Geometry, Null Hypersurfaces and New Variables

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Abstract: Hamiltonian formulations of general relativity employing null hypersurfaces as constant time hypersurfaces are discussed. New variables approaches to the canonical formalism for general relativity are reviewed.

John Stachel has contributed significantly to the study of many technical, conceptual and historical problems in general relativity. His hospitality in Boston enabled me to start investigating this approach to Hamiltonian general relativity. It is a pleasure to dedicate this paper to him.

Over the last fifty years the Hamiltonian formulation of general relativity has been the subject of many investigations. These have been intimately related to the study of initial value problems, cosmological models, asymptotically flat systems, gravitational waves and more recently black holes. However, the principal preoccupation of much of the research in this area has been the attempt to formulate a quantum theory of gravity. Attempts to construct a comprehensive quantum theory which would naturally incorporate gravity, such as M-theory [1] and its precursor supersymmetric string theory, are currently being vigorously and profitably pursued. However, in the last decade, there have also been significant advances in a new approach to the canonical quantisation of general relativity. Research initiated by Ashtekar in the 1980's, and his introduction of new canonical variables, focussed attention on connection rather than metric dynamics [2- 5]. Ashtekar's reformulation of Hamiltonian general relativity encouraged the application to gravity of techniques which had become well established in gauge theory. Development of these ideas has led to the establishment of loop quantum gravity and advances in a rigorous approach to a non-perturbative theory of quantum gravity (for recent reviews see [6], [7]). This programme has generated a significant amount of new and interesting activity in the canonical quantisation approach to quantum general relativity. Currently it provides the weightiest complement to the ambitious M-theory programme.

The new variables canonical formalism, introduced by Ashtekar, can be obtained by a complex canonical transformation from a geometrodynamical approach to Hamiltonian general relativity. Alternatively it can be constructed, via a Legendre transformation, by starting with a complex, chiral, first order Lagrangian [2-5, 31]. The new canonical variables are the components of a triple of weight one vector densities and an so(3)-valued connection on a three manifold. The Hamiltonian is a polynomial of low degree but complex-valued. It is completely constrained. The seven first class constraint terms appearing in it generate canonical transformations corresponding to space-time diffeomorphisms and internal rotations. Their Poisson bracket algebra is not a Lie algebra. In order to recover the real phase space, and real general relativity, reality conditions have to be imposed. However both the constraint equations and the reality conditions are low degree polynomial equations. When the reality conditions are satisfied and the real vector densities are linearly independent, the three manifold can be embedded as a space-like hypersurface in a real Lorentzian four manifold. The connection can then be interpreted as the self-dual part of the Levi-Civita (spin) connection of the metric. The polynomial nature of the formalism, together with the emphasis on connection rather than metric dynamics, have suggested new developments in both classical and quantum theory. In recent years, real Lorentzian and Euclidean connection based versions of the formalism have been employed in the quantum gravity programme. However only the (initially) complex formalism, which is geometrically natural in the Lorentzian context, will be discussed in this paper.

A central aim of this paper is to review a version, based on null rather than space-like hypersurfaces, of Ashtekar's approach to classical canonical general relativity [8-11]. Salient reasons for constructing the null hypersurface new variables formalism include the successful use of null hypersurfaces in canonical formulations of field theories on Minkowski space-time [12], [13]. Of course, in general relativity space-time geometry is dynamical, not merely flat and kinematical as it is Minkowski space-time. Hence it does not play merely a background role. This gives rise to technical difficulties as well as conceptual problems, such as the identification of observables and the problem of time [14]. Nevertheless the use of null hypersurfaces has also proven extremely useful in general relativity. Noteworthy and relevant here are studies of gravitational radiation and asymptotically flat systems. Useful comments on work using null hypersurfaces can be found in the recent paper by Bartnik [15]. A key reason for the successful use of null hypersurfaces in the study of field theories is that constraints arising in the relevant characteristic initial value problems can be integrated more easily than those arising in corresponding Cauchy problems. The true degrees of freedom can be exposed more explicitly when null hypersurfaces, rather than space-like hypersurfaces, are used.

Amongst the geometrical features which distinguish null hypersurface based formulations from space-like hypersurface based approaches to Hamiltonian gravity, are the following. First, the geometry of null hypersurfaces, unlike that of space-like hypersurfaces, is not metric. The singular metric induced on a null hypersurface from the space-time has maximal rank two, and the null geodesic generators of the null hypersurface are characteristic vector fields of the singular metric. Connections are also induced onto null hypersurfaces from connections on space-time, but they are not Levi-Civita connections of a three metric. In the initial value or Hamiltonian formalism the relevant structure group becomes the (4 real parameter) null rotation sub-group of the Lorentz group, rather than the three parameter special orthogonal group. Partially as a consequence, the formalism can appear less manifestly covariant than it is in the space-like case. The focussing effect of gravity causes the null geodesic generators of null hypersurfaces to cross over and to form caustics. At such points null hypersurfaces fail to be (immersed) smooth submanifolds of space-time, [16, 17]. Generically, topologically simple null hypersurfaces, such as the null planes and null cones of Minkowski space-time, are not available. Problems related to the differential and metric structure arise also in the space-like hypersurface formalisms, but they tend to be more overt in the null hypersurface case.

This paper will deal mainly with local aspects of canonical formalisms. Thus hypersurfaces will be treated as sub-manifolds with a fixed differential structure. Although the emphasis will be on local theory, it should be noted that it is usually assumed (at least implicitly) in null hypersurface Hamiltonian general relativity, that the global context is one in which the null hypersurface leaves of the space-time foliation are non-compact. It is also implicitly assumed here that the condition of asymptotic flatness (or asymptotically cosmological boundary conditions) can be imposed, and that future null infinity can be defined [18]. Normally here, fields are considered to be defined on outgoing null hypersurfaces.

In the next section a brief review of research related to null hypersurface canonical formulations of general relativity will be presented. This aims only to provide a context within which the results of later sections can be placed. It does not pretend to be comprehensive. The third section will contain an outline of Ashtekar's new variables Hamiltonian formulation of general relativity. This will be presented in such a way that the similarities to, and differences from, the null hypersurface based new variables formalism are highlighted. It will suffice in this paper to consider details of the two formalisms when there are no matter fields coupled to gravity. The null hypersurface new variables canonical formalism will be discussed in the final section. In particular a Hamiltonian system which is completely equivalent to Einstein's vacuum equations will be presented. The structure of the constraints and reality conditions will also be reviewed. Finally the space-time interpretation will be outlined. The body of results on the null hypersurface based new variables canonical formalism is neither as large, nor as developed, as that obtained within the conventional framework where the level sets of the time function are space-like. However the null hypersurface new variables Hamiltonian formulation is more complete than other null hypersurface based canonical formulations of general relativity.

The notation and conventions of Ref [9] will be followed without major exception. Lower case Latin indices i,j,k sum and range over 1,2,3 and indicate components with respect to coordinate bases. Upper case Latin indices A,B sum and run over 1,2,3 and label components of so(3) Lie -algebra valued valued geometrical objects. Lorentzian manifolds will have four metrics with signature (1,-1,-1,-1).

1 Null hypersurface canonical formalisms

Dirac pointed out in 1949, [19], that it is physically natural to use a time parameter with level sets which are null hypersurfaces. A particularly relevant example of the use of this method of describing time evolution in Minkowski space-time is provided by one approach to the canonical formalism for source free Maxwell and Yang-Mills theory. This uses a first order complex Lagrangian, with the potential (connection) and self-dual part of the field (self-dual curvature) as field variables. A Hamiltonian is constructed via a 3+1 decomposition of Minkowski space-time using outgoing null cones or null hyperplanes. It incorporates many of the features which are encountered in the general relativistic null hypersurface new variables formalism, some of which will be discussed in the fourth section. It is also sufficiently simple that it can be carried out completely which, to date, is not the case for general relativity. Consequently it is instructive to outline the results in the case of the Yang-Mills field, with internal symmetry group a real n-parameter semi-simple Lie group, [20]. A 3+1 decomposition of the first order Lagrangian, using a null hyperplane foliation, leads by inspection to a complex Hamiltonian defined on a null hyperplane. There are 2n first class constraints and 4n second class constraints. The Yang-Mills field equations are equivalent to the constraint and evolution (Hamilton's) equations. The canonical transformations which correspond to the infinitesimal internal gauge transformations are generated by the first class constraints. The first class constraint algebra represents the internal symmetry Lie algebra. Second class constraints arise because null hypersurfaces are being used. The second class constraint equations relate physically redundant variables. Solution of these leads to the elimination of an equal number of canonically conjugate pairs of phase space variables. On the resulting partially reduced phase space there remain only the first class constraints, reflecting the gauge invariance of the theory. An alternative and equivalent way of handling second class constraints is to introduce Dirac brackets, either directly or via the starred variables procedure of Bergmann and Komar, see e.g. [12]. This can be simpler than solving the second class constraint equations directly and is the procedure followed in [20]. There the complete reduced phase space is realized as a Poisson space by gauge fixing so that all constraints become second class. The Poisson structure on the completely reduced 2n dimensional phase space is defined by the Dirac bracket and the constraints can be set equal to zero in the Hamiltonian. The dynamics are determined by the Dirac brackets of the 2n independent degrees of freedom with the reduced Hamiltonian. The imposition of reality conditions enables the real theory to be recovered and a reduced phase space quantization to be effected, (for details in the similar but simpler case of the Maxwell field see [21-23]).

Early investigations of null hypersurface based canonical formulations of general relativity were carried out mainly within geometrodynamical frameworks and dealt with the vacuum equations; for example see [24-30] . Since a null hypersurface time parametrization was used there were second class constraints as well as the first class constraints associated with gauge freedom. They were non-polynomial in nature. These non-polynomial canonical formalisms, their constraint structures and their dynamics, were not as straightforward to analyse as had been hoped. The complete formulation of the dynamics on the completely reduced phase space was not satisfactorily achieved. Furthermore this work used as a coordinate t, a time function with level sets which were assumed to be null hypersurfaces. Consequently the inverse metric components g^{tt} were assumed to be zero. Hence the Einstein equations, $G_{tt} = 0$ could not be naturally derived, either in the 3+1 decomposed Lagrangian or in the Legendre transformation related Hamiltonian formalism. This research did however provide some useful technical and conceptual insights. It suffices to mention here two significant differences which were revealed between the constraint structure of spacelike hypersurface based and null hypersurface based canonical formalisms. Torre, using a 2+2 formalism in an investigation of a null hypersurface-based geometrodynamical formulation, gave a particularly clear and comprehensive discussion of these, [29]. In the space-like case the Hamiltonian is a linear combination of first class constraints. In the null-hypersurface based case it was found that, while the Hamiltonian is again a linear combination of constraints, not all of them are first class. Furthermore in the space-like case the scalar constraint which generates time translations out of the hypersurface is first class. However there are no compact deformations which map a null hypersurface to a neighbouring null hypersurface. The choice of a null time parameter is effectively a choice of gauge. As a consequence it was found that the scalar constraint was second class.

The comparative tractability of constraints on null hypersurfaces, the low degree polynomial nature of the new variables formalism first introduced by Ashtekar, and the focus on connection rather than metric dynamics, suggested that a new variables formalism based on null hypersurfaces might be more successful than previous null hypersurface Hamiltonian constructions. Unlike the Hamiltonian systems mentioned earlier, all the Einstein equations arise naturally in the new formulation, [8-11]. This is achieved by adding Lagrange multiplier terms to a Lagrangian density derived from the self-dual first order Lagrangian mentioned above, [2-5, 31]. The corresponding Euler Lagrange equations then include all the Einstein (and matter) field equations plus equations which identify level sets of constant time as null hypersurfaces. The Hamiltonian, obtained by a Legendre transformation, consequently also contains all this information. It is completely constrained but not all the constraints are first class. There are a large number of second class constraints as there always are in null hypersurface based formalisms. Analysis of the constraint structure reveals it to be rather different from that of the space-like hypersurface based Hamiltonian formalism. The first class constraints form a Lie algebra and the scalar constraint is second class. This is a distinctive feature of this formalism; the Lie algebra can be interpreted as a representation of the semi-direct product of the null rotation sub-group of the Lorentz group and the diffeomorphisms within the null hypersurface. Both constraints and reality conditions are polynomial of low degree. In addition to the work on the vacuum case, coupled Einstein-matter systems have been analysed by Soteriou, [20]. He extended the vacuum formalism to gravitational systems which included scalar fields, Yang-Mills fields and Grassmann valued Dirac fields. In the latter a two-component spinor formalism was used. Such extensions are essentially unproblematic, as had previously been shown in the space-like case, [5,32]. Boundary conditions have been examined, [10], and some progress has been made towards the elimination of the second class constraints, [11]. The construction of the completely reduced phase space, [11], has yet to be completed as successfully as it has been in, for example, the Yang-Mills canonical formalism discussed above. Consequently, the linearisation of the theory about a Minkowski space-time was investigated, using both Minkowski null hyperplanes and outgoing Minkowski null cones, [20], (for a geometrodynamical formulation see [33]). In this case, as might be expected, the programme could be completed along lines similar to those outlined above for Yang-Mills fields in Minkowski space-time. Despite the fact that not all problems have been resolved in practice, this canonical formalism has been developed more fully than any other null hypersurface based Hamiltonian formulation of general relativity; in part this is because of its polynomial nature. In the first approaches to the null hypersurface, new variables canonical formalism, [8, 9, 20], the full Dirac-Bergmann algorithm, [12, 13], was applied. A large phase space resulted. The framework used was broad enough to ensure that the complete structure of the new formulation could be properly analysed. Once this was clearly understood a streamlined version, which employs only connection and vector density triad components as canonically conjugate variables, could be used, [10, 11]. It is that formalism which will be discussed in more detail in section 4.

In recent years there have been a number of other approaches to new variables formulations of general relativity on a null hypersurface. These include the application of multisymplectic techniques, as used in jet bundle formulations of classical field theories, to the new variables null hypersurface formalism. In this work the properties of multi-momentum maps and their relation to the constraint analysis are discussed, [34, 35]. The Hamiltonian formulation of general relativity on a null hypersurface has also been considered within the context of teleparallel geometry, [36]. Finally it should be noted that a 2+2 (Lagrangian) formulation using new variables has been considered by d'Inverno and Vickers, [37]. This work focuses on the important case of a double null foliation of space-time and their discussion mirrors, to an extent, the null hypersurface new variables formalism. It includes a discussion of the relationship between different formalisms.

2 New variables and spacelike hypersurfaces

Let M be a real three dimensional manifold with local coordinates x^i. Let $\Sigma^i_A\partial_i$ be three weight one vector densities tangent to M, and let $A^A_i dx^i$ be a so(3) -valued connection 1- form on M with curvature 2-form R^A. The new variables Hamiltonian for vacuum general relativity can be written as

$$H = -2 \int d^3x (\frac{1}{2}\mathbf{N}\mathcal{H} + N^i \mathcal{H}_i + B^A \mathcal{G}_A). \qquad (3.1)$$

Here $\mathbf{N}$, N^i and B^A are Lagrange multipliers, with $\mathbf{N}$ a weight minus one density. The canonically conjugate dynamical variables are taken to be the pair $(A_i^A,\ \Sigma_A^i)$ and the only non-zero Poisson brackets between these variables are

$$\{A_i^A(x), \Sigma_B^j(y)\} = \delta_B^A \delta_i^j \delta(x,y) \qquad (3.2)$$

The Hamiltonian density, Lagrange multipliers and dynamical variables can be regarded as being complex valued so that the phase space is complex. The scalar, vector and Gauss constraint equations are

$$\begin{aligned} &\mathcal{H} \equiv i\varepsilon_A^{\ BC} R_{ij}^A \Sigma_B^i \Sigma_C^j = 0, \\ &\mathcal{H}_i \equiv -R_{ij}^A \Sigma_A^j = 0, \quad \mathcal{G}_A \equiv -iD_i \Sigma_A^i = 0. \end{aligned} \qquad (3.3)$$

Here

$$R^A = -\frac{1}{2} R_{ij}^A dx^i \wedge dx^j = (\partial_i A_j^A + \varepsilon^A_{\ BC} A_i^B A_j^C) dx^i \wedge dx^j, \qquad (3.4)$$

D_i is the covariant derivative of the connection A_i^A and $\varepsilon_{ABC} = \varepsilon_{[ABC]}$ with $\varepsilon_{123} = 1$. In the basis used the SO(3)- invariant metric is given by

$$\eta_{AB} = diag(-1,-1,-1). \qquad (3.5)$$

All these constraints are first class although the constraint algebra is not a Lie algebra. Appropriately combined they generate canonical transformations corresponding to space-time diffeomorphisms and (small internal) gauge rotations. They are transparently low degree polynomials in the canonical variables. Dynamical evolution is determined by Hamilton's equations

$$\begin{aligned} \partial_t \Sigma_A^i &= \{\Sigma_A^i, H\}, \\ \partial_t A_i^A &= \{A_i^A, H\}. \end{aligned} \qquad (3.6)$$

When the vector densities Σ^i_A are linearly independent they determine a Riemannian 3-metric on M, with inverse metric

$$g^{ij} = -\nu^{-2}\eta^{AB}\Sigma^i_A\Sigma^j_B, \tag{3.7}$$

where

$$\nu^2 = -i\varepsilon_{ijk}\Sigma^i_1\Sigma^j_2\Sigma^k_3. \quad ((3.8))$$

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In this case, on the four manifold M$\times\mathbb{R}$ with local coordinates (t, x^i), the tetrad of 4-vector fields

$$e_0 = \nu^{-1}\mathbf{N}^{-1}(\partial_t - N^i\partial_i) \;\text{ and }\; e_A = -i\nu^{-1}\Sigma^i_A\partial_i \tag{3.9}$$

forms an orthonormal basis for a 4-metric with inverse $e_0 \otimes e_0 + \eta^{AB}e_A \otimes e_B$, so

$$\begin{aligned}\partial^2/\partial s^2 &= \nu^{-2}\mathbf{N}^{-2}\partial_t \otimes \partial_t - \nu^{-2}\mathbf{N}^{-2}N^i(\partial_t \otimes \partial_i + \partial_i \otimes \partial_t) \\ &\quad + \frac{1}{2}(\nu^{-2}\mathbf{N}^{-2}N^iN^j - \nu^{-2}\eta^{AB}\Sigma^i_A\Sigma^j_B)(\partial_i \otimes \partial_j + \partial_j \otimes \partial_i).\end{aligned} \tag{3.9}$$

The volume 4-form is $\nu^2\mathbf{N}dt \wedge dx^1 \wedge dx^2 \wedge dx^3$. The 1-form,

$$\Gamma^A = A^A_i dx^i + B^A dt, \tag{3.11}$$

corresponds to the self-dual part of the Levi-Civita (spin) connection of this metric. The constraint and evolution equations given above then correspond to Einstein's vacuum equations for the metric. The pull back of the curvature of this connection, to a leaf of the foliation by level sets of t, can then be identified with the 2-form R^A. Real Lorentzian general relativity can be recovered when polynomial reality conditions, given by requiring that $\Sigma^i_A\Sigma^j_B\eta^{AB}$ and $\{\Sigma^i_A\Sigma^j_B\eta^{AB}, H\}$ be real, are imposed on the phase space variables. Agreement with the standard ADM geometrodynamical version of Hamiltonian relativity follows when the constraint, evolution and reality equations are satisfied. The standard geometrodynamical field variables can then be identified with the three metric g_{ij} (with Levi-Civita so(3)-valued connection $\gamma^A_i dx^i$), and the symmetric extrinsic curvature tensor $K_{ij} = -\nu^{-1}g_{ik}\Sigma^k_A(A^A_j + \gamma^A_j)$, on a space-like hypersurface given by a level set of t,[2-5].

3 New variables canonical formalism for general relativity on a null hypersurface

In the new variables, null hypersurface Hamiltonian formalism, presented in Ref [10], the canonically conjugate phase space variables are again defined by the components of a triple of weight one vector densities Σ_A and the components of an so(3) valued connection one form A^A on a three manifold M. The Poisson bracket relations are given as in Eq (3.2). The complex Hamiltonian is, however, now given by

$$H = \int d^3x(\mathbf{N}\mathcal{H} + N^i\mathcal{H}_i - B^A\mathcal{G}_A - \mu_i\mathcal{C}^i - \rho\alpha^2), \tag{4.1}$$

The functions $\mathbf{N}$ (a scalar density of weight one), N^i, B^A, v^i, μ_i and ρ are treated as Lagrange multipliers and

$$\begin{aligned} \mathcal{H} &\equiv \mathrm{v}^i(R^1_{ij}\Sigma^j_3 + R^2_{ij}\Sigma^j_1) \equiv \mathrm{v}^i\phi_i, \\ \mathcal{H}_i &\equiv -R^A_{ij}\Sigma^j_A, \quad \mathcal{G}_A \equiv D_i(\Sigma^i_A), \\ \mathcal{C}^i &\equiv \Sigma^i_2 + \alpha\mathrm{v}^i. \end{aligned} \tag{4.2}$$

The evolution equations are given by Hamilton's equations, as in Eq (3.6).

The Hamiltonian density is a linear sum of the product of Lagrange multipliers with terms which all vanish when constraint equations and multiplier equations are satisfied. The constraints, $\mathcal{C}^i = 0$, $\alpha^2 = 0$, and the multiplier equation $\mu_i\mathrm{v}^i = 0$, arise from Lagrange multiplier terms originating in the Legendre transformation related Lagrangian formalism. Added to a chiral Lagrangian for Einstein's vacuum equations they ensure that this is a correctly set null hypersurface based formalism, and that the canonical formalism contains all the Einstein gravitational (and, when present matter) field equations. Here α is not treated as an independent dynamical variable; to do so would merely result in further trivial second class constraints which would lead to its ultimate elimination from a reduced phase space. It is set equal to zero after Poisson brackets have been computed. Its vanishing mirrors the fact that this is a null hypersurface based formalism and implies, together with the constraints, $\mathcal{C}^i = 0$, the vector density degeneracy condition, $\Sigma^i_2 = 0$. The latter condition mirrors the degeneracy of a basis of self-dual 2-forms pulled back to a null hypersurface in space-time. The second multiplier equation above, $\mu_i \mathrm{v}^i = 0$, is required in order to obtain all the Einstein equations from the formalism. It arises naturally from the Lagrangian formalism. The vanishing of ϕ_i, the coefficients of the multipliers $\mathbf{N}\mathrm{v}^i$, results in three constraint equations. The scalar, vector and Gauss constraint equations are given by $\mathcal{H}$ =0, $\mathcal{H}_i$=0 and $\mathcal{G}_A = 0$, as in section 3. It can be seen from the above that $\mathcal{H}$ =$\mathrm{v}^i\phi_i$ and $\Sigma_1^i\phi_i = \Sigma_3^i\mathcal{H}_i$. Hence there are eleven independent constraints above. Propagation of the constraints lead to secondary conditions on Lagrange multipliers. An important example of the latter, exemplifying differences from the results of section 3, follows from the propagation of $\mathcal{C}^i = 0$, that is evolution of the vector density degeneracy condition. This leads to three secondary conditions on the multipliers v^i and B^3, given by

$$\chi^i \equiv 2\delta_2^B D_j\{\mathbf{N}v^{[i}\Sigma_A^{j]}(\delta_B^1\delta_3^A + \delta_B^2\delta_1^A)\} - 2A_j^3N^{[i}\Sigma_1^{j]} - B^3\Sigma_1^i = 0. \tag{4.3}$$

The eleven constraints comprise six second class constraints and five first class constraints. The independent first class class constraints can be identified with

$$\mathcal{G}_1 = 0;\ \mathcal{G}_2 = 0;\ \mathcal{H}_i' \equiv \mathcal{H}_i - A_i^B D_j(\Sigma_B^j) = 0. \tag{4.4}$$

There are 18 phase space variables, 6 second class constraints and 5 first class constraints. Consequently, at each point of M, there are 2 independent degrees of phase space freedom. The canonical transformations generated by the constraints functions $\mathcal{G}_1$ and $\mathcal{G}_2$ correspond to self dual null rotations, reflecting the fact that the formalism is chiral (self-dual), complex and null hypersurface based. The constraints $\mathcal{H}_i^{'}$ generate diffeomorphisms in M. The scalar constraint is second class. A null hypersurface time gauge is effectively set within the formalism and the problem of time does not appear in the way it does in the space-like formalism. The algebra of first class constraints is a genuine Lie algebra, in contrast to the spacelike based formalism in section 3. The formalism is again polynomial.

As was the case in the previous section, when a non-degeneracy condition is satisfied, the constraint, evolution and multiplier equations are equivalent to the Einstein vacuum equations. More explicitly, when the weight one vector density fields tangent to M, $\mathrm{v}^i\partial_i$, $\Sigma_1^i\partial_i$, $\Sigma_3^i\partial_i$, are linearly independent so that

$$\nu^2 \equiv \Sigma_1^i \mathrm{v}^j \Sigma_3^k \varepsilon_{ijk} \tag{4.5}$$

is non-zero, a (non-degenerate) four metric can be constructed on M$\times\mathbb{R}$. Its inverse is given by

$$\partial^2/\partial s^2 = e_0 \otimes e_1 + e_1 \otimes e_0 - e_2 \otimes e_3 - e_3 \otimes e_2, \text{ and} \tag{4.6}$$

$$\begin{aligned} \nu^2\partial^2/\partial s^2 = &-2\alpha\mathbf{N}^{-2}\partial_t \otimes \partial_t + (2\alpha\mathbf{N}^{-2}N^i - \mathbf{N}\Sigma_1^i)(\partial_t \otimes \partial_i + \partial_i \otimes \partial_t) \\ &+ (\alpha\mathbf{N}^{-2}N^iN^j - \mathrm{v}^i\Sigma_3^j - \mathbf{N}^{-1}\Sigma_1^iN^j)(\partial_i \otimes \partial_j + \partial_j \otimes \partial_i). \end{aligned} \tag{4.7}$$

Here the natural basis of vector fields is a null tetrad given by

$$\begin{aligned} &e_0 = \nu^{-1}\mathbf{N}^{-1}(\partial_t - N^i\partial_i), \quad e_1 = -\alpha\nu^{-1}\mathbf{N}^{-1}\partial_t + (\alpha\nu^{-1}\mathbf{N}^{-1}N^i - \nu^{-1}\Sigma_1^i)\partial_i, \\ &e_2 = \nu^{-1}\mathrm{v}^i\partial_i, \quad e_3 = -\nu^{-1}\Sigma_3^i\partial_i, \end{aligned} \tag{4.8}$$

and the volume 4-form is $-i\mathbf{N}\nu^2 dt \wedge dx^1 \wedge dx^2 \wedge dx^3$. When $\alpha = 0$, the level sets of t are null hypersurfaces and the null tetrad (and its null co-frame) is adapted to the null hypersurface foliation with e_1 tangent to the null geodesic generators of the null hypersurfaces. The connection 1-form, given as in Eq (3.11), is the self-dual part of the Levi-Civita (spin) connection of the metric. When the remaining multiplier equations, the constraint and the evolution equations are satisfied, the metric satisfies Einstein's vacuum equations. Real relativity can be extracted from the complex phase space description by imposing appropriate polynomial reality conditions, [9, 20]. It suffices to record here that v^i can then be chosen to be the complex conjugate of Σ_3^i; Σ_1^i, ν and $\mathbf{N}$ can be chosen to be pure imaginary; finally N^i can be chosen be chosen to be real. The null tetrad then satisfies the standard reality conditions: e_0 and e_1 real, e_2 and e_3 complex conjugates.

Starting from the Hamiltonian formalism the properties of the constructed space-times emerge from those of the phase-space geometry. The form of the Hamiltonian determines whether M should be embedded in a space-time as a space-like or null hypersurface. Space-time properties such as asymptotic flatness, and hypersurface related properties such as crossovers and caustics, arise from solutions of the differential equations of the canonical formalism subject to boundary conditions. The latter have not been discussed in this paper but extension of this work to include fields on future null infinity and hence a complete global formulation is, in principle, straightforward, (c.f. calculations by Goldberg in a geometrodynamical framework in Ref [28] and discussions of boundary conditions in Ref [10]). The formalisms discussed in sections 3 and 4 encompass theories which may be real or complex and which allow the possibility of degenerate inverse metric densities. The latter arise when $\nu^2 = 0$ and can be identified with the expressions for $\nu^2 \partial^2/\partial s^2$ in Eq (3.9) and Eq (4.7). This possiblity has been discussed in the spacelike case, e.g. in [38], and merits investigation in the null hypersurface formalism.

In order to obtain a complete and satisfactory canonical formalism the constraints must be solved and Dirac brackets calculated in one of the ways indicated in section 2. The progress that has been made in this area ([10], [11]) indicates that the canonically conjugate reduced complex phase space variables are A_2^B and Σ_D^2, with $B = 3$ and $D = 3$. In the linearised theory the imposition of the reality conditions, and the complete fixing of coordinate and gauge freedom, leads to the breaking of the chiral nature of the formalism. The completely reduced phase space and final Hamiltonian are real and the Poisson structure is provided by the Dirac bracket. These calculations, by Soteriou, [20], support the conclusion that, after removal of the gauge and coordinate freedom, the two real degrees of freedom on the completely reduced phase space can be identified with the real and imaginary parts of the connection component A_2^3. The latter has a spacetime interpretation as the shear of the null hypersurface geodesic ray generators. Work in progress, on the geometry of null hypersurfaces and connections with values in the Lie algebra of the group of null rotations, may help clarify these matters.

The approach to the new variables, null hypersurface canonical formalism presented above incorporates the components of a self-dual connection as dynamical variables. The Lagrange multiplier techniques used in [9-10] could also be employed to develop a complete, real, null hypersurface based, geometrodynamical canonical formalism. It would expected that this would be related to the above by a complex canonical transformation.

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