Generalized forms, Chern-Simons and Einstein-Yang-Mills theory

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Abstract: An integral calculus and a Stokes' theorem for generalized forms are constructed. These are used to investigate the generalized Chern class and Chern-Simons three- form. The latter is employed in the formulation of action principles for Einstein-Yang-Mills theory.

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1 Introduction

The algebra and calculus of ordinary differential forms have been extended in recent years to what have been termed generalized differential forms. Generalized forms have been used in various geometrical and classical field theories, first in the context of twistor theory, [1], [2], [3], then subsequently in other physical applications such as the Lagrangian formulations of various field theories as, for example, in references [4], [5]. The dual notion of a generalized vector field has also been investigated [6]. The general aim of this paper is to consolidate and extend the formulation and applications of generalized forms developed in a number of papers, [7]-[11]. The specific aim is to present and apply an integral calculus of generalized forms. The notions of poly-chain complexes is introduced and integration of generalized forms over poly-chains is defined. A Stokes' theorem for generalized forms is constructed by using poly-chains, and the generalized second Chern class and Chern-Simons three-form are discussed. The relationship of the latter to Lagrangian formulations of the Einstein-Yang-Mills field equations is explored.

A generalized differential form, on an n dimensional manifold M, may be described by an ordered set of ordinary differential forms. There are generalized forms of different types, each type being labelled by a non-negative integer N. The module of type N generalized $p-$forms on M is denoted $\Lambda^p_{(N)}(M)$, where for generalized forms the degree p can take integer values from $-N$ to n. Generalized $p-$forms of type $N = 0$ are just ordinary $p-$forms on M. Generalized $p-$forms of type $N = 1$ may be described by ordered pairs of ordinary p- and $(p+1)$-forms. In a similar way p-forms of type N on M may be described by ordered multiplets of ordinary forms of degrees p to $p+N$. Type $N \geq 1$ $p-$forms can be constructed iteratively in terms of ordered pairs of $p-$ and $(p+1)-$forms of type $(N-1)$. The exterior product for generalized forms makes the vector space of type N forms at a point x in M, $\Lambda^{\cdot}_{(N)}(x) = \oplus^{p=n}_{p=-N}\Lambda^p_{(N)}(x)$, into an associative algebra, in fact a super-commutative graded algebra. Generalized forms of negative degree are permitted and generalized forms of zero degree form a commutative ring with $1 \neq 0$. In this paper attention will be mainly focused on the graded modules, and super-commutative graded algebras over the ring of smooth functions on M. These are equipped with exterior derivatives, $d : \Lambda^p_{(N)}(M) \to \Lambda^{p+1}_{(N)}(M)$, super-derivations of degree one. Further discussion of generalized forms in the context of differential graded algebras and an investigation of their relationship to forms on path spaces can be found in reference [12]. Generalized forms of a given type obey the same basic algebraic and differential rules as ordinary forms, although there are some differences from the ordinary exterior algebra and calculus. For instance generalized forms of negative degree p are allowed and the generalized de Rham cohomology changes when N is non-zero.

A review of the properties of generalized forms needed in this paper is presented the next section. Formalism developed in references [7]-[11] is employed, and the conventions used in reference [11] are retained. Some simple cohomological concepts needed in later sections are also presented. In the third section type $N = 1$ generalized chains, termed poly-chains, and boundary operators in a $n-$dimensional smooth oriented manifold M, are introduced. The integration of type $N = 1$ generalized $p-$forms over poly-chains is then defined. These definitions enable Stokes' theorem for ordinary real chains, or type $N = 0$ poly-chains, and ordinary differential forms, to be extended to Stoke's theorem for poly-chains and type $N = 1$ forms. In this way a real (poly-) chain complex, dual to the co-chain complex of type $N = 1$ generalized differential forms, is constructed. These results are then broadened to encompass integration of generalized forms of any type over generalized real chains in M. Poly-chains, are defined for any $N \geq 1$, and Stoke's theorem is extended to type $N \geq 1$ generalized forms and poly-chains. Within this framework type $N \geq 0$ poly-chains, together with the boundary operators, form a real generalized (poly-) chain complex dual to the type $N \geq 0$ generalized de Rham co-chain complex. The real co-chain complex of type N generalized differential forms is bounded above by co-chains of degree n and bounded below by co-chains of degree $-N$. Similarly the dual real chain complex is not restricted to being non-negative when $N > 0$, and is bounded above by poly-chains of degree n and bounded below by poly-chains of degree $-N$. The last two sections develop in a new way previous investigations of generalized characteristic classes and Lagrangian field theories in four space-time dimensions. In references [4] and [5], generalized topological field theories were introduced and studied. Actions which included Chern-Simons terms naturally induced from the generalized topological action in the bulk were constructed for a variety of field theories by using generalized second Chern class four-forms. Further examples of the use of generalized characteristic classes to construct Lagrangians for classical field theories, such as general relativity, were presented in [11]. In the fourth section the generalizations of the ordinary second Chern class and the Chern-Simons three-form are reconsidered in the light of the results of section three. Type $N = 1$ structures are considered in detail.

An application of these latter results to gravitation and gauge theories is given in the fifth section. The Einstein vacuum and Einstein-Yang-Mills theories are reformulated as generalized Chern-Simons theories. Generalized Chern-Simons type $N = 1$ three-forms, integrated over type $N = 1$ degree three poly-chains, are identified as actions for the Einstein vacuum and Einstein-Yang-Mills equations on four dimensional manifolds with or without cosmological constant and boundary terms. The boundary, or lower dimensional, terms are not added in by hand but arise more naturally. They can include both ordinary Einstein and Yang-Mills Chern-Simons terms as well as additional terms like those added when asymptotically flat systems are studied, [13]. The geometrical framework and approach presented here differs from the large amount of previous research on Chern-Simons gravity. But it does share the same general context as those investigations. Many of them deal with Chern-Simons gravity in 2+1 dimensions with references [14] and [15] being, in different ways, highly influential. A recent review [16] discusses aspects of this research and contains many futher citations. A selection of different studies of Chern-Simons gravity in higher dimensions is contained in references [17], [18] and [19], the latter being an introduction which surveys results applying in odd dimensions. It is a straightforward matter to investigate other characteristic classes and classical field theories using the framework developed in this paper.

In the interests of clarity, and notational simplicity, details of investigations will sometimes be confined to cases where the type N is equal to one. A number of examples explore selected constructions when N is one or two, and it is easy to see how to generalize results to cases where $N \geq 2$.

2 Basic properties and cohomology of generalized forms

Here the salient properties of generalized forms on an n dimensional manifold M will be reviewed. In general the forms and manifolds considered may be real or complex but in this paper it suffices to take M to be real, smooth, orientable and oriented. Bold-face Roman letters are used to denote generalized forms, ordinary forms are denoted by Greek letters, and, where it is useful, the degree of a form is indicated above it. The exterior product of any two forms, for example α and β, is written $\alpha\beta$. As usual, any ordinary $p-$form $\overset{p}{\alpha}$, with p either negative or greater than n, is zero.

A generalized $p-$form of type $N = 0$ is an ordinary $p-$form, $\overset{p}{\mathbf{a}}_{(0)} = \overset{p}{\alpha}$ and the exterior product and derivative of type $N = 0$ forms are the ordinary exterior product and derivative. There are a number of different ways of representing generalized forms of type $N > 0$, [11], and two will be used in this paper. In the first representation, a generalized $p-$form of type $N = 1$ is an ordered pair of ordinary $p-$ and $p+1-$forms,

$$\overset{p}{\mathbf{a}}_{(1)} = (\overset{p}{\alpha}, \overset{p+1}{\alpha}). \tag{1}$$

A generalized p-form of type $N \geq 1$ is then an ordered pair generalized forms of type $N-1$, that is

$$\overset{p}{\mathbf{a}}_{(N)} = (\overset{p}{\mathbf{a}}_{(N-1)}, \overset{p+1}{\mathbf{a}}_{(N-1)}), \tag{2}$$

It follows that $\overset{p}{\mathbf{a}}_{(N)}$can be expressed, iteratively, as an ordered tuple of ordinary forms. If $\overset{q}{\mathbf{b}}_{(N)} = (\overset{q}{\mathbf{b}}_{(N-1)}, \overset{q+1}{\mathbf{b}}_{(N-1)})$ is, in a similar way, a type $N \geq 1$ generalized $q-$form, the exterior product is defined to be the ordered pair

$$\overset{p}{\mathbf{a}}_{(N)}\overset{q}{\mathbf{b}}_{(N)} = (\overset{p}{\mathbf{a}}_{(N-1)}\overset{q}{\mathbf{b}}_{(N-1)},\ \overset{p}{\mathbf{a}}_{(N-1)}\overset{q+1}{\mathbf{b}}_{(N-1)} + (-1)^{q}\overset{p+1}{\mathbf{a}}_{(N-1)}\overset{q}{\mathbf{b}}_{(N-1)}), \tag{3}$$

and this product obeys the same rules as the ordinary exterior product, in particular

$$\overset{p}{\mathbf{a}}_{(N)}\overset{q}{\mathbf{b}}_{(N)} = (-1)^{pq}\overset{q}{\mathbf{b}}_{(N)}\overset{p}{\mathbf{a}}_{(N)}. \tag{4}$$

When $N = 0$ the conditions satisfied by the ordinary exterior derivative ensure its uniqueness. However, when N is greater than one uniqueness does not necessarily follow in the same way. It was pointed out in [11], that there are naturally two distinct types of exterior derivatives when $N \geq 1$. Here they will be carried along together. The two exterior derivatives will be labeled by ϵ, where in the first case $\epsilon = 0$ and in the second case $\epsilon = 1$. The exterior derivatives, are defined, iteratively, by the ordered pairs

$$\begin{aligned} d\overset{p}{\mathbf{a}}_{(0)} &= d\overset{p}{\alpha};\ d\overset{p}{\mathbf{a}}_{(1)} = (d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}, d\overset{p+1}{\alpha}) \\ d\overset{p}{\mathbf{a}}_{(N)} &= (d\overset{p}{\mathbf{a}}_{(N-1)}, d\overset{p+1}{\mathbf{a}}_{(N-1)}); N > 1. \end{aligned} \tag{5}$$

where again d is the ordinary exterior derivative when acting on ordinary forms. The exterior derivative $d : \Lambda^{p}_{(N)}(M) \rightarrow \Lambda^{p+1}_{(N)}(M)$ is an anti-derivation,

$$d(\overset{p}{\mathbf{a}}_{(N)}\overset{q}{\mathbf{b}}_{(N)}) = d\overset{p}{\mathbf{a}}_{(N)}\overset{q}{\mathbf{b}}_{(N)} + (-1)^{p}\overset{p}{\mathbf{a}}_{(N)}d\overset{q}{\mathbf{b}}_{(N)}, \tag{6}$$

and $(\Lambda^{\cdot}_{(N)}(M), d)$ is a differential graded algebra, for each choice of ϵ.

In the second representation, which can be calculationally convenient, type N generalized forms are expanded in terms of bases. A basis for type N generalized forms consists of any basis for ordinary forms on M augmented by N linearly independent minus one-forms, $\{\mathbf{m}^{i}\}$ $(i = 1...N)$. These latter objects have the algebraic properties of ordinary exterior forms but are assigned a degree of minus one. Hence they satisfy the ordinary distributive and associative laws of exterior algebra and the exterior product rule

$$\mathbf{m}^{i}\mathbf{m}^{j} = -\mathbf{m}^{j}\mathbf{m}^{i};\ \overset{p}{\alpha}\mathbf{m}^{i} = (-1)^{p}\mathbf{m}^{i}\overset{p}{\alpha}, \tag{7}$$

together with a condition of linear independence, $\mathbf{m}^{1}\mathbf{m}^{2}....\mathbf{m}^{N} \neq 0$. Thus, a generalized p-form, $\overset{p}{\mathbf{a}}_{(N)} \in \Lambda^{p}_{(N)}$, can be written as

$$\overset{p}{\mathbf{a}}_{(N)} = \overset{p}{\alpha} + \sum_{j=1}^{j=N} \frac{1}{j!}\overset{p+j}{\alpha}_{i_1....i_j}\mathbf{m}^{i_1}.....\mathbf{m}^{i_j}, \tag{8}$$

where $\overset{p}{\alpha}$, and $\overset{p+j}{\alpha}_{i_1....i_j}$ are, respectively, ordinary $p-$ and $(p+j)-$forms; $i_1,...i_j,...,i_N$ range and sum over 1 to N; and $\overset{p+j}{\alpha}_{i_1....i_j} = \overset{p+j}{\alpha}_{[i_1....i_j]}$. From the basis expansion it can be seen that at a point x in M the generalized $p-$forms of type N, $\Lambda^{p}_{(N)}(x)$, form a real vector space of dimension $\frac{(N+n)!}{(N+p)!(n-p)!}$. The dimension of $\Lambda^{\cdot}_{(N)}(x) = \oplus_{p=-N}^{p=n}\Lambda^{p}_{(N)}(x)$ is 2^{N+n}.

A generalized p-form of type $N \geq 1$ may also be written in terms of pair generalized forms of type $N-1$,

$$\overset{p}{\mathbf{a}}_{(N)} = \overset{p}{\mathbf{a}}_{(N-1)} + \overset{p+1}{\mathbf{a}}_{(N-1)}\mathbf{m}^{N}, \tag{9}$$

the basis expansion above can then be re-obtained iteratively. When $N \geq 1$, the relationship between this basis expansion and the first representation is then given by

$$(\overset{p}{\mathbf{a}}_{(N-1)}, \overset{p+1}{\mathbf{a}}_{(N-1)}) = \overset{p}{\mathbf{a}}_{(N-1)} + \overset{p+1}{\mathbf{a}}_{(N-1)}\mathbf{m}^{N}. \tag{10}$$

In the second representation the two exterior derivatives given above can be expressed in the following way. The minus one-forms $\{\mathbf{m}^i\}, i = 1...N$ have exterior derivatives

$$d\mathbf{m}^1 = \epsilon, \qquad (11)$$
$$d\mathbf{m}^i = 0; i \geq 2.$$

Again in the first case $\epsilon = 0$, and in the second $\epsilon = 1$. They satisfy the standard rules of exterior differentiation apart from the fact that they have negative degree. For example

$$\begin{aligned} d(\mathbf{m}^j\mathbf{m}^k..\mathbf{m}^l) &= \epsilon\delta^j_1\mathbf{m}^k...\mathbf{m}^l - \mathbf{m}^j d(\mathbf{m}^k...\mathbf{m}^l), \\ d(\overset{p}{\alpha}\mathbf{m}^i) &= (d\overset{p}{\alpha})\mathbf{m}^i + (-1)^p\epsilon\overset{p}{\alpha}\delta^i_1, \\ d(\mathbf{m}^i\overset{p}{\alpha}) &= \epsilon\delta^i_1\overset{p}{\alpha} - \mathbf{m}^i d\overset{p}{\alpha}, \end{aligned} \qquad (12)$$

where i, j, k, l range from 1 to N. Altogether, in the second representation, the expressions for the exterior derivatives of a generalized $p-$form $\overset{p}{\mathbf{a}}_{(N)}$ are

$$\begin{aligned} d\overset{p}{\mathbf{a}}_{(N)} &= d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}{}_1 + [d\overset{p+1}{\alpha}{}_{i_1} + (-1)^p\epsilon\overset{p+2}{\alpha}{}_{1i_1}]\mathbf{m}^{i_1} + ... \\ &+ \frac{1}{(N-1)!}[d\overset{p+N-1}{\alpha}{}_{i_1...i_{N-1}} + (-1)^{p+N}\epsilon\overset{p+N}{\alpha}{}_{1i_1...i_{N-1}}]\mathbf{m}^{i_1}..\mathbf{m}^{i_{N-1}} \\ &+ \frac{1}{N!}d\overset{p+N}{\alpha}{}_{i_1....i_N}\mathbf{m}^{i_1}..\mathbf{m}^{i_N}. \end{aligned} \qquad (13)$$

For each $N > 0$ and each choice of ϵ, the co-chain complex, over $\mathbb{R}$,

$$0 \xrightarrow{d} \Lambda^{-N}_{(N)}(M) \xrightarrow{d} .. \xrightarrow{d} \Lambda^p_{(N)}(M) \xrightarrow{d} \Lambda^{p+1}_{(N)}(M) \xrightarrow{d} .. \xrightarrow{d} \Lambda^n_{(N)}(M) \xrightarrow{d} 0 \qquad (14)$$

is a generalization of the usual, $N = 0$, de Rham complex. The standard definitions for ordinary forms, as outlined in reference [20] for example, can be extended to generalized forms. Here vector spaces over $\mathbb{R}$ are considered. The set of type N closed generalized $p-$forms on M is the $(N, p)-$th co-cycle group and is denoted $Z^p_{(N)}(M)$, the set of type N exact generalized $p-$forms on M is the $(N, p)-$th co-boundary group and is denoted $B^p_{(N)}(M)$ and the $(N, p)-$th de Rham cohomology group is $H^p_{(N)}(M) = Z^p_{(N)}(M)/B^p_{(N)}(M)$. When $N = 0$ these are the ordinary de Rham cohomology definitions. When $N \geq 1$, in the first case where $\epsilon = 0$, $H^p_{(N)}(M) = H^p_{(N-1)}(M) \oplus H^{p+1}_{(N-1)}(M)$ but in the second case, where $\epsilon = 1$, since every closed generalized form is exact, $H^{p}_{(N)}(M)$ is trivial. The following examples provides further details which will be useful in later sections.

Example 1: Let $N = 2$ and let $\overset{p}{\mathbf{a}}_{(2)} = (\overset{p}{\mathbf{a}}_{(1)}, \overset{p+1}{\mathbf{a}}_{(1)})$, where $\overset{p}{\mathbf{a}}_{(1)} = (\overset{p}{\alpha}, \overset{p+1}{\alpha}_{1})$ and $\overset{p+1}{\mathbf{a}}_{(1)} = (\overset{p+1}{\alpha}_{2}, \overset{p+2}{\alpha})$, so that $\overset{p}{\mathbf{a}}_{(2)} = (\overset{p}{\alpha}, \overset{p+1}{\alpha}_{1}, \overset{p+1}{\alpha}_{2}, \overset{p+2}{\alpha})$. Then if in a similar way, $\overset{q}{\mathbf{b}}_{(2)} \equiv (\overset{q}{\beta}, \overset{q+1}{\beta_1}, \overset{q+1}{\beta_2}, \overset{q+2}{\beta})$, it follows that

$$\begin{aligned}
\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}} &= \overset{p+q}{\mathbf{c}} \equiv (\overset{p+q}{\gamma}, \overset{p+q+1}{\gamma_1}, \overset{p+q+1}{\gamma_2}, \overset{p+q+2}{\gamma}), \text{ where} \\
\overset{p+q}{\gamma} &= \overset{p}{\alpha}\overset{q}{\beta}, \\
\overset{p+q+1}{\gamma_1} &= \overset{p}{\alpha}\overset{q+1}{\beta}_{1} + (-1)^{q}\overset{p+1}{\alpha}_{1}\overset{q}{\beta}, \quad \overset{p+q+1}{\gamma_2} = \overset{p}{\alpha}\overset{q+1}{\beta_2} + (-1)^{q}\overset{p+1}{\alpha_2}\overset{q}{\beta}, \\
\overset{p+q+2}{\gamma} &= \overset{p}{\alpha}\overset{q+2}{\beta} + (-1)^{q+1}\overset{p+1}{\alpha}_{1}\overset{q+1}{\beta_2} + (-1)^{q}\overset{p+1}{\alpha}_{2}\overset{q+1}{\beta}_{1} + \overset{p+2}{\alpha}\overset{q}{\beta},
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
d\overset{p}{\mathbf{a}} &= (\overset{p+1}{\sigma}, \overset{p+2}{\sigma}_{1}, \overset{p+2}{\sigma}_{2}, \overset{p+3}{\sigma}), \text{ where} \\
\overset{p+1}{\sigma} &= d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}_{1}, \\
\overset{p+2}{\sigma}_{1} &= d\overset{p+1}{\alpha}_{1}, \quad \overset{p+2}{\sigma}_{2} = d\overset{p+1}{\alpha}_{2} + (-1)^{p}\epsilon\overset{p+2}{\alpha}, \\
\overset{p+3}{\sigma} &= d\overset{p+2}{\alpha}.
\end{aligned} \tag{16}$$

Example 2: Let $N = 1$ and let $\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m}$ so that $d\overset{p}{\mathbf{a}} = [d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}] + d\overset{p+1}{\alpha}\mathbf{m}$. Hence when $p = n$ the dimension of M, $\overset{n}{\mathbf{a}}$ is always closed.

In case (i), where $\epsilon = 0$, $\overset{p}{\mathbf{a}}$ is closed if and only if both $\overset{p}{\alpha}$ and $\overset{p+1}{\alpha}$ are closed. In particular when $p = -1$, so that $\overset{-1}{\mathbf{a}} = \overset{0}{\alpha}\mathbf{m}$, $\overset{-1}{\mathbf{a}}$ is closed if and only if $d\overset{0}{\alpha} = 0$. Furthermore $\overset{p}{\mathbf{a}}$ is exact if and only if both $\overset{p}{\alpha}$ and $\overset{p+1}{\alpha}$ are exact. There are, of course, no non-zero exact minus one-forms in this case.

In case (ii), where $\epsilon = 1$, $\overset{p}{\mathbf{a}}$ is closed if and only if $\overset{p+1}{\alpha} = (-1)^{p}d\overset{p}{\alpha}$ so that $\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + (-1)^{p}d\overset{p}{\alpha}\mathbf{m} = d[(-1)^{p}\overset{p}{\alpha}\mathbf{m}]$. Hence $\overset{p}{\mathbf{a}}$ is closed if and only if it is exact. In particular when $p = -1$, $\overset{-1}{\mathbf{a}}$ is closed if and only if it is zero.

In future, when the type or degree of a form is obvious from the context its explicit labeling will be omitted.

3 Poly-chains, integration and Stokes' theorem for oriented manifolds

In order to define integrals of generalized forms and a Stokes' theorem for generalized forms, on an orientable and oriented manifold M, the concepts of real chains and chain complexes need to be extended appropriately.

Recall that for ordinary forms on M, [20], Stokes theorem states that

$$\int_{c_{p+1}} d\overset{p}{\alpha} = \int_{\partial c_{p+1}} \overset{p}{\alpha} \tag{17}$$

where c_{p+1} is a real $(p+1)-$chain in M and ∂c_{p+1} is its oriented boundary. The aim in this section is to introduce the notion of a $p-$poly-chain of type N, $\mathbf{c}_p^{(N)}$, together with that of a boundary operator $\partial : \mathbf{c}_p^{(N)} \to \partial \mathbf{c}_p^{(N)}$, where $\partial \mathbf{c}_p^{(N)}$ is the boundary of $\mathbf{c}_p^{(N)}$. Then the integral of type N generalized $p-$forms, $\overset{p}{\mathbf{a}}_{(N)}$, over a poly-chain, $\mathbf{c}_p^{(N)}$, will be defined in such a way that a Stokes' theorem

$$\int_{\mathbf{c}_{p+1}^{(N)}} d\overset{p}{\mathbf{a}}_{(N)} = \int_{\partial \mathbf{c}_{p+1}^{(N)}} \overset{p}{\mathbf{a}}_{(N)}, \tag{18}$$

holds. The boundary operator ∂ will then be formally dual to the exterior derivative d in the duality product $\langle \overset{p}{\mathbf{a}}_{(N)}, \mathbf{c}_p^{(N)} \rangle$ where

$$\langle \overset{p}{\mathbf{a}}_{(N)}, \mathbf{c}_p^{(N)} \rangle = \int_{\mathbf{c}_p^{(N)}} \overset{p}{\mathbf{a}}_{(N)}. \tag{19}$$

This duality

$$\langle d\overset{p}{\mathbf{a}}_{(N)}, \mathbf{c}_{p+1}^{(N)} \rangle = \langle \overset{p}{\mathbf{a}}_{(N)}, \partial \mathbf{c}_{p+1}^{(N)} \rangle, \tag{20}$$

will enable generalized complex(es) of chains in M, dual to the generalized de Rham complex(es) of the previous section, to be constructed. The general discussion of chain complexes that follows can be treated in greater generality, but in this paper it suffices to think of ordinary real $p-$chains in M as formal sums, $c_p = \sum_A r_A M_A$, where $r_A \in \mathbb{R}$ and the M_A are smooth $p-$dimensional sub-manifolds of M, [21].

First, when $N = 0$, choose $\mathbf{c}_p^{(0)} = c_p$, where c_p is an ordinary $p-$chain in M, and let $\partial \mathbf{c}_p^{(0)}$ be the oriented boundary of the $p-$chain c_p. Here it is assumed, as usual, that the ordinary chain complex is non-negative, so that c_p is the trivial chain 0 unless p is non-negative and the boundary of c_0 is

0. Such chains will be called ordinary chains. Type $N = 1$ poly-chains are defined to be ordered pairs of ordinary $p-$ and $(p+1)-$chains, in M

$$\mathbf{c}_p^{(1)} = (c_p, c_{p+1}), \tag{21}$$

and the boundary of the poly-chain $\mathbf{c}_p^{(1)}$ is defined to be

$$\partial \mathbf{c}_p^{(1)} = (\partial c_p, \partial c_{p+1} + (-1)^p \epsilon c_p). \tag{22}$$

Here ∂c_p and ∂c_{p+1} are the ordinary boundaries of the ordinary chains c_p and c_{p+1}. Two cases, corresponding to the two classes of exterior derivative discussed in the previous section, are considered, the first where $\epsilon = 0$, and the second where $\epsilon = 1$. It is straightforward to see that in each case the boundary of a boundary, $\partial(\partial \mathbf{c}_p^{(1)})$, is zero.

Next, let $\overset{p}{\mathbf{a}}_{(1)} = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$ be a type $N = 1$ generalized $p-$form on M. Define the integral of $\overset{p}{\mathbf{a}}_{(1)}$over $\mathbf{c}_p^{(1)}$ to be

$$\int_{\mathbf{c}_p^{(1)}} \overset{p}{\mathbf{a}}_{(1)} = \int_{c_p} \overset{p}{\alpha} + \int_{c_{p+1}} \overset{p+1}{\alpha}. \tag{23}$$

From these definitions and by applying the ordinary Stokes' theorem to the ordinary forms it is a straightforward matter to see that

$$\int_{\mathbf{c}_{p+1}^{(1)}} d\overset{p}{\mathbf{a}}_{(1)} = \int_{\partial \mathbf{c}_{p+1}^{(1)}} \overset{p}{\mathbf{a}}_{(1)}. \tag{24}$$

This is Stokes' theorem for type $N = 1$ forms.

It follows from this that on an n dimensional manifold , if $\overset{n-1}{\mathbf{a}}_{(1)} = (\overset{n-1}{\alpha}, \overset{n}{\alpha})$, then since $\mathbf{c}_n^{(1)} = (c_n, 0)$ and $\partial \mathbf{c}_n^{(1)} = (\partial c_n, (-1)^n \epsilon c_n)$,

$$\int_{\mathbf{c}_n^{(1)}} d\overset{n-1}{\mathbf{a}}_{(1)} = \int_{\partial c_n} \overset{n-1}{\alpha} + (-1)^n \epsilon \int_{c_n} \overset{n}{\alpha}. \tag{25}$$

The above definitions can be straightforwardly extended to any type N in an iterative manner. Let $N > 1$ and assume that type $(N-1)-$poly-chains $\mathbf{c}_p^{(N-1)}$ have been defined, and can be non-trivial, for p ranging from $-(N-1)$ to n. A type N poly-chain $\mathbf{c}_p^{(N)}$, $p \geq -N$, is then determined by an ordered pair of type $(N-1)-$poly-chains

$$\mathbf{c}_p^{(N)} = (\mathbf{c}_p^{(N-1)}, \mathbf{c}_{p+1}^{(N-1)}). \tag{26}$$

The boundary a type $N > 1$ poly-chain $\partial \mathbf{c}_p^{(N)}$ is defined to be

$$\partial \mathbf{c}_p^{(N)} = (\partial \mathbf{c}_p^{(N-1)}, \partial \mathbf{c}_{p+1}^{(N-1)}). \tag{27}$$

Again two cases are considered, case (i) where $\epsilon = 0$ and case (ii) where $\epsilon = 1$. The boundary of a boundary is again zero, $\partial^2 \mathbf{c}_p^{(N)} = 0$. In contrast to the ordinary or type $N = 0$ chain complexes the definitions permit poly-chains with p negative and $p = 0$ poly-chains with non-empty boundaries.

Let $\overset{p}{\mathbf{a}}_{(N)} = (\overset{p}{\mathbf{a}}_{(N-1)}, \overset{p+1}{\mathbf{a}}_{(N-1)})$ be a type $N > 1$ generalized $p-$form on M. Continuing in the same iterative vein as above, define the integral of $\overset{p}{\mathbf{a}}_{(N)}$ over $\mathbf{c}_p^{(N)}$ to be

$$\int_{\mathbf{c}_p^{(N)}} \overset{p}{\mathbf{a}}_{(N)} = \int_{\mathbf{c}_p^{(N-1)}} \overset{p}{\mathbf{a}}_{(N-1)} + \int_{\mathbf{c}_{p+1}^{(N-1)}} \overset{p+1}{\mathbf{a}}_{(N-1)}. \tag{28}$$

These definitions and results, and the ordinary Stokes' theorem, lead to Stokes' theorem for generalized type N forms

$$\int_{\mathbf{c}_{p+1}^{(N)}} d\overset{p}{\mathbf{a}}_{(N)} = \int_{\partial \mathbf{c}_{p+1}^{(N)}} \overset{p}{\mathbf{a}}_{(N)}. \tag{29}$$

If the abelian group of real poly-chains $\mathbf{c}_p^{(N)}$ is denoted $\mathbf{C}_p^{(N)}$, then for each choice of ϵ, the complex

$$0 \underset{\leftarrow}{\partial} \mathbf{C}_{-N}^{(N)} \cdots \underset{\leftarrow}{\partial} \mathbf{C}_{p-1}^{(N)} \underset{\leftarrow}{\partial} \mathbf{C}_p^{(N)} \cdots \underset{\leftarrow}{\partial} \mathbf{C}_n^{(N)} \underset{\leftarrow}{\partial} 0 \tag{30}$$

is the generalized chain complex dual to the generalized de Rham complex introduced in the previous section.

Following the standard terminology for ordinary, or type $N = 0$, chains, call $\mathbf{c}_p^{(N)}$ a cycle when $\partial \mathbf{c}_p^{(1)} = 0$ and a boundary when $\mathbf{c}_p^{(1)} = \partial \mathbf{c}_{p-1}^{(1)}$. The set of type N cycles in M is the $(N, p)-$th cycle group and is denoted $Z_p^{(N)}(M)$, the set of type N boundaries on M is the $(N, p)-$th boundary group and is denoted $B_p^{(N)}(M)$ and the $(N, p)-$th homology group is $H_p^{(N)}(M) = Z_p^{(N)}(M)/B_p^{(N)}(M)$. When $N = 0$ these are the ordinary real homology definitions for groups of chains in M. The homology groups are isomorphic to the cohomological groups above for all $N \geq 0$.

In case (i), where $\epsilon = 0$, $\mathbf{c}_p^{(N)}$ is a cycle if and only if the $N - 1$ chains $\mathbf{c}_p^{(N-1)}$ and $\mathbf{c}_{p+1}^{(N-1)}$ are both cycles, and it is a boundary if and only if $\mathbf{c}_p^{(N-1)}$ and $\mathbf{c}_{p+1}^{(N-1)}$ are both boundary chains.

In case (ii), where $\epsilon = 1$, $\mathbf{c}_p^{(1)}$ is a cycle if and only if $c_p = (-1)^{p+1}\partial c_{p+1}$, and then it is also a boundary, since $\mathbf{c}_p^{(1)} = \partial \widetilde{\mathbf{c}}_{p+1}^{(1)}$, where $\widetilde{\mathbf{c}}_{p+1}^{(1)} = ((-1)^{p+1}c_{p+1}, 0)$. When $N > 1$, $\mathbf{c}_p^{(N)} = (\mathbf{c}_p^{(N-1)}, \mathbf{c}_{p+1}^{(N-1)})$ is a cycle if and only if $\mathbf{c}_p^{(N-1)}$, $\mathbf{c}_{p+1}^{(N-1)}$ are both cycles, and $\mathbf{c}_p^{(N)}$ is a boundary if and only if $\mathbf{c}_p^{(N-1)}$ and $\mathbf{c}_{p+1}^{(N-1)}$ are both boundaries. Furthermore $\mathbf{c}_p^{(N)}$ is a cycle if and only if it is a boundary.

In both cases $\partial \mathbf{c}_{-N}^{(N)} = 0$. The following examples exhibit some properties of certain type $N = 1$ and 2 poly-chains.

Example 3: Let $N = 1$ and consider a $p = -1$ poly-chain $\mathbf{c}_{-1}^{(1)} = (0, c_0)$. Then, for both the first and second cases, $\partial \mathbf{c}_{-1}^{(1)} = (0, 0)$.

On the other hand when $\epsilon = 0$, the poly-chain $\mathbf{c}_{-1}^{(1)}$ is a boundary if and only if the ordinary chain c_0 is a boundary, but if $\epsilon = 1$, $\mathbf{c}_{-1}^{(1)}$ is always the boundary of $\mathbf{c}_0^{(1)} = (c_0, 0)$.

Next consider a type $N = 1$, $p = 0$ poly-chain $\mathbf{c}_0^{(1)} = (c_0, c_1)$. When $\epsilon = 0$, $\mathbf{c}_0^{(1)}$ is a cycle if and only if c_1 is a cycle but $\mathbf{c}_0^{(1)}$ is a boundary if and only if c_0 and c_1 are both boundaries. On the other hand in case (ii), where $\epsilon = 1$, $\mathbf{c}_0^{(1)}$ is a cycle if and only if c_0 is a boundary and then $\mathbf{c}_0^{(1)} = (-\partial c_1, c_1)$. A $p = 0$ poly-chain $\mathbf{c}_0^{(1)}$ is a boundary if and only if it is a cycle.

Finally, if $\overset{-1}{\mathbf{a}}_{(1)} = (0, \overset{0}{\alpha})$ is a degree minus one-form, then $\int_{\mathbf{c}_{-1}^{(1)}} \overset{-1}{\mathbf{a}}_{(1)} = \int_{c_0} \overset{0}{\alpha}$, and by Stokes' theorem, $\int_{\mathbf{c}_0^{(1)}} d\overset{-1}{\mathbf{a}}_{(1)} = \epsilon \int_{c_0} \overset{0}{\alpha} + \int_{\partial c_1} \overset{0}{\alpha}$.

Example 4: Let $N = 2$, and consider the poly-chain $\mathbf{c}_p^{(2)} = (\mathbf{c}_p^{(1)}, \mathbf{c}_{p+1}^{(1)})$. Let $\mathbf{c}_p^{(1)} = (c_p, c_{p+1}^1)$ and $\mathbf{c}_{p+1}^{(1)} = (c_{p+1}^2, c_{p+2})$, then $\mathbf{c}_p^{(2)}$ is the ordered quadruple of ordinary chains given by one ordinary $p-$chain, two ordinary $(p+1)-$chains and an ordinary $(p+2)-$chain

$$\mathbf{c}_p^{(2)} = (c_p, c_{p+1}^1, c_{p+1}^2, c_{p+2}). \tag{31}$$

In particular

$$\begin{aligned}
\mathbf{c}_{-2}^{(2)} &= (0, 0, 0, c_0), \\
\mathbf{c}_{-1}^{(2)} &= (0, c_0^1, c_0^2, c_1), \\
\mathbf{c}_0^{(2)} &= (c_0, c_1^1, c_1^2, c_2), \\
& - - - - \\
\mathbf{c}_n^{(2)} &= (c_n, 0, 0, 0).
\end{aligned} \tag{32}$$

The boundary of $\mathbf{c}_p^{(2)}$ is given by

$$\partial \mathbf{c}_p^{(2)} = (\partial c_p, \partial c_{p+1}^1 + (-1)^p \epsilon c_p, \partial c_{p+1}^2, \partial c_{p+2} + (-1)^{p+1} \epsilon c_{p+1}^2). \qquad (33)$$

In particular

$$\begin{aligned} \partial \mathbf{c}_{-2}^{(2)} &= (0,0,0,0), \qquad (34) \\ \partial \mathbf{c}_{-1}^{(2)} &= (0,0,0,\partial c_1 + \epsilon c_0^2), \\ \partial \mathbf{c}_{0}^{(2)} &= (0,\partial c_1^1 + \epsilon c_0, \partial c_1^2, \partial c_2 - \epsilon c_1^2), \\ &---- \\ \partial \mathbf{c}_{n}^{(2)} &= (\partial c_n, (-1)^n \epsilon c_n, 0, 0). \end{aligned}$$

The poly-chain $\mathbf{c}_p^{(2)}$ is a cycle if and only if $\partial c_p = 0$, $\partial c_{p+1}^1 = (-1)^{p+1}\epsilon c_p$, $\partial c_{p+1}^2 = 0$ and $\partial c_{p+2} = (-1)^p \epsilon c_{p+1}^2$. In case (ii), where $\epsilon = 1$, if $\mathbf{c}_p^{(2)}$ is a cycle then it is also the boundary of $\widetilde{\mathbf{c}}_{p+1}^{(2)} = ((-1)^{p+1} c_{p+1}^1, 0, (-1)^p c_{p+2}, 0)$.

Finally if as in Example 1, $\overset{p}{\mathbf{a}}_{(2)} = (\overset{p}{\alpha}, \overset{p+1}{\alpha}_1, \overset{p+1}{\alpha}_2, \overset{p+2}{\alpha})$, then

$$\int_{\mathbf{c}_p^{(2)}} \overset{p}{\mathbf{a}}_{(2)} = \int_{c_p} \overset{p}{\alpha} + \int_{c_{p+1}^1} \overset{p+1}{\alpha}_1 + \int_{c_{p+1}^2} \overset{p+1}{\alpha}_2 + \int_{c_{p+2}} \overset{p+2}{\alpha}. \qquad (35)$$

4 The generalized second Chern class and the Chern-Simons form

Generalized characteristic classes such as the Chern and Pontrjagin classes have been previously introduced, by replacing ordinary forms by generalized forms in the usual defining expressions, [4], [5], [11]. Here the generalized second Chern class and the corresponding Chern-Simons form will be reconsidered using the Stokes' theorem constructed in the previous section. The results are easily extendable to other characteristic classes, such as those considered in [11] for example. In the next section, their use in the construction of Lagrangian field theories, also discussed in the latter papers, will be demonstrated in the case of actions for the Einstein-Yang-Mills equations.

Consider a type N generalized connection, with values in the Lie algebra $\mathfrak{g}$ of a matrix Lie group G. The use of matrix representations is always assumed. A generalized connection one-form, $\mathbf{A}_{(N)}$, is a $\mathfrak{g}$-valued type N generalized one-form. Its curvature two-form is

$$\mathbf{F}_{(N)} = \mathbf{d}\mathbf{A}_{(N)} + \mathbf{A}_{(N)}\mathbf{A}_{(N)}, \qquad (36)$$

The covariant exterior derivative of a type N Lie algebra valued generalized $p-$form $\mathbf{Q}_{(N)}$ is defined to be

$$D_{(N)}\mathbf{Q}_{(N)} = d\mathbf{Q}_{(N)} + \mathbf{A}_{(N)}\mathbf{Q}_{(N)} + (-1)^{p+1}\mathbf{Q}_{(N)}\mathbf{A}_{(N)}, \tag{37}$$

and the curvature satisfies the Bianchi identities

$$D_{(N)}\mathbf{F}_{(N)} = 0. \tag{38}$$

The generalized second Chern class is determined by the generalized four-form

$$\mathbf{CH}_{(N)} = \frac{1}{8\pi^2}[Tr(\mathbf{F}_{(N)}\mathbf{F}_{(N)}) - Tr(\mathbf{F}_{(N)})Tr(\mathbf{F}_{(N)})], \tag{39}$$

and is equal to the exterior derivative of the generalized Chern-Simons three-form $\mathbf{CS}_{(N)}$ where

$$\mathbf{CS}_{(N)} = \frac{1}{8\pi^2}[Tr(\mathbf{A}_{(N)}\mathbf{F}_{(N)} - \frac{1}{3}\mathbf{A}_{(N)}\mathbf{A}_{(N)}\mathbf{A}_{(N)}) - Tr(\mathbf{A}_{(N)})d(Tr\mathbf{A}_{(N)})]. \tag{40}$$

More generally, if $\mathbf{k}_{(N)}$ is any closed generalized zero-form, then

$$d(\mathbf{k}_{(N)}\mathbf{CS}_{(N)}) = \mathbf{k}_{(N)}\mathbf{CH}_{(N)}. \tag{41}$$

From the previous section it follows from Stokes' theorem that

$$\int_{\mathbf{c}_4^{(N)}} \mathbf{CH}_{(N)} = \int_{\partial\mathbf{c}_4^{(N)}} \mathbf{CS}_{(N)}. \tag{42}$$

Next these geometrical objects are explored in greater detail when $N = 1$. Let M be a manifold of dimension greater than or equal to five. Let $\mathbf{A}$ be a type $N = 1$ connection one-form on M, with the trace of $\mathbf{A}$, $Tr\mathbf{A}$, zero, and let $\mathbf{F}$ be the curvature of $\mathbf{A}$ so that

$$\begin{aligned} \mathbf{A} &= \alpha + \beta\mathbf{m}, \\ \mathbf{F} &= \mathcal{F} + \epsilon\beta + D\beta\mathbf{m}, \end{aligned} \tag{43}$$

α and β are respectively one-forms and two-forms on M with values in a matrix Lie algebra $\mathfrak{g}$, $\mathcal{F} = d\alpha + \alpha\alpha$; and $D\beta = d\beta + \alpha\beta - \beta\alpha$. The type $N = 1$ generalized second Chern class is

$$\mathbf{CH} = \frac{1}{8\pi^2}[Tr(\mathcal{F}\mathcal{F} + 2\epsilon\mathcal{F}\beta + \epsilon\beta\beta) + d\{Tr(\mathcal{F}\mathcal{F} + 2\mathcal{F}\beta + \epsilon\beta\beta)\}\mathbf{m}]. \tag{44}$$

If $\mathbf{c}_4^{(1)} = (c_4, c_5)$ then, by Stokes' theorem

$$\int_{\mathbf{c}_4^{(1)}} \mathbf{CH} = \frac{1}{8\pi^2}[\int_{c_4} Tr(\mathcal{F}\mathcal{F}) + \epsilon \int_{c_4} (2\mathcal{F}\beta + \beta\beta) + \int_{\partial c_5} Tr(\mathcal{F}\mathcal{F} + 2\mathcal{F}\beta + \epsilon\beta\beta)]. \tag{45}$$

The first term on the right hand side is just the ordinary Chern class expression written in terms of the curvature two-form $\mathcal{F}$.

When $\epsilon = 0$ this takes the form

$$\int_{\mathbf{c}_4^{(1)}} \mathbf{CH} = \frac{1}{8\pi^2}[\int_{c_4} Tr(\mathcal{F}\mathcal{F}) + 2 \int_{\partial c_5} Tr(\mathcal{F}\beta)], \tag{46}$$

but when $\epsilon = 1$ it is

$$\int_{\mathbf{c}_4^{(1)}} \mathbf{CH} = \frac{1}{8\pi^2} \int_{c_4 + \partial c_5} [Tr(\mathcal{F}\mathcal{F} + 2\mathcal{F}\beta + \beta\beta)]. \tag{47}$$

The type $N = 1$ Chern-Simons' form is the generalized three-form

$$\mathbf{CS} = \frac{1}{8\pi^2} Tr[(\alpha\mathcal{F} - \frac{1}{3}\alpha\alpha\alpha) + \epsilon\alpha\beta + (\alpha D\beta + \beta\mathcal{F} - \beta\alpha\alpha + \epsilon\beta\beta)\mathbf{m}], \tag{48}$$

where the first term on the right hand side is the ordinary Chern-Simons three-form. If $\mathbf{c}_3^{(1)} = (c'_3, c'_4)$ it can be shown, by using Stokes' theorem, that

$$\int_{\mathbf{c}_3^{(1)}} \mathbf{CS} = \frac{1}{8\pi^2}[\int_{c'_3} Tr(\alpha\mathcal{F} - \frac{1}{3}\alpha\alpha\alpha) + \epsilon \int_{c'_3} Tr(\alpha\beta) - \int_{\partial c'_4} Tr(\alpha\beta) + \int_{c'_4} Tr(2\beta\mathcal{F} + \epsilon\beta\beta)]. \tag{49}$$

The first term on the right hand side is just ordinary Chern-Simons three-form. When $\mathbf{c}_3^{(1)} = \partial\mathbf{c}_4^{(1)}$ so that $c'_3 = \partial c_4$ and $c'_4 = \partial c_5 + \epsilon c_4$, it follows from the equations above that

$$\begin{aligned} \int_{\mathbf{c}_4^{(1)}} \mathbf{CH} &= \int_{\partial\mathbf{c}_4^{(1)}} \mathbf{CS} \\ &= \frac{1}{8\pi^2}[\int_{\partial c_4} Tr(\alpha\mathcal{F} - \frac{1}{3}\alpha\alpha\alpha) + \int_{\partial c_5 + \epsilon c_4} Tr(2\beta\mathcal{F} + \epsilon\beta\beta)]. \end{aligned} \tag{50}$$

The differential forms determining the generalized Chern classes are invariant under large type $N = 1$ "generalized gauge groups", [11], but when they are used to construct Lagrangians by making particular choices of the Lie algebra valued two-forms β such symmetries are broken down to the smaller groups. An illustration of this is contained in the results of the next section.

5 Einstein-Yang-Mills as a generalized Chern-Simons theory

The specific aim here is to construct actions for the Einstein-Yang-Mills field equations by using the results of the previous sections. Rather than redoing calculations presented in [4], [5] and [11] - a straightforward matter - a generalization of ordinary Chern-Simons theory will be constructed for Einstein-Yang-Mills theory. It will be seen that these Chern-Simons actions are similar to certain first order gravitational actions which have been the subject of other investigations. Examples of such discussions, and many further references, can be found in [13], [22]-[23]. However the general approach taken here is novel and different from that taken in those papers.

Let the Yang-Mills Lie group G, assumed unimodular, have Lie algebra $\mathfrak{g}$ and let $\mathbf{A}$ be a $so(p,q) \oplus \mathfrak{g}$ valued type $N=1$ connection one-form on a manifold M. Since the immediate goal is to construct actions for four dimensional Einstein-Yang-Mills theory, $p+q=4$. The connection is represented by a $(r+4) \times (r+4)$ matrix-valued one-form

$$\mathbf{A}=\begin{pmatrix} \omega^a_b + \sigma^a_b \mathbf{m} & \mathbf{0} \\ 0 & \gamma^i_j + \rho^i_j \mathbf{m} \end{pmatrix}, \tag{51}$$

The curvatures of $\mathbf{A}$ are

$$\mathbf{F} = \begin{pmatrix} (\Omega^a_b + \epsilon\sigma^a_b) + D_\omega(\sigma^a_b)\mathbf{m} & \mathbf{0} \\ 0 & \digamma^i_j + \epsilon\rho^i_j + D_\gamma(\rho^i_j)\mathbf{m} \end{pmatrix} \tag{52}$$

where

$$\Omega^a_b = d\omega^a_b + \omega^a_c \omega^c_b,$$
$$\digamma^i_j = d\gamma^i_j + \gamma^i_k \gamma^k_j.$$

The $so(p,q)$ indices $a, b, c..$ sum and range from 1 to 4. A trace-free $r \times r$ matrix representation of $\mathfrak{g}$, the Lie algebra of the Yang-Mills group is used and the Yang-Mills indices $i, j, k..$ sum and range from 1 to r. Furthermore D_ω is the covariant exterior derivative with respect to the $so(p,q)-$valued connection one-form ω^a_b on M, D_γ is the covariant exterior derivative with respect to the $\mathfrak{g}-$valued connection one-form γ^i_j on M; σ^a_b and ρ^i_j are respectively $so(p,q)$ and $\mathfrak{g}$-valued $2-$forms on M.

Let $\mathbf{G}_{(1)}$ be the group of $(SO(p,q) \times G)-$valued type $N=1$ zero-forms, with elements $\{\mathbf{g}_{(1)} = (1 + \overset{1}{g}\mathbf{m}\}\overset{0}{g}\}$, where $\overset{0}{g}$ is an ordinary $(SO(p,q) \times G)$

valued zero-form and $\overset{1}{g}$ is an ordinary $so(p,q) \oplus \mathfrak{g}$ valued one-form. Then, as was noted in [11], the generalized Chern four-form, $\mathbf{CH} = \frac{1}{8\pi^2} Tr(\mathbf{FF})$, is invariant under the 'generalized gauge transformations'

$$\begin{aligned} &\mathbf{A} \to (\mathbf{g}_{(1)})^{-1} d\mathbf{g}_{(1)} + (\mathbf{g}_{(1)})^{-1} \mathbf{A} \mathbf{g}_{(1)}, \\ &\mathbf{F} \to (\mathbf{g}_{(1)})^{-1} \mathbf{F} \mathbf{g}_{(1)}. \end{aligned} \tag{53}$$

The generalized Chern-Simons three-forms, $\mathbf{CS} = \frac{1}{8\pi^2}[Tr(\mathbf{AF} - \frac{1}{3}\mathbf{AAA})]$, constructed from this $so(p,q) \oplus \mathfrak{g}$ valued generalized connection and its curvatures are

$$\begin{aligned} \mathbf{CS} = CS &+ \frac{1}{8\pi^2}\{\epsilon \omega^a_b \sigma^b_a + [2\sigma^a_b \Omega^b_a + \epsilon \sigma^a_b \sigma^b_a - d(\omega^a_b \sigma^b_a)]\mathbf{m}\} \\ &+ \frac{1}{8\pi^2}\{\epsilon \gamma^i_j \rho^j_i + [2\rho^i_j F^j_i + \epsilon \rho^i_j \rho^j_i - d(\gamma^i_j \rho^j_i)]\mathbf{m}\}, \end{aligned} \tag{54}$$

where

$$CS = \frac{1}{8\pi^2}(\omega^a_b \Omega^b_a - \frac{1}{3}\omega^a_b \omega^b_c \omega^c_a + \gamma^i_j F^j_i - \frac{1}{3}\gamma^i_j \gamma^j_k \gamma^k_i) \tag{55}$$

is the ordinary $so(p,q) \oplus \mathfrak{g}$ Chern-Simons three-form. If $\mathbf{c}^{(1)}_3 = (c_3, c_4)$ is a type $N = 1$ poly-chain in M then, by Stokes' theorem

$$\begin{aligned} \int_{\mathbf{c}^{(1)}_3} \mathbf{CS} = \int_{c_3} CS &+ \frac{1}{8\pi^2}\int_{\epsilon c_3 - \partial c_4} [\omega^a_b \sigma^b_a + \gamma^i_j \rho^j_i] \\ &+ \frac{1}{8\pi^2}\int_{c_4} [2(\Omega^a_b \sigma^b_a + F^i_j \rho^j_i) + \epsilon(\sigma^a_b \sigma^b_a + \rho^i_j \rho^j_i)]. \end{aligned} \tag{56}$$

In order to deal with metric geometries, four ordinary one-forms on M, $\{\theta^a\}$, are introduced and the choice

$$\sigma^a_b = \mu \theta^a \theta_b + \frac{\nu}{2}\varepsilon^a_{bcd}\theta^c \theta^d, \tag{57}$$

where μ and ν are constants with ν non-zero, is made. When the one forms $\{\theta^a\}$ are linearly independent on c_4, as will be assumed here, they form an orthonormal basis for a four-metric of signature (p,q),

$$\begin{aligned} &ds^2 = \eta_{ab}\theta^a \otimes \theta^b, \\ &(\eta_{ab}) = \begin{pmatrix} 1_{p\times p} & 0 \\ 0 & -1_{q\times q} \end{pmatrix}. \end{aligned} \tag{58}$$

In the purely gravitational case, where the Yang-Mills field $F^i_{\ j}$ and the two-forms ρ^i_j are zero, it is easy to show that now

$$\int_{\mathbf{c}_3^{(1)}} \mathbf{CS} = \int_{c_3} CS + \frac{1}{8\pi^2}\int_{\epsilon c_3 - \partial c_4} [\omega^a_b(\mu\theta^b\theta_a + \frac{\nu}{2}\varepsilon^{\ b}_{acd}\theta^c\theta^d)] \tag{59}$$
$$+ \frac{1}{8\pi^2}\int_{c_4}[\nu\varepsilon^{\ b}_{acd}\Omega^a_b\theta^c\theta^d + 2\mu\Omega^a_b\theta^b\theta_a] + \frac{\epsilon}{8\pi^2}\int_{c_4}[-4!\mu\nu V],$$

where $V = \theta^1\theta^2\theta^3\theta^4$ is the volume four-form on c_4. Here the relation, $\varepsilon_{abef}\varepsilon^{cdef} = 2s(\delta^c_a\delta^d_b - \delta^c_b\delta^d_a)$, where s is -1 when the metric signature is Lorentzian and $+1$ otherwise, has been used. With this choice of σ^a_b the gauge group is reduced to the ordinary gauge group $SO(p,q)$.

The generalized Chern-Simons integrals given by Eq.(59) can be interpreted as first order actions for the four-dimensional Einstein vacuum field equations, with and without a cosmological constant term. The different cases, and the individual terms in the integrands, are explored in the discussion of the Einstein-Yang-Mills system below.

The construction of similar results for the coupled Einstein-Yang-Mills system requires an appropriate choice of the two-forms ρ^j_i. When M is taken to be four dimensional, and the one-forms $\{\theta^a\}$ form a basis co-frame on M, the two-forms ρ^i_j can be taken to be

$$\rho^i_j = \tau F^i_{\ j} + \kappa {}^*F^i_{\ j}, \tag{60}$$

where τ and κ are constants and ${}^*F^i_{\ j}$ is the Hodge dual of the Yang-Mills field two-form $F^i_{\ j}$. With the intrinsically different choices of the two-forms σ^a_b and ρ^i_j given by Eqs.(57) and (60), the gauge group is reduced from $\mathbf{G}_{(1)}$ to $SO(p,q)\times G$. By using Eqs.(57), (58) and (60) in Eq.(54) the generalized Chern-Simons three-form, $\mathbf{CS_{EYM}}$ is obtained from $\mathbf{CS}$, and the Chern-Simons expression given by Eq.(56) becomes

$$\int_{\mathbf{c}_3^{(1)}} \mathbf{CS_{EYM}} = \int_{c_3} CS + \frac{1}{8\pi^2}\int_{\epsilon c_3 - \partial c_4} [\omega^a_b(\mu\theta^b\theta_a + \frac{\nu}{2}\varepsilon^{\ b}_{acd}\theta^c\theta^d) + \gamma^i_j(\tau F^j_{\ i} + \kappa {}^*F^j_{\ i})] \tag{61}$$
$$+ \frac{1}{8\pi^2}\int_{c_4}[\nu\varepsilon^{\ b}_{acd}\Omega^a_b\theta^c\theta^d + 2\kappa F^{i*}_{\ j}F^j_{\ i} + 2\tau F^i_{\ j}F^j_{\ i} + 2\mu\Omega^a_b\theta^b\theta_a]$$
$$+ \frac{\epsilon}{8\pi^2}\int_{c_4}[-4!\mu\nu V + 2\tau\kappa F^{i*}_{\ j}F^j_{\ i} + (\tau^2 + s\kappa^2)F^i_{\ j}F^j_{\ i}].$$

In the first case, where $\epsilon = 0$, it follows from Eq.(61) that the integral of $\mathbf{CS_{EYM}}$, over the poly-chain in M, $\mathbf{c}_3^{(1)} = (c_3, c_4)$, is

$$\int_{\mathbf{c}_3^{(1)}} \mathbf{CS_{EYM}} = \frac{1}{8\pi^2}\int_{c_4}[\nu\varepsilon^b_{acd}\Omega^a_b\theta^c\theta^d + 2\kappa F^{i*}_{\ j}F^{\ j}_{i} + 2\mu\Omega^a_b\theta^b\theta_a + 2\tau F^{\ i}_{j}F^{\ j}_{i}] \quad (62)$$
$$+\int_{c_3} CS - \frac{1}{8\pi^2}\int_{\partial c_4}[\frac{\nu}{2}\omega^a_b\varepsilon^b_{acd}\theta^c\theta^d + \mu\omega^a_b\theta^b\theta_a + \kappa\gamma^{i*}_j F^{\ j}_{i} + \tau\gamma^i_j F^{\ j}_{i}].$$

When $\partial c_4 = 0$ this has the form of a known action for the Einstein-Yang-Mills field equations, with zero cosmological constant, plus an ordinary Chern-Simons contribution given by the second integral. The first two terms in the integral over c_4, are the standard Ricci scalar and Yang-Mills term. The third term has been discussed by Holst, [22]. When the Euler-Lagrange equations are satisfied and the connection form ω^a_b is identified as the Levi-Civita spin connection, the contribution from this term vanishes by the first Bianchi identities. The fourth term, the Yang-Mills second Chern class term, makes no contribution when $\partial c_4 = 0$.

In the second case, where $\epsilon = 1$,

$$\int_{\mathbf{c}_3^{(1)}} \mathbf{CS_{EYM}} \quad (63)$$
$$= \frac{1}{8\pi^2}\int_{c_4}[\nu\varepsilon^b_{acd}\Omega^a_b\theta^c\theta^d + 2\mu\Omega^a_b\theta^b\theta_a - 4!\mu\nu V + (2\kappa + 2\tau\kappa)F^{i*}_{\ j}F^{\ j}_{i} + (2\tau + \tau^2 + s\kappa^2)F^{\ i}_{j}F^{\ j}_{i}]$$
$$+\int_{c_3} CS + \frac{1}{8\pi^2}\int_{c_3-\partial c_4}[\frac{\nu}{2}\omega^a_b\varepsilon^b_{acd}\theta^c\theta^d + \mu\omega^a_b\theta^b\theta_a + \kappa\gamma^{i*}_j F^{\ j}_{i} + \tau\gamma^i_j F^{\ j}_{i}].$$

This can also be identified as an action for the Einstein-Yang-Mills field equations. The integral over c_4 contains the usual Ricci scalar, the term included by Holst and, when $\mu\nu$ is non-zero, a non-zero cosmological constant term. It also contains the usual Yang-Mills term plus a second Chern class Yang-Mills term. The latter term could also be written as the ordinary Yang-Mills Chern-Simons three form integrated over the boundary of c_4. The integral over the three-chain c_3 is the ordinary Einstein-Yang-Mills Chern-Simons term. The integrand is, as before, given by Eq.(55). The terms in the integral over the three-chain $c_3 - \partial c_4$ all arise in formally similar ways in this approach. Of course they do not enter into consideration if either $c_3 - \partial c_4$ is trivial or they vanish because of boundary conditions. However when $c_3 = -\partial c_4$ the first term is required when first order actions and asymptotically flat systems are considered. This point is discussed in detail in reference [13].

In the approach taken in this section both the similarities and differences of the gravitational and gauge fields are highlighted. Here the four dimensional Einstein-Yang-Mills actions are intrinsically four dimensional in a way that the vacuum Einstein actions are not. In addition, in the vacuum Einstein case the bulk manifold M need not be metric.

6 Conclusion

It has been shown that by introducing the notion of poly-chains, integrals of generalized forms and a generalization of Stokes' theorem for ordinary forms can been constructed. This formalism can be exploited wherever integrals are considered. In particular, the usual approach via ordinary differential forms to characteristic classes can be extended within this new context. A generalization of the second Chern class and the Chern-Simons three-form have been considered here by using the simplest type of generalized forms. Extensions to other characteristic classes and other types of generalized forms can easily be made.

Generalized characteristic classes have been used previously to formulate actions for topological and other field theories. The relationship between such actions and generalized characteristic classes can now be investigated more completely and for different Lagrangian theories in different dimensions. In general actions will be determined by integrals on a number of chains, or manifolds, of differing dimensions, and they will include boundary integrals in a natural way.

In this paper actions for the Einstein-Yang-Mills field have been reformulated, in a novel geometrical context, by using the generalized Chern-Simons three-form. The resulting actions include integrals over both four and three dimensional manifolds. Certain well-known boundary terms appear more naturally than in the usual approaches which do not use generalized forms. Altogether, the framework provided by generalized forms and generalized characteristic classes gives a new and unified way of looking at, and hence investigating, a wide range of field theories.

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