Non-gravitating waves

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Abstract: It is pointed out that scalar-tensor theories of gravity admit solutions in which the metric is Minkowskian although the scalar and matter fields do not vanish. Explicit pp-wave solutions of the Brans-Dicke-Maxwell theory are presented. These include solutions with metrics that are flat or Ricci flat even though the Maxwell and scalar fields are non-zero.

In general relativity with vanishing cosmological constant, and with the classical matter fields minimally coupled in the standard way so there are no phantom fields, the total energy-momentum tensor vanishes if and only if all the matter fields are trivial. That is the matter fields are zero or, as in the case of zero rest mass scalar fields, constant. In addition the energy-momentum tensor of each individual classical field, such as a perfect fluid, scalar or electromagnetic field, vanishes if and only if the field is trivial. Einstein's vacuum field equations imply the absence of non-trivial matter fields. Furthermore the Minkowski metric and non-trivial solutions of the Minkowski space-time matter field equations do not together satisfy the Einstein field equations. All this is well-known and well understood. However, when the cosmological constant is non-zero or there are phantom fields the situation may be different - in obvious ways. More interestingly, when the matter coupling is non-minimal, and in gravitational theories other than general relativity, the vanishing of the Einstein tensor does not necessarily imply that the (non-metrical) matter fields are trivial. Recently there have been a number of studies of fields that are non-zero even though their energy-momentum tensors are zero [1, 2, 3]. In particular a model, with a scalar field non-minimally coupled to gravity, has been constructed whose field equations admit solutions where the metric is flat even though the scalar field is non-trivial [3]. Such fields, which together with a flat metric are solutions of gravity coupled equations of motion, may be termed non metrically gravitating, or more concisely non-gravitating. The aim of this letter is to exhibit some further examples of gravitational wave systems that admit non-gravitating fields, and matter fields which are non-vanishing even though the Einstein tensor is zero. Such systems could not be distinguished from flat space-time or Einstein vacuum space-times by curvature effects alone, even though their energy content may be quite different.

Members of a broad class of scalar-tensor theories of gravity admit the possibility of non-gravitating fields. Such theories are of interest in cosmological investigations and in the analysis of observational tests and alternatives to general relativity. They include models arising as low-energy limits of higher dimensional theories such as string and Kaluza-Klein theories. This class of theories includes those describable by Lagrangian densities of the general form (for further discussions see, e.g. [4, 5])

$$L = \sqrt{-g}[\frac{A(\Phi)}{16\pi}R - \frac{1}{2}B(\Phi)(\nabla\Phi)^2 - V(\Phi)] + L_M[e^{2\alpha(\Phi)}g_{\mu\nu}, \psi_m], \qquad (1)$$

where R is the Ricci scalar of the metric $g_{\mu\nu}$ and A, B, α and the potential V are functions of the scalar field Φ. The term L_M depends on the matter fields ψ_m and the metric $e^{2\alpha(\Phi)}g_{\mu\nu}$ which determines observable quantities such as the geodesic trajectories of freely falling test particles, proper time etc. In this letter it will suffice to focus detailed attention on the type of four-dimensional systems considered originally by Fierz, Jordan, Brans and Dicke, [6, 7, 8], since it is easy to see that similar conclusions can be drawn about solutions of more general scalar-tensor theories, such as those determined by the Lagrangians given in Eq.(1), in both four and higher dimensions.

The Lagrangian density for the Brans-Dicke-matter field equations corresponding to the choices $A = \Phi$, $B = \frac{\omega}{8\pi\Phi}$, $V = 0$ and $\alpha = 0$, is given by

$$L = \sqrt{-g}[\frac{\Phi}{16\pi}R - \frac{\omega}{16\pi\Phi}(\nabla\Phi)^2 + L_M] \tag{2}$$

where Φ is the Brans-Dicke scalar field, ω is the Brans-Dicke parameter, $\sqrt{-g}L_M(g_{\mu\nu}, \psi_m)$ is the Lagrangian density for matter fields ψ_m. The conventions of [9] are followed. The Euler-Lagrange equations obtained by varying the metric and the scalar field are

$$G_{\alpha\beta} = \frac{8\pi}{\Phi}T_{\alpha\beta} + \frac{\omega}{\Phi^2}[\nabla_\alpha\Phi\nabla_\beta\Phi - \frac{1}{2}g_{\alpha\beta}\nabla_\rho\Phi\nabla^\rho\Phi] + \frac{1}{\Phi}(\nabla_\alpha\nabla_\beta\Phi - g_{\alpha\beta}\Box\Phi), \tag{3}$$

$$\Box\Phi = \frac{8\pi}{(2\omega+3)}T, \tag{4}$$

where $T = g^{\alpha\beta}T_{\alpha\beta}$. In this letter the only additional field considered will be a source-free Maxwell field, $F_{\alpha\beta}$, so that $L_M = -\frac{1}{16\pi}F_{\alpha\beta}F_{\gamma\delta}g^{\alpha\gamma}g^{\beta\delta}$ and the remaining Euler-Lagrange equations are the Maxwell equations

$$\nabla_\alpha F^{\alpha\beta} = 0,\ \nabla_{[\gamma}F_{\alpha\beta]} = 0, \tag{5}$$

and the (Maxwell) energy-momentum tensor is

$$T_{\alpha\beta} = \frac{1}{4\pi}(F_{\alpha\rho}F^\rho_\beta - \frac{1}{4}g_{\alpha\beta}F_{\rho\sigma}F^{\rho\sigma}). \tag{6}$$

Consider now the sub-class of Kerr-Schild metrics given by the 'plane-fronted waves with parallel rays' form [10]

$$ds^2 = -2dudv + (dx^i)^2 + 2f du^2, \tag{7}$$

where f is a function of u and x^i only. Let $l_\alpha = u,_\alpha$, and $s^i_\alpha = x^i,_\alpha$, so that $g_{\alpha\beta} = \eta_{\alpha\beta} + 2fl_\alpha l_\beta$., and l_α is null with respect to both $g_{\alpha\beta}$ and the flat metric $\eta_{\alpha\beta}$. It is a straightforward matter to see that the covariant derivative of l_α satisfies the usual pp-wave condition

$$\nabla_\alpha l_\beta = 0, \tag{8}$$

and the Einstein tensor is

$$G_{\alpha\beta} = (-f,_{ij}\,\delta^{ij})l_\alpha l_\beta, \tag{9}$$

where the comma denotes partial differentiation.

Here the class of Maxwell fields considered will be given by

$$F_{\alpha\beta} = F,_i\,(l_\alpha s^i_\beta - l_\beta s^i_\alpha). \tag{10}$$

These are solutions of the source-free Maxwell equations when the function F satisfies the equations

$$F,_v = 0 \text{ and } F,_{ij}\,\delta^{ij} = 0, \tag{11}$$

and the corresponding Maxwell energy-momentum tensor reduces to

$$T_{\alpha\beta} = \frac{1}{4\pi} l_\alpha l_\beta F,_i\,F,_j\,\delta^{ij}. \tag{12}$$

It will be assumed that the scalar field Φ is a function of u only so that

$$\Box\Phi = 0 \tag{13}$$

$$\text{and } g^{\alpha\beta}\Phi,_\alpha\,\Phi,_\beta = 0, \nabla_\alpha\nabla_\beta\Phi = (\Phi,_u\,,_u)l_\alpha l_\beta.$$

It now follows that the scalar fields $\Phi(u)$, and metrics $g_{\alpha\beta}$ and Maxwell fields $F_{\alpha\beta}$ determined by Eqs. (7) and (10), satisfy all the Brans-Dicke-Maxwell field equations when Eqs.(11) and (3) are satisfied. The latter equation reduces to

$$\Phi'' + \frac{\omega(\Phi')^2}{\Phi} + 2F,_i\,F,_j\,\delta^{ij} + \Phi f,_{ij}\,\delta^{ij} = 0. \tag{14}$$

where the prime denotes differentiation with respect to u. Eqs.(11) and (14) are an underdetermined set of equations for the functions $\Phi(u)$, $F(u, x^i)$ and $f(u, x^i)$, and solutions are easy to find.

In the four dimensional case it is straightforward to see that the general solution is given, for non-zero $\Phi(u)$, by

$$
\begin{aligned}
f &= f_1 + \overline{f_1} - \frac{z\overline{z}}{4}\left(\frac{\Phi''}{\Phi} + \frac{\omega(\Phi')^2}{\Phi^2}\right) - \frac{2F_1\overline{F_1}}{\Phi}, \\
F &= F_1 + \overline{F_1}.
\end{aligned}
\tag{15}
$$

Here the bar denotes complex conjugation and f_1 and F_1 are each arbitrary complex functions of two variables u and z, where $z = x^1 + ix^2$.

The following general conclusions can be drawn from these solutions. If the metric $g_{\alpha\beta}$ is a pp-wave solution of Einstein's vacuum equations so that

$$
f_{,ij}\,\delta^{ij} = 0, \tag{16}
$$

then $g_{\alpha\beta}$ is also a solution of the Brans-Dicke-Maxwell field equations, with Maxwell field determined by

$$
F = zF_2 + \overline{z}\overline{F_2}. \tag{17}
$$

Here F_2 is a complex function of u only which, together with appropriate Φ, satisfies

$$
|F_2|^2 = -\frac{1}{8}\left(\Phi'' + \frac{\omega(\Phi')^2}{\Phi}\right). \tag{18}
$$

In particular if the metric $g_{\alpha\beta}$ is flat, with $f = 0$ say, then these equations determine non-gravitating solutions of the Brans-Dicke-Maxwell equations.

In the case where F_2 is zero, so that the Maxwell field vanishes, the solution of Eq. (18) is given by

$$
\Phi_0 = \exp(c_1 u + c_2), \tag{19}
$$

when $\omega = -1$ and

$$
\Phi_0 = (c_1 u + c_2)^{\frac{1}{\omega+1}}, \tag{20}
$$

when $\omega \neq -1$.

Hence a pp-wave metric, $g_{\alpha\beta}$, is a solution of the Einstein vacuum field equations if and only the pair $(g_{\alpha\beta}, \Phi_0)$ is a solution of the Brans-Dicke field equations. In particular when $g_{\alpha\beta}$ is the Minkowski metric the pairs $(\eta_{\alpha\beta}, \Phi_0)$ are non-gravitating Brans-Dicke solutions.

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