Generalized forms and gravitation

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Abstract: The algebra and calculus of generalized differential forms are reviewed and employed to construct a broad class of generalized connections and to investigate their properties. Ths class includes connections which are flat when Einstein's vacuum field equations are satisfied. Generalized Chern-Simons action principles are formulated and it is shown that certain of these have Einstein's vacuum field equations as Euler-Lagrange equations.

1 Introduction

Generalized differential forms have been employed in a number of different geometrically and physically interesting contexts. These include twistor theory, the construction of action integrals and the study of vector fields and path space forms, [1] - [8]. The aim of this paper is to review aspects of the algebra and calculus of generalized differential forms, as developed in [9]-[16], and to study a class of generalized connections that can be employed in the study of gravitational systems and action integrals.

A review of the basic ideas of the algebra, differential and integral calculus of generalized forms that are needed here is given in the second section. The third section contains a discussion of generalized connections and generalized Chern-Simons integrals are constructed. Certain of the formulae in this section are similar to those which arise in the formalism of higher gauge theories reviewed in [17] although the approach taken here, based on [13] and [14], is different. The fourth section deals with connections for gravitational fields without matter. In particular generalized connections with values in the Lie algebras of special orthogonal groups are discussed and flat connections are related to gravitational field equations. The generalized equations of parallel transport for these flat connections constitute a system of linear equations with these gravitational equations as integrability conditions. Section five deals with the construction of generalized Chern-Simons three-forms using the connections introduced in the previous section and the corresponding generalized Chern-Simons action integrals are discussed. In the sixth section it is shown how a particular class of these connections can be used to construct action integrals for Einstein's vacuum field equations in four dimensions with or without a cosmological constant. There is a large body of research devoted to the study of Chern-Simons gravity and related topics with two papers dating from the 1980's, [18] and [19] being particularly influential. While many of these investigations deal with gravity in 2+1 dimensions, and are reviewed in [20], results related to gravity in higher dimensions have also been obtained; some are discussed in [21] and [22]. Many further references can be found in the last three papers. Sections five and six differ from that body of work in that here use is made of generalized characteristic classes to construct action integrals for gravitational and other physical systems, an approach initiated in [4] and [5] and developed in [13] and [14].

Only type $N = 1$ forms are considered here but extensions, where appropriate, to forms of type $N > 1$ is straightforward. Discussions of the algebra and calculus of $N > 1$ forms can be found in [11]- [15] with a type $N = 2$ generalized connection which is flat when Einstein's equations are satisfied being presented in [12].

The manifolds and geometrical objects considered may be real or complex but in this paper it will be assumed that the geometry is real, all geometrical objects are smooth and M is an $n-$dimensional real, smooth, orientable and oriented manifold. Bold-face Roman letters are used to denote generalized forms and generalized form-valued vector fields, ordinary forms on M are usually denoted by Greek letters and ordinary vector fields on M by lower case Roman letters. Occasionally the degree of a form is indicated above it. The exterior product of any two forms, for example α and β, is written $\alpha\beta$, and as usual, any ordinary $p-$form $\overset{p}{\alpha}$, with p either negative or greater than n, is zero. The Einstein summation convention is used.

2 Algebra and calculus of type N=1 forms and vector fields

The algebra and calculus of generalized forms used in this paper are reviewed in this section using the notation of [13] and [14] A basis for type $N = 1$ generalized forms consists of any basis for ordinary forms on M augmented by a linearly independent minus one-form $\mathbf{m}$. Minus one forms have the algebraic properties of ordinary exterior forms but are assigned a degree of minus one. They satisfy the ordinary distributive and associative laws of exterior algebra and the exterior product rule

$$\overset{p}{\rho}\mathbf{m} = (-1)^p \mathbf{m}\overset{p}{\rho};\ \mathbf{m}^2 = 0, \tag{1}$$

together with the condition of linear independence. Thus, for a given choice of $\mathbf{m}$, a type $N = 1$ generalized p-form, $\overset{p}{\mathbf{r}} \in \Lambda^p_{(1)}$, can be written as

$$\mathbf{r} = \rho + \lambda\mathbf{m}, \tag{2}$$

where ρ, and λ are, respectively, ordinary $p-$ and $(p+1)-$forms and p can take integer values from -1 to n.

If φ is a smooth map between manifolds P and M, $\varphi : P \to M$, then the induced map of type $N = 1$ generalized forms, $\varphi^*_{(1)} : \Lambda^p_{(1)}(M) \to \Lambda^p_{(1)}(P)$, is the linear map defined by using the standard pull-back map, φ^*, for ordinary forms

$$\varphi^*_{(1)}(\mathbf{r}) = \varphi^*(\rho) + \varphi^*(\lambda)\mathbf{m}, \tag{3}$$

and $\varphi^*_{(1)}(\overset{p}{\mathbf{r}}\overset{q}{\mathbf{s}}) = \varphi^*_{(1)}(\overset{p}{\mathbf{r}})\varphi^*_{(1)}(\overset{q}{\mathbf{s}})$. Hence $\varphi^*_{(1)}(\mathbf{m}) = \mathbf{m}$.

Henceforth in this paper, in addition to assuming that the exterior derivative of generalized forms satisfies the usual properties, it is assumed, without loss of generality, [16], that

$$d\mathbf{m} = \epsilon, \tag{4}$$

where ϵ denotes a real constant.

The exterior derivatives of a type $N = 1$ generalized form $\overset{p}{\mathbf{r}}$ is then

$$d\mathbf{r} = [d\rho + (-1)^{p+1}\epsilon\lambda] + d\lambda\mathbf{m}, \tag{5}$$

where d is the ordinary exterior derivative when acting on ordinary forms. The exterior derivative $d : \Lambda^p_{(1)}(M) \rightarrow \Lambda^{p+1}_{(1)}(M)$ is an anti-derivation of degree one,

$$\begin{aligned} d(\overset{p}{\mathbf{r}}\overset{q}{\mathbf{s}}) &= (d\overset{p}{\mathbf{r}})\overset{q}{\mathbf{s}} + (-1)^p\overset{p}{\mathbf{r}}d\overset{q}{\mathbf{s}}, \\ d^2 &= 0. \end{aligned} \tag{6}$$

and $(\Lambda^\bullet_{(N)}(M), d)$ is a differential graded algebra.

The dual of a generalized one-form is a generalized form-valued vector field, [16]. Such a type $N = 1$ vector field is defined on a coordinate patch $U \subseteq M$ by

$$\mathbf{V} = \mathbf{v}^\rho\frac{\partial}{\partial x^\rho} = (v^\rho + v^\rho_\sigma dx^\sigma\mathbf{m})\frac{\partial}{\partial x^\rho} = v + (v^\rho_\sigma dx^\sigma\mathbf{m})\frac{\partial}{\partial x^\rho}, \tag{7}$$

where $\{x^\alpha\}$ are local coordinates on U and $v = v^\rho\frac{\partial}{\partial x^\rho}$ is an ordinary vector field. Globally $\mathbf{V}$ is determined by an ordinary vector field v and a $(1,1)$ type tensor field given in local coordinates by $v^\rho_\sigma\frac{\partial}{\partial x^\rho} \otimes dx^\sigma$ on M. The set of all such vector fields in M, $\{\mathbf{V}\}$, is naturally a module, $\mathcal{V}_{(1)}(M)$, over the generalized zero forms on M, $\Lambda^0_{(1)}(M)$.

Example 1: In Euclidean three-space, with Euclidean coordiates and metric $ds^2 = \delta_{\mu\nu}dx^\mu dx^\nu$ the scalar or dot product of two generalized form-valued vector fields $\mathbf{V} = \mathbf{v}^\rho\frac{\partial}{\partial x^\rho}$ and $\mathbf{W} = \mathbf{w}^\rho\frac{\partial}{\partial x^\rho} = (w^\rho + w^\rho_\sigma dx^\sigma\mathbf{m})\frac{\partial}{\partial x^\rho} = w + (w^\rho_\sigma dx^\sigma\mathbf{m})\frac{\partial}{\partial x^\rho}$ is the generalized zero form

$$\mathbf{V}.\mathbf{W} = \delta_{\mu\nu}\mathbf{v}^\mu\mathbf{w}^\nu = v.w + (\delta_{\mu\nu}v^\mu w^\nu_\sigma + \delta_{\mu\nu}w^\mu v^\nu_\sigma)dx^\sigma\mathbf{m}$$

The vector or crossproduct is the generalized form-valued vector field

$$\mathbf{V}\times\mathbf{W} = \varepsilon^{\rho\mu\nu}\mathbf{v}^{\mu}\mathbf{w}^{\nu}\frac{\partial}{\partial x^{\rho}} = v\times w + [\varepsilon^{\rho\mu\nu}(v^{\mu}w^{\nu}_{\sigma} + w^{\nu}v^{\mu}_{\sigma})dx^{\sigma}\mathbf{m}]\frac{\partial}{\partial x^{\rho}}$$

where $\varepsilon^{\rho\mu\nu}$ is the totally skew-symmetric Levi-Civita symbol.

The interior product of a generalized $p-$form $\mathbf{r} = \rho + \lambda\mathbf{m}$ with respect to $\mathbf{V}$ is given by the formula

$$i_{\mathbf{V}}\mathbf{r} = \mathbf{v}^{\alpha} i_{\frac{\partial}{\partial x^{\alpha}}}\mathbf{r}. \tag{8}$$

where $i_{\frac{\partial}{\partial x^{\alpha}}}\mathbf{r}$ is the interior product of $\mathbf{r}$ by the ordinary vector field $\frac{\partial}{\partial x^{\alpha}}$ as defined in [10], that is $i_{\frac{\partial}{\partial x^{\alpha}}}\mathbf{r} = i_{\frac{\partial}{\partial x^{\alpha}}}\rho + i_{\frac{\partial}{\partial x^{\alpha}}}\lambda\mathbf{m}$. For p equal to minus one and zero

$$i_{\mathbf{V}}\overset{-1}{\mathbf{r}} = 0. \tag{9}$$

$$i_{\mathbf{V}}\overset{0}{\mathbf{r}} = \lambda_{\alpha}v^{\alpha}\mathbf{m},$$

and for $p \geqq 1$

$$\begin{aligned} i_{\mathbf{V}}\mathbf{r} &= i_{v}\mathbf{r} + \overset{p}{\gamma}\mathbf{m} = i_{v}\rho + i_{v}\lambda\mathbf{m} + \overset{p}{\gamma}\mathbf{m}, \\ \overset{p}{\gamma} &= (-1)^{p-1}v^{\alpha}_{\beta}dx^{\beta}(i_{\frac{\partial}{\partial x^{\alpha}}}\rho) \\ &= \frac{(-1)^{p-1}}{(p-1)!}v^{\alpha}_{\lambda_1}\rho_{\alpha\lambda_2\ldots.\lambda_p}dx^{\lambda_1\cdots}dx^{\lambda_p}. \end{aligned} \tag{10}$$

This interior product satisfies the graded Leibniz rule

$$i_{\mathbf{V}}(\overset{p}{\mathbf{r}}\overset{q}{\mathbf{s}}) = (i_{\mathbf{V}}\overset{p}{\mathbf{r}})\overset{q}{\mathbf{s}} + (-1)^{p}\overset{p}{\mathbf{r}}(i_{\mathbf{V}}\overset{q}{\mathbf{s}}), \tag{11}$$

but does not in general anti-commute because the interior product on generalized zero forms need not be zero,

$$(i_{\mathbf{W}}\circ i_{\mathbf{V}} + i_{\mathbf{V}}\circ i_{\mathbf{W}})\mathbf{r} = (-1)^{p-1}\{[v^{\alpha}_{\beta}w^{\beta} + w^{\alpha}_{\beta}v^{\beta}](i_{\frac{\partial}{\partial x^{\alpha}}}\rho\}\mathbf{m}, \tag{12}$$

where $\mathbf{W} = (w^{\rho} + w^{\rho}_{\sigma}dx^{\sigma}\mathbf{m})\frac{\partial}{\partial x^{\rho}}$. The Lie derivative of generalized forms by generalized form-valued vector fields is a derivation of degree zero defined by $\pounds_{\mathbf{V}} = d\circ i_{\mathbf{V}} + i_{\mathbf{V}}\circ d$ and the Lie bracket of two generalized form-valued vector fields $\mathbf{V}$ and $\mathbf{W}$ on M is the generalized form-valued vector field, $[\mathbf{V},\mathbf{W}]$, defined by the relation $(\pounds_{\mathbf{V}}\pounds_{\mathbf{W}} - \pounds_{\mathbf{W}}\pounds_{\mathbf{V}})\mathbf{r} = \pounds_{[\mathbf{V},\mathbf{W}]}\mathbf{r}$. Type $N = 1$ generalized form-valued vector fields include, as special cases, both ordinary vector fields, [10], and the previously introduced generalized vector fields, [6] and [7].

Just as the algebra and differential calculus of ordinary differential forms on M can be expressed in terms of functions and vector fields on the reverse parity tangent bundle, ΠTM, of M, [23] and [24] generalized forms can be represented in terms of functions and vector fields on the Whitney sum of ΠTM and a trivial reverse parity line bundle over M, that is a trivial line bundle with fibre $\mathbb{R}^1$ replaced by $\mathbb{R}^{0|1}$. Further details about this and type N generalized form-valued vector fields are in [16].

Integration is defined using polychains, [14]. A $p-$polychain of type $N = 1$ in M, written here as $\mathbf{c}_p$, is an ordered pair of ordinary (real, singular) chains in M

$$\mathbf{c}_p = (c_p, c_{p+1}), \tag{13}$$

where c_p is an ordinary $p-$chain, and c_{p+1} is an ordinary $p+1-$chains. The ordinary chains have respective boundaries $\partial c_p, \partial c_{p+1}$, and the boundary of the polychain $\mathbf{c}_p$ is the $(p-1)-$polychain

$$\partial \mathbf{c}_p = (\partial c_p, \partial c_{p+1} + (-1)^p \epsilon c_p), \tag{14}$$

and

$$\partial^2 \mathbf{c}_p = 0. \tag{15}$$

Example 2: When a polychain is determined by just one ordinary chain as in the three examples

$$\begin{aligned}
\mathbf{c}_p &= (0, c_{p+1}), \\
\mathbf{c}_p &= (c_p, 0), \\
\mathbf{c}_p &= (\pm \partial c_{p+1}, c_{p+1}),
\end{aligned}$$

then the corresponding three boundaries are

$$\begin{aligned}
\partial \mathbf{c}_p &= (0, \partial c_{p+1}), \\
\partial \mathbf{c}_p &= (\partial c_p, (-1)^p \epsilon c_p), \\
\partial \mathbf{c}_p &= (0, [1 \pm (-1)^p \epsilon] \partial c_p).
\end{aligned}$$

When $N = 1$ the integral of a generalized form $\overset{p}{\mathbf{a}}$ over a polychain $\mathbf{c}_p$ is

$$\int_{\mathbf{c}_p} \overset{p}{\mathbf{r}} = \int_{c_p} \rho + \int_{c_{p+1}} \lambda. \tag{16}$$

Stokes' theorem for generalized forms and polychains in the type $N = 1$ case states that

$$\int_{\mathbf{c}_p} d\overset{p-1}{\mathbf{r}} = \int_{\partial \mathbf{c}_p} \overset{p-1}{\mathbf{r}}. \tag{17}$$

3 Generalized connections

Generalized connection with values in the Lie algebra of a matrix Lie group G are defined in essentially the same way as ordinary connections [25], except that ordinary forms, including zero forms, are replaced by generalized forms. Let P be a principal G bundle over M. If $\{U_I\}$ is a covering of an $n-$dimensional manifold M by coordinate charts, then a generalized connection $\mathbf{A}$ with values in the Lie algebra $\mathfrak{g}$ of the matrix Lie group G is an assignment of a $\mathfrak{g}$-valued generalized one-form, $\mathbf{A}_I$, to each set U_I and such that on $U_I \cap U_J$, for all I and J,

$$\mathbf{A}_J = (t_{IJ}^{-1}) dt_{IJ} + (t_{IJ}^{-1}) \mathbf{A}_I t_{IJ}, \tag{18}$$

where $t_{IJ} : U_I \cap U_J \to G$, by $p \to t_{IJ}(p)$ are transition functions satisfying the usual conditions, $t_{II}(p) = 1$, $p \in U_I$, $t_{IJ}(p) = [t_{JI}(p)]^{-1}$ $p \in U_I \cap U_J$, $t_{IJ}(p) t_{JK}(p) = t_{IK}(p)$ $p \in U_I \cap U_J \cap U_K$. Transition functions $\{t_{IJ}\}$ and $\{\widetilde{t}_{IJ}\}$ are (gauge) equivalent when $\widetilde{t}_{IJ} = (g_I)^{-1} t_{IJ} g_J$ and $g_I : U_I \to G$ and $g_J : U_J \to G$ determine gauge transformations, $\mathbf{A}_I \to (g_I^{-1}) dg_I + (g_I^{-1}) \mathbf{A}_I g_I$ and $\mathbf{A}_J \to (g_J^{-1}) dg_J + (g_J^{-1}) \mathbf{A}_J g_J$ in U_I and U_J respectively.

The curvature two-form $\mathbf{F}_I$ is the generalized form

$$\mathbf{F}_I = \mathbf{d}\mathbf{A}_I + \mathbf{A}_I \mathbf{A}_I, \tag{19}$$

and under the transformation in Eq.(18)

$$\mathbf{F}_J = (t_{IJ}^{-1}) \mathbf{F}_I t_{IJ}. \tag{20}$$

On any coordinate chart such as U_I the connection one-form $\mathbf{A}_I$ can be written as

$$\mathbf{A}_I = \alpha_I + \beta_I \mathbf{m}, \tag{21}$$

where α_I and β_I are respectively ordinary matrix valued one-forms and two-forms on M and the curvature two-form is

$$\begin{aligned} \mathbf{F}_I &= \mathcal{F}_I + \epsilon \beta_I + D\beta_I \mathbf{m}, \\ \mathcal{F}_I &= d\alpha_I + \alpha_I \alpha_I, \\ D\beta_I &= d\beta_I + \alpha_I \beta_I - \beta_I \alpha_I. \end{aligned} \tag{22}$$

It follows from the above that the locally defined ordinary one-forms α_I and curvature two-forms $\mathcal{F}_I$, patch together to define global connection and curvature forms, α and $\mathcal{F}$, of an ordinary connection. The ordinary two-forms β transform as Lie-algebra valued two-forms.

These ideas can be be broadened by extending the structure group. Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$ as above. Let $G_{(0)} = \{g\}$ be the space of G-valued (ordinary) zero-forms belonging to $\Lambda^0_{(0)}(U)$ where U is an open set $U \subseteq M$. This is a group under multiplication with identity the unit matrix 1. Let $\mathfrak{g}_{(0)} = \{h\}$ be the set of $\mathfrak{g}$-valued one-forms $\in \Lambda^1_{(0)}(U)$. There is an ad-action of of $G_{(0)}$ on $\mathfrak{g}_{(0)}$, that is a homomorphism $\Phi : G_{(0)} \to aut(\mathfrak{g}_{(o)})$, $\Phi(\ g) : h \longrightarrow gh(g)^{-1}$. Then the set of type $N = 1$ matrix-valued generalized zero-forms, $G_{(1)} = \{\mathbf{g} \mid \mathbf{g} = \mathbf{h}g = (1 + h\mathbf{m})g\}$ is a group under matrix and exterior multiplication. Calculation shows that the product of $\mathbf{g}_1 = \mathbf{h}_1 g_1 = (1 + h_1\mathbf{m})g_1$ and $\mathbf{g}_2 = \mathbf{h}_2 g_2 = (1 + h_2\mathbf{m})g_2 \in G_{(1)}$ is the element of $G_{(1)}$ given by

$$\mathbf{g}_1\mathbf{g}_2 = \{1 + [h_1 + g_1 h_2 (g_1)^{-1}]\mathbf{m}\}g_1 g_2. \tag{23}$$

The identity of $G_{(1)}$ is the unit matrix 1, and the inverse of $\mathbf{g}$ is given by

$$(\mathbf{g})^{-1} = \{1 - [(g)^{-1}hg]\mathbf{m}\}(g)^{-1}. \tag{24}$$

The Lie algebra $\mathfrak{g}_{(1)}$ of $G_{(1)}$ is given by $\{l \mid l = \lambda + \mu\mathbf{m}\}$ where λ and μ are respectively ordinary zero and one forms taking values in $\mathfrak{g}$.

Under a generalized gauge transformation by $\mathbf{g}$ where $\mathbf{g} \in G_{(1)}$

$$\begin{aligned} \mathbf{A} &\to (\mathbf{g}^{-1})d\mathbf{g} + (\mathbf{g}^{-1})\mathbf{A}\mathbf{g} = = g^{-1}dg + g^{-1}[\mathbf{h}^{-1}d\mathbf{h} + \mathbf{h}^{-1}\mathbf{A}\mathbf{h}]g \\ &= g^{-1}dg + g^{-1}(\alpha - \epsilon h)g + g^{-1}[dh - \epsilon hh + h\alpha + \alpha h + \beta]g\mathbf{m} \\ \mathbf{F} &\to (\mathbf{g}^{-1})\mathbf{F}\mathbf{g} = g^{-1}\{\mathcal{F}+\epsilon\beta + [D\beta + (\mathcal{F}+\epsilon\beta)h - h(\mathcal{F}+\epsilon\beta)]\mathbf{m}\}g. \end{aligned} \tag{25}$$

The definition of a global generalized connection given above may be generalized in the obvious way. Briefly, for a covering $\{U_I\}$ of M by coordinate charts, specify transition functions $\mathbf{t}_{IJ} \in G_{(1)}$ on $U_I \cap U_J$ by $p \to \overset{0}{\mathbf{t}}_{IJ}(p)$ satisfying $\overset{0}{\mathbf{t}}_{II}(p) = 1$, $p \in U_I$, $\overset{0}{\mathbf{t}}_{IJ}(p) = [\overset{0}{\mathbf{t}}_{JI}(p)]^{-1}$ $p \in U_I \cap U_J$, $\overset{0}{\mathbf{t}}_{IJ}(p)\overset{0}{\mathbf{t}}_{JK}(p) = \overset{0}{\mathbf{t}}_{IK}(p)$ $p \in U_I \cap U_J \cap U_K$ and local generalized connection one-forms $\mathbf{A}_I$, on each U_I, related by $\mathbf{A}_J = (\mathbf{t}_{IJ})^{-1}\mathbf{A}_I\mathbf{t}_{IJ} + (\mathbf{t}_{IJ})^{-1}d\mathbf{t}_{IJ}$ on $U_I \cap U_J$. Transition functions $\{\mathbf{t}_{IJ}\}$ and $\{\widetilde{\mathbf{t}}_{IJ}\}$ are (gauge) equivalent when $\widetilde{\mathbf{t}}_{IJ} = (\mathbf{g}_I)^{-1}\mathbf{t}_{IJ}\mathbf{g}_J$

and $\mathbf{g}_I$ and $\mathbf{g}_J$ respectively determine generalized gauge transformations in U_I and U_J as in Eq.(25) above.

Henceforth connections on M will be discussed and the subscripts corresponding to coordinate charts will be dropped.

The curvature satisfies the Bianchi identities

$$\mathbf{DF} = d\mathbf{F} + \mathbf{AF} - \mathbf{FA} = 0, \tag{26}$$

where here $\mathbf{D}$ denotes the covariant exterior derivative of a type $N = 1$ valued generalized form. For a $\mathfrak{g}-$valued p-form $\mathbf{P}$

$$\mathbf{DP} = d\mathbf{P} + \mathbf{AP} + (-1)^{p+1}\mathbf{PA}. \tag{27}$$

Let E be a $d-$dimensional vector bundle associated to P and let $\{e_i\}$, $i = 1$ to d be a a basis of sections of E over $U \subseteq M$ with the usual action of $g \in G$, $e_i \to e_j g_i^j$. A type $N = 1$ generalized form-valued vector field is given by $\mathbf{V} = \mathbf{v}^i e_i = (v^i + v^i_\sigma dx^\sigma \mathbf{m})e_i$, where the components $\mathbf{v}^i$ are generalized zero forms on U, and the action of $\mathbf{g} \in G_{(1)}$ is the extension of the action of $g \in G$ given by $e_i \to e_j \mathbf{g}_i^j$, $\mathbf{v}^i \to (\mathbf{g}^{-1})^i_j \mathbf{v}^j$. The covariant derivative of $\mathbf{V}$ is given by

$$\mathbf{DV} = \mathbf{Dv}^i \otimes e_i = (d\mathbf{v}^i + \mathbf{A}^i_j \mathbf{v}^j) \otimes e_i. \tag{28}$$

In local coordinates $\{x^\nu\}$ on M,

$$\mathbf{Dv}^i = Dv^i - \epsilon v^i_\nu dx^\nu + [D(v^i_\nu dx^\nu) + \beta^i_j v^j]\mathbf{m}, \tag{29}$$

and $\mathbf{D}$ and D are the covariant exterior derivatives with respect to $\mathbf{A}$ and α (with matrix representations with (i, j) entries $\mathbf{A}^i_j$ and α^i_j respectively). The covariant derivative with respect to a type $N = 1$ generalized form-valued vector field $\mathbf{W}$, is the generalized form-valued vector field

$$\mathbf{D_W V} = [i_{\mathbf{W}}(d\mathbf{v}^i + \mathbf{A}^i_j \mathbf{v}^i)]e_i. \tag{30}$$

The covariant derivative is extended to generalized form -valued tensor fields by using the linearity and product rules satisfied by ordinary covariant derivatives and tensor fields.

A field $\mathbf{V}$ is a parallel vector field if $\mathbf{DV} = 0$, that is

$$\begin{aligned} Dv^i - \epsilon v^i_\nu dx^\nu &= 0, \\ D(v^i_\nu dx^\nu) + \beta^i_j v^j &= 0. \end{aligned} \tag{31}$$

and such a system of equations is completely integrable if and only if the generalized curvature $\mathbf{F}$ is zero, that is when

$$\mathcal{F}^i_j+\epsilon\beta^i_j=0, \qquad D\beta^i_j = 0. \tag{32}$$

The notion of parallel transport on a sub-manifold is defined by considering the pull-backs of these equations to the sub-manifold.

When $\mathbf{F} = \mathbf{0}$ the connection $\mathbf{A} = (\mathbf{g}_c\mathbf{g})^{-1}d(\mathbf{g}_c\mathbf{g})$ for some $\mathbf{g}$ and any closed $\mathbf{g}_c \in G_{(1)}$, [13]. That is if $\mathbf{g} = (1 + h\mathbf{m})g$ and $\mathbf{g}_c = (1 + h_c\mathbf{m})g_c$ where $dg_c = \epsilon h_c g_c$ and $d(h_c g_c) = 0$,

$$\mathbf{A} = (\mathbf{g}_c\mathbf{g})^{-1}d(\mathbf{g}_c\mathbf{g}) = (g)^{-1}dg - \epsilon(g)^{-1}hg + \{(g)^{-1}[dh - \epsilon hh]g\}\mathbf{m}, \tag{33}$$

and any parallel vector field $\mathbf{V} = (\mathbf{g}^{-1})^i_j\mathbf{v}^j_0 e_i$ for some closed generalized zero-forms $\mathbf{v}^j_0$.

Henceforth in this paper only groups such that the connections $\mathbf{A}$ have, $Tr\mathbf{A}$, zero will be considered.

The generalized Chern-Pontrjagin class is determined by a generalized four-form $\mathbf{CP}$

$$\mathbf{CP} = \frac{1}{8\pi^2}Tr(\mathbf{FF}), \tag{34}$$

which is equal to the exterior derivative of the generalized Chern-Simons three-form $\mathbf{CS}$ where

$$\mathbf{CS} = \frac{1}{8\pi^2}Tr(\mathbf{AF} - \frac{1}{3}\mathbf{AAA}). \tag{35}$$

More generally, if $\mathbf{k}$ is any closed generalized zero-form, then

$$d(\mathbf{kCS}) = \mathbf{kCP}. \tag{36}$$

By Stokes' theorem, Eq.(17), for a polychain $\mathbf{c}_4$

$$\int_{\mathbf{c}_4}\mathbf{kCP} = \int_{\partial\mathbf{c}_4}\mathbf{kCS}. \tag{37}$$

Under the generalized gauge transformation given by Eq.(25)

$$\mathbf{CP} \to \mathbf{CP}, \qquad \mathbf{CS} \to \mathbf{CS} - \frac{1}{8\pi^2}d\{Tr[(d\mathbf{g})(\mathbf{g})^{-1}\mathbf{A}]\} - \frac{1}{24\pi^2}Tr[(\mathbf{g}^{-1}d\mathbf{g})^3]. \tag{38}$$

Since the last (generalized winding number) term is closed when $\mathbf{c}_3 = \partial \mathbf{c}_4$

$$\int_{\mathbf{c}_3} \mathbf{CS} \to \int_{\mathbf{c}_3} \mathbf{CS}. \tag{39}$$

For $\mathbf{A} = \alpha + \beta \mathbf{m}$ as above

$$\mathbf{CS} = \frac{1}{8\pi^2} Tr[(\alpha \mathcal{F} - \frac{1}{3}\alpha\alpha\alpha + \epsilon\alpha\beta) + (\alpha D\beta + \beta\mathcal{F} - \beta\alpha\alpha + \epsilon\beta\beta)\mathbf{m}]. \tag{40}$$

In sections five and six type $N = 1$ Chern-Simons integrals for a polychain

$$\mathbf{c}_3 = \partial \mathbf{c}_4 = \partial(c_4, c_5) = (\partial c_4, \partial c_5 + \epsilon c_4), \tag{41}$$

will be used as action integrals. In this case, [14], $\int_{\mathbf{c}_4} \mathbf{CP} = = \int_{\partial \mathbf{c}_4} \mathbf{CS} = \int_{\mathbf{c}_3} \mathbf{CS}$ and

$$\int_{\mathbf{c}_3} \mathbf{CS} = \frac{1}{8\pi^2}[\int_{\partial c_4} Tr(\alpha\mathcal{F} - \frac{1}{3}\alpha\alpha\alpha) + \int_{\partial c_5 + \epsilon c_4} Tr(2\beta\mathcal{F} + \epsilon\beta\beta)]. \tag{42}$$

The variation of $\mathbf{A} = \alpha + \beta\mathbf{m}$ is $\delta\ \mathbf{A} = \delta\alpha + \delta\beta\mathbf{m}$. Then from Eqs.(35) and (42)

$$\delta\mathbf{CS} = \frac{1}{8\pi^2}[Tr(2\delta\mathbf{AF}) + d(Tr\delta\mathbf{AA})], \tag{43}$$

$$\delta\int_{\mathbf{c}_3} \mathbf{CS} = \frac{1}{8\pi^2}\{\int_{\partial c_4} Tr[2\delta\alpha(\mathcal{F} + \epsilon\beta)] + \int_{\partial c_5 + \epsilon c_4} 2Tr[\delta\alpha D\beta + \delta\beta(\mathcal{F} + \epsilon\beta)]\}.$$

4 Connections, metrics and gravity

Consider, on an n dimensional manifold M, type $N = 1$ generalized connections represented by $(p+q+1, p+q+1)$ matrix valued generalized one-forms

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^a_b & -\sigma\mathbf{A}^a \\ \mathbf{A}_b & 0 \end{pmatrix}, \tag{44}$$

where σ is either 1, -1 or 0 and $\mathbf{A}$ takes values in the Lie algebra of $G = SO(p+1, q)$ when $\sigma = 1$, $G = SO(p, q+1)$ when $\sigma = -1$ and $G = ISO(p, q)$ when $\sigma = 0$. In the first two cases the metric is given by the $(p+q+1) \times (p+q+1)$ matrix

$$\begin{pmatrix} \eta_{ab} & 0 \\ 0 & \sigma \end{pmatrix}, \tag{45}$$

$$(\eta_{ab}) = \begin{pmatrix} 1_{p\times p} & 0 \\ 0 & -1_{q\times q} \end{pmatrix},$$

and $\mathbf{A}_b = \eta_{ba}\mathbf{A}^a$. Latin indices ranging and summing from 1 to $p+q$. The curvature of $\mathbf{A}$ is given by

$$\mathbf{F} = d\mathbf{A} + \mathbf{A}\mathbf{A} = \begin{pmatrix} \mathbf{F}^a_b & -\sigma\mathbf{F}^a \\ \mathbf{F}_b & 0 \end{pmatrix}, \tag{46}$$

where

$$\begin{aligned} \mathbf{F}^a_b &= \mathbf{dA}^a_b + \mathbf{A}^a_c\mathbf{A}^c_b - \sigma\mathbf{A}^a\mathbf{A}_b, \\ \mathbf{F}^a &= d\mathbf{A}^a + \mathbf{A}^a_b\mathbf{A}^b,\ \mathbf{F}_b = \eta_{bc}\mathbf{F}^c. \end{aligned} \tag{47}$$

Now let

$$\begin{aligned} \mathbf{A}^a_b &= \omega^a_b - \kappa^a_b\mathbf{m}, \\ \mathbf{A}^a &= \frac{1}{l}(\theta^a - \Theta^a\mathbf{m}), \end{aligned} \tag{48}$$

where l is a non-zero constant and $\omega_{ab} = -\omega_{ba}$, $\kappa_{ab} = -\kappa_{ba}$ are, respectively, an ordinary $so(p,q)$-valued one-form and two-form and θ^a and Θ^a are, respectively, ordinary one-forms and two-forms on M. Then

$$\begin{aligned} \mathbf{F}^a_b &= \Omega^a_b - \frac{\sigma}{l^2}\theta^a\theta_b - \epsilon\kappa^a_b - [D\kappa^a_b + \frac{\sigma}{l^2}(\Theta^a\theta_b - \theta^a\Theta_b)]\mathbf{m}, \\ \mathbf{F}^a &= \frac{1}{l}[D\theta^a - \epsilon\Theta^a + (\kappa^a_b\theta^b - D\Theta^a)\mathbf{m}], \\ \Omega^a_b &= d\omega^a_b + \omega^a_c\omega^c_b, \\ \mathbf{dA}^a_b + \mathbf{A}^a_c\mathbf{A}^c_b &= \Omega^a_b - \epsilon\kappa^a_b - D\kappa^a_b\mathbf{m}, \end{aligned} \tag{49}$$

The covariant exterior derivative with respect to ω^a_b is denoted D so that

$$\begin{aligned} D\theta^a &= d\theta^a + \omega^a_b\theta^b, D\Theta^a = d\Theta^a + \omega^a_b\Theta^b, \\ D\kappa^a_b &= d\kappa^a_b + \omega^a_c\kappa^c_b - \kappa^a_c\omega^c_b. \end{aligned}$$

Under a generalized gauge transformation by $\mathbf{g} = (1 + h\mathbf{m})$, where h is a $\mathfrak{g}$-valued one-form

$$h = \begin{pmatrix} h^a_b & -\sigma\frac{h^a}{l} \\ \frac{h_b}{l} & 0 \end{pmatrix},\ h_{ab} = -h_{ba}, \tag{50}$$

the generalized connection gauge transformation, Eq.(25), becomes

$$\mathbf{A} = \alpha + \beta\mathbf{m} \rightarrow (\alpha - \epsilon h) + [dh - \epsilon hh + h\alpha + \alpha h + \beta]\mathbf{m},$$

that is

$$\begin{aligned} \omega^a_b &\rightarrow \omega^a_b - \epsilon h^a_b, \\ \kappa^a_b &\rightarrow \kappa^a_b - Dh^a_b + \epsilon h^a_c h^c_b + \frac{\sigma}{l^2}(h^a\theta_b - h_b\theta^a - \epsilon h^a h_b), \\ \theta^a &\rightarrow \theta^a - \epsilon h^a, \\ \Theta^a &\rightarrow \Theta^a - Dh^a - \epsilon h^b h^a_b - h^a_b\theta^b, \end{aligned} \tag{51}$$

where

$$Dh^a_b = dh^a_b + h^a_c\omega^c_b + \omega^a_c h^c_b,\ \ Dh^a = dh^a + \omega^a_b h^b.$$

When the one-forms $\{\theta^a\}$ constitute a basis in terms of which $\Theta^a = \frac{1}{2}\Theta^a_{bc}\theta^b\theta^c$, where $\Theta^a_{bc} = \Theta^a_{[bc]}$, and $h^a_b = h^a_{bc}\theta^c$, where $h_{abc} = h_{[ab]c}$, it is straighforward to show that under the generalized gauge transformation given by Eq.(51) with

$$\begin{aligned} h_{abc} &= \frac{1}{2}(\Theta_{cab} + \Theta_{bac} + \Theta_{acb}) \\ h^a &= 0, \end{aligned} \tag{52}$$

the two-forms Θ^a transform to zero, $\Theta^a \rightarrow 0$.

The generalized gauge condition $\Theta^a = 0$ is preserved by gauge transformations given by $g \in G$ where

$$g = \begin{pmatrix} g^a_b & 0 \\ 0 & 1 \end{pmatrix},\ \ g^a_b g^c_d \eta_{ac} = \eta_{bd} \tag{53}$$

if σ is non-zero and by $g \in ISO(p,q)$ if $\sigma = 0$.

The generalized connection $\mathbf{A}$ is flat if and only if $\mathbf{F} = 0$ and then

$$\begin{aligned} D\theta^a &= \epsilon\Theta^a, \\ \Omega^a_b &= \frac{\sigma}{l^2}\theta^a\theta_b + \epsilon\kappa^a_b, \\ D\Theta^a &= \kappa^a_b\theta^b, \\ D\kappa^a_b &= \frac{\sigma}{l^2}(\theta^a\Theta_b - \Theta^a\theta_b). \end{aligned} \tag{54}$$

This is a closed differential ideal.

Suppose now that the $p+q$ ordinary one-forms $\{\theta^a\}$ are linearly independent on a $s = (p+q)-$dimensional sub-manifold $S \subseteq M$, $(p+q) \leqq n$, so that they form an orthonormal basis for a metric of signature (p, q), $ds^2 = \eta_{ab}\theta^a \otimes \theta^b$ on S. There are four cases to consider on S.

Case (ia) where $\epsilon = \sigma = 0$.

In this case it follows from the first two of Eqs.(54) that ω^a_b is the Levi-Civita connection of the metric ds^2 and the metric is flat. Hence coordinates $\{x^a\}$ can be introduced, and a gauge chosen so that $\theta^a = dx^a$ and $\omega^a_b = 0$ with the Levi-Civita covariant derivative ∇_a now the partial derivative ∂_a. The last two of Eqs.(54) then become, in terms of their components with respect to the basis one-forms,

$$\begin{aligned} \partial_{[d}\Theta^a_{bc]} &= \kappa^a_{[bcd]}, \\ \partial_{[e}\kappa^a_{|b|cd]} &= 0, \end{aligned} \tag{55}$$

where $\Theta^a = \frac{1}{2}\Theta^a_{bc}\theta^b\theta^c$, $\Theta^a_{bc} = \Theta^a_{[bc]}$, and $\kappa^a_b = \frac{1}{2}\kappa^a_{bcd}\theta^c\theta^d$, $\kappa_{abcd} = \kappa_{[ab]cd} = \kappa_{ab[cd]}$. When the additional condition

$$\Theta^a = 0 \tag{56}$$

is imposed it follows that κ^a_{bcd} has the symmetries of the Riemann tensor and by an old local result due to Trautman, [26] and [27], the solution of these equations is then given by

$$\kappa^a_{bcd} = \eta^{ae}(\partial_d\partial_b\gamma_{ec} - \partial_b\partial_c\gamma_{ed} - \partial_e\partial_d\gamma_{bc} + \partial_c\partial_e\gamma_{bd}), \tag{57}$$

where $\gamma_{ab} = \gamma_{ba}$, and κ^a_{bcd} are the components of the (linearized) Riemann tensor of a metric $\eta_{ab} + 2\gamma_{ab}$ linearized about flat space.

Case (ib) where $d\mathbf{m} = \epsilon = 0$ and σ is non-zero.

In this case it follows from the first two of Eqs.(54) that ω^a_b is the Levi-Civita connection of the metric ds^2, and this metric has constant curvature with Ricci scalar $R = \frac{\sigma}{l^2}s(s-1)$ when $s > 1$. The last two of Eqs. (54) become, in terms of their components with respect to the basis one-forms,

$$\begin{aligned} \nabla_{[d}\Theta^a_{bc]} &= \kappa^a_{[bcd]}, \\ \nabla_{[e}\kappa^a_{|b|cd]} &= \frac{\sigma}{l^2}(\Theta_{b[de}\delta^a_{c]} - \eta_{b[c}\Theta^a_{de]}), \end{aligned} \tag{58}$$

where ∇ denotes the Levi-Civita covariant derivative. In this case when the additional conditions

$$\Theta^a = 0, \kappa^a_{.bad} = 0. \tag{59}$$

are imposed it follows from Eqs.(58) that κ^a_{bcd} has the algebraic symmetries of a Weyl tensor and satisfies $\nabla_{[e}\kappa^a_{|b|cd]} = 0$. Since the metric ds^2 has constant curvature it is conformally flat and so $ds^2 = (\exp 2\mu) ds^2_F$, for the appropriate function μ, where ds^2_F is flat. In coordinates and a gauge such that $ds^2_F = \eta_{ab} dx^a \otimes dx^b$, $\theta^a =(\exp\mu) dx^a$ and the one-forms dx^a constitute the orthonormal frame for the flat metric, the flat metric Levi-Civita connection is zero and the covariant exterior derivative for the flat metric is just the ordinary exterior derivative. If $\kappa^a_{Fb} = (\exp\mu)\kappa^a_b$ then $d\kappa^a_{Fb} = (\exp\mu) D\kappa^a_b$ and so $\partial_{[e}\kappa^a_{F|b|cd]} = \kappa^a_{F[bcd]} = \kappa^a_{F.bad} = 0$. Hence, using again Trautman's result mentioned above, κ^a_{Fbcd} are the components of the (linearized) Riemann tensor of the linearized metric $\eta_{ab} + 2\gamma_{ab}$. Consequently

$$\kappa^a_{bcd} = \exp(-3\mu)\eta^{ae}(\partial_d\partial_b\gamma_{ec} - \partial_b\partial_c\gamma_{ed} - \partial_e\partial_d\gamma_{bc} + \partial_c\partial_e\gamma_{bd}) \tag{60}$$

Further more, since $\kappa^a_{.bad} = \kappa^a_{F.bad} = 0$, the linearized metric components γ_{ab} satisfy the linearized Einstein vacuum field equations with zero cosmological constant.

In summary, if $\epsilon = 0$ then the generalized connection, with $\Theta^a = 0, \kappa^a_{.bad} = 0$, is flat if and only if the metric $ds^2 = \eta_{ab}\theta^a \otimes \theta^b$ has constant curvature so a coframe can be chosen such that $\theta^a =\exp(\mu) dx^a$. Furthermore $\kappa^a_{bcd} = \exp(-3\mu)\eta^{ae}(\partial_d\partial_b\gamma_{ec} - \partial_b\partial_c\gamma_{ed} - \partial_e\partial_d\gamma_{bc} + \partial_c\partial_e\gamma_{bd})$ where $\gamma_{ab} = \gamma_{ba}$ satisfies the Einstein vacuum field equations, with zero cosmological constant, linearized about flat space.

Case (iia) where $d\mathbf{m} = \epsilon$ is non-zero but σ is zero.

In this case the generalized connection is flat if and only if the connection ω^a_b has torsion $\epsilon\Theta^a$ and curvature two-form $\epsilon\kappa^a_b$. When Eqs.(59) are imposed so that the torsion vanishes then the connection is the Levi-Civita connection of the metric ds^2 and by the second of Eqs.(59) its Ricci tensor is zero.

Case (iib) where $d\mathbf{m} = \epsilon$ and σ are both non-zero.

In this case Eqs.(54) are all satisfied, and hence the generalized connection is flat, if and only if the connection ω^a_b has torsion $\epsilon\Theta^a$ and curvature with components

$$\Omega^a_b = \frac{1}{2} R^a_{bcd}\theta^c\theta^d, \tag{61}$$
$$R^a_{bcd} = \epsilon\kappa^a_{bcd} + \frac{\sigma}{l^2}(\delta^a_c\eta_{bd} - \delta^a_d\eta_{bc}).$$

When the additional conditions given in Eq.(59) are imposed the generalized connection is flat if and only if ω^a_b is the Levi-Civita connection of the metric which satisfies the full Einstein vacuum field equations with cosmological constant given by $\frac{\sigma}{l^2}s(s-1)$.

The solutions of Eqs.(54) can be found by using Eq.(33) and the fact that the flat connection is given by $\mathbf{A} = \mathbf{g}^{-1}d\mathbf{g}$ where $\mathbf{g} \in G_{(1)}$. This is done in the following example in the cases where $\sigma = 0$. The other cases, i(b) and ii(b), where σ is non-zero can be treated similarly using the groups $G = SO(p+1, q)$ when $\sigma = 1$ and $G = SO(p, q+1)$ when $\sigma = -1$.

Example 3: Consider the cases i(a) and ii(a) above where $\sigma = 0$. In this case $G_{(1)} = ISO(p,q)_{(1)}$. The $(p+1, q+1)$ matrix representation for $\mathbf{g} = (1+h\mathbf{m})g \in G_{(1)}$ can be written in terms of the matrices

$$h= \begin{pmatrix} h^a_b & 0 \\ \frac{h_b}{l} & 0 \end{pmatrix},\ t = \begin{pmatrix} \delta^a_b & 0 \\ \frac{\tau_b}{l} & 1 \end{pmatrix},\ g_0 = \begin{pmatrix} L^a_b & 0 \\ 0 & 1 \end{pmatrix},$$

where, without loss of generality, $g = tg_0$. Here h is an ordinary one-form with values in the Lie algebra of $ISO(p,q)$, g is an $ISO(p,q)$-valued zero-form and the matrix-valued function (L^a_b) takes values in $SO(p,q)$ Working modulo g_0, $\mathbf{A} = \mathbf{g}^{-1}d\mathbf{g}$ is

$$\begin{pmatrix} -\epsilon h^a_b + (dh^a_b - \epsilon h^a_c h^c_b)\mathbf{m} & 0 \\ \frac{1}{l}\{d\tau_b + \epsilon\tau_c h^c_b - \epsilon h_b + [dh_b - \epsilon h_c h^c_b - \tau_a(dh^a_b - \epsilon h^a_c h^c_b)]\mathbf{m}\} & 0 \end{pmatrix}.$$

and this expression together with Eqs.(44) and (48) can be used to evaluate (modulo g_0 gauge transformations) ω^a_b, κ^a_b, θ^a and Θ^a,

$$\begin{aligned} &\theta^a = d\tau^a - \epsilon\tau^b h^a_b - \epsilon h^a,\ \Theta^a = -\tau^b(dh^a_b - \epsilon h^a_c h^c_b) - dh^a - \epsilon h^b h^a_b, \\ &\omega^a_b = -\epsilon h^a_b,\ \kappa^a_b = -(dh^a_b - \epsilon h^a_c h^c_b). \end{aligned}$$

The association of a linear problem with certain physically interesting non-linear partial differential equations, including those admitting soliton solutions, has enabled techniques such as the inverse scattering methods to be used to solve them, see for example [28] and [29]. These linear problems often have a geometrical interpretation, as for example is the case with the soliton connection and the equations of parallel transport of a linear connection on a principal SL(2,R) bundle, [30]. In this regard it should be noted that the (linear) equations of parallel transport, Eq.(31), for the generalized connections considered in this section have integrabilty conditions which can include, as discussed above, the Einstein vacuum field equations.

5 Generalized Chern-Simons Lagrangians

For the connection and curvature given by Eqs.(44), (46) and (47) the generalized Chern-Pontrjagin and Chern-Simons forms are

$$\mathbf{CP}=\frac{1}{8\pi^2}[\mathbf{F}^a_b\mathbf{F}^b_a-2\sigma\mathbf{F}^a\mathbf{F}_a] \tag{62}$$
$$\mathbf{CS}=\frac{1}{8\pi^2}[\mathbf{A}^a_b\mathbf{F}^b_a-\frac{1}{3}\mathbf{A}^a_b\mathbf{A}^b_c\mathbf{A}^c_a-2\sigma\mathbf{A}^a\mathbf{F}_a+\sigma\mathbf{A}^a_b\mathbf{A}^b\mathbf{A}_a].$$

Computation, using Eqs.(48) and (49), gives

$$\begin{aligned}\mathbf{CS}=&CS_0+\frac{1}{8\pi^2}\{\frac{2\sigma}{l^2}(\epsilon\theta^a\Theta_a-\theta^aD\theta_a)-\epsilon\omega^a_{.b}K^b_{.a}\\&+d(\omega^a_{.b}\kappa^b_{.a}-\frac{2\sigma}{l^2}\theta^a\Theta_a)\mathbf{m}\\&+[\epsilon\kappa^a{}_{.b}\kappa^b_{.a}-2\kappa^a_{.b}\Omega^b_{.a}+\frac{2\sigma}{l^2}(2\Theta^aD\theta_a-\kappa_{ab}\theta^a\theta^b-\epsilon\Theta^a\Theta_a)]\mathbf{m}\},\end{aligned} \tag{63}$$

where CS_0 is the ordinary Chern-Simons 3−form

$$CS_0=\frac{1}{8\pi^2}\{\omega^a_{.b}\Omega^b_{a.}-\frac{1}{3}\omega^a_{.b}\omega^b_{.c}\omega^c_{.a}\}. \tag{64}$$

When the polychain is itself a boundary with $\mathbf{c}_3=\partial\mathbf{c}_4$ as in Eq.(41), then

$$\begin{aligned}\int_{\mathbf{c}_3}\mathbf{CS}=&\frac{1}{8\pi^2}\{\int_{\partial c_4}[8\pi^2CS_0-\frac{2\sigma}{l^2}\theta^aD\theta_a]\\&+\epsilon\int_{c_4}[\epsilon\kappa^a_{.b}\kappa^b_{.a}-2\kappa^a_{.b}\Omega^b_{.a}+\frac{2\sigma}{l^2}(2\Theta^aD\theta_a-\kappa_{ab}\theta^a\theta^b-\epsilon\Theta^a\Theta_a)]\\&+\int_{\partial c_5}[\epsilon\kappa^a_{.b}\kappa^b_{.a}-2\kappa^a_{.b}\Omega^b_{.a}+\frac{2\sigma}{l^2}(2\Theta^aD\theta_a-\kappa_{ab}\theta^a\theta^b-\epsilon\Theta^a\Theta_a)]\}.\end{aligned} \tag{65}$$

Next consider this expression as a generalized Chern-Simons action integral. Computing the variation of the 3-form **CS** and this generalized Chern-Simons action gives

$$\delta\mathbf{CS}=\frac{1}{8\pi^2}\{\frac{2\sigma}{l^2}\delta\theta^a(\epsilon\Theta_a-2D\theta_a)+\frac{2\sigma\epsilon}{l^2}\delta\mathbf{\Theta^a}\theta_\mathbf{a}+\delta\omega^a_b(2\Omega^b_{.a}-\epsilon\kappa^b_a+\frac{2\sigma}{l^2}\theta_a\theta^b)-\epsilon\delta\kappa^a_{.b}\omega^b_{.a} \tag{66}$$

$$+d(\delta\omega^a_{.b}\omega^b_{.a}-\frac{2\sigma}{l^2}\delta\theta^a\theta_a)+d(\frac{2\sigma}{l^2}\delta\theta^a\Theta_a-\frac{2\sigma}{l^2}\delta\Theta^a\theta_a-\delta\omega^a_{.b}\kappa^b_{.a}+\delta\kappa^a_{.b}\omega^b_{.a})\mathbf{m}$$
$$+[\frac{4\sigma}{l^2}\delta\theta^a(D\Theta_a-\theta^b\kappa_{ab})+\frac{4\sigma}{l^2}\delta\Theta^a(D\theta_a-\epsilon\Theta_a)+2\delta\omega^a_{.b}(\frac{2\sigma}{l^2}\theta^b\Theta_a-D\kappa^b_{.a})$$
$$+2\delta\kappa^a_b(\epsilon\kappa^b_{.a}-\Omega^b_{.a}-\frac{\sigma}{l^2}\theta_a\theta^b)]\mathbf{m}\}.$$

$$\delta\int_{\mathbf{c}_3}\mathbf{CS}=\frac{1}{8\pi^2}\{\int_{\partial c_4}[\frac{4\sigma}{l^2}\delta\theta^a(\epsilon\Theta_a-D\theta_a)+\mathbf{2}\delta\omega^a_{.b}(\Omega^b_{.a}-\epsilon\kappa^b_{.a}+\frac{\sigma}{l^2}\theta_a\theta^b)] \tag{67}$$
$$+\int_{\partial c_5+\epsilon c_4}[\frac{4\sigma}{l^2}\delta\theta^a(D\Theta_a-\theta^b\kappa_{ab})+\frac{4\sigma}{l^2}\delta\Theta^a(D\theta_a-\epsilon\Theta_a)$$
$$+2\delta\omega^a_{.b}(\frac{\sigma}{l^2}\theta^b\Theta_a-\frac{\sigma}{l^2}\theta_a\Theta^b-D\kappa^b_{.a})+2\delta\kappa^a_b(\epsilon\kappa^b_{.a}-\Omega^b_{.a}-\frac{\sigma}{l^2}\theta_a\theta^b)]\}.$$

When $\sigma=1$ so that the gauge group is $SO(p+1,q)$, or $\sigma=-1$ so that the gauge group is $SO(p,q+1)$, the variation of the Chern-Simons integral vanishes for arbitrary variations of the forms $\theta^a,\Theta^a,\omega^a_{.b}$ and κ^a_b if and only if the variational equations on $(\partial c_5+\epsilon c_4)$ are given by the previously discussed Eq.(54), that is they correspond to the vanishing of the generalized curvature **F** there.

It should be noted that if the generalized connection **A** in the generalized gauge in which $\Theta^a=0$ had been used only the last three of the above variational equations (with $\Theta^a=0$) would have been obtained from the variational principle. However these imply, via the Bianchi identity, the first, that is $D\theta^a=0$.

Finally consider the case where σ is zero and the gauge group is $ISO(p,q)$. Now only the equations

$$\Omega^a_b=\epsilon\kappa^a_b, \qquad D\kappa^a_{.b}=0 \tag{68}$$

follow from the action principle on $(\partial c_5+\epsilon c_4)$. Then in case (i), where $\epsilon=0$, these variational equations reduce to $\Omega^a_b=0$ and $D\kappa^a_{.b}=0$. In case (ii), where ϵ is non-zero, Eq.(68) merely identifies Ω^a_b with $\epsilon\kappa^a_b$ on $(\partial c_5+\epsilon c_4)$.

Similarly the equations arising from non vanishing variations on ∂c_4 can be read off Eq.(67).

The variational equations and generalized gauge freedom change when constraints are imposed on either Θ^a or κ^a_b and this will be illustrated in the next section.

6 Actions and Lorentzian four-metrics

In this section an action and gravitational field equations for Lorentzian metrics and four dimensional space-times are considered. In the equations of the previous section the choice $p = 3, q = 1$ is made so that the four-metric η_{ab} has Lorentzian signature. Furthermore the two-forms κ^a_b in Eq. (48) and subsequent equations are taken to be of the form

$$\kappa^a_b = \kappa\varepsilon^a_{bcd}\theta^c\theta^d \tag{69}$$

where ε_{abcd} is the Levi-Civita symbol with $\varepsilon_{1234} = 1$, and κ is a non-zero constant, see also [14]. Hence the generalized connections considered here are given by Eqs.(44),(48) and (69) with $p + q + 1 = 5$. This restricts the variations considered in the previous section and the (gauge) structure group is reduced to $SO(3, 1)$.

With this choice of κ^a_b the Chern-Simons action integral, Eq.(65), becomes

$$\begin{aligned}\int_{\mathbf{c}_3}\mathbf{CS}=&\frac{1}{8\pi^2}\{\int_{\partial c_4}[8\pi^2 CS_0 - \frac{2\sigma}{l^2}\theta^a D\theta_a] \qquad (70)\\ &+\int_{\epsilon c_4+\partial c_5}[-2\kappa\epsilon^a_{.bcd}\theta^c\theta^d\Omega^b_{.a} - \frac{2\sigma\kappa}{l^2}\epsilon_{abcd}\theta^a\theta^b\theta^c\theta^d + \frac{2\sigma}{l^2}(2\Theta^a D\theta_a - \epsilon\Theta^a\Theta_a)].\end{aligned}$$

with variation

$$\begin{aligned}\delta\int_{\mathbf{c}_3}\mathbf{CS}=&\frac{1}{8\pi^2}\{\int_{\partial c_4}[\frac{4\sigma}{l^2}\delta\theta^a(\epsilon\Theta_a - D\theta_a)+\mathbf{2}\delta\omega^a_{.b}(\Omega^b_{.a} - \epsilon\kappa\epsilon^b_{.acd}\theta^c\theta^d + \frac{\sigma}{l^2}\theta_a\theta^b)] \qquad (71)\\ &+\int_{\partial c_5+\epsilon c_4}[4\delta\theta^a(\frac{\sigma}{l^2}D\Theta_a - \frac{2\kappa\sigma}{l^2}\kappa\epsilon_{abcd}\theta^c\theta^d - \kappa\epsilon^b_{a.cd}\Omega^c_{.b}\theta^d)\\ &+\frac{4\sigma}{l^2}\delta\Theta^a(D\theta_a - \epsilon\Theta_a) + 2\delta\omega^a_{.b}(\frac{\sigma}{l^2}\theta^b\Theta_a - \frac{\sigma}{l^2}\theta_a\Theta^b - \kappa\epsilon^b_{.acd}D(\theta^c\theta^d))]\}.\end{aligned}$$

The Euler-Lagrange equations now follow from Eq.(71) and the variation of the independent variables θ^a, Θ^a and ω^a_b . On $\epsilon c_4 + \partial c_5$, taken to be a four dimensional manifold S, they are

$$\frac{\sigma}{l^2}(D\Theta^a - 2\kappa\epsilon^a_{.bcd}\theta^b\theta^c\theta^d) + \kappa\epsilon^a_{.bcd}\Omega^{bc}\theta^d = 0, \tag{72}$$

$$\frac{\sigma}{l^2}(D\theta^a - \epsilon\Theta^a) = 0,$$

$$2\kappa D(\theta^a\theta^b) + \frac{\sigma}{l^2}\epsilon^{ab}_{..cd}\theta^c\Theta^d = 0.$$

When it is assumed that the one-forms $\{\theta^a\}$ are linearly independent on S and hence define a Lorentzian four-metric $ds^2 = \eta_{ab}\theta^a \otimes \theta^b$ on S it follows from these equations that, whether the constant ϵ is zero or not, $\Theta^a = 0$, ω^a_b is the Levi-Civita connection of the metric with curvature Ω^a_b and Einstein's vacuum field equations $G_{ab} = -6\frac{\sigma}{l^2}\eta_{ab}$, where G_{ab} are the components of the Einstein tensor $R^c_{acb} - \frac{1}{2}\eta_{ab}R^{cd}_{..cd}$, are satisfied in each of the three cases $\sigma = \pm 1$ or $\sigma = 0$.

Further equations will arise from non-vanishing variations on $\partial c_4 = \partial S$ and can be read off Eq.(71). When κ^a_b is given by Eq.(69) the vanishing of the generalized curvature implies that the four one-forms $\{\theta^a\}$ are not linearly independent except when $\epsilon = \sigma = 0$. In this case flatness implies that, when the one-forms are linearly independent and define a Lorentz metric on a four-manifold, this metric is flat, with ω^a_b being the flat Levi-Civita connection of the metric and with the two-forms$\{\Theta^a\}$ satisfying the equations $D\Theta^a = \kappa\epsilon^a_{bcd}\theta^b\theta^c\theta^d$.

In conclusion it should be noted that interesting observations on symmetry breaking and gravitational actions, from the point of view of ordinary Cartan connections and Cartan geometry, can be found in [31] and an approach to general relativity which is different from, but has aspects in common with, this work can be found in [32].

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