The real geometry of holomorphic 4-metrics

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Abstract: The real geometry of holomorphic 4-metrics is investigated. The almost product and complex structures, associated with the real 8-metrics corresponding to the real and imaginary parts of the holomorphic metrics, are studied. It is shown that half-flat holomorphic metrics, and the corresponding real 8-metrics, are associated with integrable almost product, complex and hyper-Kähler structures. Real and complex local coordinate descriptions are presented.

I. Introduction

In the approximately twenty five years since the original work, by Newman, Penrose and Plebánski on holomorphic half-flat 4-metrics[1,2,3], there have been many developments of their results[4,5,6]. There have also been numerous applications of this and related work to real 4-metrics of Riemannian and neutral (Kleinian or ultrahyperbolic) signature[7,8,9]. However the extension of the original ideas to real 4-metrics of Lorentzian signature and Ricci-flat Lorentzian 4-metrics remains to be fully realized, although some interesting results have been obtained. There have been promising developments in both the twistorial approach to this problem,[10,11] and in the approach of Newman and his collaborators[12,13]. There have also been some isolated results on combining self-dual and anti self-dual solutions to obtain Ricci flat and real metrics by using Plebánski's formalism[14,15,16,17,18]. More references to aspects of these various lines of research can be found in a recent review[19]. This paper aims to provide further background for research into the construction of real metrics from holomorphic ones by exploring, in greater detail than previously, certain aspects of the real geometry associated with holomorphic 4-metrics on complex four manifolds.

The focus in this paper is on the geometry of the underlying eight dimensional real manifold and the two real 8-metrics defined by the real and imaginary parts of holomorphic 4-metrics. Most research on such 4-metrics focuses on their holomorphic properties. While this is clearly the natural thing to do, different and interesting insights can be obtained from the study of the geometry on the real eight manifold. Investigations of this type were initiated in the 1970's by Rozga and Woodhouse[20,21]. This paper includes an extension of that line of work, with particular attention being paid to half flat holomorphic 4-metrics and the important real structures and geometry which are related to them.

The content of the paper is as follows. The second section is devoted to a discussion of the complex and almost complex structures associated with a holomorphic 4-metric, g, on a complex 4-manifold M. All the real metrics discussed explicitly in this paper will have indefinite signature, but in the interests of simplicity the term "pseudo-" as in, for example "pseudo-Kähler" will be avoided. Here, where complex four dimensional (eight real dimensional) manifolds and holomorphic 4-metrics g are considered, the real metrics, $h = \operatorname{Re} g$ and $k = \operatorname{Im} g$, have neutral signature (4,4) and this simplification in terminology is unambiguous. Some previously obtained results[21,22,23] are included in this section for the sake of completeness. Examples are the exhibition of the anti-Kähler structure of the real metrics, h and k, and the fact that these real metrics may inherit the Einstein property of a holomorphic Einstein metric g. In the particular case of four complex dimensions being considered here, additional structures occur naturally. The real metrics, h and k, obtained from a holomorphic metric g are almost hyper-Kähler, in two distinct ways. That is, there are two sets of three almost complex structures, which are compatible with the real metrics, so they are in fact almost Hermitian structures with respect to h and k, and which also satisfy quaternionic relations. These structures are determined from the holomorphic metric, certain bases of self-dual (respectively anti self-dual) holomorphic 2-forms, and their complex conjugates. Together with the complex structure of the complex 4-manifold these almost complex structures also determine two sets of almost product structures satisfying related, but different, algebraic relations. These almost hyper-Kähler and almost product structures, together with the real metrics, encode certain of the information contained in the holomorphic metric. Further aspects of the latter are exposed when differential, in addition to algebraic, relations are considered. In the third section half-flat holomorphic 4-metrics are investigated. Since these are necessarily Ricci flat the corresponding real metrics h and k are also Ricci flat. In addition, each self-dual or anti-self dual holomorphic metric determines bases (respectively anti-self dual and self-dual) of three holomorphic 2-forms with zero covariant exterior derivative. Hence, in the half-flat case there are triples of (covariantly constant) complex structures satisfying quaternionic relations and h and k are in fact hyper-Kähler. Furthermore there are now triples of almost product structures which are integrable. Unified real descriptions of these structures are presented. The particular case that is dealt with focuses on anti self-dual metrics and their real parts. However analogous constructions apply to the imaginary parts of anti self-dual metrics and the real or imaginary parts of self-dual 4-metrics.

In section four local coordinates, adapted to an integrable almost product metric structure, (P, h), of the type discussed in section three, are introduced. This coordinate formulation is like a real version of the standard local coordinate description of a Kähler metric. Next an integrable and compatible almost complex structure I is added, the system (P, h, I) is described in adapted complex coordinates, and a holomorphic metric g is naturally defined. When the condition of Ricci flatness is imposed on h the description of g is seen to coincide with a description of half-flat holomorphic 4-metrics introduced by Plebánski[3]. Hence an alternative approach to holomorphic 4-metrics, via the structure (P, h, I). is provided here.

An appendix contains a brief outline of the relationship between the construction of twistor spaces for the real hyper-Kahler 8-metrics on M[24,25], and holomorphic half-flat 4-metrics.

Lower case, Latin indices a, b, c,.. i ,j ,k.. range and sum from 1 to 4; barred indices $\overline{a}, \overline{b}, \overline{c}$... from 4+1 to 4+4; bold lower case Latin indices, **i,j,k** from 1 to 3; Greek indices from 1 to 8; upper-case Latin indices A,B,.. from 1 to 2 and barred upper case Latin indices $\overline{A}, \overline{B}$,... from 2+1, to 2+2. Complex conjugates (c.c.) are denoted with a bar over the kernel letter. Geometrical considerations are essentially local.

II. Holomorphic 4-metrics

Let M be a complex manifold with $\dim_{\mathbb{C}}$M equal to four, and let g be a holomorphic metric on M, with line element given in complex coordinates z^i by

$$ds^2 = g_{ij} dz^i \otimes dz^j, \tag{1}$$

with $\partial g_{ij}/\partial \bar{z}^k = 0$.

Let I denote the (real) complex structure tensor satisfying $I^2 = -1$ and $Idz^i = idz^i$. Then

$$g(IX, IY) = -g(X, Y) \tag{2}$$

for all vector fields X and Y tangent to M[26,27]. Since $g(X, Y)$ is zero whenever X or Y is a (0,1) vector field the metric g is degenerate. Nevertheless, when the holomorphic category only is considered, it determines in the usual way, a unique torsion free metric connection- the holomorphic Levi–Civita connection-with holomorphic curvature. It is convenient here to present this metric geometry in terms of the holomorphic Cartan structure equations, using conventions which will be used later when real Lorentzian metrics are considered. These conventions are naturally adapted to two component spinor and anti self-dual formulations[29].

Let χ^a be a basis of holomorphic 1-forms, a Cartan co-frame for g, so that the line element for g is given by

$$ds^2 = \eta_{ab} \chi^a \otimes \chi^b \tag{3}$$

where

$$\eta_{ab} = \begin{bmatrix} 0 & \epsilon_{AB} \\ -\epsilon_{AB} & 0 \end{bmatrix}, \text{ and}$$
$$\epsilon_{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{4}$$

The complex volume element is given by

$$V = i\chi^1 \wedge \chi^2 \wedge \chi^3 \wedge \chi^4. \tag{5}$$

The first and second Cartan structure equations are given by

$$\begin{aligned} d\chi^a - \chi^b \wedge A^a_b &= 0, \\ A_{ab} + A_{ba} &= 0; \end{aligned} \tag{6}$$

and

$$dA^a_b + A^a_c \wedge A^c_b = -\frac{1}{2} F^a_{bcd} \chi^c \wedge \chi^d. \tag{7}$$

Here A^a_b denotes the holomorphic Levi-Civita connection 1-form (with covariant derivative ∇), and F^a_{bcd} are the components of its curvature 2-form F^a_b. The structure group is SO(4,C) and the connection and curvature forms, which take values in the Lies algebra so(4,C), can be written as the sum of their self-dual and anti self-dual parts, ${}^+A^a_b, {}^-A^a_b, {}^+F^a_b, {}^-F^a_b$ respectively. Here, ${}^{*+}F^a_b = i^+F^a_b$, ${}^{*-}F^a_b = -i^-F^a_b$. In 4×4 matrix form

$${}^+A^a_b = \begin{bmatrix} \varpi^{0\prime}_{0\prime}1 & \varpi^{0\prime}_{1\prime}1 \\ \varpi^{1\prime}_{0\prime}1 & -\varpi^{0\prime}_{0\prime}1 \end{bmatrix} \tag{8}$$

where here 1 is the unit 2×2 matrix and $\varpi^{0\prime}_{0\prime}, \varpi^{0\prime}_{1\prime}, \varpi^{1\prime}_{0\prime}$ denote the independent components of ${}^+A^a_b$. Similarly,

$${}^-A^a_b = \begin{bmatrix} \omega^A_B & 0 \\ 0 & \omega^A_B \end{bmatrix} \tag{9}$$

where the trace of the 2×2 matrix $\left(\omega^A_B\right)$ is zero. Other self-dual and anti self-dual objects can be written similarly, for instance,

$${}^-F^a_b = \begin{bmatrix} \Omega^A_B & 0 \\ 0 & \Omega^A_B \end{bmatrix} \tag{10}$$

where

$$\Omega^A_B = d\omega^A_B + \omega^A_C \wedge \omega^C_B. \tag{11}$$

The structure group SO(4,C) is isomorphic to $\{\mathrm{SL}(2,\mathrm{C})_L \times \mathrm{SL}(2,\mathrm{C})_R\}/Z_2$. The self-dual connection and curvature take values in the Lie algebra sl(2,C)$_R$ and the anti-self dual connection and curvature take values in the Lie algebra sl(2,C)$_L$.

The basis of self dual holomorphic 2-forms given by

$$\begin{aligned} {}^+\Sigma^{\mathbf{1}} &= \chi^2 \wedge \chi^1 + \chi^4 \wedge \chi^3 \\ {}^+\Sigma^{\mathbf{2}} &= i(\chi^1 \wedge \chi^4 + \chi^3 \wedge \chi^2) \\ {}^+\Sigma^{\mathbf{3}} &= i(\chi^1 \wedge \chi^2 + \chi^4 \wedge \chi^3) \end{aligned} \tag{12}$$

defines three holomorphic tensor fields ${}^+\mathbb{J}^{\mathbf{i}}$ by

$$g(X, {}^+\mathbb{J}^{\mathbf{i}}(Y)) = {}^+\Sigma^{\mathbf{i}}(X, Y) \tag{13}$$

for any holomorphic vector fields X and Y. It is a straightforward matter to show that

$$({}^+\mathbb{J}^{\mathbf{1}})^2 = ({}^+\mathbb{J}^{\mathbf{2}})^2 = ({}^+\mathbb{J}^{\mathbf{3}})^2 = -\mathbb{I}, \tag{14}$$

where $\mathbb{I}$ is the unit operator, and

$${}^+\mathbb{J}^{\mathbf{1}}\,{}^+\mathbb{J}^{\mathbf{2}} = {}^+\mathbb{J}^{\mathbf{3}},\ {}^+\mathbb{J}^{\mathbf{2}}\,{}^+\mathbb{J}^{\mathbf{3}} = {}^+\mathbb{J}^{\mathbf{1}},\ {}^+\mathbb{J}^{\mathbf{3}}\,{}^+\mathbb{J}^{\mathbf{1}} = {}^+\mathbb{J}^{\mathbf{2}}, \tag{15}$$

$$g({}^+\mathbb{J}^{\mathbf{1}}X, {}^+\mathbb{J}^{\mathbf{1}}Y) = g({}^+\mathbb{J}^{\mathbf{2}}X, {}^+\mathbb{J}^{\mathbf{2}}Y) = g({}^+\mathbb{J}^{\mathbf{3}}X, {}^+\mathbb{J}^{\mathbf{3}}Y) = g(X, Y), \tag{16}$$

for all holomorphic vector fields X and Y. The basis of anti self-dual holomorphic 2-forms given by

$$\begin{aligned} {}^-\boldsymbol{\Sigma}^{\mathbf{1}} &= \chi^3 \wedge \chi^1 + \chi^4 \wedge \chi^2 \\ {}^-\boldsymbol{\Sigma}^{\mathbf{2}} &= i(\chi^4 \wedge \chi^1 + \chi^3 \wedge \chi^2) \\ {}^-\boldsymbol{\Sigma}^{\mathbf{3}} &= i(\chi^2 \wedge \chi^4 + \chi^3 \wedge \chi^1) \end{aligned} \tag{17}$$

define, as above, three holomorphic tensor fields, ${}^-\mathbb{J}^{\mathbf{i}}$, by

$$g(X, {}^-\mathbb{J}^{\mathbf{i}}(Y)) = {}^-\Sigma^{\mathbf{i}}(X, Y) \tag{18}$$

and these satisfy equations of the same form as Eqs.(14), (15) and (16).

The real and imaginary parts, h and k, of the holomorphic metric $g = h + ik$ are two real metrics on the real eight dimensional manifold M. In terms of the complex co-frame of eight 1-forms $(\chi^a, \bar{\chi}^a)$ on M the line element of h is given by

$$_h ds^2 = \frac{1}{2}\eta_{ab}(\chi^a \otimes \chi^b + \bar{\chi}^a \otimes \bar{\chi}^b) \tag{19}$$

and the line element of k is given by

$$_k ds^2 = \frac{i}{2}\eta_{ab}(-\chi^a \otimes \chi^b + \bar{\chi}^a \otimes \bar{\chi}^b). \tag{20}$$

The two real metrics have Kleinian (neutral or ultrahyperbolic) signatures (4,4). The covariant derivatives corresponding to the Levi-Civita connections of h and k, $_h\nabla$ and $_k\nabla$ respectively, are each determined by the same connection 1-forms[21]

$$\left[\begin{array}{cc} A^a_b & 0 \\ 0 & \overline{A}^a_b \end{array}\right]. \tag{21}$$

When g is Einstein, that is when

$$F^a_{bad} = \frac{1}{4}F\eta_{bd}, \tag{22}$$

then h is Einstein if and only if Im F is zero and k is Einstein if and only if Re F is zero, cf. references[22,23].

Both real metrics are anti-Hermitian with respect to the complex structure I, that is, for all real vector fields, X and Y on M,

$$h(IX, IY) = -h(X, Y); k(IX, IY) = -k(X, Y). \tag{23}$$

Moreover since

$$_h\nabla I =_k \nabla I = 0, \tag{24}$$

the two real metrics are, in fact, anti-Kähler[21,22,23] with respect to the complex structure I. They are of course related, with the metric k in the anti-Kähler case replacing the 2-form of the Kähler case; for any two real vector fields X and Y

$$k(X, Y) = -h(IX, Y). \tag{25}$$

The holomorphic tensor fields $^+\mathbb{J}^{\mathbf{i}}$ and $^-\mathbb{J}^{\mathbf{i}}$ defined above can be naturally extended to real tensor fields which satisfy relations like those in Eqs.(14), (15) and (16) above. Define the real tensor fields $^+J^{\mathbf{i}}$ by

$$h(X,^+ J^{\mathbf{i}}(Y)) = \frac{1}{2}\left(^+\mathbf{\Sigma}^{\mathbf{i}} +^+ \bar{\Sigma}^{\mathbf{i}}\right)(X, Y) \tag{26}$$

for X and Y any vector fields tangent to M. Here, and in the following, ${}^{+}\bar{\Sigma}^{\mathbf{i}}$ denotes the complex conjugate of ${}^{+}\Sigma^{\mathbf{i}}$. Then, if α and X are, respectively a real 1-form and vector field on M, with (1,0) parts denoted respectively by $\alpha_{(1,0)}$ and $X_{(1,0)}$, so that $\alpha = \alpha_{(1,0)} + c.c.$ and $X = X_{(1,0)} + c.c.$,

$$
{}^{+}J^{\mathbf{i}}\left(\alpha, X\right) ={}^{+}\mathbb{J}^{\mathbf{i}}\left(\alpha_{(1,0)}, X_{(1,0)}\right) + c.c. \tag{27}
$$

When h is replaced by k in the left hand side of Eq.(25) the result is the following equation

$$
k(X,{}^{+}J^{\mathbf{i}}\left(Y\right)) = \frac{i}{2}\left(-{}^{+}\Sigma^{\mathbf{i}} +{}^{+}\bar{\Sigma}^{\mathbf{i}}\right)\left(X, Y\right). \tag{28}
$$

The holomorphic tensor fields ${}^{-}\mathbb{J}^{\mathbf{i}}$ can be extended to the real tensor fields ${}^{-}J^{\mathbf{i}}$ in a similar way.

Both ${}^{+}J^{\mathbf{i}}$ and ${}^{-}J^{\mathbf{i}}$ are triplets of almost complex structures and both h and k are Hermitian with respect to each of these almost complex structures. Further more, the almost complex structures satisfy a quaternionic algebra; that is each of the triplets ${}^{+}J^{\mathbf{i}}$ and ${}^{-}J^{\mathbf{i}}$ separately satisfies the set of equations

$$
\begin{array}{c}
(J^{\mathbf{1}})^2 = (J^{\mathbf{2}})^2 = (J^{\mathbf{3}})^2 = -1 \\
J^{\mathbf{1}}J^{\mathbf{2}} = J^{\mathbf{3}}, J^{\mathbf{2}}J^{\mathbf{3}} = J^{\mathbf{1}}, J^{\mathbf{3}}J^{\mathbf{1}} = J^{\mathbf{2}}, \\
h\left(J^{\mathbf{1}}X, J^{\mathbf{1}}Y\right) = h\left(J^{\mathbf{2}}X, J^{\mathbf{2}}Y\right) = h\left(J^{\mathbf{3}}X, J^{\mathbf{3}}Y\right) = h\left(X, Y\right), \\
k\left(J^{\mathbf{1}}X, J^{\mathbf{1}}Y\right) = k\left(J^{\mathbf{2}}X, J^{\mathbf{2}}Y\right) = k\left(J^{\mathbf{3}}X, J^{\mathbf{3}}Y\right) = k(X, Y).
\end{array} \tag{29}
$$

In other words the eight dimensional real manifold M is almost quaternion and each of $(M, h, J^{\mathbf{i}})$ and $(M, k, J^{\mathbf{i}})$ are almost quaternionic metric structures[27,28].

By interchanging h and k in Eqs.(26) and (28) a second set of real tensor fields, ${}^{+}P^{\mathbf{i}}$ and ${}^{-}P^{\mathbf{i}}$ which are almost product, almost anti-Hermitian structures, rather than almost complex, almost-Hermitian structures, can be defined. Let ${}^{+}P^{\mathbf{i}}$ be defined by

$$
k(X,{}^{+}P^{\mathbf{i}}\left(Y\right)) = \frac{1}{2}\left({}^{+}\Sigma^{\mathbf{i}} +{}^{+}\bar{\Sigma}^{\mathbf{i}}\right)\left(X, Y\right) \tag{30}
$$

or equivalently

$$
{}^{+}P^{\mathbf{i}}\left(\alpha, X\right) = i{}^{+}J^{\mathbf{i}}\left(\alpha_{(1,0)}, X_{(1,0)}\right) + c.c. \tag{31}
$$

with the notation as above. It then follows, analogously to Eq. (28), that

$$h(X,{}^{+}P^{\mathbf{i}}(Y)) = \frac{i}{2}\left({}^{+}\Sigma^{\mathbf{i}} - {}^{+}\bar{\Sigma}^{\mathbf{i}}\right)(X,Y)\,. \tag{32}$$

${}^{-}P^{\mathbf{i}}$ can be defined in a similar way and satisfies equations parallel to Eqs.(30), (31) and (32). In terms of their components with respect to the co-frame $(\chi^{a},\chi^{\overline{a}})$ and its dual basis of vector fields, these operators are given by

$$I = i\begin{bmatrix} \delta^{a}_{b} & 0 \\ 0 & -\delta^{a}_{b} \end{bmatrix}, {}^{+}J^{\mathbf{i}} = \begin{bmatrix} {}^{+}\mathbb{J}^{\mathbf{i}} & 0 \\ 0 & {}^{+}\bar{\mathbb{J}}^{\mathbf{i}} \end{bmatrix}, {}^{+}P^{\mathbf{i}} = i\begin{bmatrix} {}^{+}\mathbb{J}^{\mathbf{i}} & 0 \\ 0 & -{}^{+}\bar{\mathbb{J}}^{\mathbf{i}} \end{bmatrix}, \tag{33}$$

and similarly for ${}^{-}J^{\mathbf{i}}$ and ${}^{-}P^{\mathbf{i}}$. The algebraic relationships between these real tensors, for short I, $J^{\mathbf{i}}$ and $P^{\mathbf{i}}$, can be summarised in the following equations

$$\begin{aligned} IJ^{\mathbf{i}} &= J^{\mathbf{i}}I = P^{\mathbf{i}}, \\ IP^{\mathbf{i}} &= -J^{\mathbf{i}}, \\ J^{\mathbf{i}}P^{\mathbf{j}} &= P^{\mathbf{i}}J^{\mathbf{j}} = -I\delta^{\mathbf{ij}} + \varepsilon^{\mathbf{ijk}}P^{\mathbf{k}}, \end{aligned} \tag{34}$$

where in each equation $J^{\mathbf{i}}$ and $P^{\mathbf{i}}$ are both self-dual or both anti-self dual. The self-dual and anti-self dual operators commute. Both ${}^{+}P^{\mathbf{i}}$ and ${}^{-}P^{\mathbf{i}}$ are almost product rather than almost complex structures[27], and satisfy algebraic equations which are generalizations of the quaternionic equations. They also satisfy almost anti-Hermitian rather than almost-Hermitian relations with the real metrics; altogether

$$\begin{aligned} (P^{\mathbf{1}})^2 = (P^{\mathbf{2}})^2 = (P^{\mathbf{3}})^2 = 1, \\ P^{\mathbf{1}}P^{\mathbf{2}} = IP^{\mathbf{3}}, P^{\mathbf{2}}P^{\mathbf{3}} = IP^{\mathbf{1}}, P^{\mathbf{3}}P^{\mathbf{1}} = IP^{\mathbf{2}}, \\ h\left(P^{\mathbf{1}}X, P^{\mathbf{1}}Y\right) = h\left(P^{\mathbf{2}}X, P^{\mathbf{2}}Y\right) = h\left(P^{\mathbf{3}}X, P^{\mathbf{3}}Y\right) = -h\left(X,Y\right), \\ k\left(P^{\mathbf{1}}X, P^{\mathbf{1}}Y\right) = k\left(P^{\mathbf{2}}X, P^{\mathbf{2}}Y\right) = k\left(P^{\mathbf{3}}X, P^{\mathbf{3}}Y\right) = -k\left(X,Y\right). \end{aligned} \tag{35}$$

Since g is a *4-metric*, it always follows that

$$\nabla^{+}\mathbb{J}^{\mathbf{1}} = -{}^{+}\mathcal{A}^{\mathbf{1}+}_{\mathbf{i}}\mathbb{J}^{\mathbf{i}}, \nabla^{+}\mathbb{J}^{\mathbf{2}} = -{}^{+}\mathcal{A}^{\mathbf{2}+}_{\mathbf{i}}\mathbb{J}^{\mathbf{i}}, \nabla^{+}\mathbb{J}^{\mathbf{3}} = -{}^{+}\mathcal{A}^{\mathbf{3}+}_{\mathbf{i}}\mathbb{J}^{\mathbf{i}} \tag{36}$$

where the so(3,C) -valued 1-form ${}^{+}\mathcal{A}^{\mathbf{i}}_{\mathbf{j}}$, is the so(3,C)-valued representation of the self-dual part of the Levi-Civita spin connection. Similar equations holds for the anti-self dual objects. From these equations it follows that, for i = 1,2,3 separately,

$${}_{h}\nabla_{\gamma}({}^{+}J^{\mathbf{i}\alpha}_{\beta}) = {}_{k}\nabla_{\gamma}({}^{+}J^{\mathbf{i}\alpha}_{\beta}) = \begin{bmatrix} -{}^{+}\mathcal{A}^{\mathbf{i}}_{\mathbf{j}\gamma}{}^{+}\mathbb{J}^{\mathbf{j}a}_{b} & 0 \\ 0 & c.c. \end{bmatrix}, \tag{37}$$

(in components with respect to the complex bases and where c.c. denotes the complex conjugate of $-({}^+\mathcal{A}^{\mathbf{i}}_{\mathbf{j}\gamma}\ {}^+\mathbb{J}^{\mathbf{j}a}_{b})$). Similar results hold in the anti self-dual case. Hence when a gauge can be found in which the components of ${}^+\mathcal{A}^{\mathbf{i}}_{\mathbf{j}}$ are real (and hence constant) so that ${}^+\mathcal{A}^{\mathbf{i}}_{\mathbf{j}}$ takes values in so(3,R), the triples (M, h, ${}^+J^{\mathbf{i}}$) and (M, k, ${}^+J^{\mathbf{i}}$) constitute a quaternion Kähler structure and the metrics are Einstein metrics[28,30]. Similar results hold, of course, when the anti-self dual case is considered.

III. Half-flat anti-self dual 4-metrics

In this section half-flat, anti self-dual 4-metrics, g, on M will be considered. (In both the self-dual and the anti-self dual case, the Levi-Civita connection of the corresponding real metrics h and k is determined by the real so(1,3) connection given by twice the real part of the holomorphic Levi-Civita connection of g. There is only one real connection.) Since the self-dual part of the curvature of a holomorphic half-flat anti self-dual metric is zero, ${}^+F^a_b = 0$, a co-frame can be chosen in which ${}^+A^a_b = 0$. In this gauge $d^+\Sigma^{\mathbf{i}} = 0$, and the tensors, defined by Eq.(13), are covariantly constant,

$$\nabla^+\mathbb{J}^{\mathbf{i}} = 0. \tag{38}$$

It then follows from Eqs.(36) and (37), and then (24) and (34) that the real tensors, defined by Eqs.(26) and (30), are covariantly constant with respect to the Levi-Civita covariant derivatives of the Ricci flat 8-metrics h and k. Relationships between the various structures discussed in this section can be summarized in three propositions which it is a straightforward matter to prove. The first is due to Woodhouse[21–23].

Proposition 1. Let h be a real 8-metric of signature (4,4) on a real eight dimensional manifold M. Let h be anti-Hermitian with respect to an almost complex structure I. Then I is a complex structure and the tensor g given by the equation

$$g(X,Y) = h(X,Y) - ih(IX,Y),$$

is holomorphic (a holomorphic metric) with respect to the complex structure I if and only if ${}_h\nabla I = 0$.

Proposition 2. If the conditions and conclusions of Lemma 1 hold, and in addition there is a hyper-Kähler structure $(\mathrm{M}, h, J^{\mathbf{i}})$ with $IJ^{\mathbf{i}} = J^{\mathbf{i}}I$, then

(a) $J^{\mathbf{i}} = \mathbb{J}^{\mathbf{i}j}_{k}\partial/\partial z^j \otimes dz^k + c.c.$, for some complex $\mathbb{J}^{\mathbf{i}j}_{k}$, where $I\partial/\partial z^j = i\partial/\partial z^j$,

(b) for each $\mathbf{i} = 1, 2, 3$, ${}_g\nabla \mathbb{J}^{\mathbf{i}} = 0$, and $g(\mathbb{J}^{\mathbf{i}}\partial/\partial z^j, \mathbb{J}^{\mathbf{i}}\partial/\partial z^k) = g(\partial/\partial z^j, \partial/\partial z^k)$
,

(c) the holomorphic metric g is half-flat.

(d) (M, $k = \operatorname{Im} g$, $J^{\mathbf{i}}$) is also hyper-Kähler, $\nabla_k J^{\mathbf{i}} = 0$.

(e) The hyper-Kähler structures are Ricci flat.

The results of Proposition 2 can be reformulated in terms of three almost product structures $P^{\mathbf{i}}$.

Proposition 3. If the conditions and conclusions of Lemma 1 hold, and in addition there are three almost product structures $P^{\mathbf{i}}$ which satisfy the conditions $P^{\mathbf{i}}P^{\mathbf{j}} = \epsilon^{\mathbf{ijk}} I P^{\mathbf{k}}$, $h(P^{\mathbf{i}}X, P^{\mathbf{i}}Y) = -h(X, Y)$, $\nabla_h P^{\mathbf{i}} = 0$ and $IP^{\mathbf{i}} = P^{\mathbf{i}}I$; then the tensors $J^{\mathbf{i}} = -IP^{\mathbf{i}}$ satisfy the conditions of Lemma 2 and hence determine a holomorphic half-flat metric g and Ricci-flat hyper-Kähler structures $(\mathrm{M}, h, J^{\mathbf{i}})$ and (M, k, $J^{\mathbf{i}}$) on the eight dimensional real manifold M.

Results from this and the previous section will be used later and it is useful to summarize them in a convenient form, which starts here with the almost product structures. For the sake of explicitness structures arising from anti self-dual half-flat holomorphic metrics, $g = h + ik$, will be given, but there are similar equations for self-dual holomorphic metrics.

First consider the almost product structures ${}^+P^{\mathbf{i}}$, with components given in Eq.(33),

$$ {}^+P^{\mathbf{i}} = i \begin{bmatrix} {}^+\mathbb{J}^{\mathbf{i}} & 0 \\ 0 & -{}^+\bar{\mathbb{J}}^{\mathbf{i}} \end{bmatrix} \tag{39} $$

Let P denote the almost product structure (strictly the two complex parameter family of almost product structures)

$$ P = i \begin{bmatrix} a_{\mathbf{i}}^{+}\mathbb{J}^{\mathbf{i}} & 0 \\ 0 & -\overline{a}_{\mathbf{i}}^{+}\bar{\mathbb{J}}^{\mathbf{i}} \end{bmatrix} \tag{40} $$

for any *complex* numbers $a_{\mathbf{i}}$ satisfying the equation

$$ (a_{\mathbf{1}})^2 + (a_{\mathbf{2}})^2 + (a_{\mathbf{3}})^2 = 1. \tag{41} $$

Then the results given in the propositions above for almost product structures ${}^+P^{\mathbf{i}}$, and complex structure I, associated with an anti-self dual holomorphic half-flat metric g, can be summarized by the equations

$$(P)^2 = 1, \tag{42}$$
$$h\,(PX, PY) = -h\,(X, Y)\,,\ \ k\,(PX, PY) = -k\,(X, Y)\,, \tag{43}$$
$$_h\nabla P = 0,\ \ _k\nabla P = 0, \tag{44}$$
$$(I)^2 = -1, PI = IP, \tag{45}$$
$$h(IX, IY) = -h(X, Y), k(IX, IY) = -k(X, Y), \tag{46}$$
$$_h\nabla I = 0,\ \ _k\nabla I = 0, \tag{47}$$

where X and Y are any vectors tangent to M. Furthermore h and k are Ricci flat. Conversely, Eqs.(42)-(47) imply that the complex metric $g = h + ik$ is a holomorphic 4-metric and half-flat, and the 8-metrics h and k are Ricci-flat. A local coordinate description of the geometry determined by a single triple consisting of an almost product structure P, an almost complex structure I and a metric h, satisfying Eqs.(42)-(47), will be given in Sec. IV.

It follows from Eqs.(42)-(47) that the almost complex structure J (here strictly a two complex parameter family) defined by

$$J = -IP, \tag{48}$$

satisfies the equations

$$\begin{aligned} J^2 &= -1, \\ JP &= PJ,\ \ JI = IJ, \\ h(JX, JY) &= h(X, Y), k(JX, JY) = k(X, Y), \\ _h\nabla J &= 0,\ \ _k\nabla J = 0. \end{aligned} \tag{49}$$

A two complex parameter family of Kähler structures, (J, h), so defined, determines the hyper-Kähler structures mentioned above. These results, with the parameters $a_{\mathbf{i}}$ real, are used in the twistor construction reviewed in the Appendix.

Finally, it is interesting to note that all the Eqs.(42)-(49) can be re-written in terms of a four real parameter family of tensors $T(\alpha_{\mathbf{i}}, \beta_{\mathbf{j}})$. Define this family of real tensors by

$$T \equiv (\alpha_{\mathbf{i}} + \beta_{\mathbf{i}} I) J^{\mathbf{i}}, \tag{50}$$

where $\alpha_{\mathbf{i}}$ and $\beta_{\mathbf{i}}$ are real parameters satisfying the conditions

$$\begin{aligned} \alpha_{\mathbf{i}}\beta_{\mathbf{j}}\delta^{\mathbf{ij}} &= 0, \\ \varepsilon \equiv \alpha_{\mathbf{i}}\alpha_{\mathbf{j}}\delta^{\mathbf{ij}} &- \beta_{\mathbf{i}}\beta_{\mathbf{j}}\delta^{\mathbf{ij}}, \end{aligned}$$

for fixed ε; or, equivalently, the conditions

$$\text{if } a_{\mathbf{i}} = \alpha_{\mathbf{i}} + i\beta_{\mathbf{i}}, \text{then } (a_{\mathbf{1}})^2 + (a_{\mathbf{2}})^2 + (a_{\mathbf{3}})^2 = \varepsilon. \tag{51}$$

Eqs.(41)-(49) imply, for all vector fields X and Y, that

$$\begin{aligned} T^2 &= -\varepsilon, \\ h(TX, TY) &= \varepsilon h(X, Y), \\ k(TX, TY) &= \varepsilon k(X, Y), \\ {}_h\nabla T &= 0, \ {}_k\nabla T = 0. \end{aligned} \tag{52}$$

It follows from these equations that, when ε is non-zero that

$$\begin{aligned} I &= -\varepsilon^{-1} T(\alpha_{\mathbf{i}}, \beta_{\mathbf{j}}) T(-\beta_{\mathbf{i}}, \alpha_{\mathbf{j}}), \\ \text{for each i} : J^{\mathbf{i}} &= T(\alpha^{\mathbf{i}} = 1, \beta^{\mathbf{i}} = 0) = -IT(\alpha^{\mathbf{i}} = 0, \beta^{\mathbf{i}} = 1). \end{aligned} \tag{53}$$

Structures of similar types have been considered by Dunajski[33,34].

IV. Local coordinate description of a structure (P, h, I)

In this section local coordinate descriptions are given of the geometry represented by a *single* triple (P, h, I) which satisfies Eqs.(42)-(47) of Sec.III. It is shown that when the Ricci tensor of h is zero such a geometry defines a holomorphic half-flat metric. The aim here is to show, using local coordinates, the effect of a sequential introduction of these structures and the implementation of the equations. Eqs.(42)-(44) are used first to introduce coordinates adapted to the almost product structure P, and to derive a description of the integrable almost product structure P, and the 8-metric h, in these coordinates. Eqs.(45)-(47) are then used to introduce complex coordinates adapted to both P and I, and then the triple (P, h, I) is expressed in terms of these coordinates. (Similar calculations could be carried out using the metric k and/or the parallel results associated with self-dual holomorphic metrics.) Finally a holomorphic half-flat metric is identified.

Eq.(44), the vanishing of the covariant derivative of the almost product structure P, implies that the Nijenhuis tensor of P is also zero[27,31]. Hence P is integrable. Standard arguments, analogous to those which can be used for Kähler metrics, can now be applied to construct a local coordinate system adapted to the geometrical structure determined by Eqs.(42)-(44). First, it follows from Eqs.(42) that the eigenvalues of P are 1 or -1. At each point of M the eigenvectors corresponding respectively to the eigenvalues 1 and -1 span complementary distributions, D_1 and D_2, of dimensions d_1 and d_2 say, where $d_1 + d_2 =$dim M. The dimensions of the two distributions can be different but here it suffices to consider the case where they are equal. The ideas then apply to even dimensional manifolds but here of courses dim M equal eight. Let $(e_\alpha) = (e_a,\, e_{\overline{a}})$ be a basis of real tangent vectors to M, adapted to the almost product structure so that (e_a) span D_1 and $(e_{\overline{a}})$ span D_2; lower case barred Latin indices range and sum over 4+1 to 4+4. It follows from Eqs.(42) and (43) that a real 2-form, Π, is defined by the equations

$$\Pi(X,Y) = h(X,PY), \tag{54}$$

for all vector fields X and Y, and that the components of h, with respect to the basis of vector fields $(e_a,\, e_{\overline{a}})$, are given by

$$h = \left[\begin{array}{cc} 0 & h_{a\bar{b}} \\ h_{\overline{a}b} & 0 \end{array} \right] \tag{55}$$

where $h_{\overline{a}b} = h_{b\overline{a}}$. The vanishing of the Nijenhuis tensor of P is equivalent to integrability of the two complementary distributions[27,31], D_1 and D_2 . Hence adapted local coordinates $u^\alpha = (u^a, u^{\overline{a}})$ can be introduced on M and the adapted bases chosen to be coordinate bases so that $e_a = \partial/\partial u^a$ and $e_{\,\overline{a}} = \partial/\partial u^{\overline{a}}$. The vanishing of the covariant derivative of P implies that Π is closed and therefore a local potential function, H, exists such that

$$h_{a\bar{b}} = \partial^2 H/\partial u^a \partial u^{\bar{b}} \equiv H_{,a\bar{b}}. \tag{56}$$

Consequently the line element of h is given, in these coordinates, by

$${}_h ds^2 = h_{\alpha\beta} du^\alpha \otimes du^\beta = H_{,a\bar{b}}\,(du^a \otimes du^{\bar{b}} + du^{\bar{b}} \otimes du^a). \tag{57}$$

Neither the coordinates, nor the potential are unique. In these coordinates

$$\Pi = -2H_{,a\bar{b}}\, du^a \wedge du^{\bar{b}}, \tag{58}$$

and

$$P^\alpha_\beta = \left[\begin{array}{cc} \delta^a_b & 0 \\ 0 & -\delta^{\overline{a}}_{\overline{b}} \end{array} \right]. \tag{59}$$

It is a straight forward matter to compute the curvature tensor of the metric given by Eq.(57). Here it is convenient to employ the first and second sets of Cartan structure equations using the adapted coordinate basis of 1-forms $du^\alpha = (du^a, du^{\overline{a}})$. The first set of Cartan structure equations

$$\begin{aligned} \Gamma^\alpha_{\beta\gamma} du^\gamma &= \Gamma^\alpha_{\gamma\beta} du^\gamma, \\ dh_{\alpha\beta} &= h_{\alpha\sigma}\Gamma^\sigma_{\ \beta} + h_{\sigma\beta}\Gamma^\sigma_{\ \alpha}, \end{aligned} \tag{60}$$

reduces to the equation

$$dh_{a\overline{b}} = h_{c\overline{b}}\Gamma^c_{\ a} + h_{a\overline{c}}\Gamma^{\overline{c}}_{\ \overline{b}}, \tag{61}$$

and the only non-zero Levi-Civita connection 1-forms are

$$\begin{aligned} \Gamma^a_{\ b} &= \Gamma^a_{bc} du^c = h^{a\overline{d}} H,_{bc\overline{d}}\, du^c, \\ \Gamma^{\overline{a}}_{\ \overline{b}} &= \Gamma^{\overline{a}}_{\overline{b}\overline{c}} du^{\overline{c}} = h^{d\overline{a}} H,_{d\overline{b}\overline{c}}\, du^{\overline{c}}. \end{aligned} \tag{62}$$

Here $h^{a\overline{d}} = h^{\overline{d}a}$, and $h^{a\overline{d}} h_{\overline{d}b} = \delta^a_b$, $h^{d\overline{a}} h_{\overline{b}d} = \delta^{\overline{a}}_{\overline{b}}$.

The second set of Cartan structure equations

$$d\Gamma^\alpha_{\ \beta} + \Gamma^\alpha_{\ \sigma} \wedge \Gamma^\sigma_{\ \beta} = -\frac{1}{2} R^\alpha_{\beta\gamma\delta} du^\gamma \wedge du^\delta \tag{63}$$

reduce to

$$\begin{aligned} d\Gamma^a_{\ b} + \Gamma^a_{\ c} \wedge \Gamma^c_{\ b} &= -R^a_{bc\overline{d}} du^c \wedge du^{\overline{d}}, \\ d\Gamma^{\overline{a}}_{\ \overline{b}} + \Gamma^{\overline{a}}_{\ \overline{c}} \wedge \Gamma^{\overline{c}}_{\ \overline{b}} &= -R^{\overline{a}}_{\overline{b}\overline{c}d} du^{\overline{c}} \wedge du^d. \end{aligned} \tag{64}$$

The non-zero components of the curvature tensor in this basis are given by

$$\begin{aligned} R^a_{bc\overline{d}} &= -R^a_{b\overline{d}c} = \Gamma^a_{bc,\overline{d}}, \\ R^{\overline{a}}_{\overline{b}\overline{c}d} &= -R^{\overline{a}}_{\overline{b}d\overline{c}} = \Gamma^{\overline{a}}_{\overline{b}\overline{c},d}. \end{aligned} \tag{65}$$

The only components of the Ricci tensor, $R_{\beta\delta} = R^\alpha_{\beta\alpha\delta}$, of this curvature tensor which are not identically zero are

$$\begin{aligned} R_{b\overline{d}} &= R^a_{ba\overline{d}} = [h^{a\overline{c}}(H,_{a\overline{c}b})],_{\overline{d}} = [\ln|\det(H,_{a\overline{c}})|],_{b\overline{d}}\,, \\ R_{\overline{b}d} &= R^{\overline{a}}_{\overline{b}\overline{a}d} = [h^{\overline{a}d}(H,_{\overline{a}d\overline{b}})],_d = [\ln|\det(H,_{a\overline{c}})|],_{d\overline{b}}\,. \end{aligned} \tag{66}$$

In the Ricci-flat case the function H must satisfy a Monge-Ampère equation, $\det(H,_{a\overline{b}}) = \exp[B(u^{\overline{a}}) + C(u^a)]$, where B and C are arbitrary functions of their arguments. By choice of coordinates the right hand side can be set equal to a constant so that the condition of Ricci-flatness of the metric h becomes

$$\det(H_{,a\overline{b}}) = const. \tag{67}$$

It should be noted that the condition that the covariant derivative of P be zero is now also satisfied.

(It is also interesting to note, as an aside, the following. Let N be a four dimensional real sub-manifold of M, given locally by the level sets $u^a - u^{\overline{a}} = 0$. It follows from Eqs.(60) that the connection 1-forms Γ^a_b and $\Gamma^{\overline{a}}_{\overline{b}}$ both pull- back to define symmetric affine connections on N. When the holonomy group of the pulled-back connections is contained in SO(p,q), the connections are the Levi-Civita connections of 4-metrics on N of signature (p,q)[32]. Furthermore, Eqs.(64)–(66), and the appropriate pull-backs to N, imply that these connections are Ricci-flat when h is Ricci-flat. In the special cases where H is a function only of the four real variables $u^a + u^{\overline{a}}$, that is h has four Killing vector fields, $\partial/\partial u^a - \partial/\partial u^{\overline{a}}$, and h is also Ricci-flat, the 8-metric h pull back to 4-metrics on N given, in local coordinates u^a on N, by ${}_h ds^2_N = \frac{\partial^2 H}{\partial u^a \partial u^b}(du^a \otimes du^b + du^b \otimes du^a)$, where det $(\frac{\partial^2 H}{\partial u^a \partial u^b})$ is a constant which can be chosen to be one. Such 4-metrics belong to a class first considered by Calabi[35,36], and are formally similar to a class of metrics which are of interest in stochastic geometry[37]. In general they will not be Ricci-flat.)

So far only Eqs.(42)-(44) have been used in the main discussion in this section. Now consider the implementation of Eqs.(45)-(47). It follows from Eq.(45) that, in adapted local coordinates the almost complex structure I must have components of the form

$$I^\alpha_\beta = \begin{bmatrix} I^a_b & 0 \\ 0 & I^{\overline{a}}_{\overline{b}} \end{bmatrix}, \tag{68}$$

where $I^a_b I^b_c = -\delta^a_c$ and $I^{\overline{a}}_{\overline{b}} I^{\overline{b}}_{\overline{c}} = -\delta^{\overline{a}}_{\overline{c}}$. It follows from Eq.(46) that $h_{\alpha\gamma} I^\gamma_\beta = I_{\alpha\beta} = I_{\beta\alpha}$, where

$$I_{\alpha\beta} = \begin{bmatrix} 0 & I^c_a h_{c\overline{b}} \\ I^{\overline{c}}_{\overline{a}} h_{\overline{c}b} & 0 \end{bmatrix}. \tag{69}$$

Eqs.(47) now imply that the four dimensional integral manifolds of D_1 and D_2 each admit integrable complex structures I_1 and I_2 with components given respectively by I^a_b and $I^{\overline{a}}_{\overline{b}}$. In fact Eqs.(47) take the form

$$\begin{aligned} &\partial_c I^a_b - I^a_s \Gamma^s_{bc} + I^s_b \Gamma^a_{sc} = 0,\ \partial_{\overline{c}} I^a_b = 0, \\ &\partial_{\overline{c}} I^{\overline{a}}_{\overline{b}} - I^{\overline{a}}_{\overline{s}} \Gamma^{\overline{s}}_{\overline{b}\overline{c}} + I^{\overline{s}}_{\overline{b}} \Gamma^{\overline{b}}_{\overline{s}\overline{c}} = 0,\ \partial_c I^{\overline{a}}_{\overline{b}} = 0. \end{aligned} \tag{70}$$

Hence the four coordinates on each of D_1 and D_2 can respectively be chosen to be pairs of complex coordinates, $(z^A, \overline{z}^{A'})$ and $(z^{\overline{A}}, \overline{z}^{\overline{A'}})$, where $I_1\partial/\partial z^A = i\partial/\partial z^A$ and $I_2\partial/\partial z^{\overline{A}} = i\partial/\partial z^{\overline{A}}$, and $\overline{z}^{A'}$ and $\overline{z}^{\overline{A'}}$ denote the complex conjugate of z^A and $z^{\overline{A}}$ respectively. In order to fully satisfy Eqs.(42)-(47) Eqs.(69) and (70) must now be completely implemented. Eq.(69) and the requirement that the matrix $(I_{\alpha\beta})$ be symmetric imply that

$$H,_{A\overline{B'}} = H,_{A'\overline{B}} = 0. \tag{71}$$

It follows that ,without loss of generality in this context, the function H can be taken to be of the form

$$H = \mathbb{H} + c.c., \tag{72}$$

where $\mathbb{H}$ is a holomorphic function of z^A and $z^{\overline{A}}$ only. In these adapted complex coordinates, the metric h can now be written

$$_h ds^2 = \mathbb{H},_{A\overline{B}}\,(dz^A \otimes dz^{\overline{B}} + dz^{\overline{B}} \otimes dz^A) + c.c., \tag{73}$$

and the components of the complex structure I take the form

$$I^\alpha_\beta = \begin{bmatrix} i\delta^A_B & 0 & 0 & 0 \\ 0 & -i\delta^{A'}_{B'} & 0 & 0 \\ 0 & 0 & i\delta^{\overline{A}}_{\overline{B}} & 0 \\ 0 & 0 & 0 & -i\delta^{\overline{A}'}_{\overline{B}'} \end{bmatrix}. \tag{74}$$

It follows from Eq.(70) that, in these complex coordinates, the only non-zero components of the Christoffel symbols Γ^a_{bc} and $\Gamma^{\overline{s}}_{\overline{bc}}$ are the respective pairs of components (Γ^A_{BC} , $\Gamma^{A'}_{B'C'}$) and($\Gamma^{\overline{A}}{}_{\overline{BC}}$, $\Gamma^{\overline{A'}}_{\overline{B'C'}}$) where

$$\begin{aligned} \Gamma^A_B &= h^{A\overline{D}}\mathbb{H},_{B\overline{D}C}\, dz^C, \\ \Gamma^{\overline{A}}_{\overline{BC}} &= h^{\overline{A}D}\mathbb{H},_{\overline{B}D\overline{C}}\, dz^{\overline{C}}, \\ h^{A\overline{D}}\mathbb{H},_{B\overline{D}} &= \delta^A_B, h^{\overline{A}D}\mathbb{H},_{D\overline{B}} = \delta^{\overline{A}}_{\overline{B}}. \end{aligned} \tag{75}$$

and similarly for the complex conjugates $\overline{\Gamma}^{A'}_{B'C'}$ and$\overline{\Gamma}^{\overline{A'}}_{\overline{B'C'}}$. The curvature of the connection can be simply computed from these expressions as in Eqs.(64)-(66).

Since, by Eq.(25), the components of the metric k, $k_{\alpha\beta}$, are equal to $-I_{\alpha\beta}$, the metric k is given by

$$ {}_k ds^2 = -i\mathbb{H},_{A\overline{B}}\,(dz^A \otimes dz^{\overline{B}} + dz^{\overline{B}} \otimes dz^A) + c.c., \tag{76} $$

and $g = h + ik$ is a holomorphic metric with line element

$$ ds^2 = 2\mathbb{H},_{A\overline{B}}\,(dz^A \otimes dz^{\overline{B}} + dz^{\overline{B}} \otimes dz^A). \tag{77} $$

When the condition of Ricci-flatness, Eq.(67), is imposed on the metric h, it follows that

$$ \det(\mathbb{H},_{A\overline{B}}\,) = const. \tag{78} $$

Hence, as expected, g is also Ricci-flat. In fact, the last two equations correspond to a description of holomorphic half-flat 4-metrics given by Plebanski[3]. Therefore it follows that a triple (P, h, I) which satisfies Eqs.(42)-(47) and the condition of Ricci-flatness of h imply that the metric $g = h + ik$ is a holomorphic half-flat 4-metric.

Finally here, note the description of the Kähler structure $(J = -IP, h)$, implied by the structure (P, h, I), in these complex coordinates. It follows from Eqs.(59) and (74) that the components of J are given by

$$ J^{\alpha}_{\beta} = \begin{bmatrix} -i\delta^A_B & 0 & 0 & 0 \\ 0 & i\delta^{A'}_{B'} & 0 & 0 \\ 0 & 0 & i\delta^{\overline{A}}_{\overline{B}} & 0 \\ 0 & 0 & 0 & -i\delta^{\overline{A}'}_{\overline{B}'} \end{bmatrix}. \tag{79} $$

Consequently the Kähler 2-form is given by $[i(\mathbb{H},_{A\overline{B}} dz^A \wedge dz^{\overline{B}}) + c.c.]$; the metric h is given by Eq.(73).

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A Twistor spaces

In this appendix the relation between twistor spaces for hyper-Kähler metrics, 8-metrics on M, and half-flat holomorphic metrics on M is outlined.

Salamon's generalization of Penrose's non-linear graviton construction[24] (see also Hitchin et al[25]) to hyper-Kähler structures, can be used to encode the hyper-Kähler structures presented in section three in the complex structure of a twistor space, Z, where $\dim_{\mathbb{C}} Z$ is five. As a real manifold Z is $\mathrm{M}\times\mathrm{S}^2$. In the case of the real or imaginary parts of the anti self-dual half flat metrics on M, considered above, the complex structure on the corresponding twistor space Z_{ASD} is given by

$$I_{ASD} = (a_{\mathbf{i}}({}^{+}J^{\mathbf{i}}),\ I_0),$$
$$(a_{\mathbf{1}})^2 + (a_{\mathbf{2}})^2 + (a_{\mathbf{3}})^2 = 1,$$

and, following references 24 and 25, $\{a_{\mathbf{i}}\} \in \mathbb{R}$ here.

Here the tangent space at the point (m, ς) in Z_{ASD} has been expressed as the direct sum $T_m \oplus T_\varsigma$, and I_0 is the operation of multiplication by i on the tangent space T_ς of $\varsigma \in \mathrm{S}^2$. Furthermore S^2 is identified with CP^1, with local complex coordinate ζ, so that at $(a_{\mathbf{1}}, a_{\mathbf{2}}, a_{\mathbf{3}}) \in \mathrm{S}^2$, $(a_{\mathbf{1}}, a_{\mathbf{2}}, a_{\mathbf{3}}) = (\frac{1-\varsigma\bar{\varsigma}}{1+\varsigma\bar{\varsigma}}, \frac{\varsigma+\bar{\varsigma}}{1+\varsigma\bar{\varsigma}}, \frac{i(\varsigma-\bar{\varsigma})}{1+\varsigma\bar{\varsigma}})$. It is shown in detail in references 24 and 25 that the integrability conditions for the tensor I_{ASD} to define a complex structure on $\mathrm{M}\times\mathrm{S}^2$ are indeed satisfied, and furthermore that the hyper-Kähler structure on M can be recovered from the holomorphic properties of Z_{ASD}. The twistor space Z_{ASD} can also be identified with the primed projective spin bundle over M

In a similar way, now using ${}^{-}J^{\mathbf{i}}$, the hyper-Kahler structures associated with any half-flat self-dual holomorphic metric on M can be encoded in, and extracted from, a complex structure, I_{SD} say on $\mathrm{M}\times\mathrm{S}^2$, (this time the unprimed projective spin bundle over M), and the holomorphic properties of the resulting twistor space, Z_{SD}. As before the latter can be extracted from the former.

Although the results contained in references 24 and 25 enable real hyper-Kähler metrics to be derived from the structure of a twistor space Z this is not quite the same as deriving a holomorphic half-flat metric g from the properties of a twistor space of the above type. An additional complex structure on M satisfying certain properties is also required, as was noted in Propositions 1 and 2 of Sec. III. In summary, the holomorphic structure on the five complex dimensional twistor space Z (which determines a hyper-Kähler structure, say, $(\mathrm{M}, J^{\mathbf{i}}, h)$), and a complex structure I, on the four complex dimensional manifold M, which together satisfy the compatibility relations $h(IX, IY) = -h(X, Y)$, ${}_{h}\nabla I = 0$ and $IJ^i = J^{\mathbf{i}}I$, determine a half-flat holomorphic metric g. Hence they also determine the real metric k and the hyper-Kähler structure $(\mathrm{M}, J^{\mathbf{i}}, k)$.

By using results from Sec. III these statements can be re-expressed in terms of (integrable) almost product structures.

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