}\overline{2}&\overline{0}\\\overline{0}&\overline{2}\end{smallmatrix}),\overline{2})\rangle$, denoted $SL_2(\mathbb{F}_3) * \mathbb{Z}/4\mathbb{Z}$. This group has precisely six irreducible two-dimensional characters: $\chi_1, \check{\chi}_1$ (corresponding to the representations denoted σ and $\bar{\sigma}$ in [13], which each have character field $\mathbb{Q}(i)$) and $\chi_2, \check{\chi}_2, \chi_2', \check{\chi}_2'$ (corresponding to the representations $\rho, \bar{\rho}, \rho'$ and $\bar{\rho}'$, which each have character field $\mathbb{Q}(\zeta_{12})$). Since for any σ in Σ_F the element τ_σ is a non-central element of order 2, hence belongs to the conjugacy class A_0A_3 in the character table [13, Table I], we see that $\chi(1) - \chi(\tau_\sigma) = 2$ for every such character χ.

We now describe F and our identification of G with $SL_2(\mathbb{F}_3) * \mathbb{Z}/4\mathbb{Z}$. Note that $PGL_2(\mathbb{F}_3) \simeq S_4$ by means of its action on $\mathbb{P}^1_{\mathbb{F}_3}$, which also gives $PSL_2(\mathbb{F}_3) \simeq A_4$. As $SL_2(\mathbb{F}_3)$ has a unique element of order 2 one sees easily that it is generated by any two non-commuting elements λ_1 and λ_2 of order 3. The disjoint conjugacy classes of $(\begin{smallmatrix}\overline{1}&\overline{1}\\\overline{0}&\overline{1}\end{smallmatrix})$ and its inverse contain all eight elements of order 3, so we can take the λ_i in the same conjugacy class. Then $SL_2(\mathbb{F}_3) * \mathbb{Z}/4\mathbb{Z}$ is generated by $(\lambda_1, \overline{0})$, $(\lambda_2, \overline{0})$, and either element of order 4 in its centre, and has at most 48 automorphisms. In fact all 48 automorphisms can be obtained by letting $PGL_2(\mathbb{F}_3)$ act on $SL_2(\mathbb{F}_3)$ by conjugation and $\{\pm 1\}$ act on $\mathbb{Z}/4\mathbb{Z}$ by multiplication. We may therefore identify G with $SL_2(\mathbb{F}_3) * \mathbb{Z}/4\mathbb{Z}$ by specifying two distinct conjugate elements g_1, g_2 of order 3 as well as a central element h of order 4, and letting them correspond to $((\begin{smallmatrix}\overline{1}&\overline{1}\\\overline{0}&\overline{1}\end{smallmatrix}),\overline{0})$, $((\begin{smallmatrix}\overline{0}&\overline{1}\\\overline{2}&\overline{2}\end{smallmatrix}),\overline{0})$ and $((\begin{smallmatrix}\overline{1}&\overline{0}\\\overline{0}&\overline{1}\end{smallmatrix}),\overline{1})$ respectively.

We note F is the splitting field of the irreducible polynomial $f(x) = x^{16} - x^{15} + 4x^{14} + x^{13} + 2x^{12} + 2x^{11} - 9x^{10} + 3x^9 + 19x^8 + 23x^7 + 13x^6 + x^5 + 2x^4 + 3x^3 + 3x^2 + 3x + 1$ in $\mathbb{Q}[x]$, so $F = \mathbb{Q}(a)(b)$ where $f(a) = 0$ and b is a root of the factor

$$g(x) = x^3 - \left(\tfrac{21430423}{9679918}a^{15} + \cdots + \tfrac{27880963}{4839959}\right)x^2 + \cdots + \left(\tfrac{10585549}{4839959}a^{15} + \cdots + \tfrac{43460997}{9679918}\right)$$

of $f(x)$ in $\mathbb{Q}(a)[x]$. Let σ be the embedding of F with $\sigma(a) = 1.254\ldots + 0.583\ldots \cdot i$ and $\sigma(b) = 0.849\ldots - 1.939\ldots \cdot i$, one of the roots of $g^\sigma(x) = x^3 - (0.222\ldots - 2.544\ldots \cdot i)x^2 - (1.658\ldots - 1.108\ldots \cdot i)x - (0.827\cdots + 0.252\ldots \cdot i)$. We take distinct conjugate elements g_1, g_2 of order three with $g_1(a) = a$, $\sigma(g_1(b)) = -0.612\ldots + 0.058\ldots \cdot i$, $\sigma(g_2(a)) = -0.612\ldots - 0.058\ldots \cdot i$, $\sigma(g_2(b)) = 1.254\ldots - 0.583\ldots \cdot i$, and h the element of order four in the centre given by $\sigma(h(a)) = 0.586\ldots - 0.409\ldots \cdot i$ and $\sigma(h(b)) = -0.612\ldots - 0.058\ldots \cdot i$. The field F is computationally difficult but fortunately we were able to find elements in $K_3(F)_{\mathrm{tf}}$ by searching in $\ker(\delta_{2,F'}) \subset \ker(\delta_{2,F})$ for $F' = \mathbb{Q}(c)$ where $\sigma(c) = 1.472\ldots + 0.900\ldots \cdot i$. (This field F' is one of six conjugate subfields of degree 24 over $\mathbb{Q}$, the only other subfield of this degree being the fixed field of the normal subgroup $\{1, h^2\}$.)

The characters χ_1 and $\check{\chi}_1$ form an orbit under the action of $G_{\mathbb{Q}(i)/\mathbb{Q}}$. MAGMA gives $L'(-1, \check{\chi}_1) \doteq -64.577\ldots + 631.991\ldots \cdot i$, and its complex conjugate for $L'(-1, \chi_1)$. We found that, for both γ in $G_{\mathbb{Q}(i)/\mathbb{Q}}$,

$$(2\pi i)^{-1}\mathrm{reg}_\sigma(\beta^\gamma) \doteq 12L'(-1, \check{\chi}_1^\gamma)$$

with $\beta = \sum_{g\in G} \check{\chi}_1(g)(1 - \tau_\sigma)g\varphi_F(\xi)$ for some ξ in $\ker(\delta_{2,F'})$, so that $e_{\sigma,\chi_1}\beta = \beta$. We may therefore take $\beta_\sigma(\chi_1, S_\infty) = f\beta$ with $f = 96$ in $\mathcal{O} = \mathbb{Z}[i]$.

The remaining characters $\chi_2, \check{\chi}_2, \chi_2'$ and $\check{\chi}_2'$ form an orbit under the action of $G_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}$. The corresponding L-values are $L'(-1, \check{\chi}_2) \doteq -2.5823\ldots + 4.4538\ldots \cdot i$ and $L'(-1, \check{\chi}_2') \doteq -3.1252\ldots + 4.8866\ldots \cdot i$, as well as their complex conjugates $L'(-1, \chi_2)$ and $L'(-1, \chi_2')$. In this case we found that, for all γ in $G_{\mathbb{Q}(\zeta_{12})/\mathbb{Q}}$,

$$(2\pi i)^{-1}\mathrm{reg}_\sigma(\beta^\gamma) \doteq 12L'(-1, \check{\chi}_2^\gamma)$$

with $\beta = \sum_{g\in G} \check{\chi}_2(g)(1 - \tau_\sigma)g\varphi_F(\xi)$ for the same ξ in $\ker(\delta_{2,F'})$, so that $e_{\sigma,\chi_2}\beta = \beta$. We may therefore take $\beta_\sigma(\chi_2, S_\infty) = f\beta$ with $f = 96$ in $\mathcal{O} = \mathbb{Z}[\zeta_{12}]$.

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