Holomorphic 4-metrics and Lorentzian structures

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Abstract: Methods of constructing Lorentzian metrics on real four dimensional manifolds from complex and holomorphic 4-metrics are presented. In particular half-flat holomorphic 4-metrics are used to construct real Lorentzian 4-metrics. Holomorphic and real solutions of Einstein's equations, and relations between them, are discussed.

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1 Introduction

The aim of this paper is to present some methods of constructing Lorentzian 4-metrics from holomorphic 4-metrics and to discuss the construction of certain real solutions of Einstein's equations from holomorphic 4-metrics. The early work, by Newman, Penrose and Plebänski on holomorphic half-flat 4-metrics, [1-3], was followed quickly by many developments, [4-6], including significant results about real 4-metrics of Riemannian and neutral (Kleinian or ultrahyperbolic) signatures, [7-9]. However results about real metrics of Lorentzian signature, satisfying Einstein's equations, have been more limited. The twistorial approach towards this latter problem is reviewed in references [10, 11] and examples of the work of Newman and his collaborators are contained in references [12] and [13]. Isolated results on combining self-dual and anti self-dual solutions to obtain Ricci flat and real metrics by using Plebański's formalism are contained in references [14-18]. More references to these these various lines of research can be found in a recent review, [19]. In this paper the general formalism used by Plebański and his collaborators is employed to obtain further results on constructing real solutions of Einstein's equations from holomorphic metrics. Real 4-geometries are constructed from the non-linear superposition of holomorphic geometries and their complex conjugates. Earlier work, on combining self-dual and anti self-dual half flat holomorphic metrics, [14], to form real and complex solutions of Einstein's vacuum field equations, is extended to include solutions with pure radiation energy-momentum tensors. Real $p-$forms, which are naturally defined by holomorphic half-flat metrics, are used to construct real Lorentzian 4-metrics on real 4-manifolds.

The structure of the paper is as follows. In the second section the basic formalism is introduced. This includes coordinate and frame presentations of metrics, and the Cartan structure equations, on a holomorphic 4-manifold M. The two component spinor formalism is also introduced. In the third section certain one (complex) parameter families of holomorphic 4- metrics on M are discussed. When the parameters are unity the metrics in the families are half-flat. Examples of these metrics are given which correspond to solutions of the holomorphic Einstein equations with pure radiation energy-momentum tensors. They are generalizations of known half-flat solutions, see e.g. [14]. The formalism used here is adapted from one introduced by Plebański in his discussion of half-flat metrics, [3]. The fourth section of the paper deals with methods of combining, locally on M, holomorphic 4-metrics from section three, and deriving from such superpositions real Lorentzian metrics on a real four dimensional sub-manifold, N, of M. Classes of vacuum solutions and solutions with pure radiation energy-momentum tensors are constructed; in particular real Lorentzian 4-metrics in the Kundt class of algebraically special metrics are reobtained within this context. These results generalize previous calculations contained in reference [14]. It is also shown in this section that the 8-metric, corresponding to the real part of the holomorphic 4-metrics of this section, can be pulled back to a Lorentzian metric on a real four sub-manifold, N, of the complex 4-manifold. Although the one parameter family that includes anti-self dual metrics is used here, it is clear that the family that includes self-dual metrics could be employed in a similar manner. The fifth section of the paper contains an investigation of real $p-$forms on M, where $2\leq p \leq 8$, which are constructed from the co-frame for an anti self-dual metric. Again, the latter are used for the sake of definiteness, frames for self-dual metrics could equally well be used. The method of construction of these forms ensures that they have vanishing exterior covariant derivative with respect to a so(1,3)-valued connection. This connection is constructed by adding the anti-self dual part of the connection of the anti-self dual metric to its complex conjugate. The real $p-$forms, which arise naturally, satisfy equations on M which are formally analogous to the equations satisfied by the $1-$forms of a Cartan frame for a real Ricci-flat Lorentzian 4-metric. By considering the pull-backs of these equations and differential forms to a real four dimensional submanifold, N, of M, real geometrical structures are constructed on N. Only the case where $p = 3$ is investigated in detail in this section although a brief discussion of the $p = 2$ case is given in an appendix. When $p = 3$, a class of real Lorentzian 4-metrics which encode, in part at least, the properties of the holomorphic anti-self dual metric can be calculated relatively simply. Lorentzian metrics are constructed from the $3-$forms for broad classes of embeddings of real 4-dimensional submanifolds, in contrast to the results of previous sections. All these metrics linearize to real solutions of the linearized vacuum Einstein equations. However, the aim of dealing with Einstein's equations for Lorentzian 4-metrics, by using this construction is incompletely realized. To achieve this geometrical conditions on the embedding maps which ensure that the Lorentzian 4-metrics are solutions of Einstein's equations would have to be formulated. Nevertheless, this approach to deriving Lorentzian 4-metrics extends both the other approaches and the "real slices" approach [20, 21], to constructing Lorentzian 4-metrics from holomorphic metrics.

The emphasis in much of this paper is placed on holomorphic metrics, [34, 22-28]. These are particularly interesting because holomorphic half-flat metrics can be obtained by using Penrose's non-linear graviton twistor construction. However it should be noted that essential features of the method of construction of Lorentzian 4-metrics given in sections four and five do not necessarily require analyticity and holomorphic metrics. For instance, one can take Plebański's approach to half-flat metrics, or a generalization, and carry it out on a real 4-manifold N. Plebański's equation, or its generalization, is viewed, in this particular context, as a non-linear extension of the complex scalar wave equation in Minkowski space-time. Its solutions are then viewed as non-linear versions of Penrose's complex Hertz potentials for spin two zero rest-mass fields in Minkowski space-time. (All solutions of the vacuum Einstein equations, linearized about the Minkowski solution, can be generated by such potentials [33].) Now, the "metrics", half-flat or otherwise, constructed in this way are not real metrics on N but formally similar complex objects determined by complex, but not necessarily analytic, solutions of the complex field equation. When the constructions of the type discussed in sections four and five are carried out within this context the real Lorentzian 4-metrics obtained can be regarded as having been constructed by "superposing" the complex "metrics" and their complex conjugates. Hence, Lorentzian metrics obtained in this way, which also satisfy the vacuum equations, can be regarded as having being constructed by an extension of the method of constructing solutions of the linearized Einstein vacuum field equations from complex Hertz potentials and their complex conjugates.

In this paper all ordinary lower case Latin indices a, b, c, i, j, k and bold lower case Latin indices **a**, **b**, **c** range and sum from 1 to 4; bold lower case Latin indices, **i, j**, **k** from 1 to 3; upper-case Latin indices from 0 to 1 and Greek indices from 1 to 8. Complex conjugates (c.c.) are denoted with a bar. Geometrical considerations are essentially local.

2 Holomorphic 4-metrics

Let M be a complex manifold with $\dim_{\mathbb{C}}$M equal to four, and let g be a holomorphic metric on M, with line element given in complex coordinates z^i

by

$$ds^2 = g_{ij} dz^i \otimes dz^j, \tag{1}$$

with $\partial g_{ij}/\partial \bar{z}^k = 0$, (see also [34], and references [22-28, 30] on complex, holomorphic and hyper-Kahler structures).

It is convenient here to present this metric geometry in terms of the holomorphic Cartan structure equations, using conventions which will be used later when real Lorentzian metrics are considered. These conventions are naturally adapted to two component spinor and anti self-dual formulations, [32].

Let χ^a be a basis of holomorphic 1-forms, a Cartan co-frame for g, with dual basis of vector fields e_a, so that the line element for g is given by

$$ds^2 = \eta_{ab} \chi^a \otimes \chi^b \tag{2}$$

where

$$\eta_{ab} = \begin{bmatrix} 0 & \epsilon_{AB} \\ -\epsilon_{AB} & 0 \end{bmatrix}, \text{ and}$$
$$\epsilon_{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3}$$

The complex volume element is given by

$$V = i\chi^1 \wedge \chi^2 \wedge \chi^3 \wedge \chi^4. \tag{4}$$

The real and imaginary parts, $\mathbf{h}$ and $\mathbf{k}$, of the holomorphic metric $g = \mathbf{h} + i\mathbf{k}$ are each real 8-metrics on the real eight dimensional manifold M. In terms of the complex co-frame of eight 1-forms $\chi^\alpha = (\chi^a, \bar{\chi}^a)$ on M (with dual basis of vector fields $e_\alpha = (e_a, \bar{e}_a)$), the line element of $\mathbf{h}$ is given by

$${}_{\mathbf{h}}ds^2 = h_{\alpha\beta} \chi^\alpha \otimes \chi^\beta \equiv \frac{1}{2} \eta_{ab} (\chi^a \otimes \chi^b + \bar{\chi}^a \otimes \bar{\chi}^b) \tag{5}$$

and the line element of $\mathbf{k}$ is given by

$${}_{\mathbf{k}}ds^2 = k_{\alpha\beta} \chi^\alpha \otimes \chi^\beta \equiv \frac{i}{2} \eta_{ab} (-\chi^a \otimes \chi^b + \bar{\chi}^a \otimes \bar{\chi}^b). \tag{6}$$

The two real metrics have Kleinian (neutral or ultrahyperbolic) signatures (4,4).

The holomorphic Cartan structure equations are given by

$$\begin{aligned} d\chi^a - \chi^b \wedge \Gamma^a_b &= 0, \\ \Gamma_{ab} + \Gamma_{ba} &= 0, \\ d\Gamma^a_b + \Gamma^a_c \wedge \Gamma^c_b &= -\frac{1}{2} F^a_{bcd} \chi^c \wedge \chi^d, \end{aligned} \tag{7}$$

where Γ^a_b denotes the holomorphic Levi-Civita connection 1-form (with covariant derivative ∇), and F^a_{bcd} are the components of its curvature 2-form F^a_b. The structure group is SO(4,C) and the connection and curvature forms, which take values in the Lies algebra so(4,C), can be written as the sum of their self-dual and anti-self-dual parts, ${}^+\Gamma^a_b, {}^-\Gamma^a_b, {}^+F^a_b, {}^-F^a_b$ respectively. Here, $*{}^+F^a_b = i{}^+F^a_b$, $*{}^-F^a_b = -i{}^-F^a_b$. In 4×4 matrix form

$${}^+\Gamma^a_b = \begin{bmatrix} \varpi^{0'}_{0'}1 & \varpi^{0'}_{1'}1 \\ \varpi^{1'}_{0'}1 & -\varpi^{0'}_{0'}1 \end{bmatrix} \tag{8}$$

where here 1 is the unit 2×2 matrix and $\varpi^{0'}_{0'}, \varpi^{0'}_{1'}, \varpi^{1'}_{0'}$ denote the independent components of ${}^+\Gamma^a_b$. Similarly,

$${}^-\Gamma^a_b = \begin{bmatrix} \omega^{\mathbf{A}}_{\mathbf{B}} & 0 \\ 0 & \omega^{\mathbf{A}}_{\mathbf{B}} \end{bmatrix} \tag{9}$$

where the trace of the 2×2 matrix $\left(\omega^{\mathbf{A}}_{\mathbf{B}}\right)$ is zero. Other self-dual and anti self-dual objects can be written similarly, for instance,

$${}^-F^a_b = \begin{bmatrix} \Omega^{\mathbf{A}}_{\mathbf{B}} & 0 \\ 0 & \Omega^{\mathbf{A}}_{\mathbf{B}} \end{bmatrix} \tag{10}$$

where

$$\Omega^{\mathbf{A}}_{\mathbf{B}} = d\omega^{\mathbf{A}}_{\mathbf{B}} + \omega^{\mathbf{A}}_{\mathbf{C}} \wedge \omega^{\mathbf{C}}_{\mathbf{B}}. \tag{11}$$

The structure group SO(4,C) is isomorphic to $\{\mathrm{SL(2,C)}_L \times \mathrm{SL(2,C)}_R\}/\mathbb{Z}_2$. The self-dual connection and curvature take values in the Lie algebra $\mathrm{sl(2,C)}_R$ and the anti-self dual connection and curvature take values in the Lie algebra $\mathrm{sl(2,C)}_L$.

It will also be convenient subsequently to use a two-component spinor approach to the geometry, [32]. This can be summarized as follows, using notation which is compatible with the above, [29].

In a conventional two component spinor formulation the line element, given by equation (2), can be written

$$ds^2 = \epsilon_{\mathbf{AB}}\epsilon_{A'B'}\chi^{\mathbf{A}A'} \otimes \chi^{\mathbf{B}B'}, \tag{12}$$

where the basis of holomorphic co-frames is represented by a 2×2 matrix $\chi^{\mathbf{A}A'}$,

$$\chi^{\mathbf{A}A'} = \begin{bmatrix} \chi^1 & \chi^3 \\ \chi^2 & \chi^4 \end{bmatrix}. \tag{13}$$

The Cartan structure equations (7) take the form

$$d\chi^{\mathbf{A}A'} - \chi^{\mathbf{A}B'} \wedge \omega^{\mathbf{A}}_{\mathbf{B}} - \chi^{\mathbf{B}A'} \wedge \varpi^{A'}_{B'} = 0,$$

$$\begin{aligned} \Omega^{\mathbf{A}}_{\mathbf{B}} &\equiv d\omega^{\mathbf{A}}_{\mathbf{B}} + \omega^{\mathbf{A}}_{\mathbf{C}} \wedge \omega^{\mathbf{C}}_{\mathbf{B}}, \\ \tilde{\Omega}^{A'}_{B'} &\equiv d\varpi^{A'}_{B'} + \varpi^{A'}_{C'} \wedge \varpi^{C'}_{B'}. \end{aligned} \tag{14}$$

Here, the anti self-dual and self-dual components of the Levi-Civita spin connection are given, respectively, by $\omega^{\mathbf{A}}_{\mathbf{B}}$ and $\varpi^{A'}_{B'}$, in agreement with equations (8) and (9). Bold upper case Latin indices and primed ordinary upper case Latin indices represent, respectively, transformation properties under $SL(2,C)_L$ and $SL(2,C)_R$ the anti self-dual and self-dual subgroups of the structure group $SO(4,C) = \{SL(2,C)_L \times SL(2,C)_R\}/\mathbb{Z}_2$. The components of the curvature 2-forms are given by

$$\begin{aligned} \Omega^{\mathbf{A}}_{\mathbf{B}} &= \Psi^{\mathbf{A}}_{\mathbf{BCD}}\Sigma^{\mathbf{CD}} + 2\Lambda\Sigma^{\mathbf{A}}_{\mathbf{B}} + \Phi^{\mathbf{A}}_{\mathbf{B}C'D'}\Sigma^{C'D'}, \\ \tilde{\Omega}^{A'}_{B'} &= \tilde{\Psi}^{A'}_{B'C'D'}\Sigma^{C'D'} + 2\Lambda\Sigma^{A'}_{B'} + \Phi_{\mathbf{CD}}{}^{A'}_{B'}\Sigma^{\mathbf{CD}}, \end{aligned} \tag{15}$$

where $\Sigma^{\mathbf{A}}_{\mathbf{B}} = 1/2\chi^{\mathbf{A}}_{A'} \wedge \chi^{A'}_{\mathbf{B}}$ and $\Sigma^{A'}_{B'} = 1/2\chi^{A'}_{\mathbf{A}} \wedge \chi^{\mathbf{A}}_{B'}$. The the anti self-dual and self-dual components of the Weyl spinor are given, respectively, by the totally symmetric spinors $\Psi_{\mathbf{ABCD}}$ and $\tilde{\Psi}_{A'B'C'D'}$ and $-2\Phi^{\mathbf{A}}_{\mathbf{B}C'D'}$ and 24Λ correspond respectively to the trace free part of the Ricci tensor and the Ricci scalar.

In the half-flat case, for example here chosen to be the anti-self dual or right flat case,

$$\begin{aligned} \Omega^{\mathbf{A}}_{\mathbf{B}} &= \Psi^{\mathbf{A}}_{\mathbf{BCD}}\Sigma^{\mathbf{CD}}, \\ \tilde{\Omega}^{A'}_{B'} &= 0. \end{aligned} \tag{16}$$

A half-flat metric is automatically Ricci flat, that is

$$\Omega^{\mathbf{A}}_{\mathbf{B}} \wedge \chi^{\mathbf{B}B'} = 0. \tag{17}$$

3 One (complex) parameter family of holomorphic 4- metrics including half-flat metrics

In this section one parameter families of holomorphic metrics which reduce, when the parameter is equal to one, to half-flat, anti self-dual 4-metrics (respectively self-dual metrics) on M will be considered. The coordinates used are such that the volume element, given in equation (4), is equal to $idz^1 \wedge dz^2 \wedge dz^3 \wedge dz^4$.

Using Plebański's "second" type of local coordinate description, the line element of the first one- complex parameter (p) family of metrics to be considered here is

$$\begin{aligned} ds^2 = dz^1 \otimes dz^4 + dz^4 \otimes dz^1 - dz^2 \otimes dz^3 - dz^3 \otimes dz^2 - 2\beta_{,33}\, dz^1 \otimes dz^1 \\ - 2\beta_{,34}\,(dz^1 \otimes dz^2 + dz^2 \otimes dz^1) - 2\beta_{,44}\, dz^2 \otimes dz^2, \end{aligned} \tag{18}$$

where the holomorphic function β satisfies a generalization of Plebański's second equation

$$\beta_{,14} - \beta_{,23} + p(\beta_{,33}\,\beta_{,44} - (\beta_{34})^2) = 0. \tag{19}$$

When $p = 1$ equations (18) and (19) define half-flat anti self-dual metrics and equation (19) reduces to Plebański's equation. A co-frame for the metric, relative to which the self dual part of the Levi-Civita connection is zero when $p = 1$, is given by

$$\begin{aligned} &\chi^1 = dz^1,\ \chi^2 = dz^2,\ \chi^3 = dz^3 + \beta_{,34}\, dz^1 + \beta_{,44}\, dz^2, \\ &\chi^4 = dz^4 - \beta_{,33}\, dz^1 - \beta_{,34}\, dz^2, \end{aligned} \tag{20}$$

The representation of these holomorphic metrics takes the following spinorial form. The metric given by equation (18) is

$$ds^2 = (\epsilon_{AB}\epsilon_{A'B'} - 2\iota_{A'}\iota_{B'}\beta_{AB})dz^{AA'} \otimes dz^{BB'}. \tag{21}$$

where $\iota^{0'} = 0$ and $\iota^{1'} = 1$, and

$$z^{II'} = \begin{bmatrix} z^1 & z^3 \\ z^2 & z^4 \end{bmatrix}. \tag{22}$$

The generalization of Plebanski's second equation, that is equation (19), is

$$\Box\beta + p\beta_{AB}\beta^{AB} = 0, \tag{23}$$

where $\Box \equiv \epsilon^{AB}\epsilon^{A'B'}\partial^2/\partial z^{AA'}\partial z^{BB'}$. Here and henceforth the differential operator $\iota^{A'}\partial/\partial z^{AA'}$ will be denoted ∂_A, and

$$\beta_{AB} \equiv \partial_A\partial_B(\beta). \tag{24}$$

The co-frame given by equation (20) is (compare [6, 14]),

$$\chi^{\mathbf{A}A'} = (\delta_I^{\mathbf{A}}\delta_{I'}^{A'} - \iota^{A'}\iota_{I'}\beta_I^{\mathbf{A}})dz^{II'}, \tag{25}$$

This co-frame is a basis in which the connection 1-forms are

$$\begin{aligned} \omega_{\mathbf{B}}^{\mathbf{A}} &= -\partial_C(\beta_{\mathbf{B}}^{\mathbf{A}})\iota_{C'}dz^{CC'}, \\ \varpi_{B'}^{A'} &= 1/2(1-p)\iota^{A'}\iota_{B'}\partial_C(\beta_{PQ}\beta^{PQ})\iota_{C'}dz^{CC'}, \end{aligned} \tag{26}$$

and the curvature components are

$$\begin{aligned} \Psi_{\mathbf{ABCD}} &= -\partial_{\mathbf{A}}\partial_{\mathbf{B}}\beta_{\mathbf{CD}}, \\ \Psi_{A'B'C'D'} &= \iota_{A'}\iota_{B'}\iota_{C'}\iota_{D'}\Psi, \\ \Psi &= 1/4[p-1][\Box(\beta_{PQ}\beta^{PQ}) + 2\beta^{CD}\partial_C\partial_D(\beta_{PQ}\beta^{PQ})], \\ \Phi_{A'B'\mathbf{CD}} &= 1/2(1-p)\iota_{A'}\iota_{B'}\partial_{\mathbf{C}}\partial_{\mathbf{D}}(\beta_{PQ}\beta^{PQ}), \\ \Lambda &= 0. \end{aligned} \tag{27}$$

The self-dual part of the Weyl tensor is Petrov type N or zero. When $p = 1$ the co-frame above defines a basis in which the components of the self-dual part of the connection $\varpi_{B'}^{A'}$ are zero and the metrics are half-flat. When p is not equal to one, the Ricci tensor corresponds to that of a Maxwell field with self-dual part admitting $\iota_{A'}$ as a repeated principal null spinor.

Using a similar notation the analogous equations generalizing results for self-dual metrics can be written down. These correspond to interchanging the indices 2 and 3, and replacing the function β by a function α in equations (18) to (20). The spinor form of the equations is the following. The metric and co-frame are given by

$$ds^2 = (\epsilon_{AB}\epsilon_{A'B'} - 2\iota_A\iota_B\alpha_{A'B'})dz^{AA'} \otimes dz^{BB'}, \tag{28}$$

$$\chi^{\mathbf{A}A'} = (\delta_B^{\mathbf{A}}\delta_{B'}^{A'} - \iota^{\mathbf{A}}\iota_B\alpha_{B'}^{A'})dz^{BB'}, \tag{29}$$

where $\alpha_{A'B'} \equiv \partial_{A'}\partial_{B'}(\alpha)$, and $\partial_{A'} \equiv \iota^A\partial/\partial z^{AA'}$. The equations satisfied by these metrics are obtained from those satisfied by self-dual metrics by generalizing the Plebanski equation for self-dual metrics, to

$$\Box\alpha + q\alpha_{A'B'}\alpha^{A'B'} = 0. \tag{30}$$

When the complex parameter $q = 1$, self-dual metrics and the corresponding Plebański equation are recovered. It is the case of course, that all solutions of the above differential equations for α and β can be generated from solutions of the corresponding equations where the parameters are unity merely by rescaling the relevant dependent and/or independent variables. However the curvatures of the metrics depend on the values of p and q and hence it is convenient to explicitly use the above parameter dependent differential equations.

The co-frame given by equation (29) defines a basis in which the connection 1-forms are

$$\begin{aligned}\omega^{\mathbf{A}}_{\mathbf{B}} &= 1/2(1-q)\iota^{\mathbf{A}}\iota_{\mathbf{B}}\iota_C\partial_{C'}(\alpha_{P'Q'}\alpha^{P'Q'})dz^{CC'},\\ \varpi^{A'}_{B'} &= -\iota_C\partial_{C'}(\alpha^{A'}_{B'})dz^{CC'},\end{aligned} \tag{31}$$

and the curvature components are

$$\begin{aligned}\Psi_{\mathbf{ABCD}} &= \iota_{\mathbf{A}}\iota_{\mathbf{B}}\iota_{\mathbf{C}}\iota_{\mathbf{D}}\tilde{\Psi},\\ \tilde{\Psi} &= 1/4[q-1][\Box(\alpha_{P'Q'}\alpha^{P'Q'}) + 2\alpha^{C'D'}\partial_{C'}\partial_{D'}(\alpha_{P'Q'}\alpha^{P'Q'})],\\ \Psi_{A'B'C'D'} &= -\partial_{A'}\partial_{B'}\alpha_{C'D'},\\ \Phi_{\mathbf{AB}C'D'} &= 1/2(1-q)\iota_{\mathbf{A}}\iota_{\mathbf{B}}\partial_{C'}\partial_{D'}(\alpha_{P'Q'}\alpha^{P'Q'}),\\ \Lambda &= 0.\end{aligned} \tag{32}$$

Some exact solutions of these equations, which will be used in examples in the next section, can easily be obtained from the above equations by making a simplifying assumptions.[1]

[1]It is interesting to note that if β satisfies equation *(23),* and in addition $\beta_{,2} = \beta_{,3}$ then β also satisfies equation (30), with $q = p$. The corresponding metrics (21) and (28) have holomorphic Killing vector fields $K = \partial/\partial z^2 - \partial/\partial z^3$, and can both be expressed in terms of β, the holomorphic function of three variables$\{z^1, z^3, z^4\}$, which satisfies the equation $\beta_{,14} - \beta_{,33} + p(\beta_{,33}\beta_{,44} - (\beta_{,34})^2) = 0$.

M Dunajski (private communication) has pointed out this equation can be shown to be is equivalent to the three dimensional wave equation in the appropriate coordinates. The

Example 1: Consider first the class of metrics (equation 18 or 21) satisfying the additional condition

$$\iota^A \iota^B \beta_{AB} = \beta_{,44} = 0. \tag{33}$$

The solutions of equation (23) which also satisfy this equation (33) are given by

$$\beta_{AB} = \iota_A \iota_B [A_{,3}\, z^4 + (A_{,1} - 2pAA_{,3})z^2 + B] - A(\iota_A o_B + \iota_B o_A), \tag{34}$$

where A and B are arbitrary holomorphic functions of z^1 and z^3 only. The only non-zero curvature components are now

$$\begin{aligned} \Psi_{0000} &= -[A_{,333}\, z^4 + (A_{,133} - p(A^2)_{,333})z^2 + B_{,33}], \\ \Psi_{0001} &= -A_{,33}\,, \\ \Phi_{\mathbf{AB}C'D'} &= \iota_{\mathbf{A}} \iota_{\mathbf{B}} \iota_{C'} \iota_{D'} (p-1)(A^2)_{,33}\,. \end{aligned} \tag{35}$$

The Weyl tensor is right flat and of Petrov type III, N or 0 and, if non-zero the Ricci tensor is of pure radiation form.

The analogous simplifying assumption on the function α is

$$\alpha_{A'B'} \iota^{A'} \iota^{B'} = \alpha_{,44} = 0. \tag{36}$$

Solutions of equation (30) which also satisfy equation (36) are given by

$$\alpha_{A'B'} = \iota_{A'} \iota_{B'} [\tilde{A}_{,2}\, z^4 + (\tilde{A}_{,1} - 2q\tilde{A}\tilde{A}_{,2})z^3 + \tilde{B}] - \tilde{A}(\iota_{A'} o_{B'} + \iota_{B'} o_{A'}), \tag{37}$$

where $\tilde{A}$ and $\tilde{B}$ are arbitrary holomorphic functions of z^1 and z^2 only. The only non-zero curvature components in this case are

$$\begin{aligned} \Psi_{0'0'0'0'} &= -[\tilde{A}_{,222}\, z^4 + (\tilde{A}_{,122} - q(\tilde{A}^2)_{,222})z^3 + \tilde{B}_{,22}], \\ \Psi_{0'0'0'1'} &= -\tilde{A}_{,22} \quad , \\ \Phi_{\mathbf{AB}C'D'} &= \iota_{\mathbf{A}} \iota_{\mathbf{B}} \iota_{C'} \iota_{D'} (q-1)(\tilde{A}^2)_{,22}\,. \end{aligned} \tag{38}$$

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solutions determine metrics which, when p=1, correspond to holomorphic versions of the SD or ASD Gibbons-Hawking solutions, [31], [9]. In appropriate local complex coordinates p, s, w, t the SD and ASD metrics, ${}^+g$ and ${}^-g$, determined by the above equation with p=1, are respectively given by ${}^{+}_{-}ds^2 = V(dp^2 - 4dwds) - V^{-1}(\pm dt + \Omega)^2$
where $V_{,pp} - V_{,sw} = 0$, and $*dV = -d\Omega$.

4 Lorentzian 4-metrics from holomorphic metrics

In this section examples will be given of 4-metrics which are constructed from certain of the holomorphic metrics introduced in section three. In the first two examples previously obtained results for Ricci flat metrics, [14], will be generalized to produce solutions of Einstein's equations with pure radiation energy-momentum tensors. In the third example the pull-back of the real part of a holomorphic 4-metric to a real four dimensional manifold will be discussed.

Example 2: Here the construction of new holomorphic metrics on M by the superposition of the metrics of Example 1 will be presented. These results extend those in reference [14] where self-dual and anti-self dual solutions were superposed to form Ricci flat, but not half-flat, holomorphic metrics. As in that reference, consider holomorphic metrics of the form

$$ds^2 = (\epsilon_{AB}\epsilon_{A'B'} - 2\iota_A\iota_B\alpha_{A'B'} - 2\iota_{A'}\iota_{B'}\beta_{AB})dz^{AA'} \otimes dz^{BB'}, \qquad (39)$$

that is

$$\begin{aligned} ds^2 &= dz^1 \otimes dz^4 + dz^4 \otimes dz^1 - dz^2 \otimes dz^3 - dz^3 \otimes dz^2 \\ &\quad - 2(\beta,_{33} + \alpha,_{22})dz^1 \otimes dz^1 - 2\beta,_{34}(dz^1 \otimes dz^2 + dz^2 \otimes dz^1) \\ &\quad - 2\alpha,_{24}(dz^1 \otimes dz^3 + dz^3 \otimes dz^1). \end{aligned} \qquad (40)$$

where α satisfies equations (30) and (36), and hence is given by equation (37), and β satisfies equations (23) and (33), and hence (34). When either α or β is zero the results of Example 1 are re-obtained. In the gauge in which the Cartan co-frame for the metrics given by equations (39) or (40) is chosen to be, as in reference [14],

$$\chi^{\mathbf{A}A'} = (\delta^{\mathbf{A}}_B\delta^{A'}_{B'} - \iota^{\mathbf{A}}\iota_B\alpha^{A'}_{B'} - \iota^{A'}\iota_{B'}\beta^{\mathbf{A}}_B + \iota_B\iota_{B'}\iota_C\beta^{\mathbf{A}C}\iota_{C'}\alpha^{A'C'})dz^{BB'}, \qquad (41)$$

it follows, using equations (34) and (37), that the connection 1-forms are given by

$$\begin{aligned} \omega^{\mathbf{A}}_{\mathbf{B}} &= -\partial_C(\beta^{\mathbf{A}}_{\mathbf{B}})\iota_{C'}dz^{CC'} + (1-q)\iota^{\mathbf{A}}\iota_{\mathbf{B}}\iota_C\iota_{C'}(\tilde{A}^2),_2 dz^{CC'}, \\ \varpi^{A'}_{B'} &= -\iota_{C'}\partial_C(\alpha^{A'}_{B'})dz^{CC'} + (1-p)\iota^{A'}\iota_{B'}\iota_C\iota_{C'}(A^2),_3 dz^{CC'}. \end{aligned} \qquad (42)$$

The non-zero Weyl curvature components are, the same as they are in equations (35) and (38), and $\Lambda = 0$, but now

$$\Phi_{\mathbf{AB}C'D'} = \iota_{\mathbf{A}}\iota_{\mathbf{B}}\iota_{C'}\iota_{D'}[(p-1)(A^2)_{,33} + (q-1)(\tilde{A}^2)_{,22}]. \tag{43}$$

The Weyl curvature is consequently Petrov type III⊕III, or more (left and/or right) degenerate and the Einstein tensor has a pure radiation source. This class of metrics can be regarded as being obtained by a superposition of the metrics considered in Example 1. The metrics reduce to the vacuum metrics considered in reference [14] when the parameters p and q are each one.

Example 3: In this example *real* 4-metrics, with Lorentzian signature, which are solutions of Einstein's equations are obtained from the metrics in Example 2.

Consider the holomorphic metrics constructed in Example 2. Following reference [14], it is clear that real, pure radiation Lorentzian metric solutions of Einstein's equations (or vacuum solutions when $p = q = 1$) can be constructed from them. Let the four dimensional real submanifold N, of M be defined, by

$$\rho : N \to M \text{ by } z^{AA'} = x^{AA'}. \tag{44}$$

where the local coordinates on N are given by

$$x^{AA'} = \begin{bmatrix} x^1 & x^3 \\ x^2 & x^4 \end{bmatrix}, \tag{45}$$

x^1 and x^4 are real and x^2 is the complex conjugate of x^3.

When $q = \overline{p}$ and the function α is chosen, as it always can be, so that on the pull-back to N, $\bar{\alpha} = \beta = \psi$, the pull-back of equation (40) to N is the real Lorentzian metric

$$\begin{aligned} ds^2 = {} & dx^1 \otimes dx^4 + dx^4 \otimes dx^1 - dx^2 \otimes dx^3 - dx^3 \otimes dx^2 \\ & - 2(\psi_{,33} + \bar{\psi}_{,22})dx^1 \otimes dx^1 - 2\psi_{,34}(dx^1 \otimes dx^2 + dx^2 \otimes dx^1) \\ & - 2\bar{\psi}_{,24}(dx^1 \otimes dx^3 + dx^3 \otimes dx^1). \end{aligned} \tag{46}$$

where the comma now denotes partial derivatives with respect to the coordinates x^i. By construction, the complex function ψ satisfies the equations, now with independent variables x^i rather than z^i,

$$\begin{aligned} \psi_{,14} - \psi_{,23} + p(\psi_{,33}\psi_{,44} - \psi_{,34}\psi_{,34}) &= 0, \\ \psi_{,44} &= 0, \end{aligned} \tag{47}$$

or equivalently

$$
\begin{aligned}
\Box\psi + p\psi_{AB}\psi^{AB} &= 0, \\
\iota^A\iota^B\psi_{AB} &= 0.
\end{aligned} \tag{48}
$$

From equation (34) it follows that

$$
\psi_{AB} = \iota_A\iota_B[A_{,3}\, x^4 + (A_{,1} - p(A^2)_{,3}\,)x^2 + B] - A(\iota_A o_B + \iota_B o_A), \tag{49}
$$

where A and B are arbitrary complex functions of x^1 and x^3 only. The connection forms and curvature components can be obtained straightforwardly from equations (42) and (43). The Weyl curvature is Petrov type III or more degenerate and the pure radiation source is zero when $p = 1$.

Example 4: (See also, for comparison, earlier work on *real slices* of M, [20, 21].) It is also straightforward to construct real Lorentzian 4- metrics, on real four dimensional sub- manifolds, N, of M, from the real or imaginary parts of holomorphic metrics like those in equations (21) and (28). This point can be illustrated by considering the former metrics, although the the latter can equally well be used. As mentioned in section two, a holomorphic metric g, here determined by a holomorphic function β, can be split into its real and imaginary parts, **h** and **k**. Let N be a real four dimensional submanifold of M, with local coordinates x^i, with x^1 and x^4 real and x^2 and x^3 complex conjugates and let ρ :N→M be defined by $z^i = x^i$, as in the previous example. Let

$$
\rho^*\beta = \phi. \tag{50}
$$

The pull-back of **h** to N is then given by

$$
\begin{aligned}
ds^2_{\mathbf{h}} = {}& dx^1 \otimes dx^4 + dx^4 \otimes dx^1 - dx^2 \otimes dx^3 - dx^3 \otimes dx^2 \\
& - (\phi_{,33} + \bar{\phi}_{,22})dx^1 \otimes dx^1 - \phi_{,44}dx^2 \otimes dx^2 - \bar{\phi}_{,44}dx^3 \otimes dx^3 \\
& - \phi_{,34}(dx^1 \otimes dx^2 + dx^2 \otimes dx^1) - \bar{\phi}_{,24}(dx^1 \otimes dx^3 + dx^3 \otimes dx^1),
\end{aligned} \tag{51}
$$

where the sub-scripts denote partial derivatives with repect to the coordinates x^i. Since the determinant of this real metric is given by $1 - \phi_{,44}\bar{\phi}_{,44}$, its signature is Lorentzian as long as $1 >\mid \phi_{,44} \mid^2$. The pull-back of **k**, the imaginary part of g, to N is degenerate with $\partial/\partial x^4$ being an eigenvector of g with eigenvalue zero. The curvature of the pull-back of **h** will depend on the equations satisfied by β and hence ϕ. It does not necessarily follow, of course, that this real 4-metric will be Ricci flat when the holomorphic metric and the real 8-metrics are Ricci flat. However there can be a simple relation between their curvatures. A case which illustrates this latter point is simply obtained by assuming that ϕ also satisfies the additional condition

$$\phi_{,44} = 0. \tag{52}$$

In this case the metrics given by equation (51) can be immediately seen to be identical to the metrics given by equation (46) when $\phi = 2\psi$. This latter observation leads directly to the following two results in this special case where equation (52) is also satisfied:

(a) when the holomorphic metric g is anti-self dual and β satisfies equation (23), with p=1, the pull-back of the metric $\mathbf{h}$ to N is of the same form as the pure radiation metric, with $p = 2$, given in Example 3 above.

(b) when the holomorphic metric g is not half-flat but belongs to the class of metrics given by equations (21)-(23), with $p = 1/2$, then the pull-back of $\mathbf{h}$ to N is a vacuum solution belonging to the class given in Example 3, with $p = 1$.

5 Real p-forms and Lorentzian 4-metrics

In this section real differential forms on the holomorphic four-manifold M will be constructed from half-flat holomorphic solutions, satisfying equations (12) to (16), and their complex conjugates. The forms will have transformation properties determined by the structure groups of the half-flat metrics. Lorentzian metrics, on real four dimensional manifolds, will be constructed from these real forms.

Real forms on M can be constructed from co-frames, $\chi^{\mathbf{A}A'}$, for anti-self dual holomorphic metrics and their complex conjugates as follows. (Similar calculations could be carried out using co-frames determining self-dual geometries.) Let the complex conjugate of $\chi^{\mathbf{A}A'}$ be denoted by $\bar{\chi}^{A\mathbf{A}'}$; bold primed indices indicate transformation properties under $\overline{SL(2,C)_L}$ and ordinary unprimed indices indicating transformation properties under $\overline{SL(2,C)_R}$. Consider the Hermitian matrix-valued p-form $\sigma^{\mathbf{A}\mathbf{A}'}$, $2 \leq p \leq 8$ defined on M by

$$\sigma^{\mathbf{A}\mathbf{A}'} = i\chi^{\mathbf{A}A'} \wedge \bar{\chi}^{A\mathbf{A}'} \wedge \kappa_{AA'},$$
$$\kappa_{AA'} = \begin{bmatrix} \kappa_1 & \kappa_3 \\ \kappa_2 & \kappa_4 \end{bmatrix}. \tag{53}$$

Here $\kappa_{AA'}$ is chosen to be a Hermitian matrix-valued $(p-2)$-form so that it corresponds to a real vector-valued $(p-2)-$form κ_a. In addition, let κ_a be chosen to be covariantly constant with respect to the real *flat* connection given by

$$\varpi^a_b \leftrightarrow \varpi^{AA'}_{BB'} = \delta^A_B \varpi^{A'}_{B'} + \delta^{A'}_{B'} \bar{\varpi}^A_B. \tag{54}$$

This flat connection takes values in the Lie algebra of $SO(1,3)_R = \{SL(2,C)_R \times \overline{SL(2,C)_R}\}/\mathbb{Z}_2$, and κ_a has been chosen so that its covariant exterior derivative with respect to this flat connection is zero, that is

$$d\kappa_a - (-1)^p \kappa_b \wedge \varpi^b_a = 0; \text{ or equivalently}$$
$$d\kappa_{AA'} - (-1)^p \kappa_{BA'} \wedge \bar{\varpi}^B_A - (-1)^p \kappa_{AB'} \wedge \varpi^{B'}_{A'} = 0. \tag{55}$$

The p-form $\sigma^{\mathbf{AA'}}$corresponds in the usual way to four real p-forms

$$\sigma^{\mathbf{AA'}} = \begin{bmatrix} \sigma^{\mathbf{1}} & \sigma^{\mathbf{3}} \\ \sigma^{\mathbf{2}} & \sigma^{\mathbf{4}} \end{bmatrix}, \tag{56}$$

where $\sigma^{\mathbf{1}}$ and $\sigma^{\mathbf{4}}$ are real and $\sigma^{\mathbf{2}}$ and $\sigma^{\mathbf{3}}$ are complex conjugates. These are compatible with the real so(1,3)$_L$-valued connection

$$\omega^{\mathbf{a}}_{\mathbf{b}} =^- \bar{\Gamma}^a_b +^- \Gamma^a_b \leftrightarrow \delta^{\mathbf{A}}_{\mathbf{B}} \bar{\omega}^{\mathbf{A'}}_{\mathbf{B'}} + \delta^{\mathbf{A'}}_{\mathbf{B'}} \omega^{\mathbf{A}}_{\mathbf{B}}, \tag{57}$$

in the sense that it follows from the method of construction that the covariant exterior derivative of $\sigma^{\mathbf{a}}$, with respect to the latter connection, is zero i.e.

$$d\sigma^{\mathbf{a}} + (-1)^p \sigma^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} = 0, \text{or equivalently}$$
$$d\sigma^{\mathbf{AA'}} + (-1)^p \sigma^{\mathbf{BA'}} \wedge \omega^{\mathbf{A}}_{\mathbf{B}} + (-1)^p \sigma^{\mathbf{AB'}} \wedge \bar{\omega}^{\mathbf{A'}}_{\mathbf{B'}} = 0. \tag{58}$$

Lower case bold Latin indices, a,b,c.. represent transformation properties under $SO(1,3)_L = \{SL(2,C)_L \times \overline{SL(2,C)_L}\}/\mathbb{Z}_2$ and range and sum over 1 to 4.

Furthermore, it also follows from the method of construction that

$$\Omega^{\mathbf{A}}_{\mathbf{B}} \wedge \sigma^{\mathbf{BA'}} = 0 \tag{59}$$

and similarly for the complex conjugate equation. In other words, if $\Omega^{\mathbf{a}}_{\mathbf{b}}$ is the curvature 2-form of $\omega^{\mathbf{a}}_{\mathbf{b}}$ then

$$\mathbf{\Omega}^{\mathbf{a}}_{\mathbf{b}} \wedge \sigma^{\mathbf{b}} = \mathbf{\Omega}^{\mathbf{a}}_{\mathbf{b}} \wedge \epsilon^{\mathbf{b}}_{\mathbf{acd}} \wedge \sigma^{\mathbf{c}} = 0, \tag{60}$$

with Levi-Civita tensor given by

$$\epsilon_{\mathbf{abcd}} = \epsilon_{[\mathbf{abcd}]} \leftrightarrow i(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{A'C'}\epsilon_{B'D'}\epsilon_{AD}\epsilon_{BC}), \epsilon_{\mathbf{1234}} = i. \quad (61)$$

These equations for differential forms can be pulled back to sub-manifolds of M, in particular to four dimensional real manifolds, N↪M, so that they define real geometrical structures on N, [18]. Here only the cases where the forms $\sigma^{\mathbf{a}}$ are 3-forms, and the construction of Lorentzian metrics from them, will be discussed further. The 2-form case is briefly discussed in the appendix.

In the case where the real forms $\sigma^{\mathbf{a}}$ are 3-forms it is natural to construct Lorentzian 4-metrics on a four dimensional submanifold N of M by first pulling back the forms to N and then using the duality of vector densities and 3-forms in four dimensions When the four real 3-forms on M, $\sigma^{\mathbf{a}}$, can be pulled back to a basis of real 3-forms on N, (also written $\sigma^{\mathbf{a}}$), then there exists a co-frame of 1-forms $\theta^{\mathbf{a}} \leftrightarrow \theta^{\mathbf{AA'}}$ on N such that

$$\sigma^{\mathbf{a}} = \frac{1}{6}\epsilon^{\mathbf{a}}_{\mathbf{bcd}}\theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}, \quad (62)$$

$$\theta^{\mathbf{AA'}} = \begin{bmatrix} \theta^{\mathbf{1}} & \theta^{\mathbf{3}} \\ \theta^{\mathbf{2}} & \theta^{\mathbf{4}} \end{bmatrix}, \quad (63)$$

where $\theta^{\mathbf{1}}$ and $\theta^{\mathbf{4}}$ are real, and $\theta^{\mathbf{2}} = \bar{\theta}^{\mathbf{3}}$. It then follows that a Lorentzian metric, $\mathbf{g}$, is defined on N by

$$ds^2 = \eta_{\mathbf{ab}}\theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}}. \quad (64)$$

The Cartan structure equation which follows from the pull-back to N of equation (55) (with $p = 3$) relates the real Lorentzian metric, $\mathbf{g}$, on N to the pull back of the so(1,3)-valued connection $\omega^{\mathbf{a}}_{\mathbf{b}}$ (also written $\omega^{\mathbf{a}}_{\mathbf{b}}$) and its torsion 2-form $\Theta^{\mathbf{a}}$. It encodes, at least in part, the starting anti-self dual geometry on M, and is given by

$$d\theta^{\mathbf{a}} - \theta^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} = \Theta^{\mathbf{a}}, \quad (65)$$

where $\Theta^a = \frac{1}{2}\Theta^{\mathbf{a}}_{\mathbf{bc}}\theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}}$, $\Theta^{\mathbf{a}}_{\mathbf{bc}} = -\Theta^{\mathbf{a}}_{\mathbf{cb}}$. Equation (55) implies that the trace-free condition, $\Theta^{\mathbf{a}}_{\mathbf{ba}} = 0$, must be satisfied. If the *Levi-Civita* connection of

$\mathbf{g}$ is written $A^{\mathbf{a}}_{\mathbf{b}}$ with curvature 2-form $F^{\mathbf{a}}_{\mathbf{b}}$, then on N

$$\begin{aligned} d\theta^{\mathbf{a}} - \theta^{\mathbf{b}} \wedge A^{\mathbf{a}}_{\mathbf{b}} &= 0, \\ A^{\mathbf{a}}_{\mathbf{b}} &= \omega^{\mathbf{a}}_{\mathbf{b}} + \tau^{\mathbf{a}}_{\mathbf{b}}, \\ \tau_{\mathbf{ab}} &= \frac{1}{2}(\Theta_{\mathbf{abc}} - \Theta_{\mathbf{bac}} - \Theta_{\mathbf{cab}})\theta^{\mathbf{c}}, \\ F^{\mathbf{a}}_{\mathbf{b}} &= \Omega^{\mathbf{a}}_{\mathbf{b}} + D\tau^{\mathbf{a}}_{\mathbf{b}} + \tau^{\mathbf{a}}_{\mathbf{c}} \wedge \tau^{\mathbf{c}}_{\mathbf{b}}, \end{aligned} \tag{66}$$

where D denotes the covariant exterior derivative determined by $\omega^{\mathbf{a}}_{\mathbf{b}}$.

From a calculational point of view it is simplest to use the above results to compute the inverse metric density of weight two corresponding to the metric $\mathbf{g}$ on N, as in the following. Let x^i be local coordinates on N. Then on N the (pull-back of the) 3-forms $\sigma^{\mathbf{AA'}}$ can be written

$$\sigma^{\mathbf{AA'}} = (1/3!)E^{\mathbf{AA'}i}\eta_{ijkl}dx^j \wedge dx^k \wedge dx^l, \tag{67}$$

where the weight minus one tensor density $\eta_{ijkl} = \eta_{[ijkl]}$, and $\eta_{1234} = i$. When the weight one vector densities $E^{\mathbf{AA'}i}\partial/\partial x^i$ are linearly independent, the inverse metric density is given by

$$(\det \mathbf{g})\mathbf{g}^{II'JJ'} = \epsilon^{\mathbf{AB}}\epsilon^{\mathbf{A'B'}}E^{II'}_{\mathbf{AA'}}E^{JJ'}_{\mathbf{BB'}}, \tag{68}$$

and $\mathbf{g}$ is given by

$$ds^2 = \mathbf{g}_{ij}dx^i \otimes dx^j = \mathbf{g}_{II'JJ'}dx^{II'} \otimes dx^{JJ'}. \tag{69}$$

The vector fields $\{(\det \mathbf{g})^{-1/2}E^{II'}_{\mathbf{AA'}}\partial/\partial x^{II'}\}$ form a basis dual to the basis given by the co-frame $\{\theta^{\mathbf{AA'}}\}$.

The details of the Lorentzian geometry on N depend both on the way N is embedded in M and on the choice of the forms $\kappa_{AA'}$. When these calculations are carried out using the co-frame given by equation (25), equation (52) reduces in this case to the requirement that the 1-forms $\kappa_{AA'}$ be closed. Then, in a star-like region on M these forms can be chosen, without loss of generality, to be given by

$$\kappa^{II'} = df^{II'}, \tag{70}$$

where the Hermitian $f^{II'} \leftrightarrow f^i$, corresponds to four real functions, f^i, on M. With this choice the 3-forms on M are given by

$$\sigma^{\mathbf{AA'}} = i(\delta^{\mathbf{A}}_M\delta^{K'}_{M'} - \iota^{K'}\iota_{M'}\beta^{\mathbf{A}}_M)dz^{MM'} \wedge (\delta^{\mathbf{A'}}_{N'}\delta^K_N - \iota^K\iota_N\bar{\beta}^{\mathbf{A'}}_{N'})d\bar{z}^{NN'} \wedge df_{KK'}. \tag{71}$$

The dependence of the real geometry on N on the choice of the embedding mapping $\rho : \mathrm{N} \rightarrow \mathrm{M}$ can be exhibited by again denoting local coordinates on N by x^i , and by writing the local coordinate presentation of this map in terms of functions $\rho^{II'}$, with $z^{II'} = \rho^{II'}(x^{JJ'})$. Then if

$$\rho^*(\beta_I^{\mathbf{A}}) = \varphi_I^{\mathbf{A}}, \tag{72}$$

the pull back of $\sigma^{\mathbf{AA'}}$ to N is given by

$$i(\delta_M^{\mathbf{A}}\delta_{M'}^{K'} - \iota^{K'}\iota_{M'}\varphi_M^{\mathbf{A}})(\delta_{N'}^{\mathbf{A}'}\delta_N^K - \iota^K\iota_N\overline{\varphi}_{N'}^{\mathbf{A}'})\frac{\partial\rho^{MM'}}{\partial x^{II'}}\frac{\partial\bar{\rho}^{NN'}}{\partial x^{JJ'}}\frac{\partial f_{KK'}}{\partial x^{LL'}}dx^{II'}\wedge dx^{JJ'}\wedge dx^{LL'}. \tag{73}$$

Subject to the appropriate transversality condition being satisfied a natural choice of coordinates on N is given by the functions f^i. With this choice the remaining incompletness in the specification of the Lorentzian 4-metric on N lies in the specification of the embedding map ρ. A way in which to make natural choices of this map and hence obtain solutions of interesting equations on N, such as the vacuum Einstein equations, is a key missing feature of this construction. Nevertheless the method does enable (subject to the above conditions such as the linear independence of the vector densities) Lorentzian metrics to be constructed in a coordinate independent way on any real four dimensional sub-manifold from half-flat holomorphic metrics on M.

An illustration of the use of the above method to construct a class of Lorentzian 4- metrics on N is given in the final example. Here particularly simple choices of the local coordinates on N and the mapping ρ are made that enable comparisions with previous examples to be drawn.

Example 5: Let N be given by the particularly simple mapping $\rho : \mathrm{N} \rightarrow \mathrm{M}$ with local coordinate presentation $x^{II'} \rightarrow z^{II'} = x^{II'}$. In other words, as in previous examples, N is given by $z^i = x^i$. Let f^i be chosen to be x^i, i.e. $f^i = x^i$. Then, the weight one vector density components are given by

$$E_{\mathbf{AA'}}^{II'} = (3\delta_{\mathbf{A}}^I\delta_{\mathbf{A'}}^{I'} + \iota_{\mathbf{A'}}\iota^{I'}\varphi_{\mathbf{A}}^I + \iota_{\mathbf{A}}\iota^I\bar{\varphi}_{\mathbf{A'}}^{I'} - (\varphi_{\mathbf{A}J}\iota^J)(\bar{\varphi}_{\mathbf{A'}J'}\iota^{J'})\iota^I\iota^{I'}), \tag{74}$$

and, in terms of components with respect to the coordinates $x^i \longleftrightarrow x^{II'}$ the inverse metric density is given by

$$\begin{aligned}(\det \mathbf{g})\mathbf{g}^{II'JJ'} &= [9\epsilon^{IJ}\epsilon^{I'J'} + 6\iota^{I'}\iota^{J'}\varphi^{IJ} + 6\iota^{I}\iota^{J}\bar{\varphi}^{I'J'} - 3\iota^{J}\iota^{J'}(\varphi^{I}_{P}\iota^{P})(\bar{\varphi}^{I'}_{P'}\iota^{P'}) \\ &- 3\iota^{I}\iota^{I'}(\varphi^{J}_{P}\iota^{P})(\bar{\varphi}^{J'}_{P'}\iota^{P'}) - \iota^{I'}\iota^{J}(\varphi^{I}_{A}\iota^{A})(\bar{\varphi}^{J'}_{\acute{A}}\iota^{A'}) \\ &- \iota^{J'}\iota^{I}(\varphi^{J}_{A}\iota^{A})(\bar{\varphi}^{I'}_{\acute{A}}\iota^{A'}) - 3\iota^{I}\iota^{I'}\iota^{J}\iota^{J'}(\varphi_{AB}\varphi^{AB})(\bar{\varphi}_{A'B'}\iota^{A'}\iota^{B'}) \\ &- 3\iota^{I}\iota^{I'}\iota^{J}\iota^{J'}(\bar{\varphi}_{A'B'}\bar{\varphi}^{A'B'})(\varphi_{AB}\iota^{A}\iota^{B})].\end{aligned} \tag{75}$$

Furthermore $\varphi_{AB} = \partial^2\varphi/\partial x^{A1'}\partial x^{B1'}$ or

$$\varphi_{AB} = \begin{bmatrix} \varphi_{,33} & \varphi_{,34} \\ \varphi_{,34} & \varphi_{'44} \end{bmatrix}, \tag{76}$$

and on N

$$\varphi_{,41} - \varphi_{,32} + [\varphi_{,33}\varphi_{,44} - (\varphi_{,34})^2] = 0. \tag{77}$$

Athough these expressions lead to curvature forms which are to complicated to be usefully discussed in this example, two special and simpler cases are worth noting. First, the linearized version of these equations and geometrical quantities determine all the real linearized solutions of Einstein's vacuum equations, with the linearized field φ satisfying the complex wave equation on Minkowski space-time and corresponding to Penrose's Hertz potential for spin two fields, [33]. Second when the function φ is also assumed to satisfy the simplifying equation

$$\varphi_{,44} = 0, \tag{78}$$

the metrics constructed in this example can be compared directly with the metrics in the previous examples. It is a straightforward matter to see that these metrics (up to a constant scale factor) are the same as the metrics given by equations (46)-(49) when $p = 3$. Hence the method of combining half-flat metrics using the real 3-forms produces, in the special case of this example, Lorentzian 4-metrics which are, when $\varphi_{,34}$ is non-zero, pure radiation solutions and not vacuum solutions, of the real Einstein's equations.

The generation of real solutions of Einstein's equations with non-zero energy-momentum tensors from half-flat holomorphic solutions is not without interest, [16]. However Example 5 also makes it clear that further development of the approach using real 3-forms, introduced in this section, requires the clarification of the geometrical conditions which should be imposed on the embedding map ρ.

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6 Appendix: Real 2-forms and Lorentzian metrics

This appendix contains a brief discussion of a possible proceedure for the construction of Lorentzian 4-metrics when the differential forms constructed in section five, $\sigma^{\mathbf{a}}$, are four real 2-forms and the 0-forms κ_a are $SO(1,3)_R$ - gauge related to constants. It is natural in this case to construct four real 1-forms from the 2-forms by taking the inner product with a real vector field, V, on M. It is then a straight forward matter to use the results in section five to prove the following proposition.

Proposition:

Suppose that a real vector field on M, V, exists which satisfies the gauge covariant conditions:

$$\begin{aligned} \pounds_V \sigma^{\mathbf{a}} + \sigma^{\mathbf{b}}(V \rfloor \omega^{\mathbf{a}}_{\mathbf{b}}) &= 0, \\ \epsilon^{\mathbf{a}}_{\mathbf{bcd}}(V \rfloor \Omega^{\mathbf{b}}_{\mathbf{a}}) \wedge \sigma^{\mathbf{c}} &= 0, \end{aligned} \tag{79}$$

where £ and $\rfloor$ respectively denote the Lie derivative and inner product.

Then, if four real 1-forms are defined by $\theta^{\mathbf{a}} \equiv V \rfloor \sigma^{\mathbf{a}}$, it follows that

$$\begin{aligned} d\theta^{\mathbf{a}} - \theta^{\mathbf{b}} \wedge \omega^{\mathbf{a}}_{\mathbf{b}} &= 0, \\ \epsilon^{\mathbf{a}}_{\mathbf{bcd}} \Omega^{\mathbf{b}}_{\mathbf{a}} \wedge \theta^{\mathbf{c}} &= 0. \end{aligned} \tag{80}$$

Such 1-forms define a degenerate metric on M given by

$$ds^2 = \eta_{\mathbf{ab}} \theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}} = \epsilon_{\mathbf{AB}} \epsilon_{\mathbf{A'B'}} \theta^{\mathbf{AA'}} \otimes \theta^{\mathbf{BB'}}, \tag{81}$$

$$\theta^{\mathbf{AA'}} = W^{\mathbf{A}}_{B} \overline{\chi}^{\mathbf{A'}B} + \overline{W}^{\mathbf{A'}}_{B'} \chi^{\mathbf{A}B'}, \tag{82}$$

where, if $E_{\mathbf{A}A'}$ is the basis of holomorphic vector fields dual to $\chi^{\mathbf{A}A'}$, and $V = V^{\mathbf{A}A'} E_{\mathbf{A}A'} + c.c.$, then $W^{\mathbf{A}}_{B} = i V^{\mathbf{A}A'} \kappa_{BA'}$.

Corollary:

Let N be a real four dimensional sub-manifold of M. Then, if the pull-backs of the 1-forms $\theta^{\mathbf{a}}$ to N are linearly independent, they define a (non-degenerate) Ricci-flat Lorentzian metric on N given by the pull-back of the above degenerate metric.

It should be noted that the result contained in Corollary 1 follows even if the left-hand sides of equation (79) are required to be zero only on the pull-back to N. The definition of the 1-forms $\theta^{\mathbf{a}}$ implies that on M, $V \rfloor \theta^{\mathbf{a}} =$

0. Hence linear independence of the 1-forms on N requires that V not be tangent to N. When the forms on N are linearly dependent they may still satisfy constraint equations on sub-manifolds of N or be extended to obtain solutions. The main difficulty in using this proposition lies in finding an appropriate vector field V. Nevertheless known non-twisting type N vacuum solutions can be re-obtained by using this proposition.

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