Generalized forms, vector fields and superspace

D C Robinson
Mathematics Department
King's College London
Strand, London WC2R 2LS
United Kingdom
email:david.c.robinson@kcl.ac.uk

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Abstract: Vector fields with components which are generalized zero-forms are constructed. Inner products with generalized forms, Lie derivatives and Lie brackets are computed. The results are shown to generalize those reported for generalized vector fields. Generalized affine connections and metrics are defined and the fundamental theorem of Riemannian geometry is extended. The global structure of the exterior derivative of generalized forms is investigated.

1 Introduction

Generalized forms have been applied in a number of geometrically related areas of physics. By extending the algebra and calculus of ordinary differential forms new points of view about a number of different geometrical and physical systems have been obtained. For example in twistor theory forms of negative degree were introduced in order to try to extend twistor results on half-flat space-times and to associate an abstract twistor space with general analytic solutions of Einstein's vacuum field equations [1] [2] [3], field theories, such as BF, Yang- Mills and gravity theories have been reformulated as generalized topological field theories with generalized Chern-Pontrjagin and Chern-Simons forms as Lagrangians [4] [5], [6], [7] ,and generalized differential forms have been related to forms on path space [8]. Further applications are contained in a series of papers, [9] to [12], [6] and [7], devoted to the development of the formalism of generalized forms. Generalized differential forms were extensively studied in these latter papers but they dealt only with ordinary vector fields. Some interesting progress going beyond ordinary vector fields was made in [13] and [14] where the concept of a generalized vector field was introduced. In this paper the study of vector fields is continued and their work is extended. First a dictionary between the algebra and calculus of certain functions and vector fields on a superspace and the algebra and calculus of generalized forms and vector fields is established. This dictionary is not only useful in its own right but it also facilitates the introduction of the concept of a generalized form-valued vector field. In the case considered here such an object is determined by an ordered pair consisting of of an ordinary vector field and a $(1, 1)$ type tensor field. This concept includes generalized vector fields as a special case and provides an improved understanding of their properties. The introduction of generalized form-valued vector fields also enables generalized metrics and affine connections to be defined by constructions which can be extended to more general connections.

A brief review of the algebra and differential calculus of generalized forms needed in this paper is given in the second section. Different types of generalized differential forms, on an n dimensional manifold M, are labelled by a non-negative integer N. In this paper only the case where $N = 1$ is considered but the results are easily extendible to $N \geqq 2$. A type $N = 1$ generalized $p-$form is defined by an ordered pair consisting of two ordinary forms of degrees p and $p + 1$ respectively, where $-1 \leqq p \leqq n$. The module of type $N = 1$ generalized $p-$forms on M is denoted $\Lambda^p_{(1)}(M)$. The exterior product for generalized forms makes the vector space of type $N = 1$ forms at a point x in M, $\Lambda^{\bullet}_{(1)}(x) = \oplus_{p=-1}^{p=n} \Lambda^p_{(1)}(x)$, into an associative algebra, in fact a super-commutative graded algebra. Generalized forms of degree zero form a commutative ring with $1 \neq 0$. The graded module, and super-commutative graded algebra over the ring of smooth functions, on M is equipped with an exterior derivatives, $d : \Lambda^p_{(1)}(M) \to \Lambda^{p+1}_{(1)}(M)$, a super-derivation of degree one. While both the exterior algebra and differential calculus satisfied by generalized forms are similar to the algebra and calculus of ordinary forms there are some differences. For instance, generalized forms of degree $p = -1$ are allowed and the generalized de Rham cohomology can be different from the de Rham cohomology of ordinary forms.

The actions of ordinary vector fields on generalized forms on M presented previously, [10], are summarized in section three. In the fourth section the algebra and calculus of generalized forms and the actions of ordinary vector fields on M are represented on the Whitney sum of a reverse parity trivial line bundle and the reverse parity tangent bundle over M. This extends to generalized forms a known approach to ordinary differential forms, [16]. This point of view is employed in the fifth section where generalized form-valued vector fields are introduced and their properties explored. The definitions of the interior products and Lie derivatives of generalized forms with respect to such vector fields and the definition of a Lie bracket are given, extending the results of section three from ordinary vector fields to generalized form-valued vector fields. Generalized vector fields, which were introduced in [13] and [14] and applied to the Hamiltonian formalism for a free relativistic particle, are discussed and shown to form a sub-class of generalized-form valued vector fields. Two examples of the use of generalized form-valued vector fields are presented, one introducing generalized form-valued Hamiltonian vector fields. An application is given in the sixth section where generalized form-valued vector fields are used in the construction of the tensor calculus of generalized affine connections and metrics. The compatibility conditions of generalized affine connections and generalized metrics are presented and an extension of the fundamental theorem of Riemannian geometry is obtained. The seventh section contains a brief summary of the results and an outline of ways in which they can be used and developed. Finally there is an appendix in which the global structure of exterior derivatives of type $N = 1$ forms is discussed.

The results in this paper can apply to manifolds and geometrical objects that are real or complex but in this paper it will be assumed that the geometry is real, all geometrical objects are smooth and M is an $n-$dimensional real, smooth, orientable and oriented manifold. Bold-face Roman letters are used to denote generalized forms and generalized vector fields, ordinary forms on M are usually denoted by Greek letters and ordinary vector fields on M by lower case Roman letters. Occasionally the degree of a form is indicated above it. The exterior product of any two forms, for example α and β, is written $\alpha\beta$, and as usual, any ordinary $p-$form $\overset{p}{\alpha}$, with p either negative or greater than n, is zero. The Einstein summation convention is used.

2 Algebra and calculus of generalized forms

The algebraic and differential properties of generalized forms are outlined in this section using the notation of [6] and [7]. In this paper generalized forms will be expressed in terms of a minus one-form which is linearly independent of ordinary forms on M, [6]. Hence a basis for type $N = 1$ generalized forms consists of any basis for ordinary forms on M augmented by a minus one-form $\mathbf{m}$. Apart from having a degree of minus one the latter has the same algebraic properties as an ordinary exterior form. It satisfies the ordinary distributive and associative laws of exterior algebra and the exterior product rule

$$\overset{p}{\alpha}\mathbf{m} = (-1)^p \mathbf{m}\overset{p}{\alpha};\ \mathbf{m}^2 = 0, \tag{1}$$

together with the condition of linear independence. Thus, for a given choice of $\mathbf{m}$, a generalized p-form, $\overset{p}{\mathbf{a}} \in \Lambda^p_{(1)}$, can be written as

$$\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m}, \tag{2}$$

where $\overset{p}{\alpha}$, and $\overset{p+j}{\alpha}$ are, respectively, ordinary $p-$ and $(p+1)-$forms and p can take integer values from -1 to n. At a point x in M the generalized $p-$forms of type N=1, $\Lambda^p_{(1)}(x)$, form a real vector space of dimension $\frac{(1+n)!}{(1+p)!(n-p)!}$. The dimension of $\Lambda^\bullet_{(1)}(x) = \oplus_{p=-1}^{p=n}\Lambda^p_{(1)}(x)$ is 2^{1+n}.

If φ is a smooth map between manifolds P and M, $\varphi : P \to M$, then the induced map of type $N = 1$ generalized forms, $\varphi^*_{(1)} : \Lambda^p_{(1)}(M) \to \Lambda^p_{(1)}(P)$, is the linear map defined by using the standard pull-back map, φ^*, for ordinary forms

$$\varphi^*_{(1)}(\overset{p}{\mathbf{a}}) = \varphi^*(\overset{p}{\alpha}) + \varphi^*(\overset{p+1}{\alpha})\mathbf{m}, \tag{3}$$

and $\varphi^*_{(1)}(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = \varphi^*_{(1)}(\overset{p}{\mathbf{a}})\varphi^*_{(1)}(\overset{q}{\mathbf{b}})$. Hence $\varphi^*_{(1)}(\mathbf{m}) = \mathbf{m}$.

Henceforth in this paper, in addition to assuming that the exterior derivative of generalized forms satisfies the usual properties, it is assumed that

$$d\mathbf{m} = \epsilon, \tag{4}$$

where ϵ denotes a real constant. If $\mathbf{m} \mapsto \widetilde{\mathbf{m}} = \mu\mathbf{m}$, where μ is a non-zero function on M, then $\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m} = \overset{p}{\alpha} + \overset{p+1}{\widetilde{\alpha}}\widetilde{\mathbf{m}}$, where $\overset{p+1}{\widetilde{\alpha}} = \mu^{-1}\overset{p+1}{\alpha}$. Furthermore $d\widetilde{\mathbf{m}} = \widetilde{\epsilon}$, where $\widetilde{\epsilon}$ is also a real constant, if and only if $d\mu = 0$ and then $\widetilde{\epsilon} = \mu\epsilon$.

The exterior derivative of a type $N = 1$ generalized form $\overset{p}{\mathbf{a}}$ is then

$$d\overset{p}{\mathbf{a}} = [d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}] + d\overset{p+1}{\alpha}\mathbf{m}, \tag{5}$$

where d is the ordinary exterior derivative when acting on ordinary forms. The exterior derivative $d : \Lambda^p_{(1)}(M) \to \Lambda^{p+1}_{(1)}(M)$ is an anti-derivation of degree one,

$$\begin{aligned} d(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) &= d\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}} + (-1)^p\overset{p}{\mathbf{a}}d\overset{q}{\mathbf{b}}, \\ d^2 &= 0. \end{aligned} \tag{6}$$

and $(\Lambda^\bullet_{(N)}(M), d)$ is a differential graded algebra. The exterior derivative is discussed in more detail in the appendix.

3 Vector fields and type $N = 1$ forms

In this section the definitions of the inner product and Lie derivative of type $N = 1$ forms by ordinary vector fields introduced in [10] will be summarized. Let v be an ordinary vector field tangent to M, $v \in \mathcal{V}_{(o)}(M)$, where $\mathcal{V}_{(o)}(M)$ is the module of ordinary vector fields over $C^\infty(M)$, the real valued functions on M. Let the generalized p-form $\overset{p}{\mathbf{a}}$ and q-form $\overset{q}{\mathbf{b}}$ be given, respectively, by $\overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m}$ and $\overset{q}{\beta} + \overset{q+1}{\beta}\mathbf{m}$. The inner product or contraction operator on generalized forms, $i_v : \Lambda^p_{(1)} \to \Lambda^{p-1}_{(1)}$, for $-1 \leqq p \leqq n$, is defined in terms of the inner product for ordinary forms by

$$i_v\overset{p}{\mathbf{a}} = i_v\overset{p}{\alpha} + (i_v\overset{p+1}{\alpha})\mathbf{m}. \tag{7}$$

Since $\overset{-1}{\alpha} = 0$, and $i_v\overset{0}{\alpha} = 0$, $i_v\overset{-1}{\mathbf{a}} = 0$ and $i_v\overset{0}{\mathbf{a}} = (i_v\overset{1}{\alpha})\mathbf{m}$. Furthermore, for any two vector fields v and $w \in \mathcal{V}_{(o)}(M)$

$$i_w(i_v\overset{p}{\mathbf{a}}) + i_v(i_w\overset{p}{\mathbf{a}}) = 0. \tag{8}$$

It is a straight forward matter to show that Eq.(7) implies that

$$i_v(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = (i_v\overset{p}{\mathbf{a}})\overset{q}{\mathbf{b}} + (-1)^p\overset{p}{\mathbf{a}}(i_v\overset{q}{\mathbf{b}}), \tag{9}$$

that is

$$i_v(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = i_v(\overset{p}{\alpha}\overset{q}{\beta}) + [i_v(\overset{p}{\alpha}\overset{q+1}{\beta}) + (-1)^q i_v(\overset{p+1}{\alpha}\overset{q}{\beta})]\mathbf{m},$$

The Lie derivative with respect to v, $\pounds_v$ is defined by

$$\pounds_v\overset{p}{\mathbf{a}} = i_v d\overset{p}{\mathbf{a}} + d(i_v\overset{p}{\mathbf{a}}), \tag{10}$$

from which it follows that $d \circ \pounds_v\overset{p}{\mathbf{a}} = \pounds_v \circ d\overset{p}{\mathbf{a}}$.

A calculation then shows that

$$\pounds_v\overset{p}{\mathbf{a}} = \pounds_v\overset{p}{\alpha} + (\pounds_v\overset{p+1}{\alpha})\mathbf{m}. \tag{11}$$

A direct consequence of Eqs.(9) and (10) above is that $\pounds_v$ satisfies the Leibniz rule

$$\pounds_v(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = (\pounds_v\overset{p}{\mathbf{a}})\overset{q}{\mathbf{b}} + \overset{p}{\mathbf{a}}\pounds_v(\overset{q}{\mathbf{b}}), \tag{12}$$

A couple of important differences from results for ordinary forms should be noted,

$$\pounds_v\overset{-1}{\mathbf{a}} = i_v(d\overset{0}{\alpha})\mathbf{m}, \tag{13}$$

and

$$i_v(d\overset{\mathbf{0}}{\mathbf{a}}) = i_v d\overset{0}{\alpha} - \epsilon i_v\overset{1}{\alpha} + i_v d\overset{1}{\alpha}\mathbf{m}, \tag{14}$$

In contrast to the case for ordinary zero-forms, the latter is not equal to the Lie derivative for

$$\pounds_v\overset{0}{\mathbf{a}} = i_v d\overset{0}{\alpha} + [i_v d\overset{1}{\alpha} + d(i_v\overset{1}{\alpha})]\mathbf{m}. \tag{15}$$

If v and w are vector fields in M it follows from the definitions above that

$$(\pounds_v \circ i_w - i_w \circ \pounds_v)\overset{p}{\mathbf{a}} = i_{[v,w]}\overset{p}{\mathbf{a}}, \tag{16}$$

$$(\pounds_v \circ \pounds_w - \pounds_w \circ \pounds_v)\overset{p}{\mathbf{a}} = \pounds_{[v,w]}\overset{p}{\mathbf{a}}, \tag{17}$$

where $[v, w]$ denotes the Lie bracket of the vector fields v and w. The latter type of equation will be used in section five to define a generalization of the Lie bracket.

In summary, with these definitions, $\Lambda_{(1)}$ is a graded algebra and there is a natural grading of these linear operators on $\Lambda(M)$, d is degree 1, $\pounds_v$ is of degree 0 and ι_v is of degree -1. These derivations span a super Lie algebra and satisfy the H.Cartan formulae, [15],

$$\begin{gathered} d \circ d = 0,\, i_v \circ i_w + i_w \circ i_v = 0, \\ \pounds_v \equiv d \circ i_v + i_v \circ d \Rightarrow\; d \circ \pounds_v - \pounds_v \circ d = 0. \\ \pounds_v \circ \pounds_w - \pounds_w \circ \pounds_v = \pounds_{[v,w]},\, \pounds_v \circ i_w - i_w \circ \pounds_v = i_{[v,w]}, \end{gathered} \tag{18}$$

for all vector fields v and $w \in \mathcal{V}_{(o)}(M)$.

4 Representation of the algebra and calculus of generalized forms

The algebra and calculus of ordinary differential forms on M can be expressed in terms of functions and vector fields on the reverse parity tangent bundle, ΠTM, of M, [16]. A recent exposition containing further references can be found in [17]. A sample of texts where superspace calculations are discussed is [18], [19], [20].

The reverse parity tangent bundle is just the ordinary tangent bundle with the parity reversed in the fibre directions. If x^α , $\alpha = 1....n = \dim M$, denote local coordinates on M then local coordinates on ΠTM are obtained by adding to these n anticommuting fibre coordinates. The latter are obtained by replacing the natural tangent bundle fibre coordinates with n anticommuting (fermionic) coordinates with the same transformation properties. These can be denoted by the symbols dx^α or, as will be done here for notational clarity by ζ^a. Then an ordinary $p-$form ρ on M with coordinate basis components $\rho_{\alpha_1....\alpha_p}(x^\alpha)$.

$$\rho = \frac{1}{p!}\rho_{\alpha_1....\alpha_p}(x^\alpha)dx^{\alpha_1.....}dx^{\alpha_p}, \tag{19}$$

corresponds to, r, a homogeneous polynomial of degree p in the anticommuting fibre coordinates on ΠTM

$$r = \frac{1}{p!}\rho_{\alpha_1 \ldots \alpha_p}(x^\alpha)\zeta^{\alpha_1} \ldots \zeta^{\alpha_p}. \tag{20}$$

The exterior product of ordinary forms $p-$ and $q-$forms on M corresponds to the product of such functions (which are homogeneous polynomials of respective degrees p and q in the anticommuting coordinates) on ΠTM. The exterior derivative of ordinary $p-$forms on M, where $p > 0$, corresponds to the action of the odd vector field $\zeta^\alpha \frac{\partial}{\partial x^\alpha}$ on the corresponding functions on ΠTM and the correspondence can be written as

$$d\rho \leftrightarrow \zeta^\alpha \frac{\partial r}{\partial x^\alpha}. \tag{21}$$

The interior product, i_v, of an ordinary $p-$ form ρ on M by a vector field $v= v^\alpha \frac{\partial}{\partial x^\alpha} \in \mathcal{V}_{(o)}(M)$, corresponds to the action of the odd vector field $v^\alpha \frac{\partial}{\partial \zeta^\alpha}$ on the function r in ΠTM; the correspondence can be written as

$$i_v\rho \leftrightarrow v^\alpha \frac{\partial r}{\partial \zeta^\alpha} \tag{22}$$

For example, the local coordinate expression for the action of a vector field v on a zero-form ρ on M is

$$v(\rho) = i_v d\rho, \tag{23}$$

and using the correspondences above

$$v(\rho) \leftrightarrow v^\alpha \frac{\partial}{\partial \zeta^\alpha}(\zeta^b \frac{\partial r}{\partial x^b}). \tag{24}$$

The Lie derivative $\pounds_v$ on M corresponds to the even vector field, $[d, i_v]$, on ΠTM where $[.,.]$ denotes the super Lie bracket of the odd vector fields d and i_v or equivalently the supercommutator of the differential operators on ΠTM,

$$\pounds_v \leftrightarrow [d, i_v] = d \circ (i_v) + i_v \circ d, \tag{25}$$

$$[d, i_v]r = v^\alpha \frac{\partial r}{\partial x^\alpha} + \frac{\partial v^\alpha}{\partial x^\beta}\zeta^\beta \frac{\partial r}{\partial \zeta^\alpha}. \tag{26}$$

Henceforth the same notation will be used for corresponding operators on M and reverse parity bundles. It will be clear from the context which is meant, for example on ΠTM

$$\pounds_v = [d, i_v]$$
$$\pounds_v r = v^\alpha \frac{\partial r}{\partial x^\alpha} + \frac{\partial v^\alpha}{\partial x^\beta}\zeta^\beta \frac{\partial r}{\partial \zeta^\alpha}$$

and correspondingly on M

$$\pounds_v \rho = [d \circ (i_v) + i_v \circ d]\rho.$$

If $w = w^\alpha \frac{\partial}{\partial x^\alpha}$ is a another vector field $\in \mathcal{V}_{(o)}(M)$, calculation of the super commutator on ΠTM of $\pounds_v$ and $\pounds_w$ gives

$$[\ \pounds_v, \pounds_w] = \ \pounds_v \circ \pounds_w - \pounds_w \circ \pounds_v = \pounds_{[v,w]}. \tag{27}$$

Similarly computing the supercommutator on ΠTM gives

$$[\pounds_v, i_w] = \pounds_v \circ i_w - i_w \circ \pounds_v = i_{[v,w]}, \tag{28}$$

where $[v, w]$ is the Lie bracket of v and w. By the correspondences the same results hold for forms on M.

It is a straightforward matter to extend these ideas to generalized forms of all types on M. Here only type $N = 1$ forms will be considered in detail. Let $\widetilde{M}$ be the Whitney sum of ΠTM and a trivial reverse parity line bundle over M, that is a trivial line bundle with fibre $\mathbb{R}^1$ replaced by $\mathbb{R}^{0|1}$, and let $\mathbb{R}^{0|1}$ have anti-commuting coordinate μ . Local coordinates for $\widetilde{M}$ can then be chosen to be the commuting coordinates x^α together with the anti-commuting coordinates $\zeta^a = dx^\alpha$ and μ. If

$$\mathbf{r} = \rho + \sigma\mathbf{m} \tag{29}$$

is a generalized $p-$form on M and the ordinary $p-$ and $(p+1)-$forms ρ and σ have respective coordinate basis components $\rho_{\alpha_1 \dots \alpha_p}$ and $\sigma_{\alpha_1 \dots \alpha_{p+1}}$ and

$$d\mathbf{m} = \epsilon, \tag{30}$$

where ϵ is a constant, then $\mathbf{r}$ corresponds to the function

$$\mathfrak{r} = \frac{1}{p!}\rho_{\alpha_1 \dots \alpha_p}(x^\alpha)\zeta^{\alpha_1} \cdots \zeta^{\alpha_p} + \frac{1}{(p+1)!}\sigma_{\alpha_1 \dots \alpha_{p+1}}(x^\alpha)\zeta^{\alpha_1} \cdots \zeta^{\alpha_{P+1}}\mu \tag{31}$$

on $\widetilde{M}$. The exterior product of generalized forms in M corresponds to the product of such functions in $\widetilde{M}$. The exterior derivative of a generalized form $\mathbf{r}$ in M, $d\mathbf{r}$, corresponds to the action of the odd vector field on the corresponding function in $\widetilde{M}$

$$d : \mathfrak{r} \to (\zeta^\alpha \frac{\partial}{\partial x^\alpha} + \epsilon \frac{\partial}{\partial \mu})\mathfrak{r}. \tag{32}$$

The interior product, $i_v\mathbf{r}$, of a generalized $p-$form $\mathbf{r}$ on M by a vector field $v= v^\alpha \frac{\partial}{\partial x^\alpha} \in \mathcal{V}_{(o)}(M)$ is the generalized $(p-1)-$form

$$i_v\mathbf{r} = i_v\rho + (i_v\sigma)\mathbf{m} \tag{33}$$

which corresponds to $v^\alpha \frac{\partial \mathfrak{r}}{\partial \zeta^\alpha}$ on $\widetilde{M}$, i.e. on $\widetilde{M}$

$$i_v : \mathfrak{r} \to v^\alpha \frac{\partial \mathfrak{r}}{\partial \zeta^\alpha}. \tag{34}$$

($d\mathfrak{r} = (\zeta^\alpha \frac{\partial}{\partial x^\alpha} + \epsilon \frac{\partial}{\partial \mu})\mathfrak{r}$ and $i_v\mathfrak{r} = v^\alpha \frac{\partial \mathfrak{r}}{\partial \zeta^\alpha}$ in accordance with the convention established above.)

5 Generalized form-valued vector fields

These ideas of the previous sections can be extended to include vector fields with generalized form-valued components. Define such a type N vector field on M by $\mathbf{V} = \mathbf{v}^\rho \frac{\partial}{\partial x^\rho}$ where the components $\mathbf{v}^\rho$ are type N zero-forms which transform as the components of a vector field . In the case considered in this paper $N = 1$ and

$$\mathbf{V} = \mathbf{v}^\rho \frac{\partial}{\partial x^\rho} = (v^\rho + v^\rho_\sigma dx^\sigma \mathbf{m})\frac{\partial}{\partial x^\rho} = v + (v^\rho_\sigma dx^\sigma \mathbf{m})\frac{\partial}{\partial x^\rho}, \tag{35}$$

where $v = v^\rho \frac{\partial}{\partial x^\rho}$. Hence $\mathbf{V}$ is determined by an ordinary vector field v and a $(1, 1)$ type tensor field $v^\rho_\sigma \frac{\partial}{\partial x^\rho} \otimes dx^\sigma$ on M. The set of all such vector field in M, $\mathbf{V}$, is naturally a module, $\mathcal{V}_{(1)}(M)$, over the generalized zero forms on M, $\Lambda^0_{(1)}(M)$.

The interior product of such a vector field $\mathbf{V}$ with a generalized $p-$form $\mathbf{r}$ is a generalized $(p-1)-$form on M, denoted $i_{\mathbf{V}}\mathbf{r}$. Its definition is obtained by using the approach of the previous section and extending the formulae there by considering the action of the odd vector field

$$i_{\mathbf{V}} = (v^{\rho} + v^{\rho}_{\sigma}\zeta^{\sigma}\mu)\frac{\partial}{\partial\zeta^{\rho}} \tag{36}$$

on the function $\mathfrak{r}$ on $\widetilde{M}$ given in Eq.(31); that is $i_{\mathbf{V}}\mathbf{r}$ on M is defined to correspond to the function on $\widetilde{M}$ given by

$$i_{\mathbf{V}}\mathfrak{r} = (v^{\rho}+v^{\rho}_{\sigma}\zeta^{\sigma}\mu)\frac{\partial}{\partial\zeta^{\rho}}[\frac{1}{p!}\rho_{\alpha_1....\alpha_p}(x^{\alpha})\zeta^{\alpha_1....}\zeta^{\alpha_p}+\frac{1}{(p+1)!}\sigma_{\alpha_1....\alpha_{p+1}}(x^{\alpha})\zeta^{\alpha_1....}\zeta^{\alpha_{p+1}}\mu]. \tag{37}$$

It follows that on M the interior product with respect to $\mathbf{V}$ is given by the formula

$$i_{\mathbf{V}}\mathbf{r} = \mathbf{v}^{\rho} i_{\frac{\partial}{\partial x^{\rho}}}\mathbf{r}. \tag{38}$$

For p equal to minus one and zero

$$i_{\mathbf{V}}\overset{-1}{\mathbf{r}} = 0. \tag{39}$$

$$i_{\mathbf{V}}\overset{0}{\mathbf{r}} = \sigma_{\alpha}v^{\alpha}\mathbf{m},$$

and for $p \geqq 1$

$$i_{\mathbf{V}}\mathbf{r} = i_v\mathbf{r}+\overset{p}{\gamma}\mathbf{m} = i_v\rho + i_v\sigma\mathbf{m}+\overset{p}{\gamma}\mathbf{m}, \tag{40}$$

$$\overset{p}{\gamma} = (-1)^{p-1}v^{\alpha}_{\beta}dx^{\beta}(i_{\frac{\partial}{\partial x^{\alpha}}}\rho)$$

$$= \frac{(-1)^{p-1}}{(p-1)!}v^{\alpha}_{\lambda_1}\rho_{\alpha\lambda_2....\lambda_p}dx^{\lambda_1\cdots}dx^{\lambda_p}.$$

This interior product satisfies the graded Leibniz rule

$$i_{\mathbf{V}}(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = (i_{\mathbf{V}}\overset{p}{\mathbf{a}})\overset{q}{\mathbf{b}} + (-1)^{p}\overset{p}{\mathbf{a}}(i_{\mathbf{V}}\overset{q}{\mathbf{b}}), \tag{41}$$

but does not in general anti-commute because the interior product on generalized zero forms need not be zero,

$$(i_{\mathbf{W}}\circ i_{\mathbf{V}} + i_{\mathbf{V}}\circ i_{\mathbf{W}})\mathbf{r}=(-1)^{p-1}\{[v^{\alpha}_{\beta}w^{\beta} + w^{\alpha}_{\beta}v^{\beta}](i_{\frac{\partial}{\partial x^{\alpha}}}\rho\}\mathbf{m}, \tag{42}$$

where $\mathbf{W}=(w^{\rho} + w^{\rho}_{\sigma}dx^{\sigma}\mathbf{m})\frac{\partial}{\partial x^{\rho}}$. However if

$$\mathbf{V}=v + i_v\Xi^{\alpha}\mathbf{m}\frac{\partial}{\partial x^{\alpha}}, \mathbf{W} =w + i_w\Xi^{\alpha}\mathbf{m}\frac{\partial}{\partial x^{\alpha}} \tag{43}$$

$$\Xi^{\alpha} = \frac{1}{2}\Xi^{\alpha}_{\beta\gamma}dx^{\beta}dx^{\gamma},$$

where $v, w \in \mathcal{V}_{(0)}(M)$ and Ξ is an ordinary vector-valued two-form, then

$$(i_{\mathbf{W}} \circ i_{\mathbf{V}} + i_{\mathbf{V}} \circ i_{\mathbf{W}}) = 0. \tag{44}$$

The Lie derivative of generalized forms with respect to a generalized form-valued vector field $\mathbf{V}$, which will be denoted $\pounds_{\mathbf{V}}$, is defined by

$$\pounds_{\mathbf{V}} = d \circ i_{\mathbf{V}} + i_{\mathbf{V}} \circ d. \tag{45}$$

Calculation of the corresponding supercommutator on $\widetilde{M}$ gives the even vector field $\pounds_{\mathbf{V}}$ where

$$\begin{aligned} \pounds_{\mathbf{V}}\mathfrak{r} = & \ (v^{\alpha}\frac{\partial}{\partial x^{\alpha}} + \frac{\partial v^{\alpha}}{\partial x^{\beta}}\zeta^{\beta}\frac{\partial}{\partial \zeta^{\alpha}} - \epsilon v^{\alpha}_{\beta}\zeta^{\beta}\frac{\partial}{\partial \zeta^{\alpha}})\mathfrak{r} \\ & + (v^{\alpha}_{\beta}\zeta^{\beta}\mu\frac{\partial}{\partial x^{\alpha}} + \frac{\partial v^{\alpha}_{\beta}}{\partial x^{\gamma}}\zeta^{\gamma}\zeta^{\beta}\mu\frac{\partial}{\partial \zeta^{\alpha}})\mathfrak{r}. \end{aligned} \tag{46}$$

It follows that on M the Lie derivative of a generalized $p-$form $\mathbf{r}$ on with respect to a generalized form valued vector field $\mathbf{V} = \mathbf{v}^{\alpha}\frac{\partial}{\partial x^{\alpha}}$ is

$$\begin{aligned} \pounds_{\mathbf{V}}\mathbf{r} = & \ (d \circ \mathbf{i}_{\mathbf{V}} + \mathbf{i}_{\mathbf{V}} \circ d)\mathbf{r} \\ = & \ (\mathbf{v}^{\alpha}\frac{\partial}{\partial x^{\alpha}} + d(\mathbf{v}^{\alpha})i_{\frac{\partial}{\partial x^{\alpha}}})\mathbf{r}, \end{aligned} \tag{47}$$

where $\mathbf{v}^{\alpha}\frac{\partial \mathbf{r}}{\partial x^{\alpha}}$ denotes the expression

$$\mathbf{v}^{\alpha}\{\frac{1}{p!}\frac{\partial}{\partial x^{a}}[\rho_{\alpha_1 \dots \alpha_p}(x^{\alpha})]dx^{\alpha_1} \dots dx^{\alpha_p} + \frac{1}{(p+1)!}\frac{\partial}{\partial x^{a}}[\sigma_{\alpha_1 \dots \alpha_{p+1}}(x^{\alpha})]dx^{\alpha_1} \dots dx^{\alpha_{p+1}}\mathbf{m}\}. \tag{48}$$

Hence when $p = -1$ and $\mathbf{r} = \sigma\mathbf{m}$

$$\pounds_{\mathbf{V}}\mathbf{r} = \mathbf{v}^{\alpha}\frac{\partial \sigma}{\partial x^{\alpha}}\mathbf{m}, \tag{49}$$

when $p = 0$ and $\mathbf{r} = \rho + \sigma\mathbf{m}$,

$$\pounds_{\mathbf{V}}\mathbf{r} = \pounds_{v}\rho + [\pounds_{v}\sigma + v^{\alpha}_{\beta}(\frac{\partial \rho}{\partial x^{\alpha}} - \epsilon\sigma_{\alpha})dx^{\beta}]\mathbf{m}, \tag{50}$$

when $p \geqq 1$ and $\mathbf{r} = \rho + \sigma\mathbf{m}$,

$$\begin{aligned} \pounds_{\mathbf{V}}\mathbf{r} = & \ \pounds_{v}\rho - \frac{\epsilon}{(p-1)!}v^{\alpha}_{\beta}\rho_{\alpha\lambda_2 \dots \lambda_p}dx^{\beta}dx^{\lambda_2} \dots dx^{\lambda_p} \\ & + [\pounds_{v}\sigma + \frac{(-1)^p}{p!}v^{\alpha}_{\beta}\frac{\partial}{\partial x^{\alpha}}\rho_{\lambda_1 \dots \lambda_p} + \frac{(-1)^p}{(p-1)!}\frac{\partial v^{\alpha}_{\beta}}{\partial x^{\lambda_1}}\rho_{\alpha\lambda_2 \dots \lambda_p} \\ & - \frac{\epsilon}{p!}v^{\alpha}_{\beta}\sigma_{\alpha\lambda_1 \dots \lambda_p}]dx^{\beta}dx^{\lambda_1} \dots dx^{\lambda_p}\mathbf{m}. \end{aligned} \tag{51}$$

It follows from Eqs.(41) and (45) that $\pounds_{\mathbf{V}}$ satisfies the Leibniz rule and is a derivation of degree zero.

The Lie bracket of two generalized form-valued vector fields $\mathbf{V} = \mathbf{v}^{\alpha} \frac{\partial}{\partial x^{\alpha}}$ and $\mathbf{W} = \mathbf{w}^{\alpha} \frac{\partial}{\partial x^{\alpha}}$ on M is the generalized form-valued vector field, $[\mathbf{V}, \mathbf{W}]$, defined by the relation

$$(\pounds_{\mathbf{V}} \pounds_{\mathbf{W}} - \pounds_{\mathbf{W}} \pounds_{\mathbf{V}})\mathbf{r} = \pounds_{[\mathbf{V},\mathbf{W}]}\mathbf{r}. \tag{52}$$

Then

$$[\mathbf{V}, \mathbf{W}] = [\mathbf{V}, \mathbf{W}]^{\gamma} \frac{\partial}{\partial x^{\gamma}} \tag{53}$$

$$[\mathbf{V}, \mathbf{W}]^{\gamma} = [v, w]^{\gamma} + \{v^{\beta} \frac{\partial}{\partial x^{\beta}} w^{\gamma}_{\alpha} - w^{\beta} \frac{\partial}{\partial x^{\beta}} v^{\gamma}_{\alpha} + w^{\gamma}_{\beta} \frac{\partial}{\partial x^{\alpha}} v^{\beta} - v^{\gamma}_{\beta} \frac{\partial}{\partial x^{\alpha}} w^{\beta}$$
$$+ v^{\beta}_{\alpha} \frac{\partial}{\partial x^{\beta}} w^{\gamma} - w^{\beta}_{\alpha} \frac{\partial}{\partial x^{\beta}} v^{\gamma} + \epsilon v^{\gamma}_{\beta} w^{\beta}_{\alpha} - \epsilon w^{\gamma}_{\beta} v^{\beta}_{\alpha}\} dx^{\alpha}\mathbf{m},$$

where $[v, w]^{\gamma} \frac{\partial}{\partial x^{\gamma}}$ is the ordinary Lie bracket of v and w. It follows from Eq.(52) that the Lie bracket satisfies the Jacobi identity, if $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ are generalized form-valued vector fields $\in \mathcal{V}_{(1)}(M)$

$$[\mathbf{U},[\mathbf{V}, \mathbf{W}]] + [\mathbf{V},[\mathbf{W}, \mathbf{U}]] + [\mathbf{W},[\mathbf{U}, \mathbf{V}]] = 0. \tag{54}$$

All the operators d, $i_{\mathbf{V}}$, $\pounds_{\mathbf{V}}$ and $[\mathbf{V}, \mathbf{W}]$ reduce to the usual operators when acting on ordinary forms and vector fields.

Henceforth generalized form-valued vector fields will be referred to as type N vector fields with $N = 1$ here. Ordinary vector fields are therefore type $N = 0$ vector fields.

In [13] and [14] the concept of a generalized vector field was introduced and explored. Such a vector field V is determined by a pair consisting of an ordinary vector field $v \in \mathcal{V}_{(o)}(M)$ and a scalar field v_0 on M. it is straightfoward to see that such a generalized vector field is a type $N = 1$ vector field $\mathbf{V} = v + (v^{\rho}_{\sigma} dx^{\sigma}\mathbf{m}) \frac{\partial}{\partial x^{\rho}}$ when the special choice

$$v^{\alpha}_{\beta} = \delta^{\alpha}_{\beta} v_0. \tag{55}$$

is made. Moreover if V and W are two generalized vector fields, the inner product, I_V , Lie derivative $\mathfrak{L}_V$ and Lie bracket $\{V, W\}$ introduced in [13] and [14] are the same as the inner product $i_{\mathbf{V}}$, Lie Derivative $\pounds_{\mathbf{V}}$ and Lie bracket $[\mathbf{V}, \mathbf{W}]$ for the generalized form-valued vector fields $\mathbf{V} = (v^{\rho} + v_0 dx^{\rho}\mathbf{m}) \frac{\partial}{\partial x^{\rho}}$ and $\mathbf{W} = (w^{\rho} + w_0 dx^{\rho}\mathbf{m}) \frac{\partial}{\partial x^{\rho}}$.

In concluding this section it should be noted that in [13] and [14] it was observed that it was not possible, in general, to define a generalized vector field which was a Lie derivative, $\mathfrak{L}_V W$, by using the equation

$$\mathfrak{L}_V \circ i_W - i_W \circ \mathfrak{L}_V = i_{\mathfrak{L}_V W}. \tag{56}$$

This result also applies to general vector fields $\in \mathcal{V}_{(1)}(M)$ and is not surprising in the light of the failure, as was noted above, of inner products to anti-commute. A modified Lie derivative,$\widehat{\mathfrak{L}}_V$ of generalized forms which could be used in Eq.(56) to define a (modified) Lie derivative of generalized vector fields, $\widehat{\mathfrak{L}}_V W$, was introduced. That construction will not be pursued here but the following general observation can be made. Eqs.(31) and (32) suggests that for generalized forms the exterior derivative, d, splits naturally into two exterior derivatives

$$d = d_{(0)} + \epsilon d_{(1)} \tag{57}$$

where $d = d_{(0)}$ when $\epsilon = 0$ and $d_{(1)}\mathbf{m} = 1$, $d_{(1)}\alpha = 0$ for any ordinary form α. Similarly any type $N = 1$ vector field can be naturally written as the sum of two vector fields v and $\mathbf{V}_{(1)}$

$$\mathbf{V} = v + (v^\rho_\sigma dx^\sigma \mathbf{m}) \frac{\partial}{\partial x^\rho} = v + \mathbf{V}_{(1)}. \tag{58}$$

Operators, such as the modified Lie derivative operator, can be constructed by making use of these splittings. In fact the modified Lie derivative operator of [13] and [14] is given by

$$\widehat{\mathfrak{L}}_{\mathbf{V}}\mathbf{r} = \pounds_{\mathbf{V}}\mathbf{r} - (d_{(0)} \circ i_{\mathbf{V}_{(1)}} + i_{\mathbf{V}_{(1)}} \circ d_{(0)})\mathbf{r}, \tag{59}$$

where $\mathbf{V} = (v^\rho + v_0 dx^\rho \mathbf{m})\frac{\partial}{\partial x^\rho}$.

These ideas are illustrated in the following simple examples. First an example using "pure" type $N = 1$ vector fields, for which $\mathbf{V} = (v^\rho_\sigma dx^\sigma \mathbf{m})\frac{\partial}{\partial x^\rho} = \mathbf{V}_{(1)}$. is given. Second Hamiltonian type $N = 1$ vector fields are briefly introduced and a simple special case is discussed.

Example 1 Pure type $N = 1$ vector fields:

Let J_i be three $(1,1)$ type tensor fields which satisfy, as for example with hyperkähler metrics, the conditions $J_i J_j = \varepsilon_{ijk} J_k$ where i, j, k range from one to three and ε_{ijk} is the totally skew symmetric Levi=Civita symbol. If $\mathbf{V}_i = \mathbf{V}_{i(1)} = \frac{1}{2} J^\alpha_{i\beta} dx^\alpha \mathbf{m} \frac{\partial}{\partial x^\alpha}$ then $[\mathbf{V}_i, \mathbf{V}_j] = \epsilon\varepsilon_{ijk}\mathbf{V}_k$. Hence when ϵ is zero the vector fields commute and when ϵ is non-zero the Lie brackets of the three pure type $N = 1$ vector fields $\mathbf{V}_i = \mathbf{V}_{i(1)} = \frac{1}{2\epsilon} J^{\alpha}_{i\beta} dx^{\alpha} \mathbf{m} \frac{\partial}{\partial x^{\alpha}}$ satisfy the so(3) Lie algebra condition, $[\mathbf{V}_i, \mathbf{V}_j] = \varepsilon_{ijk} \mathbf{V}_k$.

Example 2 Type $N = 1$ Hamiltonian vector fields

Let M be an even dimensional manifold with local coordinates $\{x^{\alpha}\}$. Let $\mathbf{s} = \Omega + \Upsilon \mathbf{m}$ be a generalized symplectic two -form on M, that is a closed, non-degenerate generalized two-form. Since $\mathbf{s}$ is required to be closed Ω and Υ must both be closed when $\epsilon = 0$ and when ϵ is non-zero $\Upsilon = \frac{1}{\epsilon} d\Omega$ and $\mathbf{s} = \Omega + \frac{1}{\epsilon} d\Omega \mathbf{m}$. The two-form $\mathbf{s}$ is defined to be non-degenerate if and only if Ω is non-degenerate. In this case both the components of $\mathbf{s}$ and the ordinary two-form Ω are invertible (written as square matrices) and Ω defines an isomorphism between TM and T^*M; if $\Omega = \frac{1}{2} \Omega_{\alpha\beta} dx^{\alpha} dx^{\beta}$ then conventionally $\Omega^{\alpha\gamma} \Omega_{\beta\gamma} = \delta^{\alpha}_{\beta}$, [21]. This definition permits non-zero pure generalized form-valued vector field solutions $\mathbf{W} = \mathbf{W}_{(1)}$, here termed kernel vector fields, to the equation

$$i_{\mathbf{W}} \mathbf{s} = 0.$$

Let $\mathbf{H} = h + k\mathbf{m}$ be a generalized zero-form, then $\mathbf{V}_H$ is by definition a generalized form-valued Hamiltonian vector field corresponding to $\mathbf{H}$ when

$$i_{\mathbf{V}_H} \mathbf{s} = -d\mathbf{H}.$$

Such a Hamiltonian vector field is also a Hamiltonian vector field for the generalized zero-form $\mathbf{H}+d\mathbf{L}$ where $\mathbf{L}$ is any generalized minus one-form $l\mathbf{m}$. When $\mathbf{H} \rightarrow \mathbf{H}+d\mathbf{L}$, h$\rightarrow h + \epsilon l$ and $k \rightarrow k + dl$.

Employing the decomposition of Eq.(58) and writing $\mathbf{V}_H = v_H + \mathbf{V}_{H(1)}$ the solutions of this equation, modulo arbitrary kernel vector fields, are given in terms of components by

$$v^{\alpha}_H = \Omega^{\alpha\beta} (\epsilon k_{\beta} - h_{,\beta}),$$
$$v^{\alpha}_{H\beta} = \Omega^{\gamma\alpha} (k_{[\beta,\gamma]} - \frac{1}{2} v^{\mu}_H \Upsilon_{\mu\beta\gamma}),$$

where $\Upsilon = \frac{1}{3!} \Upsilon_{\alpha\beta} dx^{\alpha} dx^{\beta} dx^{\gamma}$, partial differentiation is denoted by a comma and square brackets denote the totally skew part. By Eq.(52), and the fact that $\pounds_{\mathbf{V}_H} \mathbf{s} = 0$, the Lie bracket of two generalized form-valued Hamiltonian vector fields is also a generalized form-valued Hamiltonian vector field.

A case where there is much simplification arises when the ordinary two-form Ω is itself symplectic, $\mathbf{s} = \Omega$, and only generalized vector fields, defined in Eq.(55), are considered. Then there are no kernel generalized vector fields.

The consistency of the solutions above now requires that $2v_0\Omega = dk$ and that, in dimension greater than two, v_0 must be a constant. In symplectic coordinates (p_a, q^a), with $\Omega = dp_a dq^a$ and with the choices v_0 a constant and $k = 2v_0 p_a dq^a$, the Hamiltonian generalized vector field is

$$\begin{aligned}\mathbf{V}_H &= v_H + \mathbf{V}_{H(1)} \\ &= \frac{\partial h}{\partial p_a}\frac{\partial}{\partial q^a} - (\frac{\partial h}{\partial q^a} - 2\epsilon v_0 p_a)\frac{\partial}{\partial p_a} - \mathbf{m}v_0(dq^a\frac{\partial}{\partial q^a} + dp_a\frac{\partial}{\partial p_a}).\end{aligned}$$

Integral curves for v_H satisfy the generalized Hamilton's equations

$$\frac{dq^a}{dt} = \frac{\partial h}{\partial p_a}, \frac{dp_a}{dt} = -(\frac{\partial h}{\partial q^a} - 2\epsilon v_0 p_a).$$

For example, if $h = \Sigma_{a=1}^{l}\frac{1}{2}[(q^a)^2 + (p_a)^2]$ the solutions of the generalized Hamilton's equations are determined by the solutions of the differential equations

$$\frac{d^2}{dt^2}q^a - 2\epsilon v_0\frac{d}{dt}q^a + q^a = 0.$$

The appearance of the damping (or anti-damping) term when ϵ is non-zero appears to parallel the appearance of a mass term in field equations when ϵ is non-zero, as for example in [10].

The ideas above can be straightforwardly extended to type $N \geqq 2$ generalized forms and vector fields by considering functions on the Whitney sum, $\widetilde{M}^N$ of ΠTM and a reverse parity $\mathbb{R}^N$ bundle over M, that is a trivial vector bundle with fibre $\mathbb{R}^N$ replaced by $\mathbb{R}^{0|N}$. Natural local coordinates on $\widetilde{M}^N$ are $(x^\alpha, \zeta^\alpha, \mu^i)$, where μ^i $(i = 1..N)$ are anti-commuting coordinates on $\mathbb{R}^{0|N}$. Type N generalized forms on M correspond, in the obvious generalization of the type $N = 1$ case, to functions on $\widetilde{M}^N$ which are polynomial in the anticommuting coordinates. The exterior product of type N forms on M corresponds to the product of such functions on $\widetilde{M}^N$. When the exterior derivatives of the basis minus one-forms on M are given by $d\mathbf{m}^i = \epsilon^i$, where ϵ^i are constants, the exterior derivative of type N generalized forms on M corresponds to the action of the vector field $\zeta^\alpha\frac{\partial}{\partial x^\alpha} + \epsilon^i\frac{\partial}{\partial \mu^i}$ on such functions on $\widetilde{M}^N$. The interior product $i_{\mathbf{V}}$ of a generalized type N form by a type N generalized vector field $\mathbf{V} = \mathbf{v}^\alpha\frac{\partial}{\partial x^\alpha}$, where the components $\mathbf{v}^\alpha$ are type N generalized zero-forms, corresponds to the action of the vector field $\mathbf{v}^\alpha\frac{\partial}{\partial \zeta^\alpha}$ on the corresponding functions on $\widetilde{M}^N$.

The ideas of the previous sections also extend straightforwardly to generalized form-valued tensor fields and geometrical objects. Such an extension is outlined in the next section.

6 Type $N = 1$ generalized affine and metric connections

In this section the formalism above will be applied and the tensor calculus of generalized affine connections and metrics, when $N = 1$, will be outlined. The definition of a generalized affine connection is the same as the definition of an ordinary affine connection except that ordinary forms, including zero forms, are replaced by generalized forms. If $\{U_I\}$ is a covering of an $n-$dimensional manifold M by coordinate charts, each with coordinates $\{x_I^\alpha\}$ then a generalized affine connection $\mathbf{A}$ is an assignment of a $n \times n$ matrix-valued generalized one-form, with (μ, ν) entry $\mathbf{A}^{\mu}_{I\nu}$, to each set U_I and such that on $U_I \cap U_J$, for all I and J,

$$\mathbf{A}^{\mu}_{J\nu} = (G^{-1}_{IJ})^{\mu}_{\gamma} dG_{IJ}{}^{\gamma}_{\nu} + (G^{-1}_{IJ})^{\mu}_{\gamma} \mathbf{A}^{\gamma}_{I\lambda} G_{IJ}{}^{\lambda}_{\nu}, \tag{60}$$

where

$$G_{IJ}{}^{\mu}_{\nu} = \frac{\partial x^{\mu}_{I}}{\partial x^{\nu}_{J}}. \tag{61}$$

The curvature two-form $\mathbf{F}_I$ is the generalized form

$$\mathbf{F}^{\mu}_{\mathbf{I}\nu} = \mathbf{d}\mathbf{A}^{\mu}_{I\nu} + \mathbf{A}^{\mu}_{I\rho}\mathbf{A}^{\rho}_{I\nu}, \tag{62}$$

and under the transformation in Eq.(60)

$$\mathbf{F}^{\mu}_{\mathbf{J}\nu} = (G^{-1}_{IJ})^{\mu}_{\gamma} \mathbf{F}^{\gamma}_{I\lambda} G_{IJ}{}^{\lambda}_{\nu}.$$

On any coordinate chart such as U_I the connection one-form $\mathbf{A}^{\mu}_{I\nu}$ can be written as

$$\mathbf{A}^{\mu}_{I\nu} = \alpha^{\mu}_{I\nu} + \beta^{\mu}_{I\nu}\mathbf{m}, \tag{63}$$

where $\alpha^{\mu}_{I\nu}$ and $\beta^{\mu}_{I\nu}$ are respectively ordinary matrix valued one-forms and the curvature two-form is then

$$\mathbf{F}^{\mu}_{I\nu} = \mathcal{F}^{\mu}_{I\nu} + \epsilon\beta^{\mu}_{I\nu} + D\beta^{\mu}_{I\nu}\mathbf{m}, \tag{64}$$

where

$$\mathcal{F}^{\mu}_{I\nu} = d\alpha^{\mu}_{I\nu} + \alpha^{\mu}_{I\rho}\alpha^{\rho}_{I\nu}, \qquad (65)$$
$$D\beta^{\mu}_{I\nu} = d\beta^{\mu}_{I\nu} + \alpha^{\mu}_{I\rho}\beta^{\rho}_{I\nu} - \beta^{\mu}_{I\rho}\alpha^{\rho}_{I\nu}.$$

It follows from the above that the locally defined ordinary one-forms $\alpha^{\mu}_{I\nu}$ and curvature two-forms $\mathcal{F}^{\mu}_{I\nu}$, patch together to define global connection and curvature forms, α and $\mathcal{F}$, of an ordinary affine connection D. The ordinary two-forms $\beta^{\mu}_{I\nu}$ transform as $(1,1)$ type tensor valued two-forms. Henceforth connections on M will be discussed and the subscripts corresponding to coordinate charts will be dropped.

The curvature satisfies the Bianchi identities

$$\mathbf{DF}^{\mu}_{\ \nu} = d\mathbf{F}^{\mu}_{\ \nu} + \mathbf{A}^{\mu}_{\ \lambda}\mathbf{F}^{\lambda}_{\ \nu} - \mathbf{F}^{\mu}_{\ \lambda}\mathbf{A}^{\lambda}_{\ \nu} = 0, \qquad (66)$$

where here $\mathbf{D}$ denotes the covariant exterior derivative of a type $N = 1$ valued generalized form. For a $\binom{1}{1}-$tensor valued generalized p-form $\mathbf{P}$

$$\mathbf{DP}^{\mu}_{\ \nu} = d\mathbf{P}^{\mu}_{\ \nu} + \mathbf{A}^{\mu}_{\ \lambda}\mathbf{P}^{\lambda}_{\ \nu} + (-1)^{p+1}\mathbf{P}^{\mu}_{\ \lambda}\mathbf{A}^{\lambda}_{\ \nu}. \qquad (67)$$

The covariant derivative of a generalized zero-form is the exterior derivative. If $\mathbf{V} = \mathbf{v}^{\rho}\frac{\partial}{\partial x^{\rho}} = (v^{\rho} + v^{\rho}_{\sigma}dx^{\sigma}\mathbf{m})\frac{\partial}{\partial x^{\rho}}$, the covariant derivative is

$$\nabla\mathbf{V} = \mathbf{Dv}^{\mu}\otimes\frac{\partial}{\partial x^{\mu}} = (d\mathbf{v}^{\mu} + \mathbf{A}^{\mu}_{\nu}\mathbf{v}^{\nu})\otimes\frac{\partial}{\partial x^{\mu}}, \qquad (68)$$

where

$$\mathbf{Dv}^{\mu} = Dv^{\mu} - \epsilon v^{\mu}_{\nu}dx^{\nu} + [D(v^{\mu}_{\nu}dx^{\nu}) + \beta^{\mu}_{\nu}v^{\nu}]\mathbf{m}, \qquad (69)$$

and $\mathbf{D}$ and D are the covariant exterior derivatives with respect to $\mathbf{A}$ and α respectively. The covariant derivative with respect to a type $N = 1$ vector field $\mathbf{W}$ is the generalized form-valued vector field

$$\nabla_{\mathbf{W}}\mathbf{V} = [\mathbf{i}_{\mathbf{W}}((d\mathbf{v}^{\alpha} + \mathbf{A}^{\alpha}_{\beta}\mathbf{v}^{\beta})]\frac{\partial}{\partial x^{\alpha}}. \qquad (70)$$

The covariant derivative is extended to type $N = 1$ generalized form -valued tensor fields by using the linearity and product rules satisfied by ordinary covariant derivatives and tensor fields.

A field $\mathbf{V}$ is a parallel vector field if $\nabla\mathbf{V} = 0$, that is

$$
\begin{aligned}
Dv^{\mu} - \epsilon v^{\mu}_{\nu} dx^{\nu} &= 0, \\
D(v^{\mu}_{\nu} dx^{\nu}) + \beta^{\mu}_{\nu} v^{\nu} &= 0.
\end{aligned} \tag{71}
$$

and such a system of equations is completely integrable if and only if the generalized curvature $\mathbf{F}$ is zero, that is when

$$
\begin{aligned}
\mathcal{F}^{\mu}_{\nu} &= -\epsilon \beta^{\mu}_{\nu}. \\
D\beta^{\mu}_{\nu} &= 0.
\end{aligned} \tag{72}
$$

A generalized metric can be defined by straightforwardly extending the definition of an ordinary metric to encompass generalized forms. A generalized metric $\mathbf{g}$ is a smooth symmetric bilinear function on $\mathcal{V}_{(1)\ p}(M)$ at each $p \in M$

$$
\begin{aligned}
\mathbf{g}(\mathbf{V}, \mathbf{W}) &= \mathbf{g}_{\mu\nu} \mathbf{v}^{\mu} \mathbf{w}^{\nu}, \\
\mathbf{g}_{\mu\nu} &= \mathbf{g}_{v\mu},
\end{aligned} \tag{73}
$$

where the components are generalized zero-forms and $\mathbf{V}$ and $\mathbf{W}$ are any type $N = 1$ vector fields as above. If

$$
\mathbf{g}_{\mu\nu} = \gamma_{\mu\nu} + \chi_{\mu\nu} \mathbf{m}, \tag{74}
$$

where $\gamma_{\mu\nu}$ and $\chi_{\mu\nu}$ are ordinary zero and one-forms respectively, the generalized metric is said to be non-degenerate when $det(\gamma_{\mu\nu})$ is non-zero. A non-degenerate metric has inverse

$$
\mathbf{g}^{\mu\nu} = \gamma^{\mu\nu} - \chi^{\mu\nu} \mathbf{m} \tag{75}
$$

where $\gamma^{\mu\nu}\gamma_{\nu\rho} = \delta^{\mu}_{\rho}$ and $\chi^{\mu\nu} = \gamma^{\mu\rho}\gamma^{\mu\sigma}\chi_{\rho\sigma}$. Henceforth only non-degenerate metrics will be considered.

The expanded form of $\mathbf{g}_{\mu\nu}\mathbf{v}^{\mu}\mathbf{w}^{\nu}$ is

$$
\mathbf{g}_{\mu\nu}\mathbf{v}^{\mu}\mathbf{w}^{\nu} = \gamma_{\mu\nu} v^{\mu} w^{\nu} + (v_{\mu} w^{\mu}_{\rho} + w_{\mu} v^{\mu}_{\rho} + \chi_{\mu\nu\rho} v^{\mu} w^{\nu}) dx^{\rho} \mathbf{m}, \tag{76}
$$

where here and henceforth indices are lowered (and raised) by using $\gamma_{\mu\nu}$ and its inverse and $\chi_{\mu\nu} = \chi_{\mu\nu\rho} dx^{\rho}$.

If $\mathbf{A}$ is a generalized connection and $\mathbf{g}$ then $\mathbf{A}$ is a generalized metric connection when the covariant derivative of $\mathbf{g}$ is zero. This compatibility condition may be expressed as the vanishing of the generalized non-metricity one-form $\mathbf{Q}_{\mu\nu}$ where

$$\mathbf{Dg}_{\mu\nu} = d\mathbf{g}_{\mu\nu} - \mathbf{g}_{\mu\lambda}\mathbf{A}^{\lambda}_{\nu} - \mathbf{g}_{\lambda\nu}\mathbf{A}^{\lambda}_{\mu} = \mathbf{Q}_{\mu\nu}. \tag{77}$$

Here

$$\begin{aligned}\mathbf{Q}_{\mu\nu} &= q_{\mu\nu} - \epsilon\chi_{\mu\nu} + [D\chi_{\mu\nu} - (\beta_{\mu\nu} + \beta_{\nu\mu})]\mathbf{m}, \\ D\chi_{\mu\nu} &= d\chi_{\mu\nu} - \alpha^{\lambda}_{\mu}\chi_{\lambda\nu} - \alpha^{\lambda}_{\nu}\chi_{\mu\lambda},\end{aligned} \tag{78}$$

and $q_{\mu\nu}$ is the non-metricity one-form for α^{μ}_{ν} and $\gamma_{\mu\nu}$,

$$q_{\mu\nu} = D\gamma_{\mu\nu} = d\gamma_{\mu\nu} - \gamma_{\mu\lambda}\alpha^{\lambda}_{\nu} - \gamma_{\lambda\nu}\alpha^{\lambda}_{\mu}. \tag{79}$$

When $\epsilon = 0$, $\mathbf{Q}_{\mu\nu} = 0$ if and only if

$$\begin{aligned}q_{\mu\nu} &= 0, \\ D\chi_{\mu\nu} &= (\beta_{\mu\nu} + \beta_{\nu\mu}),\end{aligned} \tag{80}$$

that is α is a metric connection for the metric $\gamma_{\mu\nu}dx^{\mu}dx^{\nu}$ and

$$\mathbf{A}^{\mu}_{\nu} = \alpha^{\mu}_{\nu} + (\widetilde{\beta}^{\mu}_{\cdot\nu} + \frac{1}{2}D\chi^{\mu}_{\nu})\mathbf{m} \tag{81}$$

where $\widetilde{\beta}^{\mu}_{\nu} = \frac{1}{2}\gamma^{\mu\lambda}(\beta_{\lambda\nu} - \beta_{\nu\lambda})$.

When $\epsilon \neq 0$, $\mathbf{Q}_{\mu\nu} = 0$ if and only if

$$\begin{aligned}\mathbf{g}_{\mu\nu} &= \gamma_{\mu\nu} + \epsilon^{-1}q_{\mu\nu}\mathbf{m}, \\ \mathbf{A}^{\mu}_{\nu} &= \alpha^{\mu}_{\nu} + [\widetilde{\beta}^{\mu}_{\nu} - \frac{1}{2\epsilon}(\mathcal{F}^{\mu}_{\cdot\nu} + \mathcal{F}^{\cdot\mu}_{\nu})]\mathbf{m}.\end{aligned} \tag{82}$$

Hence there is the following extension of the fundamental theorem of Riemannian geometry.

Let $\mathbf{g}_{\mu\nu} = \gamma_{\mu\nu} + \chi_{\mu\nu}\mathbf{m}$ be a generalized metric. Then if $\mathbf{A}^{\mu}_{\nu} = \alpha^{\mu}_{\nu} + \beta^{\mu}_{\nu}\mathbf{m}$ is a generalized connection where α^{μ}_{ν} has zero torsion and $\beta_{\mu\nu} = \beta_{\nu\mu}$:

(i) When $\epsilon = 0$ the only such connection which is metric, that is $\mathbf{Dg}_{\mu\nu} = 0$, is $\mathbf{A}^{\mu}_{\nu} = \alpha^{\mu}_{\nu} + \frac{1}{2}D\chi^{\mu}_{\nu}\mathbf{m}$ where α^{μ}_{ν} is the unique Levi-Civita connection for the metric $\gamma_{\mu\nu}dx^{\mu}dx^{\nu}$. In this case the generalized curvature form is $\mathbf{F}^{\mu}_{\nu} = \mathcal{F}^{\mu}_{\nu} + \frac{1}{2}(\mathcal{F}^{\mu}_{\lambda}\chi^{\lambda}_{\nu} - \chi^{\mu}_{\lambda}\mathcal{F}^{\lambda}_{\nu})\mathbf{m}$.

(ii) When $\epsilon \neq 0$ the only such connections which are metric are given by $\mathbf{A}^\mu_\nu = \alpha^\mu_\nu - \frac{1}{2\epsilon}(\mathcal{F}^\mu_{.\nu} + \mathcal{F}_\nu^{.\mu})\mathbf{m}$ with curvature two-forms $\mathbf{F}^\mu_\nu = \frac{1}{2}(\mathcal{F}^\mu_{.\nu} - \mathcal{F}_\nu^{.\mu}) - \frac{1}{2\epsilon}(q_{\nu\lambda}\mathcal{F}^{\lambda\mu} + q^{\mu\lambda}\mathcal{F}_{\nu\lambda})\mathbf{m}$.

Note that in the latter case if the generalized metric is an ordinary metric, that is $\mathbf{g}_{\mu\nu} = \gamma_{\mu\nu}$, then the only such generalized connection which is metric is $\mathbf{A}^\mu_\nu = \alpha^\mu_\nu$ with generalized curvature $\mathbf{F}^\mu_\nu = \mathcal{F}^\mu_{.\nu}$, where α^μ_ν is the unique Levi-Civita connection for the metric $\gamma_{\mu\nu}dx^\mu dx^\nu$.

7 Discussion

In this paper type $N = 1$ generalized form-valued vector fields have been constructed and it has been shown that generalized vector fields constitute a sub-class of such fields Generalized affine connections and metrics have also been introduced. It is a straightforward matter to extend the results in this paper to general vector bundles, generalized form-valued sections and generalized connections. The latter, discussed in earlier papers, bear a formal similarity to connections used in the higher gauge theories reviewed in [22]. Those generalized connections have been used to formulate Lagrangian field theories and a similar use can be made of the generalized affine connections and metrics introduced here.

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8 Appendix: Exterior derivatives of type N=1 forms

The exterior derivative $d : \Lambda^p_{(0)}(M) \rightarrow \Lambda^{p+1}_{(0)}(M)$ for ordinary forms is uniquely determined by the four conditions [15]

(i) $d(\alpha + \beta) = d\alpha + d\beta$.

(ii) for $f \in f \in \Lambda^0_{(0)}(M)$, df has its usual meaning as the differential of f,

(iii) $d \circ d = 0$,

(iv) $d(\alpha\beta) = d\alpha\beta + (-1)^p \alpha d\beta$, where α is a $p-$form.

Exterior derivatives, $d : \Lambda^p_{(N)}(M) \rightarrow \Lambda^{p+1}_{(N)}(M)$, for generalized forms of type N greater than zero, also satisfy these conditions but they are not uniquely determined by them. The aim of this appendix is to discuss this point by developing previous work, [6], and constructing global solutions of the differential ideal defining the exterior derivative. This will be done here for type $N = 1$ forms since they can be treated most easily and completely.

Let $d : \Lambda^p_{(1)}(M) \rightarrow \Lambda^{p+1}_{(1)}(M)$ be an exterior derivative for type $N = 1$ forms on an on a real $n-$dimensional differentiable manifold M. Assume that $\mathbf{m}$ is a non-zero minus one-form and that any type $N = 1$ generalized $p-$form, $\overset{p}{\mathbf{a}} \in \Lambda^p_{(1)}(M)$, may be expressed as

$$\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m}, \tag{83}$$

where the ordinary forms $\overset{p}{\alpha}$ and $\overset{p+1}{\alpha}$ are respectively of degree p and $p+1$ on M, and p can take integer values from -1 to n. All the forms are assumed to obey the usual rules of exterior algebra and calculus and the exterior derivative of $\mathbf{m}$ is required to be a type $N = 1$ generalized zero-form.

It follows that

$$d\mathbf{m} = \vartheta - \varphi\mathbf{m} \tag{84}$$

where ϑ is an ordinary zero-form and φ is an ordinary one-form on M. Then $d^2\mathbf{m} = 0$ if and only if

$$\begin{aligned} d\vartheta + \vartheta\varphi &= 0, \\ d\varphi &= 0. \end{aligned} \tag{85}$$

The solutions of this closed differential ideal of ordinary forms determine the possible exterior derivatives d. The exterior derivative of any type $N = 1$ form $\overset{p}{\mathbf{a}}$ is then given by

$$d\overset{p}{\mathbf{a}} = d\overset{p}{\alpha} + (-1)^{p+1}\vartheta\overset{p+1}{\alpha} + [d\overset{p+1}{\alpha} - \varphi\overset{p+1}{\alpha}]\mathbf{m}. \tag{86}$$

Consider now the consequences of these global assumptions. In a contractible open set U on M the closed form φ is exact. Therefore in U,

$$\begin{aligned} \varphi &= d\xi, \\ \vartheta &= \tau\exp(-\xi), \end{aligned} \tag{87}$$

for some constant τ and some function ξ. Hence, in U

$$d\mathbf{m} = \tau\exp(-\xi) - d\xi\mathbf{m}. \tag{88}$$

The pair (τ,ξ) is not unique since there is the freedom $\tau \rightarrow \tau\exp\chi$, $\xi \rightarrow \xi+\chi$, where χ is a constant.

Consider next a good covering of M by a family of (contractible) open sets $\{U_I\}$. By Eqs.(87) and (88) there are constants and functions $(\tau_I,\, \xi_I)$ such that

$$\begin{aligned} \varphi &= d\xi_I, \\ \vartheta &= \tau_I \exp(-\xi_I), \\ d\mathbf{m} &= \tau_I \exp(-\xi_I) - d\xi_I \mathbf{m}, \end{aligned} \tag{89}$$

on U_I and similarly on each set in the covering. On the intersection of any two sets in the covering, U_I and U_J say, it follows from Eq.(89) that

$$\begin{aligned} \tau_I \exp(-\xi_I) &= \tau_J \exp(-\xi_J), \\ d\xi_I &= d\xi_J. \end{aligned} \tag{90}$$

Hence on any intersection such as $U_I \cap U_J$

$$\begin{aligned} \xi_I - \xi_J &= \tau_{IJ}, \\ \tau_I &= \tau_J \exp \tau_{IJ}, \end{aligned} \tag{91}$$

for constants τ_{IJ} satisfying $\tau_{IJ} = -\tau_{JI}$. Consistency on triple intersections, $U_I \cap U_J \cap U_K$ requires that

$$\tau_{IJ} + \tau_{JK} + \tau_{KI} = 0. \tag{92}$$

Therefore, on U_I

$$d\overset{p}{\mathbf{a}} = d\overset{p}{\alpha} + (-1)^{p+1} \tau_I \exp(-\xi_I) \overset{p+1}{\alpha} + [d\overset{p+1}{\alpha} - d\xi_I \overset{p+1}{\alpha}]\mathbf{m}, \tag{93}$$

and similarly on all the sets in the open covering.

From Eq.(90) it follows that if τ_I is zero so is τ_J and then on each set U_I in the open cover

$$d\mathbf{m} = -d\xi_I \mathbf{m} \tag{94}$$

and $\vartheta = 0$ on M. Call this case (i). On the other hand if τ_I is non-zero in U_I then, from Eq.(90) τ_J is non-zero in U_J and hence ϑ must be non-zero in M. Call this case (ii).

Now consider rescalings of $\mathbf{m}$ and $\overset{p+1}{\alpha}$. On each open set of the cover such as U_I let c_I be a non-zero constant, and on any intersection such as $U_I \cap U_J$ let these constants be related by

$$c_I = c_J \exp \tau_{IJ}. \tag{95}$$

On U_I let

$$\widetilde{\mathbf{m}}_I = c_I^{-1} \exp(\xi_I)\mathbf{m}, \tag{96}$$

$$\overset{p+1}{\widetilde{\alpha}}_I = c_I \exp(-\xi_I) \overset{p+1}{\alpha}, \tag{97}$$

with similar scalings on the other sets in the open cover. Then

$$\begin{aligned} \overset{p}{\mathbf{a}} &= \overset{p}{\alpha} + \overset{p+1}{\widetilde{\alpha}}_I \widetilde{\mathbf{m}}_I, \\ d\widetilde{\mathbf{m}}_I &= \tau_I c_I^{-1}, \\ d\overset{p}{\mathbf{a}} &= [d\overset{p}{\alpha} + (-1)^{p+1} \tau_I c_I^{-1} \overset{p+1}{\widetilde{\alpha}}_I] + d\overset{p+1}{\widetilde{\alpha}}_I \widetilde{\mathbf{m}}_I, \end{aligned} \tag{98}$$

on U_I and similarly on all the sets in the cover. It follows from Eq.(90) and the following equations that on any intersection such as $U_I \cap U_J$

$$\begin{aligned} \widetilde{\mathbf{m}}_I = \widetilde{\mathbf{m}}_J,\ & \overset{p+1}{\widetilde{\alpha}}_I = \overset{p+1}{\widetilde{\alpha}}_J, \\ d\widetilde{\mathbf{m}}_I = d\widetilde{\mathbf{m}}_J,\ & d\overset{p+1}{\widetilde{\alpha}}_I = d\overset{p+1}{\widetilde{\alpha}}_J, \end{aligned} \tag{99}$$

and there is consistency on triple intersections.

In case (i), for all c_I

$$d\widetilde{\mathbf{m}}_I = 0, \tag{100}$$

$$d\overset{p}{\mathbf{a}} = d\overset{p}{\alpha} + d\overset{p+1}{\widetilde{\alpha}}_I \widetilde{\mathbf{m}}_I \tag{101}$$

In case (ii), making the choice of constants c_I

$$c_I = \tau_I \epsilon^{-1} \tag{102}$$

in U_I , where ϵ is a real non-zero constant, gives

$$\begin{aligned} d\widetilde{\mathbf{m}}_I &= \epsilon, \\ d\overset{p}{\mathbf{a}} &= [d\overset{p}{\alpha} + (-1)^{p+1} \epsilon \overset{p+1}{\widetilde{\alpha}}_I] + d\overset{p+1}{\widetilde{\alpha}}_I \widetilde{\mathbf{m}}_I, \end{aligned} \tag{103}$$

and similarly for all the open sets in the cover. When $\epsilon = 1$ this choice corresponds to the choice of what has been termed a canonical basis on M.

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