Investigation of a superspace approach to generalized forms and vector fields

D C Robinson
Mathematics Department
King's College London
Strand, London WC2R 2LS
United Kingdom
email: david.c.robinson@kcl.ac.uk

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Abstract

The relationship between generalized differential forms and generalized vector fields to certain differential forms and vector fields in superspace is explored. The investigations in this technical report have some interesting aspects but a more satisfactory approach to generalized forms and vector fields has now been developed in the paper Generalized forms, vector fields and superspace.

1 Introduction

Generalized forms have been applied in diverse areas of geometrically related physics, including twistor theory [1] [2] [3], Lagrangian field theories, such as BF theory, Yang- Mills, general relativity, Chern-Simons theories [4] [5] and path space [8]. This paper extends the formal developments and applications of generalized forms contained in [9] to [14]. Its aim is to explore relationships between the exterior algebra and differential calculus of generalized differential forms on manifolds and the algebra and calculus of exterior forms on supermanifolds. In particular, generalized vector fields, introduced in [6] and [7], are related to certain vector fields on superspace and the actions of vector fields on differential forms are discussed.

A review of the algebra and differential calculus of generalized forms needed in this paper is given in the second section. Different types of generalized differential forms, on an n dimensional manifold M, are labelled by a non-negative integer N. In this paper the case where $N = 1$ is considered. A type $N = 1$ generalized $p-$form is defined by an ordered pair consisting of two ordinary forms of degrees p and $p + 1$ respectively, where $-1 \leqq p \leqq n$. The module of type $N = 1$ generalized $p-$forms on M is denoted $\Lambda^p_{(1)}(M)$. The exterior product for generalized forms makes the vector space of type $N = 1$ forms at a point x in M, $\Lambda^\bullet_{(1)}(x) = \oplus^{p=n}_{p=-1}\Lambda^p_{(1)}(x)$, into an associative algebra, in fact a super-commutative graded algebra. Generalized forms of degree zero form a commutative ring with $1 \neq 0$. The graded module, and super-commutative graded algebra over the ring of smooth functions, on M is equipped with an exterior derivatives, $d : \Lambda^p_{(1)}(M) \to \Lambda^{p+1}_{(1)}(M)$, a super-derivation of degree one. While both the exterior algebra and differential calculus satisfied by generalized forms are similar to the algebra and calculus of ordinary forms there are some differences. For instance, generalized forms of degree $p = -1$ are allowed and the generalized de Rham cohomology can be different from the de Rham cohomology of ordinary forms.

Properties and actions of generalized vector fields are summarized in section three. These objects are ordered pairs of ordinary vector fields and functions on M. Their actions on generalized forms are extensions of the actions of ordinary vector fields considered in [10]. The interior products, Lie derivatives and Lie brackets constructed in [6] are listed in this section.

In section four possible relations between the algebra and differential calculus of type $N = 1$ generalized $p-$forms and the exterior algebra and differential calculus of certain classes of differential forms on supermanfolds are discussed. Mappings between generalized forms on M and certain superspace forms and between a class of vector fields on superspace and generalized vector fields on M are constructed. The compatibility of these mappings with exterior products, exterior derivatives, commutators, interior products and Lie derivatives are considered. Incompatibilities which arise and the definitions in [6] and [7] are discussed. All these calculations are local so the manifold M is taken to be $\mathbb{R}^n$ and the corresponding superspace to be $\mathbb{R}^{n|1}$. Since the approach to the integration of generalized forms, as developed in [14], is different from that of superspace integration the integral calculi are not considered.

The fifth section contains a summary and discussion of the results. Finally there is an appendix in which some of the results of section four are illustrated by using them to formulate the Dirac equation in Minkowski space-time.

In general the forms and manifolds considered may be real or complex but in this paper it will be assumed, unless it is otherwise explicitly stated, that the geometry is real, all geometrical objects are smooth and that M is an $n-$dimensional real, smooth, orientable and oriented manifold. Bold-face Roman letters are used to denote generalized forms and generalized vector fields, ordinary forms on M are denoted by Greek letters and vector fields on M by lower case Roman letters. The exterior product of any two forms, for example α and β, is written $\alpha\beta$, and as usual, any ordinary $p-$form $\overset{p}{\alpha}$, with p either negative or greater than n, is zero. The degree of a form is indicated above it and the Einstein summation convention is used

2 Algebra and calculus of generalized forms

The algebraic and differential properties of generalized forms are outlined in this section using the notation of [13] and [14]. In this paper generalized forms will be expressed in terms of a minus one-form which is linearly independent of ordinary forms on M, [13]. Hence a basis for type $N = 1$ generalized forms consists of any basis for ordinary forms on M augmented by a minus one-forms $\mathbf{m}$. These latter type of objects have the algebraic properties of ordinary exterior forms but are assigned a degree of minus one. They satisfy the ordinary distributive and associative laws of exterior algebra and the exterior product rule

$$\overset{p}{\alpha}\mathbf{m} = (-1)^p \mathbf{m}\overset{p}{\alpha};\ \mathbf{m}^2 = 0, \tag{1}$$

together with the condition of linear independence. Thus, for a given choice of $\mathbf{m}$, a generalized p-form, $\overset{p}{\mathbf{a}} \in \Lambda^p_{(1)}$, can be written as

$$\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m}, \tag{2}$$

where $\overset{p}{\alpha}$, and $\overset{p+j}{\alpha}$ are, respectively, ordinary $p-$ and $(p+1)-$forms and p can take integer values from -1 to n. At a point x in M the generalized $p-$forms of type N=1, $\Lambda^p_{(1)}(x)$, form a real vector space of dimension $\frac{(1+n)!}{(1+p)!(n-p)!}$. The dimension of $\Lambda^{\bullet}_{(1)}(x) = \oplus_{p=-1}^{p=n}\Lambda^p_{(1)}(x)$ is 2^{1+n}.

If φ is a smooth map between manifolds P and M, $\varphi : P \to M$, then the induced map of type $N = 1$ generalized forms, $\varphi^*_{(1)} : \Lambda^p_{(1)}(M) \to \Lambda^p_{(1)}(P)$, is the linear map defined by using the standard pull-back map, φ^*, for ordinary forms

$$\varphi^*_{(1)}(\overset{p}{\mathbf{a}}) = \varphi^*(\overset{p}{\alpha}) + \varphi^*(\overset{p+1}{\alpha})\mathbf{m}, \tag{3}$$

and $\varphi^*_{(1)}(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = \varphi^*_{(1)}(\overset{p}{\mathbf{a}})\varphi^*_{(1)}(\overset{q}{\mathbf{b}})$. Hence $\varphi^*_{(1)}(\mathbf{m}) = \mathbf{m}$.

Henceforth in this paper, in addition to assuming that the exterior derivative of generalized forms satisfies the usual properties, it is assumed that

$$d\mathbf{m} = \epsilon, \tag{4}$$

where ϵ denotes a real constant. If $\mathbf{m} \mapsto \widetilde{\mathbf{m}} = \mu\mathbf{m}$, where μ is a non-zero function on M, then $\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m} = \overset{p}{\alpha} + \overset{p+1}{\widetilde{\alpha}}\widetilde{\mathbf{m}}$, where $\overset{p+1}{\widetilde{\alpha}} = \mu^{-1}\overset{p+1}{\alpha}$. Furthermore $\mathrm{d}\widetilde{\mathbf{m}} = \widetilde{\epsilon}$, where $\widetilde{\epsilon}$ is also a real constant, if and only if $d\mu = 0$ and then $\widetilde{\epsilon} = \mu\epsilon$.

The exterior derivative of a type $N = 1$ generalized form $\overset{p}{\mathbf{a}}$ is then

$$d\overset{p}{\mathbf{a}} = [d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}] + d\overset{p+1}{\alpha}\mathbf{m}, \tag{5}$$

where d is the ordinary exterior derivative when acting on ordinary forms. The exterior derivative $d : \Lambda^p_{(1)}(M) \to \Lambda^{p+1}_{(1)}(M)$ is an anti-derivation of degree one,

$$\begin{aligned} d(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) &= d\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}} + (-1)^p\overset{p}{\mathbf{a}}d\overset{q}{\mathbf{b}}, \\ d^2 &= 0. \end{aligned} \tag{6}$$

and $(\Lambda^{\bullet}_{(N)}(M), d)$ is a differential graded algebra. The exterior derivative is discussed in more detail in the appendix.

3 Generalized vector fields and type $N = 1$ forms

The ordinary Cartan calculus consists of three linear operators, the exterior derivative d, the interior product (or contraction) i_v, and the Lie derivative

$\pounds_v$,where v is an ordinary vector field on M. The H.Cartan formulae, [15], are satisfied, that is

$$\begin{aligned} d^2 = 0,\ d\pounds_v - \pounds_v d = 0, \\ d\iota_v + \iota_v d = \pounds_v,\ \pounds_v\pounds_w - \pounds_w\pounds_v = \pounds_{[v,w]}, \\ \pounds_v\iota_w - \iota_w\pounds_v = \iota_{[v,w]},\ \iota_v\iota_w + \iota_w\iota_v = 0, \end{aligned} \tag{7}$$

for all vector fields v and w on M. They span a Lie superalgebra. These linear operators were used in the definitions of similar operators acting on generalized forms. They were introduced and defined in [10] for type $N = 1$ forms as follows (for $N \geqq 1$ see [11]). Let two type $N = 1$ generalized forms be given by $\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m}$ and $\overset{q}{\mathbf{b}} = \overset{q}{\beta} + \overset{q+1}{\beta}\mathbf{m}$. The interior (contraction or inner) product for type $N = 1$ generalized forms is defined for $-1 \leqq p \leqq n$ by

$$\begin{aligned} i_v : \Lambda^p_{(1)} \to \Lambda^{p-1}_{(1)} \\ i_v\overset{p}{\mathbf{a}} = i_v\overset{p}{\alpha} + i_v\overset{p+1}{\alpha}\mathbf{m}, \end{aligned} \tag{8}$$

where $\overset{-1}{\alpha} = 0$, and $i_v\overset{0}{\alpha} = 0$. Then

$$i_v(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = (i_v\overset{p}{\mathbf{a}})\overset{q}{\mathbf{b}} + (-1)^p\overset{p}{\mathbf{a}}(i_v\overset{q}{\mathbf{b}}), \tag{9}$$

that is

$$i_v(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = i_v(\overset{p}{\alpha}\overset{q}{\beta}) + [i_v(\overset{p}{\alpha}\overset{q+1}{\beta}) + (-1)^q i_v(\overset{p+1}{\alpha}\overset{q}{\beta})]\mathbf{m}. \tag{10}$$

The Lie derivative with respect to v of a generalized form, $\pounds_v$, is defined by a Cartan-like formula

$$\pounds_v\overset{p}{\mathbf{a}} = i_v(d\overset{p}{\mathbf{a}}) + d(i_v\overset{p}{\mathbf{a}}), \tag{11}$$

or equivalently

$$\pounds_v\overset{p}{\mathbf{a}} = \pounds_v\overset{p}{\alpha} + (\pounds_v\overset{p+1}{\alpha})\mathbf{m}. \tag{12}$$

Then

$$\pounds_v(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = (\pounds_v\overset{p}{\mathbf{a}})\overset{q}{\mathbf{b}} + \overset{p}{\mathbf{a}}\pounds_v(\overset{q}{\mathbf{b}}). \tag{13}$$

$\Lambda_{(1)}M)$ is a graded algebra and there is a natural grading of these linear operators on $\Lambda(M)$, d is degree 1, $\pounds_v$ is of degree 0 and ι_v is of degree -1. These derivations span a super Lie algebra and satisfy the H.Cartan type relations above.

These results were extended when generalized vector fields were introduced in [6]. In the remainder of this section definitions from that paper will be listed so that they can be compared with results obtained in subsequent sections by using standard superspace results.

A generalized vector field is defined as an ordered pair of an ordinary vector field on M and a function on M. A generalized vector field $\mathbf{V}$ can be written as

$$\mathbf{V} = (v_1, v_0) \tag{14}$$

where v_1 is a vector field and v_0 is a function on M. The product of a generalized zero-form, $\overset{0}{\mathbf{a}} = \overset{0}{\alpha} + \overset{1}{\alpha}\mathbf{m}$, with a generalized vector field is defined to be

$$\overset{0}{\mathbf{a}}\mathbf{V} = (\overset{0}{\alpha}v_1, \overset{0}{\alpha}v_0 + i_{v_1}\overset{1}{\alpha}). \tag{15}$$

It follows from this definition that $(\overset{0}{\mathbf{b}}\overset{0}{\mathbf{a}})\mathbf{V} = \overset{0}{\mathbf{b}}(\overset{0}{\mathbf{a}}\mathbf{V})$ where $\overset{0}{\mathbf{b}}$ is a second generalized zero-form. Generalized vector fields form a module over the ring of generalized zero-forms on M.

The generalized contraction or interior product of a generalized vector field and a generalized $p-$form $\overset{p}{\mathbf{a}}$ is denoted $I_{\mathbf{V}}\overset{p}{\mathbf{a}}$ and is a generalized $(p-1)-$form given by

$$I_{\mathbf{V}}\overset{p}{\mathbf{a}} = i_{v_1}\overset{p}{\alpha} + [i_{v_1}\overset{p+1}{\alpha} + \kappa p(-1)^{p-1}v_0\overset{p}{\alpha}]\mathbf{m}, \tag{16}$$

where κ is an arbitrary constant. This is a (graded) derivation of degree minus one and is constructed to satisfy the Leibniz rule

$$I_{\mathbf{V}}(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = I_{\mathbf{V}}(\overset{p}{\mathbf{a}})\overset{q}{\mathbf{b}} + (-1)^p\overset{p}{\mathbf{a}}I_{\mathbf{V}}(\overset{q}{\mathbf{b}}), \tag{17}$$

where $\overset{q}{\mathbf{b}}$ is any generalized $q-$ form. In their papers the authors set $\kappa = 1$. However this will not be done here so that later results can be easily compared with results in this section.

The generalized Lie derivative, $\mathfrak{L}_{\mathbf{V}}\overset{p}{\mathbf{a}}$ of a generalized form $\overset{p}{\mathbf{a}}$ with respect to a generalized vector field $\mathbf{V}$ is defined by Cartan-like formula

$$\begin{aligned} \mathfrak{L}_{\mathbf{V}}\overset{p}{\mathbf{a}} &= I_{\mathbf{V}}d\overset{p}{\mathbf{a}} + \mathbf{d}(I_{\mathbf{V}}\overset{p}{\mathbf{a}}) \\ &= (\pounds_{v_1}\overset{p}{\alpha} - \epsilon\kappa p v_0\overset{p}{\alpha}) + \\ &+ \{\pounds_{v_1}\overset{p+1}{\alpha} - \epsilon(p+1)\kappa v_0\overset{p+1}{\alpha} + \kappa p(-1)^{p-1}(dv_0)\overset{p}{\alpha} + (-1)^p\kappa v_0 d\overset{p}{\alpha}\}\mathbf{m}. \end{aligned} \tag{18}$$

It is a derivation of degree zero and satisfies the Leibniz rule

$$\mathfrak{L}_{\mathbf{V}}(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = \mathfrak{L}_{\mathbf{V}}(\overset{p}{\mathbf{a}})\overset{q}{\mathbf{b}} + \overset{p}{\mathbf{a}}\mathfrak{L}_{\mathbf{V}}(\overset{q}{\mathbf{b}}). \tag{19}$$

The generalized commutator of two generalized vector fields $\mathbf{V}$ and $\mathbf{W}$ is a generalized vector field and is defined by

$$\{\mathbf{V}, \mathbf{W}\} = ([v_1, w_1], \pounds_{v_1} w_0 - \pounds_{w_1} v_0). \tag{20}$$

Here $\mathbf{W} = (w_1, w_0)$, and the ordinary commutator or Lie bracket is denoted with square brackets. The Jacobi identity is satisfied so generalized vector fields form a Lie algebra. The generalized commutator is constructed so that for any generalized form

$$(\mathfrak{L}_{\mathbf{V}}\mathfrak{L}_{\mathbf{W}} - \mathfrak{L}_{\mathbf{W}}\mathfrak{L}_{\mathbf{V}})\overset{p}{\mathbf{a}} = \mathfrak{L}_{\{\mathbf{V},\mathbf{W}\}}\overset{p}{\mathbf{a}}. \tag{21}$$

A modified Lie derivative $\widehat{\mathfrak{L}}_{\mathbf{V}}$, which like the Lie derivative, $\mathfrak{L}_{\mathbf{V}}$, is a derivation of degree zero satisfying the Leibniz rule and

$$(\widehat{\mathfrak{L}}_{\mathbf{V}}\widehat{\mathfrak{L}}_{\mathbf{W}} - \widehat{\mathfrak{L}}_{\mathbf{W}}\widehat{\mathfrak{L}}_{\mathbf{V}})\overset{p}{\mathbf{a}} = \widehat{\mathfrak{L}}_{\{\mathbf{V},\mathbf{W}\}}\overset{p}{\mathbf{a}}, \tag{22}$$

is given by

$$\widehat{\mathfrak{L}}_{\mathbf{V}}\overset{p}{\mathbf{a}} = (\pounds_{v_1}\overset{p}{\alpha} - p\epsilon v_0\overset{p}{\alpha}) + (\pounds_{v_1}\overset{p+1}{\alpha} - (p+1)\epsilon v_0\overset{p+1}{\alpha})\mathbf{m}. \tag{23}$$

The modified Lie derivative of a generalized vector field is defined by

$$\widehat{\mathfrak{L}}_{\mathbf{V}}\mathbf{W} = ([v_1, w_1] + \epsilon v_0 w_1, \pounds_{v_1} w_0). \tag{24}$$

With this definition $\widehat{\mathfrak{L}}$, unlike the unmodified generalized Lie derivative $\mathfrak{L}$, satisfies

$$\widehat{\mathfrak{L}}_{\mathbf{V}} I_{\mathbf{W}} - I_{\mathbf{W}}\widehat{\mathfrak{L}}_{\mathbf{V}} = I_{\widehat{\mathfrak{L}}_{\mathbf{V}}\mathbf{W}}. \tag{25}$$

However, $\widehat{\mathfrak{L}}_{\mathbf{V}}\mathbf{W}$ is not equal to the generalized commutator and the modified Lie derivative of a generalized form is not given by a Cartan-like formula. The modified Lie derivative will not be considered further in this paper.

4 Superspace, generalized forms and vector fields

First in this section relations between the exterior algebra and calculus of type $N = 1$ generalized $p-$forms on a manifold M and the exterior algebra and calculus of certain classes of $[p+1+r(p)]-$forms, $-1 \leqq p \leqq n$, on a supermanifold [16] [17] [18], are constructed and explored. Here $r(p)$ denotes an integer-valued function of p which will be discussed below. Then vector fields on superspace and generalized vector fields are related and the action of the vector fields on the differential forms is discussed. All the considerations here are local ones so the real manifold M is taken to be $\mathbb{R}^n$ and the supermanifold to be the $\mathbb{Z}_2$ graded vector space $\mathbb{R}^{n|1}$. Complex manifolds can be dealt with in a similar way In this section the superspace definitions and sign conventions generally follow those used in [18].

Let coordinates on $\mathbb{R}^{n|1}$ be the (even) coordinates of $\mathbb{R}^n$ and the odd variable y which commutes with ordinary functions and forms on $\mathbb{R}^n$. The following relations and conventions hold,

$$\begin{aligned} y^2 &= 0,\ ydy = -(dy)y, \\ dydy &\neq 0, \\ \overset{p}{\alpha}dy &= (-1)^p(dy)\overset{p}{\alpha}, \\ d(y\overset{p}{\alpha}) &= dy\overset{p}{\alpha} + yd\overset{p}{\alpha}, \end{aligned} \tag{26}$$

where $\overset{p}{\alpha}$ is any ordinary one-form on $\mathbb{R}^n$.

Let $\overset{p}{\mathbf{a}}$ and $\overset{q}{\mathbf{b}}$ be generalized type $N = 1$ forms on $\mathbb{R}^n$

$$\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m} \text{ and } \overset{q}{\mathbf{b}} = \overset{q}{\beta} + \overset{q+1}{\beta}\mathbf{m}, \tag{27}$$

where $d\mathbf{m} = \epsilon$ and ϵ is a real constant. If their exterior product is the degree $(p+q)$ generalized form $\overset{p+q}{\mathbf{c}}$,then

$$\begin{aligned} \overset{p+q}{\mathbf{c}} &= \overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}, \\ &= \overset{p}{\alpha}\overset{q}{\beta} + [\overset{p}{\alpha}\overset{q+1}{\beta} + (-1)^q\overset{p+1}{\alpha}\overset{q}{\beta}]\mathbf{m}, \end{aligned} \tag{28}$$

Now, for given ϵ, k and $r(p)$ introduce the map $\Phi : \Lambda^{\bullet}_{(1)}(\mathbb{R}^n) \to \Lambda^{\bullet}(\mathbb{R}^{n|1})$ by $\overset{p}{\mathbf{a}} \in \Lambda^{\bullet}_{(1)}(\mathbb{R}^n) \to \overset{p+1+r(p)}{\mathfrak{a}} \in \Lambda^{\bullet}(\mathbb{R}^{n|1})$ where

$$\overset{p}{\mathbf{a}} \mapsto \Phi(\overset{p}{\mathbf{a}}) = \overset{p+1+r(p)}{\mathfrak{a}} = [\overset{p}{\alpha} dy + (\epsilon y + k)\overset{p+1}{\alpha}][dy]^{r(p)}. \tag{29}$$

Here k is an odd Grassmann parameter, satisfying

$$\begin{aligned} dk &= 0, \\ k^2 &= 0,\ ky = -yk,\ kdy = -(dy)k, \end{aligned} \tag{30}$$

and for $s \geqq 0$ $[dy]^s$ denotes the $s-$fold exterior product of dy with itself, with $[dy]^0 = 1$. Furthermore $r : p \mapsto r(p)$ is an integer-valued function of p as above.

Under a (super) diffeomorphism $y = \mu\widetilde{y} + l$, where μ is of even type and l is odd, $d\mu = dl = 0$, $\overset{p+1+r(p)}{\mathfrak{a}} = [\overset{p}{\widetilde{\alpha}} d\widetilde{y} + (\widetilde{\epsilon}\widetilde{y} + \widetilde{k})\overset{p+1}{\widetilde{\alpha}}][d\widetilde{y}]^{r(p)}$, where $\overset{p}{\widetilde{\alpha}} = \overset{p}{\alpha}\mu^{r(p)+1}$, $\overset{p+1}{\widetilde{\alpha}} = \overset{p+1}{\alpha}\mu^{r(p)}, \widetilde{\epsilon} = \epsilon\mu, \widetilde{k} = \epsilon l + k$. For fixed $\mathbf{m}$ and y and k the map $\Lambda^{\bullet}_{(1)}(\mathbb{R}^n) \to \Phi[\Lambda^{\bullet}_{(1)}(\mathbb{R}^n)]$ is bijective.

Similarly $\Phi(\overset{q}{\mathbf{b}})$ is the $[q+1+r(q)]-$form $\overset{q+1+r}{\mathfrak{b}}$ on $\mathbb{R}^{n|1}$ given by

$$\Phi(\overset{q}{\mathbf{b}}) = \overset{q+1+r(q)}{\mathfrak{b}} = [\overset{q}{\beta} dy + (\epsilon y + k)\overset{q+1}{\beta}][dy]^{r(q)},$$

and $\Phi(\overset{p+q}{\mathbf{c}})$ is the $[p+q+1+r(p+q)]-$form on $\mathbb{R}^{n|1}$, given by

$$\overset{p+q+1+r(p+q)}{\mathfrak{c}} = \{\overset{p}{\alpha}\overset{q}{\beta} dy + [\epsilon y + k][\overset{p}{\alpha}\overset{q+1}{\beta} + (-1)^q\overset{p+1}{\alpha}\overset{q}{\beta}]\}\{dy\}^{r(p+q)}. \tag{31}$$

On the other hand the exterior product of of the superspace forms $\Phi(\overset{p}{\mathbf{a}})$ and $\Phi(\overset{q}{\mathbf{b}})$ is given by the $[p+q+2+r(p)+r(q)]-$form

$$\begin{aligned} \Phi(\overset{p}{\mathbf{a}})\Phi(\overset{q}{\mathbf{b}}) &= \overset{p+1+r(p)}{\mathfrak{a}}\overset{q+1+r(q)}{\mathfrak{b}} \\ &= (-1)^{q[r(p)+1]}\{\overset{p}{\alpha}\overset{q}{\beta} dy + [\epsilon y + k][\overset{p}{\alpha}\overset{q+1}{\beta} + (-1)^q\overset{p+1}{\alpha}\overset{q}{\beta}]\}\{dy\}^{1+r(p)+r(q)}. \end{aligned} \tag{32}$$

Hence the exterior product of $\Phi(\overset{p}{\mathbf{a}})$ and $\Phi(\overset{q}{\mathbf{b}})$ is related to $\Phi(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = \Phi(\overset{p+q}{\mathbf{c}})$ by

$$\Phi(\overset{p}{\mathbf{a}})\Phi(\overset{q}{\mathbf{b}}) = (-1)^{q[r(p)+1]}\Phi(\overset{p+q}{\mathbf{c}})(dy)^{1+r(p)+r(q)-r(p+q)}, \text{ that is} \tag{33}$$

$$\overset{p+1+r(p)}{\mathfrak{a}}\overset{q+1+r(q)}{\mathfrak{b}} = (-1)^{q[r(p)+1]}(\overset{p+q+1+r(p+q)}{\mathfrak{c}})(dy)^{1+r(p)+r(q)-r(p+q)}.$$

Consider now the exterior calculus. The exterior derivative of the generalized $p-$form $\overset{p}{\mathbf{a}}$ is given by the $(p+1)-$form

$$d\overset{p}{\mathbf{a}} = (d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}) + d\overset{p+1}{\alpha}\mathbf{m}, \tag{34}$$

with image under Φ given by the $[p+2+r(p+1)]-$form

$$\Phi(d\overset{p}{\mathbf{a}}) = \{[d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}]dy + (\epsilon y + k)d\overset{p+1}{\alpha}\}\{dy\}^{r(p+1)}. \tag{35}$$

However the exterior derivative of $\Phi(\overset{p}{\mathbf{a}})=\overset{p+1+r(p)}{\mathfrak{a}}$ is given by

$$d[\Phi(\overset{p}{\mathbf{a}})] = d\overset{p+1+r(p)}{\mathfrak{a}} = \{[d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}]dy + (\epsilon y + k)d\overset{p+1}{\alpha}\}\{dy\}^{r(p)}. \tag{36}$$

From Eq.(33) it follows that Φ is a homomorphism of the exterior algebras of generalized forms and forms on superspace, that is

$$\Phi(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = \Phi(\overset{p}{\mathbf{a}})\Phi(\overset{q}{\mathbf{b}}),$$

if and only if

$$\begin{aligned} 1 + r(p) + r(q) - r(p+q) &= 0, \\ (-1)^{q[r(p)+1]} &= 1. \end{aligned} \tag{37}$$

The solution of Eq.(37) is given by

$$r(p) = 2cp - 1, \tag{38}$$

where c is an integer. On the other hand, it can be seen from Eqs.(35) and(36) that the map Φ is compatible with the exterior derivatives, that is

$$\Phi(d\overset{p}{\mathbf{a}}) = d[\Phi(\overset{p}{\mathbf{a}})],$$

if and only if

$$r(p) = r(p+1). \tag{39}$$

This equation requires r to be the constant function. This difference between the compatibility conditions given by Eqs. (37) and (39) has consequences which will be discussed below. Before that a relationship between vector fields on $\mathbb{R}^{n|1}$ and generalized vector fields on $\mathbb{R}^n$, discussed in section three, will be exhibited.

Let $\chi_{(1)}(\mathbb{R}^{n|1})$ denote the module, over the ring of real functions on $\mathbb{R}^n$, of superspace vector fields of the form

$$\mathbb{V} = v_1 + v_0(\epsilon y + k)\frac{\partial}{\partial y}, \tag{40}$$

where v_1 and v_0 are respectively vector fields and functions on $\mathbb{R}^n$. The Lie supercommutator, of $\mathbb{V}$ and $\mathbb{W} = w_1 + w_0(\epsilon y + k)\frac{\partial}{\partial y}$ is (with the Lie superbracket sign convention opposite to [18])

$$[\mathbb{V}, \mathbb{W}] = [v_1, w_1] + [v_1(w_0) - w_1(v_0)](\epsilon y + k)\frac{\partial}{\partial y}. \tag{41}$$

Hence the module of such vector fields is an algebra, in fact a Lie algebra. Furthermore the Lie derivative of $\mathbb{W}$ with respect $\mathbb{V}$ is

$$\pounds_{\mathbb{V}}\mathbb{W} = [\mathbb{V}, \mathbb{W}] = \pounds_{v_1} w_1 + [v_1(w_0) - w_1(v_0)](\epsilon y + k)\frac{\partial}{\partial y}. \tag{42}$$

where, as above, $\pounds_{v_1}$ denotes the ordinary Lie derivative when acting on tensor fields on $\mathbb{R}^n$.

Let Ψ denote the mapping of $\chi_{(1)}(\mathbb{R}^{n|1})$ to the module of generalized vector fields on $\mathbb{R}^n$, $\chi_G(\mathbb{R}^n)$ by

$$\begin{aligned} &\Psi : \mathbb{V} \mapsto \Psi(\mathbb{V}) = \mathbf{V}, \text{ where} \\ &\mathbb{V} = v_1 + v_0(\epsilon y + k)\frac{\partial}{\partial y} \text{ and } \mathbf{V} = (v_1, v_0). \end{aligned} \tag{43}$$

For fixed ϵ, y and k the map $\chi_{(1)}(\mathbb{R}^{n|1}) \to \Psi(\chi_{(1)}(\mathbb{R}^{n|1}))$ is a bijection.

It follows from Eqs.(20) and (41) that this mapping is compatible with the Lie supercommutator on $\mathbb{R}^{n|1}$ and the generalized commutator of generalized vector fields on $\mathbb{R}^n$, that is

$$\Psi([\mathbb{V}, \mathbb{W}]) = \{\Psi(\mathbb{V}), \Psi(\mathbb{W})\} = \{\mathbf{V}, \mathbf{W}\}, \tag{44}$$

where the generalized vector fields $\Psi(\mathbb{V}) = \mathbf{V} = (v_1, v_0)$ and $\Psi(\mathbb{W}) = \mathbf{W} = (w_1, w_0)$.

Such compatibility does not extend in a straightforward way to the interior product and the Lie derivatives of forms. This is unsurprising considering the compatibility results above for the map Φ and the exterior products on the one hand and the exterior derivatives on the other hand First consider the interior product.

The superspace interior product for such a vector field $\mathbb{V}$,$i_{\mathbb{V}}$, is a graded derivation of degree minus one and and for any superspace forms, say $\overset{p}{\mathfrak{u}}$ and $\overset{q}{\mathfrak{v}}$, satisfies

$$i_{\mathbb{V}}(\overset{p}{\mathfrak{u}}\overset{q}{\mathfrak{v}}) = (i_{\mathbb{V}}\overset{p}{\mathfrak{u}})\overset{q}{\mathfrak{v}} + (-1)^p\overset{p}{\mathfrak{u}}(i_{\mathbb{V}}\overset{q}{\mathfrak{v}}). \tag{45}$$

The interior product of $\mathbb{V}$ with the superspace $[p+1+r(p)]-$form, $\overset{p+1+r(p)}{\mathfrak{a}}$, $p+1+r(p)>0$, is the $[p+r(p)]-$degree form $i_{\mathbb{V}}\overset{p+1+r(p)}{\mathfrak{a}}$ where

$$\begin{aligned} \overset{p+1+r(p)}{\mathfrak{a}} &= [\overset{p}{\alpha}dy + (\epsilon y + k)\overset{p+1}{\alpha}][dy]^{r(p)} \\ i_{\mathbb{V}}\overset{p+1+r(p)}{\mathfrak{a}} &= \{(i_{v_1}\overset{p}{\alpha})dy + [\epsilon y + k][i_{v_1}\overset{p+1}{\alpha} + (-1)^p(r(p)+1)v_0\overset{p}{\alpha}]\}(dy)^{r(p)}. \end{aligned} \tag{46}$$

It follows from Eqs.(16) and (46) above that there is compatibility of the interior products $i_{\mathbb{V}}$ and $I_{\mathbf{V}}$ above with the maps Φ and Ψ, that is

$$\Phi(I_{\mathbf{V}}\overset{p}{\mathbf{a}}) = \Phi(I_{\Psi(\mathbb{V})}\overset{p}{\mathbf{a}}) = i_{\mathbb{V}}\overset{p+1+r(p)}{\mathfrak{a}}, \tag{47}$$

if and only if, for integer c,

$$\begin{aligned} \kappa &= 2c, \\ r(p) &= 2cp - 1. \end{aligned} \tag{48}$$

The superspace Lie derivative of the superspace form $\overset{p+1+r(p)}{\mathfrak{a}}$ with respect to the vector field $\mathbb{V}$ is denoted $\pounds_{\mathbb{V}}$ is a graded derivation of degree zero and is defined by the Cartan-type formula

$$\pounds_{\mathbb{V}}\overset{p+1+r(p)}{\mathfrak{a}} = d(i_{\mathbb{V}}\overset{p+1+r(p)}{\mathfrak{a}}) + i_{\mathbb{V}}(d\overset{p+1+r(p)}{\mathfrak{a}}). \tag{49}$$

Calculation gives

$$\begin{aligned} \pounds_{\mathbb{V}}\overset{p+1+r(p)}{\mathfrak{a}} &= \{[\pounds_{v_1}\overset{p}{\alpha} + \epsilon(r+1)v_0\overset{p}{\alpha}]dy+ \\ &+ [\epsilon y + k][\pounds_{v_1}\overset{p+1}{\alpha} + \epsilon(r+1)v_0\overset{p+1}{\alpha} + (-1)^p(r+1)(dv_0)\overset{p}{\alpha}]\}(dy)^{r(p)}. \end{aligned} \tag{50}$$

Hence the Lie derivatives $\pounds_{\mathbb{V}}$ and $\mathfrak{L}_{\mathbf{V}}$ are compatible with the maps Φ and Ψ, that is

$$\Phi(\mathfrak{L}_{\mathbf{V}}\overset{p}{\mathbf{a}}) = \Phi(\pounds_{\Psi(\mathbb{V})}\overset{p}{\mathbf{a}}) = \pounds_{\mathbb{V}}\Phi(\overset{p}{\mathbf{a}}) = \pounds_{\mathbb{V}}\overset{p+1+r(p)}{\mathfrak{a}}, \tag{51}$$

if and only if

$$\mathfrak{L}_{\mathbf{V}}\overset{p}{\mathbf{a}} = [\pounds_{v_1}\overset{p}{\alpha} + \epsilon(r+1)v_0\overset{p}{\alpha}] + [\pounds_{v_1}\overset{p+1}{\alpha} + \epsilon(r+1)v_0\overset{p+1}{\alpha} + (-1)^p(r+1)(dv_0)\overset{p}{\alpha}]\mathbf{m}. \tag{52}$$

By comparing this equation with the expression for $\mathfrak{L}_{\mathbf{V}}\overset{p}{\mathbf{a}}$ in Eq.(18) it is seen that compatibility under the maps Φ and Ψ holds for the superspace Lie derivative and the generalized Lie of generalized forms if and only if

$$\kappa = r(p) + 1 = 0. \tag{53}$$

Hence the only solution of both Eqs.(38) and (39) compatible with Eq.(51) is

$$c = 0,\ r(p) = -1. \tag{54}$$

This is not surprising as the Lie derivative, unlike the interior product, involves both the latter, with compatibility condition Eq.(48) consistent with Eq.(38) and the exterior derivative with compatibility condition given by Eq.(39). When Eq.(54) is satisfied the definitions of the interior product and the Lie derivative of a generalized form by a generalized vector field (v_1, v_o) on $\mathbb{R}^n$ reduce to the definitions of the interior product and Lie derivative by the ordinary vector field v_1 as in [10].

Furthermore compatibility of both the exterior product and exterior derivative with the maps Φ and Ψ and the consequent compatibility of the interior products and Lie derivatives requires the introduction of a formal superspace quantity $(dy)^{-1}$ so that the map Φ becomes $\Phi : \Lambda^{\bullet}_{(1)}(\mathbb{R}^n) \rightarrow \Lambda^{\bullet}(\mathbb{R}^{n|1})$ by $\overset{p}{\mathbf{a}} \in \Lambda^{\bullet}_{(1)}(\mathbb{R}^n) \rightarrow \overset{p+1}{\mathfrak{a}} \in \Lambda^{\bullet}(\mathbb{R}^{n|1})$ where

$$\overset{p}{\mathbf{a}} \mapsto \Phi(\overset{p}{\mathbf{a}}) = \overset{p}{\mathfrak{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}[(\epsilon y + k)dy^{-1}]. \tag{55}$$

In order that the earlier calculations using $(dy)^s$ continue to hold when $s = -1$ the formal object $(dy)^{-1}$ is assumed to be of odd type, have formal degree minus one, satisfy the usual linearity properties of differential forms and have the same type of properties that $(dy)^s$ for non-negative s, that is

$$\begin{aligned} dy(dy)^{-1} &= (dy)^{-1}dy = (dy)^0 = 1, \\ (\epsilon y + k)(dy)^{-1} &= -(dy)^{-1}(\epsilon y + k),\ \overset{p}{\alpha}(dy)^{-1} = (-1)^p(dy)^{-1}\overset{p}{\alpha}, \\ d[d(y)^{-1}] &= 0,\ d[\overset{p}{\alpha}(dy)^{-1}] = d\overset{p}{\alpha}(dy)^{-1}, d[y(dy)^{-1}] = dy(dy)^{-1} = 1. \end{aligned} \tag{56}$$

Hence, for example, if f is a function on superspace $df(dy)^{-1} = \partial f/\partial\xi^i[d\xi^i(dy)^{-1}] + \partial f/\partial y$, where $\{\xi^i\}$ are n coordinates on $\mathbb{R}^n$.

It is natural, in this context, to define a generalized degree minus one form $\mathfrak{m}$ on $\mathbb{R}^{n|1}$ by

$$\mathfrak{m} = (\epsilon y + k)(dy)^{-1}. \tag{57}$$

so that, from Eq.(55),

$$\begin{aligned} &\mathbf{m} \mapsto \mathbf{\Phi}(\mathbf{m}) = \mathfrak{m}, \\ &\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m} \mapsto \Phi(\mathbf{a}) = \overset{p}{\mathfrak{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathfrak{m}. \end{aligned} \tag{58}$$

5 Conclusion

The first aim of this work was to discover if the algebra and differential calculus of generalized forms could be realized via a mapping into the algebra and calculus of superspace forms. The second aim was to see if generalized vector fields could be satisfactorily related to superspace vector fields. Formulae that appeared in [13] were developed and employed, in a superspace context, to construct mappings between generalized forms and superspace forms. The compatibility conditions between these maps and the exterior products on the one hand and the exterior derivatives on the other were found to be different. A mapping of superspace vector fields to generalized vector fields was also constructed. This map was compatible with the commutators of these vector fields. In addition, when the mapping of forms satisfied the compatibility conditions for the exterior product, it was compatible with the interior products of vector fields and forms However there was no compatibility between the mappings and the Lie derivatives of forms. This was not surprising as the computation of the Lie derivative involves the use of the exterior derivative which required a different compatibility condition from the exterior product. The incompatibilities could be resolved by enlarging the concept of superspace differential forms with the introduction of a formally defined negative degree superspace form. When this was done it was found that the interior products and Lie derivatives with respect to generalized vector fields reduced to the ones introduced for ordinary vector fields in [10]. From the point of view taken in this paper the need for the modification of the Lie derivative, as discussed and introduced in [6] and exhibited in section three above, can be traced back to the difference in the compatibility conditions for the exterior product and exterior derivative. To sum up, while some interesting results and insights were obtained the original aims of this investigation were not fully realized. What the investigation did indicate was that exterior algebra and calculus of generalized forms are indeed generalizations of the ordinary manifold and supermanifold algebras and calculi.

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6 Appendix

6.1 The classical Dirac equation

In this appendix the classical Dirac equation in Minkowski space-time is discussed. This will provide a simple example where the map Φ, defined in section four, that is used is compatible with the exterior derivative but not the exterior product. Consider four dimensional Minkowski space-time $\mathbb{R}^{1,3}$ with line element given, using two-component spinors[19] , by $ds^2 = \varepsilon_{AB}\varepsilon_{A'B'}d\xi^{AA'} \otimes d\xi^{BB'}$, where $\xi^{AA'}$ is related to Minkowski coordinates by

$$\xi^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi^0 + \xi^3 & \xi^1 - i\xi^2 \\ \xi^1 + i\xi^2 & \xi^0 - \xi^3 \end{pmatrix},$$

and spinor suffixes sum and range over zero to one. Using two-component spinors the classical Dirac equation takes the form

$$\partial^{AA'}\sigma_A = -\frac{i\mu}{\sqrt{2}}\rho^{A'}; \partial_{AA'}\rho^{A'} = -\frac{i\mu}{\sqrt{2}}\sigma_A,$$

where $\partial_{AA'} = \partial/\partial\xi^{AA'}$ and μ denotes the rest mass, [20]. Introduce the four-spinor valued type $N = 1$ generalized three-form

$$\mathbf{S} = \begin{pmatrix} \chi^{A'} + \zeta^{A'}\mathbf{m} \\ \alpha_A + \beta_A\mathbf{m} \end{pmatrix}$$

on four dimensional Minkowski space-time $\mathbb{R}^{1,3}$ where $\alpha_A, \chi^{A\prime}$ and $\beta_A, \zeta^{A\prime}$ are respectively the classical spinor-valued ordinary three-forms and four-forms

$$\chi^{A'} = \sigma_A \eta^{AA'}, \zeta^{A'} = \frac{i}{\sqrt{2}} \rho^{A'} \upsilon,$$
$$\alpha_A = \rho^{A'} \eta_{AA'}, \beta_A = \frac{i}{\sqrt{2}} \sigma_A \upsilon.$$

Here υ is the Minkowski volume four-form and $\eta^{AA'} = \frac{i}{3} d\xi^{AB'} d\xi^{BA'} d\xi_{BB'}$ is the basis of three-forms dual to the one-forms basis $d\xi^{AA'}$. The exterior derivative of the minus one-form $\mathbf{m}$ is given by $d\mathbf{m} = \epsilon$, with $\epsilon = \mu$.

Consider first the case where the rest-mass is non-zero. The four-spinor $\psi = \begin{pmatrix} \rho^{A'} \\ \sigma_A \end{pmatrix}$ satisfies the classical Dirac equation if and only if the four-spinor valued generalized three-form $\mathbf{S}$ is closed

$$d\mathbf{S} = 0.$$

By the results of section four, when the choice $r(p) = 0$ is made $\mathbf{S}$ corresponds to the spinor-valued four-form $\Phi(\mathbf{S})$ on $\mathbb{R}^{1,3|1}$,

$$\Phi(\mathbf{S}) = \begin{pmatrix} \chi^{A'} dy + (\mu y + k) \zeta^{A'} \\ \alpha_A dy + (\mu y + k) \beta_A \end{pmatrix},$$

which is also closed,

$$d[\Phi(\mathbf{S})] = 0,$$

if and only if the classical Dirac equation is satisfied by ψ on $\mathbb{R}^{1,3}$. When the Dirac equation is satisfied these forms are not only closed they are also exact and

$$\mathbf{S} = -\frac{1}{\mu} d \begin{pmatrix} \chi^{A'} \mathbf{m} \\ \alpha_A \mathbf{m} \end{pmatrix}; \Phi(\mathbf{S}) = -\frac{1}{\mu} d \begin{pmatrix} (\mu y + k) \chi^{A'} \\ (\mu y + k) \alpha_A \end{pmatrix}.$$

Consider next the case where the rest-mass μ is zero so that the Dirac equations reduce to a pair of decoupled Weyl equations

$$\partial^{AA'} \sigma_A = 0; \partial_{AA'} \rho^{A'} = 0.$$

Since now $d\mathbf{m} = 0$, and therefore $\zeta^{A'} \mathbf{m}$, $\beta_A \mathbf{m}$, $k \zeta^{A'}$ and $k \beta_A$ are closed independently of the field equations, the forms $\mathbf{S}$ and $\Phi(\mathbf{S})$ can be simplified to $\widehat{\mathbf{S}} = \begin{pmatrix} \chi^{A'} \\ \alpha_A \end{pmatrix}$ and $\Phi(\widehat{\mathbf{S}}) = \begin{pmatrix} \chi^{A'}dy \\ \alpha_A dy \end{pmatrix}$ respectively, and the Weyl equations are satisfied if and only if the simplified forms $\widehat{\mathbf{S}}$ and $\Phi(\widehat{\mathbf{S}})$ are closed. When the Weyl equations are satisfied these closed forms can be written as the exact forms

$$\widehat{\mathbf{S}}{=}d\begin{pmatrix} \vartheta^{A'} \\ \gamma_A \end{pmatrix}; \Phi(\widehat{\mathbf{S}}) = d\begin{pmatrix} \vartheta^{A'}dy \\ \gamma_A dy \end{pmatrix}.$$

The spinor-valued two-forms $\vartheta^{A'}$ and γ_A are respectively defined up to the gauge freedoms $\vartheta^{A'} \longmapsto \vartheta^{A'} + d\tau^{A'}$, $\gamma_A \longmapsto \gamma_A + d\iota_A$ where $\tau^{A'}$ and ι_A are arbitrary spinor-valued one-forms.

Now the spinor valued two-form $\vartheta^{A'}$ can be expressed in terms of its components (potentials) with respect to the coordinate basis

$$\vartheta^{A'} = \vartheta^{A'}_{BC}\Sigma^{BC} + \vartheta^{A'}_{B'C'}\Sigma^{B'C'} + (\delta^{A'}_{B'}\varphi_{C'})\Sigma^{B'C'}$$

where $\Sigma^{BC} = \frac{1}{2}d\xi^{B}_{B'}d\xi^{CB'}$, $\Sigma^{B'C'} = \frac{1}{2}d\xi^{B'}_{B}d\xi^{BC'}$, $\vartheta^{A'}_{BC} = \vartheta^{A'}_{(BC)}$, $\vartheta_{A'B'C'} = \vartheta_{(A'B'C')}$, and similarly for γ_A. By using this expansion and the equality $\sigma_A\eta^{AA'} = d\vartheta^{A'}$ the solutions of the Weyl equations can be expressed in the form

$$\sigma_A = \frac{i}{2}[\partial_{BA'}(\vartheta^{A'B}_{\ \ ..A}) - \frac{3}{2}\partial_{AA'}\varphi^{A'}],$$

when the potentials satisfy the equations

$$\partial^{C}_{D'}\vartheta^{D'}_{A'C'} = \partial_{D(C'}\vartheta^{CD}_{A')} + \frac{1}{2}\partial^{C}_{(A'}\varphi_{C')}.$$

The solutions $\rho^{A'}$ may be similarly obtained from the complex conjugates of such equations. This example illustrates a correspondence between generalized forms and superspace forms in a case where the exterior product is not employed. However it should be noted that, because of the linearity in this particular example, if the odd superspace coordinate y and parameter k are replaced respectively by an even coordinate, say on a line bundle over Minkowski space-time, and a real number, the formulae above, and a closed form expression for the Dirac equation, still hold.

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