Generalized Chern-Simons action principles for gravity

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Abstract: Generalized differential forms are employed to construct generalized connections. Lorentzian four-metrics determined by certain of these connections satisfy Einstein's vacuum field equations when the connections are flat. Generalized Chern-Simons action principles with Einstein's equations as Euler-Lagrange equations are constructed by using these connections.

1 Introduction

Chern-Simons gravity and related topics have been the subject of extensive investigation since the 1980's. Following pioneering papers such as [1]- [4], most of that research has dealt with gravity in 2+1 dimensions. In three dimensions source free general relativity, with or without a cosmological constant, can be interpreted as Chern-Simons theories of the relevant structure group and the field equations correspond to the vanishing of the relevant curvature tensor. Reviews of that line of research can be found in [5] and a recent broad ranging discussion is given in [6]. Chern-Simons approaches to gravity in higher dimensions have also been discussed, although to a lesser extent, as for example, in [7], and aspects of that work are reviewed in [8] and [9]. In this paper a different approach is followed in that the formalism of generalized forms is used to construct generalized Chern-Simons actions for the four dimensional Einstein vacuum field equations with non-zero cosmological constant. This type of approach, using generalized generalized characteristic classes and generalized Chern-Simons forms, was initiated in [10] and [11] and was subsequently developed in [12]- [14]. The main new result in this paper is the construction of a Palatini type Lagrangian for gravity from a generalized Chern-Simons integral by using a generalized connection which is flat when the field equations are satisfied. In order to do this results presented in [14] are extended from type $N = 1$ to type $N = 2$ generalized forms.

In sections two and three properties of type $N = 2$ forms, generalized connections and generalized Chern-Simons integrals are outlined. Much of the material in these sections has been presented elswhere but it is included in order to make this paper reasonably self-contained. In section four attention is concentrated on type $N = 2$ generalized connections with values in the Lie algebras of $SO(p, q)$, where $p + q = 5$. These connections are defined on manifolds of dimension six or greater. When they are pulled back to (boundary) four dimensional manifolds, and a regularity condition is satisfied, they define Lorentian metrics there. In that case when the generalized curvature of the $SO(p, q)$ connections vanishes these metrics satisfy Einstein's vacuum field equations with non-zero cosmological constant. The connections are also used to construct a parametrized family of gravitational actions from generalized Chern-Simons integrals. These actions have Einstein's equations as Euler-Lagrange equations.

It will be assumed that all geometrical objects are smooth and M is an $n-$dimensional real, smooth, orientable and oriented manifold. Bold-face Roman letters are used to denote generalized forms and ordinary forms on M are usually denoted by Greek letters. Sometimes the degree of a form is indicated above it. The exterior product of any two forms, for example α and β, is written $\alpha\beta$, and as usual, any ordinary $p-$form $\overset{p}{\alpha}$, with p either negative or greater than n, is zero. The Einstein summation convention is used.

2 Type N=2 generalized differential forms

In this section the properties of type $N = 2$ differential forms on an n dimensional manifold M that are needed in this paper are reviewed. The notation of [12] and [13] is again used. Further discussion of type $N = 2$ forms can be found in [15] and [16].

Type $N = 2$ generalized forms constitute a module $\Lambda^{\bullet}_{(2)} = \Sigma^{n}_{p=-2} \Lambda^{p}_{(2)}$ and obey the same algebraic and differential relations as ordinary forms. In particular if $\overset{p}{\mathbf{r}}$and $\overset{q}{\mathbf{s}} \in \Lambda^{p}_{(2)}$ and $\Lambda^{q}_{(2)}$ are respectively a $p-$form and a $q-$form, then $\overset{p}{\mathbf{r}}\overset{q}{\mathbf{s}} = (-1)^{pq}\overset{q}{\mathbf{s}}\overset{p}{\mathbf{r}}$. A basis for type $N = 2$ generalized forms consists of any basis for ordinary forms on M augmented by a pair of linearly independent minus one-forms $\{\mathbf{m}^i\}$ $(i, j = 1, 2)$. Minus one-forms have the algebraic properties of ordinary exterior forms but are assigned a degree of minus one. They satisfy the ordinary distributive and associative laws of exterior algebra and the exterior product rule

$$\overset{p}{\rho}\mathbf{m}^i = (-1)^p\mathbf{m}^i\overset{p}{\rho};\ \mathbf{m}^i\mathbf{m}^j = -\mathbf{m}^j\mathbf{m}^i. \tag{1}$$

together with the condition of linear independence. For a given choice of $\{\mathbf{m}^i\}$, a type $N = 2$ generalized p-form, $\overset{p}{\mathbf{r}}$, can be written as

$$\overset{p}{\mathbf{r}} = \overset{p}{\rho} + \overset{p+1}{\rho}{}_i\mathbf{m}^i + \overset{p+2}{\rho}\mathbf{m}^1\mathbf{m}^2, \tag{2}$$

where $\overset{p}{\rho}, \overset{p+1}{\rho}{}_i, \overset{p+1}{\rho}{}_i, \overset{p+2}{\rho}$ are ordinary forms, respectively a $p-$form, two $(p+1)-$forms and a $(p+2)-$form. Hence, given a linearly independent pair $\{\mathbf{m}^i\}$, $\overset{p}{\mathbf{r}}$ is determined by an ordered quadruple of ordinary differential forms

$$\overset{p}{\mathbf{r}} = (\overset{p}{\rho}, \overset{p+1}{\rho}{}_1, \overset{p+1}{\rho}{}_2, \overset{p+2}{\rho}). \tag{3}$$

When it is assumed that the exterior derivative, d, of generalized forms satisfies the usual properties, in particular $d^2 = 0$, and that the exterior derivative of any basis minus one form is a type $N = 2$ generalized zero form, that is

$$d\mathbf{m}^i = \mu^i - \nu^i_j\mathbf{m}^j + \rho^i\mathbf{m}^1\mathbf{m}^2$$

where μ^i, ν^i_j and ρ^i are respectively zero- one- and two-forms, it is a straightforward matter to show that the freedom in the choice of basis minus one-forms,

$$\mathbf{m}^i \mapsto (\Lambda^{-1})^i_j\mathbf{m}^j + \Upsilon^i\mathbf{m}^1\mathbf{m}^2,$$

where the determinant of the matrix-valued function Λ is non-zero and Υ^i are one-forms, can be used to construct a basis of minus one-forms satisfying

$$d\mathbf{m}^i = \epsilon^i, \tag{4}$$

where ϵ^1 and $\epsilon^2$are constants, [12].

In this paper bases satisfying Eq.(4), with at least one of the constants non-zero, will be used. It then follows that the exterior derivative of a type $N = 2$ generalized $p-$form $\overset{p}{\mathbf{r}}$ is the $(p+1)-$form

$$\begin{aligned} d\overset{p}{\mathbf{r}} &= d\overset{p}{\rho} + (-1)^{p+1}\rho_i\epsilon^i + (d\overset{p+1}{\rho}_1 + (-1)^{p+1}\epsilon^2\overset{p+2}{\rho})\mathbf{m}^1 + \\ &+ (d\overset{p+1}{\rho}_2 + (-1)^p\epsilon^1\overset{p+2}{\rho})\mathbf{m}^2 + d\overset{p+2}{\rho}\mathbf{m}^1\mathbf{m}^2 \end{aligned} \tag{5}$$

where d is the ordinary exterior derivative when acting on ordinary forms. The exterior derivative $d : \Lambda^p_{(2)}(M) \to \Lambda^{p+1}_{(2)}(M)$ is an anti-derivation of degree one,

$$\begin{aligned} d(\overset{p}{\mathbf{r}}\overset{q}{\mathbf{s}}) &= (d\overset{p}{\mathbf{r}})\overset{q}{\mathbf{s}} + (-1)^p\overset{p}{\mathbf{r}}d\overset{q}{\mathbf{s}}, \\ d^2 &= 0. \end{aligned} \tag{6}$$

and $(\Lambda^{\bullet}_{(2)}(M), d)$ is a differential graded algebra.

If φ is a smooth map between manifolds P and M, $\varphi : P \to M$, then the induced map of type $N = 2$ generalized forms, $\varphi^*_{(2)} : \Lambda^p_{(2)}(M) \to \Lambda^p_{(2)}(P)$, is the linear map defined by using the standard pull-back map, φ^*, for ordinary forms

$$\varphi^*_{(2)}(\overset{p}{\mathbf{r}}) = \varphi^*(\overset{p}{\rho}) + \varphi^*(\overset{p+1}{\rho}_i)\mathbf{m}^i + \varphi^*(\overset{p+2}{\rho})\mathbf{m}^1\mathbf{m}^2, \tag{7}$$

and $\varphi^*_{(2)}(\overset{p}{\mathbf{r}}\overset{q}{\mathbf{s}}) = \varphi^*_{(2)}(\overset{p}{\mathbf{r}})\varphi^*_{(2)}(\overset{q}{\mathbf{s}})$. Hence $\varphi^*_{(2)}(\mathbf{m}^i) = \mathbf{m}^i$.

Integration is defined using polychains, [13]. A $p-$polychain of type $N = 2$ in M, denoted $\mathbf{c}_p$ is an ordered quadruple of ordinary (real, singular) chains in M

$$\mathbf{c}_p = (c_p, c^1_{p+1}, c^2_{p+1}, c_{p+2}), \tag{8}$$

where c_p is an ordinary $p-$chain, c^1_{p+1} and c^2_{p+1} are ordinary $p+1-$chains and c_{p+2} is and ordinary ordinary $p+2-$chain. The ordinary chains boundaries are denoted by ∂ and the boundary of the polychain $\mathbf{c}_p$ is the $(p-1)-$polychain $\partial\mathbf{c}_p$ given by

$$(\partial c_p, \partial c^1_{p+1} + (-1)^p\epsilon^1 c_p, \partial c^2_{p+1} + (-1)^p\epsilon^2 c_p, \partial c_{p+2} + (-1)^p\epsilon^2 c^1_{p+1} + (-1)^{p-1}\epsilon^1 c^2_{p+1}), \tag{9}$$

and

$$\partial^2 \mathbf{c}_p = 0. \tag{10}$$

When $N = 2$ the integral of a generalized form $\overset{p}{\mathbf{r}}$ over a polychain $\mathbf{c}_p$ is

$$\int_{\mathbf{c}_p} \overset{p}{\mathbf{r}} = \int_{c_p} (\overset{p}{\rho} + \int_{c^1_{p+1}} \overset{p+1}{\rho}{}_1 + \int_{c^2_{p+1}} \overset{p+1}{\rho}{}_2 + \int_{c_{p+2}} \overset{p+2}{\rho}. \tag{11}$$

and Stokes' theorem applies

$$\int_{\mathbf{c}_p} d\overset{p-1}{\mathbf{r}} = \int_{\partial \mathbf{c}_p} \overset{p-1}{\mathbf{r}}. \tag{12}$$

Under a change of basis minus one-forms $\mathbf{m}^1$ and $\mathbf{m}^2$

$$\mathbf{m}^i \mapsto T^i_j \mathbf{m}^j, \tag{13}$$

where $\left(T^i_j\right)$ is a constant matrix and $T = \det\left(T^i_j\right)$ is non-zero,

$$\epsilon^i \mapsto T^i_j \epsilon^j. \tag{14}$$

and the components of $\overset{p}{\mathbf{r}}$ transform as

$$(\overset{p}{\rho}, \overset{p+1}{\rho}{}_1, \overset{p+1}{\rho}{}_2, \overset{p+2}{\rho}) \mapsto (\overset{p}{\rho}, (T^{-1})^j_1 \overset{p+1}{\rho}{}_j, (T^{-1})^j_2 \overset{p+1}{\rho}{}_j, (T^{-1}\overset{p+2}{\rho}). \tag{15}$$

The form of the right hand side of Eq.(11) is then preserved if the components of $\mathbf{c}_p$ transform as

$$(c_p, c^1_{p+1}, c^2_{p+1}, c_{p+2}) \mapsto (c_p, (T)^1_j c^j_{p+1}, (T)^2_j c^j_{p+1}, T c_{p+2}). \tag{16}$$

In the following sections the usual definitions will be extended to admit complex coefficients and the complex (and complex conjugate) combinations

$$\begin{aligned} \mathbf{m} &= \epsilon^{-1}(\mathbf{m}^1 + i\mathbf{m}^2), \\ \overline{\mathbf{m}} &= \overline{\epsilon}^{-1}(\mathbf{m}^1 - i\mathbf{m}^2), \\ \epsilon &= \epsilon^1 + i\epsilon^2, \end{aligned} \tag{17}$$

which satisfy

$$d\mathbf{m} = d\overline{\mathbf{m}} = 1, \tag{18}$$

will be used.

Just as the algebra and differential calculus of ordinary differential forms on M can be expressed in terms of functions and vector fields on the reverse parity tangent bundle, ΠTM, of M, [17], [18], generalized forms can be represented in terms of functions and vector fields on the Whitney sum of ΠTM and a trivial reverse parity line bundle over M. For type $N = 2$ forms the latter is a trivial bundle with fibre $\mathbb{R}^2$ replaced by $\mathbb{R}^{0|2}$. Further details about this and type N generalized form-valued vector fields are in [19].

3 Type N=2 generalized connections

A generalized connection $\mathbf{A}$ with values in the Lie algebra, $\mathfrak{g}$, of a matrix Lie group G is defined in essentially the same way as ordinary connections, as for example described in [20], except that ordinary forms are replaced by generalized forms. In this paper it will be sufficient to use matrix representations of Lie groups and Lie algebras and connections will be represented by matrix-valued generalized forms on M. The primary focus will be on real generalized connection forms. $\mathbf{A}$, but generalization to complex generalized connection forms is trivial.

Under a gauge transformation by $g : M \rightarrow G$ a $\mathfrak{g}$-valued generalized connection one-form transforms in the usual way

$$\mathbf{A} \rightarrow (g^{-1})dg + (g^{-1})\mathbf{A}g \tag{19}$$

The generalized curvature two-form $\mathbf{F}$ is the generalized two-form

$$\mathbf{F} = \mathbf{dA} + \mathbf{AA}, \tag{20}$$

Under the transformation in Eq.(19)

$$\mathbf{F} \rightarrow (g^{-1})\mathbf{F}g. \tag{21}$$

In terms of the complex basis introduced in Eq.(17) above a type $N = 2$ connection one-form can be written as

$$\mathbf{A} = \alpha - \phi\mathbf{m} - \overline{\phi}\overline{\mathbf{m}} + i\chi\mathbf{m}\overline{\mathbf{m}}, \tag{22}$$

where for real $\mathbf{A}$, α and τ are, respectively, a real ordinary $\mathfrak{g}$-valued one-form and three-form on M and ϕ is an ordinary complex $\mathfrak{g}$-valued two-form with complex conjugate $\overline{\phi}$. The curvature two-form is

$$\mathbf{F} = \mathcal{F} - \phi - \overline{\phi} - (D\phi - i\chi)\mathbf{m} - (D\overline{\phi} + i\chi)\overline{\mathbf{m}} + (iD\chi + \phi\overline{\phi} - \overline{\phi}\phi)\mathbf{m}\overline{\mathbf{m}}, \tag{23}$$

$$\mathcal{F} = d\alpha + \alpha\alpha,$$
$$D\overset{p}{\rho} = d\overset{p}{\rho} + \alpha\overset{p}{\rho} + (-1)^{p+1}\overset{p}{\rho}\alpha.$$

The generalized curvature two-form is zero if and only if the generalized connection can be written in the form

$$\mathbf{A} = \alpha - \frac{1}{2}(\mathcal{F} + i\tau)\mathbf{m} - \frac{1}{2}(\mathcal{F} - i\tau)\overline{\mathbf{m}} + \frac{i}{2}D\tau\mathbf{m}\overline{\mathbf{m}}, \tag{24}$$

where, for real $\mathbf{A}$, τ is an arbitrary real $\mathfrak{g}-$valued two-form.

Henceforth in this paper it will be assumed, unless stated otherwise, that a connection $\mathbf{A}$ has zero trace, $Tr\mathbf{A} = 0$.

The generalized Chern-Pontrjagin class is determined by a generalized four-form $\mathbf{CP}$

$$\mathbf{CP} = \frac{1}{8\pi^2}Tr(\mathbf{FF}), \tag{25}$$

which is equal to the exterior derivative of the generalized Chern-Simons three-form $\mathbf{CS}$

$$\mathbf{CS} = \frac{1}{8\pi^2}Tr(\mathbf{AF} - \frac{1}{3}\mathbf{AAA}). \tag{26}$$

By Stokes' theorem, Eq.(12), for a polychain $\mathbf{c}_4$

$$\int_{\mathbf{c}_4}\mathbf{CP} = \int_{\partial\mathbf{c}_4}\mathbf{CS} \tag{27}$$

and when $\mathbf{c}_3 = \partial\mathbf{c}_4$ under the gauge transformation given by Eq.(19)

$$\int_{\mathbf{c}_3}\mathbf{CS} \rightarrow \int_{\mathbf{c}_3}\mathbf{CS}.$$

Using the generalized connection $\mathbf{A}$ given in Eq.(22)

$$\mathbf{CS} = \Pi + \Delta\mathbf{m} + \overline{\Delta}\overline{\mathbf{m}} + \Xi\mathbf{m}\overline{\mathbf{m}}$$

where

$$\Pi = \frac{1}{8\pi^2}Tr\{\alpha\mathcal{F} - \frac{1}{3}\alpha\alpha\alpha - \alpha(\phi + \overline{\phi})\}, \tag{28}$$
$$\Delta = \frac{1}{8\pi^2}Tr\{\phi(\phi + \overline{\phi})) + d(\alpha\phi) - 2\phi\mathcal{F} + i\alpha\chi\},$$
$$\overline{\Delta} = \frac{1}{8\pi^2}Tr\{\overline{\phi}(\phi + \overline{\phi})) + d(\alpha\overline{\phi}) - 2\overline{\phi}\mathcal{F} - i\alpha\chi\},$$
$$\Xi = \frac{1}{8\pi^2}Tr\{\overline{\phi}D\phi - \phi D\overline{\phi} + 2i(\mathcal{F} - \phi - \overline{\phi})\chi\},$$

and

$$\int_{\partial\mathbf{c}_4}\mathbf{CS} = \int_{\partial c_4}\Pi + \int_{c_4}(\Delta + \overline{\Delta}) + \tag{29}$$
$$+ (\epsilon\overline{\epsilon})^{-1}[\int_{\partial c_5^1}(\overline{\epsilon}\Delta + \epsilon\overline{\Delta}) + i\int_{\partial c_5^2}(\overline{\epsilon}\Delta - \epsilon\overline{\Delta}) + 2\int_{\partial c_6 + \epsilon^2 c_5^1 - \epsilon^1 c_5^2}\Xi].$$

In the following section type $N = 2$ Chern-Simons integrals for a boundary polychain $\mathbf{c}_3 = \partial\mathbf{c}_4$ will be used as action integrals.

4 Connections, metrics and gravity

In this section certain type $N = 2$ generalized connections $\mathbf{A}$, on an $n \geqq 6$ dimensional manifold M will be considered. The general formalism follows that in [14] where type $N = 1$ forms were used. The connections will be represented by a 5×5 matrix-valued generalized one-forms

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^a_b & -\sigma\mathbf{A}^a \\ \mathbf{A}_b & 0 \end{pmatrix}. \tag{30}$$

and take values in the Lie algebra, $\mathfrak{g}$, of G, where G is $SO(3+1,1)$ when $\sigma = 1$; $SO(3,1+1)$ when $\sigma = -1$ and $ISO(3,1)$ when $\sigma = 0$. In the first two cases, which will be of primary interest here, the metric is given by the 5×5 matrix

$$\begin{pmatrix} \eta_{ab} & 0 \\ 0 & \sigma \end{pmatrix}, \tag{31}$$
$$\eta_{ab} = diag(-1, 1, 1, 1)$$

and $\mathbf{A}_b = \eta_{ba}\mathbf{A}^a$. Latin indices ranging and summing from 1 to 4. The generalized curvature of $\mathbf{A}$ is given by

$$\mathbf{F} = d\mathbf{A} + \mathbf{A}\mathbf{A} = \begin{pmatrix} \mathbf{F}^a_b & -\sigma\mathbf{F}^a \\ \mathbf{F}_b & 0 \end{pmatrix}, \tag{32}$$

where

$$\begin{aligned} \mathbf{F}^a_b &= \mathbf{d}\mathbf{A}^a_b + \mathbf{A}^a_c\mathbf{A}^c_b - \sigma\mathbf{A}^a\mathbf{A}_b, \\ \mathbf{F}^a &= d\mathbf{A}^a + \mathbf{A}^a_b\mathbf{A}^b,\ \mathbf{F}_b = \eta_{bc}\mathbf{F}^c. \end{aligned} \tag{33}$$

Now let

$$\begin{aligned} \mathbf{A}^a_b &= \omega^a_b - ({}^-\Omega^a_b - \frac{\sigma}{l^2}{}^-\Sigma^a_b)\mathbf{m} - ({}^+\Omega^a_b - \frac{\sigma}{l^2}{}^+\Sigma^a_b)\overline{\mathbf{m}}, \\ \mathbf{A}^a &= \frac{1}{l}\theta^a, \end{aligned} \tag{34}$$

where l is a non-zero constant, $\omega_{ab} = -\omega_{ba}$ are ordinary real one-forms and θ^a are four real ordinary one-forms on M. Furthermore if

$$\Omega^a_b = d\omega^a_b + \omega^a_c\omega^c_b \tag{35}$$

denotes the curvature of ω^a_b (regarded as an ordinary connection) then ${}^\pm\Omega_{ab}$ are respectively the $so(3,1)$ self-dual and anti-self dual parts of Ω^a_b. That is

$${}^\pm\Omega_{ab} = \frac{1}{2}(\Omega_{ab} \mp i^*\Omega_{ab}), \tag{36}$$

where ${}^*\Omega_{ab} = \frac{1}{2}\varepsilon_{abcd}\Omega^{cd}$ and the totally skew symmetric Levi-Civita symbol satisfies $\varepsilon_{1234} = 1$. Furthermore $\Sigma^{ab} = \theta^a\theta^b$ and ${}^\pm\Sigma^{ab}$ denote its $so(3,1)$ self and anti-self dual parts ${}^\pm\Sigma^{ab} = \frac{1}{2}(\Sigma^{ab} \mp i^*\Sigma^{ab})$.

Then for this connection the curvature $\mathbf{F}$ in Eq.(32) is given by

$$\begin{aligned} \mathbf{F}^a_b &= \frac{\sigma}{l^2}(D^-\Sigma^a{}_b\mathbf{m} + D^+\Sigma^a{}_b\overline{\mathbf{m}}), \\ \mathbf{F}^a &= \frac{1}{l}[D\theta^a + ({}^-\Omega^a_b - \frac{\sigma}{l^2}{}^-\Sigma^a_b)\theta^b\mathbf{m} + ({}^+\Omega^a_b - \frac{\sigma}{l^2}{}^+\Sigma^a_b)\theta^b\overline{\mathbf{m}}], \end{aligned} \tag{37}$$

Here the covariant exterior derivative with respect to ω^a_b is denoted D so that

$$\begin{aligned} D\theta^a &= d\theta^a + \omega^a_b\theta^b, \\ D\,{}^\pm\Sigma^a_b &= d^\pm\Sigma^a_b + \omega^a_c\,{}^\pm\Sigma^c_b - {}^\pm\Sigma^a_c\omega^c_b \\ &= d^\pm\Sigma^a_b +{}^\pm\omega^a_c\,{}^\pm\Sigma^c_b -{}^\pm\Sigma^a_c\,{}^\pm\omega^c_b, \end{aligned} \tag{38}$$

where ${}^{\pm}\omega^a_b = \frac{1}{2}(\omega^{ab} \mp i^*\omega^{ab})$ respectively denote the $so(3,1)$ self-dual and anti-self dual parts of ω^a_b.

The generalized connection $\mathbf{A}$ is flat if and only if $\mathbf{F} = 0$ and then

$$D\theta^a = 0, \tag{39}$$
$$({}^{-}\Omega^a_b - \frac{\sigma}{l^2}\ {}^{-}\Sigma^a_b)\theta^b = 0,$$
$$({}^{+}\Omega^a_b - \frac{\sigma}{l^2}\ {}^{+}\Sigma^a_b)\theta^b = 0.$$

Suppose now that when the 4 ordinary one-forms $\{\theta^a\}$ are pulled back to a four dimensional sub-manifold $S \subseteq M$ they are linearly independent and so form an orthonormal basis for a Lorentzian metric $ds^2 = \eta_{ab}\theta^a \otimes \theta^b$ there. Then when Eq.(39) is satisfied it follows that this metric satisfies Einstein's vacuum field equations on S with cosmological constant $\Lambda = \frac{3\sigma}{l^2}$.

In passing it is interesting to note the sub-case of the complex connection ${}_{asd}\mathbf{A}$ given by Eq.(30) with $\sigma = 0$ and

$${}_{asd}\mathbf{A}^a_b = {}^{-}\omega^a_b - {}^{-}\Omega^a_b\mathbf{m};\ {}_{asd}\mathbf{A}_a = \theta_a;\ {}^{-}\Omega^a_b = d\ {}^{-}\omega^a_b + {}^{-}\omega^a_c\ {}^{-}\omega^c_b \tag{40}$$

with curvature ${}_{asd}\mathbf{F}$ given by

$${}_{asd}\mathbf{F}^a_b = 0, \tag{41}$$
$${}_{asd}\mathbf{F}_a = d\theta_a + {}^{-}\omega_{ab}\theta^b + {}^{-}\Omega_{ab}\theta^b\mathbf{m},$$

If this generalized connection is flat the (complex) metric, $ds^2 = \eta_{ab}\theta^a\otimes\theta^b$, determined analogously to the real four-metric above, is half flat with anti-self dual curvature ${}^{-}\Omega^a_b$.

For the connection and curvature given by Eqs.(30), (32) and (33) the generalized Chern-Pontrjagin and Chern-Simons forms are

$$\mathbf{CP}=\frac{1}{8\pi^2}[\mathbf{F}^a_b\mathbf{F}^b_a - 2\sigma\mathbf{F}^a\mathbf{F}_a] \tag{42}$$
$$\mathbf{CS}=\frac{1}{8\pi^2}[\mathbf{A}^a_b\mathbf{F}^b_a - \frac{1}{3}\mathbf{A}^a_b\mathbf{A}^b_c\mathbf{A}^c_a - 2\sigma\mathbf{A}^a\mathbf{F}_a + \sigma\mathbf{A}^a_b\mathbf{A}^b\mathbf{A}_a].$$

Computation, using Eqs.(34) and (37), gives

$$\mathbf{CS}=\frac{1}{8\pi^2}\{-\frac{1}{3}\omega^a_b\omega^b_c\omega^c_a + \frac{\sigma}{l^2}\omega^a_b\theta^b\theta_a - \frac{2\sigma}{l^2}\theta_a D\theta^a + \tag{43}$$
$$+[\frac{1}{3}d(\ {}^{-}\omega^a_{.b}\ {}^{-}\omega^b_c\ {}^{-}\omega^c_a - 3\frac{\sigma}{l^2}\ {}^{-}\omega^a_b\ {}^{-}\Sigma^b_{.a}) - \frac{2\sigma}{l^2}\ {}^{-}\Omega_{ab}\Sigma^{ab} + \frac{i\sigma^2}{4l^4}\varepsilon_{abcd}\theta^a\theta^b\theta^c\theta^d]\mathbf{m}+$$
$$+[\frac{1}{3}d(\ {}^{+}\omega^a_b\ {}^{+}\omega^b_c\ {}^{+}\omega^c_a - 3\frac{\sigma}{l^2}\ {}^{+}\omega^a_b\ {}^{+}\Sigma^b_{.a}) - \frac{2\sigma}{l^2}\ {}^{+}\Omega_{ab}\Sigma^{ab} + \frac{i\sigma^2}{4l^4}\varepsilon_{abcd}\theta^a\theta^b\theta^c\theta^d]\overline{\mathbf{m}}\}.$$

Using Eq.(17) and integrating over a boundary polychain $\mathbf{c}_3 = \partial \mathbf{c}_4$, as in Eq.(9) with $p = 4$, gives

$$\int_{\mathbf{c}_3} \mathbf{CS} = \frac{\sigma}{4\pi^2 l^2}\{-\int_{\partial c_4} \theta^a D\theta_a - \int_{c_4} \Omega_{ab}\Sigma^{ab} + \tag{44}$$
$$-\int_{\partial c_5^1} [i\kappa^2({}^{+}\Omega_{ab}\Sigma^{ab} - {}^{-}\Omega_{ab}\Sigma^{ab} - \frac{\sigma}{4l^2}\varepsilon_{abcd}\theta^a\theta^b\theta^c\theta^d) + \kappa^1\Omega_{ab}\Sigma^{ab}]$$
$$+\int_{\partial c_5^2} [i\kappa^1({}^{+}\Omega_{ab}\Sigma^{ab} - {}^{-}\Omega_{ab}\Sigma^{ab} - \frac{\sigma}{4l^2}\varepsilon_{abcd}\theta^a\theta^b\theta^c\theta^d) - \kappa^2\Omega_{ab}\Sigma^{ab}]\}.$$

where

$$\kappa^1 = \frac{\epsilon^1}{\epsilon\bar{\epsilon}};\ \kappa^2 = \frac{\epsilon^2}{\epsilon\bar{\epsilon}}.$$

Now consider this expression as a generalized Chern-Simons action integral and the case where the four ordinary one-forms $\{\theta^a\}$ are linearly independent on the four dimensional sub-manifolds (chains), $c_4, \partial c_5^1$and ∂c_5^2. Since the one forms constitute an orthonormal basis there for a Lorentzian metric $ds^2 = \eta_{ab}\theta^a \otimes \theta^b$ this action can now be rewritten as

$$\int_{\mathbf{c}_3} \mathbf{CS} = \frac{\sigma}{4\pi^2 l^2}\{-\int_{\partial c_4} \theta^a D\theta_a - \int_{c_4} \Omega_{ab}\theta^a\theta^b + \tag{45}$$
$$-\int_{\partial c_5^1} [\kappa^2(R - 2\Lambda)V + \kappa^1\Omega_{ab}\Sigma^{ab}] +$$
$$+\int_{\partial c_5^2} [\kappa^1(R - 2\Lambda)V - \kappa^2\Omega_{ab}\Sigma^{ab}]\},$$

where on $\partial c_5^1$and ∂c_5^2

$$\Omega^a_b = \frac{1}{2}R^a_{bcd}\theta^c\theta^d, R = \eta^{bd}R^a_{bad}, \tag{46}$$
$$V = \theta^1\theta^2\theta^3\theta^4, \Lambda = \frac{3\sigma}{l^2}$$

and the action terms there correspond to the usual first order (Palatini) Einstein-Hilbert action terms augmented by the term proportional to $\Omega_{ab}\Sigma^{ab}$, [21]

On the boundary manifolds $\partial c_5^1$and ∂c_5^2 the Euler-Lagrange equations are Einstein's vacuum field equations, $(G_{ab} + \Lambda\eta_{ab})\theta^a \otimes \theta^b = 0$, with cosmological constant $\Lambda = \frac{3\sigma}{l^2}$. In addition the variation of the first two terms gives

$$\delta(-\int_{\partial c_4} \theta^a D\theta_a - \int_{c_4} \Omega_{ab}\theta^a\theta^b) = -2\int_{\partial c_4} \delta\theta^a D\theta_a + \int_{c_4} 2\delta\theta^b\Omega_{ab}\theta^a - \delta\omega^a_b D(\theta^a\theta_b). \tag{47}$$

When the geometry of the submanifolds is specified in more detail these results can be interpreted more completely.

In conclusion it should be noted that the use of an anti-deSitter/deSitter connection, invariant only under the Lorentz group, in an action principle with the Einstein field equations as Euler Lagrange equations dates from the late 1970's, [22], [23] and[24]. However the approach, initiated in these papers, which has recently been interpreted in terms of Cartan geometries, [25], is different from the one taken here.

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