Generalized differential forms

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Abstract:

The algebra and calculus of generalized differential forms are reviewed and developed. Bases of minus one-forms are studied and used in the investigation of groups of generalized forms and generalized connections. Different representations of generalized forms are discussed. Physical and mathematical applications of generalized forms are presented in a number of examples.

1 Introduction

In recent years an extension of E. Cartan's formulation of the algebra and calculus of differential forms to what have been termed *generalized differential forms* has been developed. Ordinary forms of different degrees are combined into more general objects in such a way that these objects, the *generalized forms*, obey the basic algebraic and differential rules of the ordinary exterior algebra and Cartan calculus. However there are some differences from standard exterior algebra and calculus, for example the analogue of the Poincaré lemma can be different. Furthermore the definitions allow generalized forms of negative degree. Forms of this type were first introduced by Sparling in an attempt to construct twistor spaces for real analytic space-times with Ricci-flat Lorentzian metrics. The use of negative degree forms enabled him to overcome the standard obstacle to the construction of such spaces, at least to the extent that he was able to construct what he termed 'abstract twistor spaces' [1], [2]. Subsequently there have been interesting applications of generalized forms to various field theories, including BF theory, Yang-Mills and gravity, [3], [4]. Recently analogous ideas have been explored in the dual context, and the notion of a generalized vector field has been introduced and studied, [5]. In a series of papers both the mathematical formalism and a variety of applications of generalized forms to physical theories have been developed [6]-[9]. Here the mathematical results contained in this series are extended and used. Generalized forms, like ordinary differential forms, can be used as a tool in diverse areas of geometrically related physics. This paper contains new applications in a number of different physically important contexts.

Generalized differential forms on an n dimensional manifold M may be described in terms of their *type,* labelled by a non-negative integer N. Generalized p-forms of type $N = 0$ are just ordinary p-forms. Generalized p-forms of type $N = 1$ may be described in terms of ordered pairs of ordinary p- and $(p+1)$-forms. A p-form of type N may be described by an ordered multiplet of ordinary forms of degrees p to $p+N$. Forms of higher type may be defined iteratively. A p-form of type N,where the integer $N \geq 1$, may be defined in terms of an ordered pair of p- and (p+1)-forms of type $(N - 1)$. It follows from the definitions that generalized forms of degree p, where $n \geq p \geq -N$, are permitted. Generalized forms of all different degrees and types obey the same basic rules of exterior multiplication and differentiation as those governing the algebra and calculus of ordinary differential forms. However, when $N > 0$ there are some differences from the standard results for ordinary, that is type $N = 0$, forms. For instance, as mentioned previously, the analogue of the Poincaré lemma can be different when N is greater than zero.

In the second section the essential aspects of the basic formalism are presented and explored more fully than previously. Here generalized forms are expressed in terms of expansions in bases. Type N forms are assumed to admit unique expansions in terms of ordinary forms and negative degree forms, the latter being constructed from N linearly independent minus one-forms. Such bases are not unique and their properties, and changes of bases, are discussed. The multiplet description of generalized forms, mentioned above, corresponds to considering expansions in terms of one fixed basis of negative degree forms. In previous work it has been assumed (implicitly or explicitly) that the exterior derivatives of the basis minus one-forms are constant ordinary zero-forms. This is a reasonable choice because the exterior derivative of a $p = -1$ form must be a zero-form with vanishing derivative. A more general initial assumption is that the exterior derivative is a nilpotent differential operator, with the usual exterior derivative properties, and that the exterior derivatives of type N basis minus one-forms are type N generalized zero-forms. The consequences of this assumption are explored in detail when N is one and two. It is found that the exterior derivative determines, and is determined by, a closed differential ideal of ordinary forms. Different classes of solutions of the differential ideal determine different exterior derivatives. It is then shown how to construct two distinct exterior derivatives. The first exterior derivative admits bases of minus one-forms all whose members have vanishing exterior derivative. The second exterior derivative admits bases of minus one-forms with the property that the exterior derivative of only one basis minus one form is non-zero. Its value can be chosen to be one. The construction of the first type of basis is local in contrast to the construction of the second. Such bases are termed canonical bases. When $N = 1$ the second type is unique, but canonical bases are not unique when $N > 1$. Throughout the paper calculations are carried out using both exterior derivatives but the second is more interesting as far as the applications are concerned. Since calculations are simplest when carried out using canonical bases these are always used in subsequent sections. Section three is devoted to algebraic considerations. The basic formalism of groups of matrix-valued generalized forms is presented more generally than previously. The special orthogonal group of generalized forms and its action as a transformation group are provided as concrete examples. Included here are the generalizations of the Lorentz group and Lorentz transformations that arise naturally when generalized forms are used. The calculations can be extended straightforwardly to other groups. The fourth section contains an outline of the local properties of generalized connections and the construction of generalized characteristic classes. Examples dealing with type $N = 1$ connections for metric-connection geometries are given, and it is shown that the classical Dirac equation corresponds to the vanishing of the covariant exterior derivative of a spinor valued generalized form. The generalized Euler class, the generalized first Pontrjagin and the generalized second Chern classes in four dimensions are introduced and used to construct Lagrangians for Einstein's vacuum field equations, with and without a cosmological constant, and for the Yang-Mills field. The use of the generalized Euler class gives a Lagrangian which necessarily contains both a non-zero cosmological constant and a Gauss-Bonnet term. The non-trivial role that the latter term can play, even in four dimensions, has been investigated in recent years, for example in references [12] and [13]. The calculations using generalized characteristic classes were motivated by previous investigations of 'generalized topological field theory' and the use of the generalized second Chern class to construct Lagrangians for various physically interesting field theories, [3], [4]. Finally, in an appendix, two representations of generalized forms solely in terms of ordinary forms are exhibited. In the first, generalized forms are represented by matrices with ordinary forms of different degrees as entries. The exterior product of generalized forms corresponds to the matrix product, and the exterior derivative corresponds to the derivative of these matrices by a new nilpotent differential operator. This representation has been presented before, [8], but the discussion here includes some small differences and minor corrections. The second representation[1] replaces the exterior derivative of a type N generalized p-form on M by the ordinary exterior derivative of an ordinary $(p+N)-$form on (locally) $M \times R^N$. An application of this result is made to the mathematically and physically interesting Beltrami vector fields. These vector fields have been widely discussed in a number of different contexts such as magnetohydrodynamics and plasma physics; various examples are included in references [14] and [15]. The second representation is used to show that Beltrami vector fields, on three dimensional manifolds with Euclidean signature metrics, determine symplectic structures on four

[1]private communication from P-A Nagy

dimensional manifolds.

Details of investigations of $\Lambda_{(N)}(M) = \oplus_{p=-N}^{p=n} \Lambda_{(N)}^{p}(M)$ the module of generalized forms of type N, will sometimes be confined to the cases where N is less than or equal to two. There can be differences between the cases where $N = 1$ and $N > 1$, and these are well illustrated by exhibiting the results for $N = 1$ and $N = 2$. Doing this will keep the notation simple and it is easy to see what are the appropriate generalizations of the calculations and arguments to forms of higher type. In order to make this paper reasonably self contained a selection of salient results from previous papers are reviewed. There are a small number of obvious changes of notation from previous papers, but most of the conventions of references [6]-[9] are retained. In particular bold-face Roman letters are again used to denote generalized forms, including, in this paper, basis minus one-forms. Ordinary forms are again denoted by Greek letters, and, where it is useful, the degree of a form is indicated above it. The exterior product of any two forms, α and β, is written $\alpha\beta$. Any ordinary form $\overset{q}{\alpha}$, with q either negative or greater than n, the dimension of the manifold, is zero. Sometimes it is helpful to indicate the type of a generalized form by a subscript, for example by writing $\overset{p}{\mathbf{a}}_{(N)}$, where $\overset{p}{\mathbf{a}}_{(N)} \in \Lambda_{(N)}^{p}(M)$ - the module of generalized p-forms of type N. When the type or degree is obvious from the context this will not be indicated so explicitly. The forms and manifold may be real or complex.

2 Basic properties and formalism

This section is devoted to a discussion of the exterior algebra and calculus of generalized differential forms, on a manifold M of dimension n. A generalized p-form of type $N = 0$ is an ordinary differential p-form. The exterior algebra of type $N = 0$ forms is the ordinary exterior algebra. Any generalized p-form of type $N \geq 1$ can, by assumption, be uniquely expressed in terms of a basis constructed from any basis of ordinary forms on M augmented by N linearly independent minus one-forms, $\{\mathbf{m}^i\}$ $(i = 1, 2, .., N)$. These latter quantities are assigned the same algebraic properties as ordinary exterior p-forms, apart from p taking the value minus one in standard formulae. In particular they are assumed to satisfy the ordinary distributive and associative laws of exterior algebra; the product rules, $\mathbf{m}^i\mathbf{m}^j = -\mathbf{m}^j\mathbf{m}^i$ and $\overset{p}{\alpha}\mathbf{m}^i = (-1)^p\mathbf{m}^i\overset{p}{\alpha}$, where $\overset{p}{\alpha}$ is any ordinary p-form; together with the condition of linear independence, $\mathbf{m}^1\mathbf{m}^2....\mathbf{m}^N \neq 0$. A generalized p-form of type $N \geq 1$, $\overset{p}{\mathbf{a}}_{(N)} \in \Lambda^p_{(N)}(M)$, is thus a geometrical object with a unique expansion of the form

$$\overset{p}{\mathbf{a}}_{(N)} = \overset{p}{\alpha} + \overset{p+1}{\alpha}{}_{i_1}\mathbf{m}^{i_1} + \frac{1}{2!}\overset{p+2}{\alpha}{}_{i_1 i_2}\mathbf{m}^{i_1}\mathbf{m}^{i_2} + ... + \frac{1}{j!}\overset{p+j}{\alpha}{}_{i_1....i_j}\mathbf{m}^{i_1}.....\mathbf{m}^{i_j} + ... \tag{1}$$

$$+ \frac{1}{N!}\overset{p+N}{\alpha}{}_{i_1....i_N}\mathbf{m}^{i_1}.....\mathbf{m}^{i_N}.$$

Here $\overset{p}{\alpha}, \overset{p+1}{\alpha}_{i_1}, ..., \overset{p+j}{\alpha}{}_{i_1....i_j} = \overset{p+j}{\alpha}{}_{[i_1....i_j]}, ..., \overset{p+N}{\alpha}{}_{i_1....i_N}$ are, respectively, ordinary p-, $(p+1)$-,..., $(p+j)$-,...$(p+N)$- ordinary forms; j ranges from 1 to N and $i_{1},...i_j, ..., i_N$ range and sum over $1, 2, ..\ N$. It then follows that generalized forms satisfy the usual distributive and associative laws of exterior algebra, together with the product rule $\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}} = (-1)^{pq}\overset{q}{\mathbf{b}}\overset{p}{\mathbf{a}}$, and generalized exterior forms are defined for $n \geq p \geq -N$. With the basis of minus one forms fixed, generalized p-forms can be identified with ordered tuples of ordinary forms - the approach that has been used in most previous papers. For each $p = -N, -N+1, .., n-1, n$, $\Lambda^p_{(N)}(M)$ is a module over the ring of function on M and $\Lambda_{(N)}(M) = \oplus_{p=-N}^{p=n}\Lambda^p_{(N)}(M)$ is a graded algebra with pointwise exterior product $\Lambda^p_{(N)}(M) \times \Lambda^q_{(N)}(M) \rightarrow \Lambda^{p+q}_{(N)}(M)$ as defined above.

In calculations it will be useful to use the fact, noted in reference [8], that any p-form of type $N \geq 1$ can be expressed as a of a pair generalized forms of type $N-1$, that is

$$\overset{p}{\mathbf{a}}_{(N)} \equiv \overset{p}{\mathbf{a}}_{(N-1)} + \overset{p+1}{\mathbf{a}}{}_{(N-1)}\mathbf{m}^N, \tag{2}$$

where, when $N > 1$, each of the type $(N-1)$ forms can be expressed in terms of an ordered pair of $(N-2)$ forms, and so on. By using this type of expression iteratively, formulae for higher order forms can often be quickly deduced from results for type $N = 1$ forms. If $\overset{q}{\mathbf{b}}_N = \overset{q}{\mathbf{b}}_{(N-1)} + \overset{q+1}{\mathbf{b}}{}_{(N-1)}\mathbf{m}^N$ is a q-form of type $N \geq 1$, then the exterior product of $\overset{p}{\mathbf{a}}_{(N)}$ and $\overset{q}{\mathbf{b}}_{(N)}$ is the p+q-form of type N given (recursively) by

$$\overset{p}{\mathbf{a}}_{(N)}\overset{q}{\mathbf{b}}_{(N)} = \overset{p}{\mathbf{a}}_{(N-1)}\overset{q}{\mathbf{b}}_{(N-1)} + [\overset{p}{\mathbf{a}}_{(N-1)}\overset{q+1}{\mathbf{b}}{}_{(N-1)} + (-1)^q\overset{p+1}{\mathbf{a}}{}_{(N-1)}\overset{q}{\mathbf{b}}_{(N-1)}]\mathbf{m}^N. \tag{3}$$

A general change of basis minus one-forms, $\mathbf{m}^i \mapsto \widetilde{\mathbf{m}}^i$, is of the form

$$\widetilde{\mathbf{m}}^i = (\Lambda^{-1})^i_j\mathbf{m}^j + \frac{1}{2!}\overset{1}{\Upsilon}{}^i{}_{i_1 i_2}\mathbf{m}^{i_1}\mathbf{m}^{i_2} + ... + \frac{1}{N!}\overset{N-1}{\Upsilon}{}^i{}_{i_1 i_2..i_N}\mathbf{m}^{i_1}\mathbf{m}^{i_2}...\mathbf{m}^{i_N}. \tag{4}$$

Linear independence of the new basis minus one forms implies that Δ, the determinant of the $N \times N$ matrix-valued function with entries the zero-forms Λ^i_j, must be non-zero, justifying the notation.

Example 2.1:

Consider the case where $N = 2$. Corresponding conclusions for $N = 1$ forms can be read off from the results for $N = 2$ forms. Let

$$\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}{}_i \mathbf{m}^i + \overset{p+2}{\alpha} \mathbf{m}^1 \mathbf{m}^2, \tag{5}$$

where the indices i, j range and sum over 1-2.

The most general change of basis of the minus one forms $(\mathbf{m}^1, \mathbf{m}^2) \mapsto (\widetilde{\mathbf{m}}^1, \widetilde{\mathbf{m}}^2)$ is of the form

$$\widetilde{\mathbf{m}}^i = (\Lambda^{-1})^i_j \mathbf{m}^j + \Upsilon^i \mathbf{m}^1 \mathbf{m}^2, \tag{6}$$

with inverse transformation given by

$$\mathbf{m}^i = \Lambda^i_j \widetilde{\mathbf{m}}^j - \Delta \Lambda^i_j \Upsilon^j \widetilde{\mathbf{m}}^1 \widetilde{\mathbf{m}}^2. \tag{7}$$

The basis transformations, acting on the right here, form a group with the composition of transformations given by

$$(\Lambda_2, \Upsilon_2) \circ (\Lambda_1, \Upsilon_1) = (\Lambda_1 \Lambda_2, \Lambda_2^{-1} \Upsilon_1 \ + \Delta_1^{-1} \Upsilon_2).$$

If the representation of the generalized p-form $\overset{p}{\mathbf{a}}$ in this new basis is given by

$$\overset{p}{\mathbf{a}} = \overset{p}{\widetilde{\alpha}} + \overset{p+1}{\widetilde{\alpha}}{}_i \widetilde{\mathbf{m}}^i + \overset{p+2}{\widetilde{\alpha}} \widetilde{\mathbf{m}}^1 \widetilde{\mathbf{m}}^2, \tag{8}$$

then it follows that

$$\begin{aligned} \overset{p}{\widetilde{\alpha}} &= \overset{p}{\alpha}, \\ \overset{p+1}{\widetilde{\alpha}}{}_i &= \Lambda^j_i \overset{p+1}{\alpha}{}_j, \\ \overset{p+2}{\widetilde{\alpha}} &= \Delta[\overset{p+2}{\alpha} - \Lambda^j_i \overset{p+1}{\alpha}{}_j \Upsilon^i]. \end{aligned} \tag{9}$$

Consider now the exterior calculus of generalized forms. It is assumed that exterior derivative operators $d : \Lambda^p_{(N)} \to \Lambda^{p+1}_{(N)}$ agree with the usual exterior derivative when they act on type $N = 0$ (ordinary) forms and satisfies the usual exterior derivative rules, that is for type $N \geq 0$ forms

$$\begin{aligned} d(\overset{p}{\mathbf{a}} + \overset{p}{\mathbf{b}}) &= d\overset{p}{\mathbf{a}} + d\overset{p}{\mathbf{b}}, \ d\varphi(X) = X(\varphi), \\ d(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) &= (d\overset{p}{\mathbf{a}})\overset{q}{\mathbf{b}} + (-1)^p \overset{p}{\mathbf{a}} d\overset{q}{\mathbf{b}}, \\ d^2 \overset{p}{\mathbf{a}} &= 0. \end{aligned} \tag{10}$$

where X is any vector field and φ any function on M. Since the exterior derivative of each basis minus one-form must be a generalized zero-form, it follows then that there are ordinary zero-, one-, two-,..., N-forms, respectively μ^i, $\nu^i_{i_1}$, $\rho^i_{i_1 i_2}$,..., $\iota^i_{i_1 \ldots i_N}$., such that

$$d\mathbf{m}^i = \mu^i - \nu^i_{i_1} \mathbf{m}^{i_1} + \frac{1}{2!}\rho^i_{i_1 i_2} \mathbf{m}^{i_1} \mathbf{m}^{i_2} + \ldots + \frac{1}{N!}\iota^i_{i_1 \ldots i_N} \mathbf{m}^{i_1} \mathbf{m}^{i_2} \ldots . \mathbf{m}^{i_N}. \tag{11}$$

The ordinary forms in this expression are restricted by the requirements above, in particular $d^2\mathbf{m}^i = 0$. These lead to a differential ideal composed of the ordinary forms, each solution of which determines an exterior derivative. The next example illustrates this. Type $N = 2$ forms are considered and such a differential ideal is constructed explicitly. Its solutions, and the corresponding exterior derivatives which they define, are then investigated.

Example 2.2:

Let $N = 2$ and let a basis of two minus one-forms and an exterior derivative be given for which

$$d\mathbf{m}^i = \mu^i - \nu^i_j \mathbf{m}^j + \rho^i \mathbf{m}^1 \mathbf{m}^2, \tag{12}$$

where μ^i, ν^i_j and ρ^i are respectively ordinary zero-, one- and two-forms. Applying the rules of exterior algebra and calculus, as above, in particular, $d^2\mathbf{m}^i = 0$, it follows that

$$\begin{aligned} \Theta^i &\equiv d\mu^i + \mu^j \nu^i_j = 0, \\ \Phi^i_j &\equiv d\nu^i_j + \nu^i_k \nu^k_j - \rho^i \mu_j = 0, \\ \Psi^i &\equiv d\rho^i + \nu^i_j \rho^j - \rho^i \nu^j_j = 0. \end{aligned} \tag{13}$$

The differential ideal determined by Θ^i, Φ^i_j, Ψ^i is closed. Here and in the following the skew symmetric matrices ε_{ij} and ε^{ij}, where $\varepsilon_{12} = \varepsilon^{12} = 1$, are used to raise and lower the Latin indices, so that $\mu^i \varepsilon_{ij} = \mu_j$.

Conversely any set of ordinary differential forms μ^i, ν^i_j, ρ^i satisfying Eqs.(13) determines an exterior derivative, via Eq.(12), satisfying Eq.(10). Such ordinary forms determine the exterior derivative of a generalized p-form to be

$$
\begin{aligned}
d\overset{p}{\mathbf{a}} = d\overset{p}{\alpha} + (-1)^{p+1}\overset{p+1}{\alpha}{}_i\mu^i + [d\overset{p+1}{\alpha}{}_i - \nu^j_i\overset{p+1}{\alpha}{}_j + (-1)^p\mu_i\overset{p+2}{\alpha}]\mathbf{m}^i \\
+ [d\overset{p+2}{\alpha} - \nu^i_i\overset{p+2}{\alpha} + (-1)^{p+1}\rho^i\overset{p+1}{\alpha}{}_i]\mathbf{m}^1\mathbf{m}^2. \qquad (14)
\end{aligned}
$$

Under a change of basis given by Eq.(6) it follows that

$$d\widetilde{\mathbf{m}}^i = \widetilde{\mu}^i - \widetilde{\nu}^i_j\widetilde{\mathbf{m}}^j + \widetilde{\rho}^i\widetilde{\mathbf{m}}^1\widetilde{\mathbf{m}}^2, \qquad (15)$$

where

$$
\begin{aligned}
\widetilde{\mu}^i &= (\Lambda^{-1})^i_j\mu^j, \qquad (16)\\
\widetilde{\nu}^i_j &= (\Lambda^{-1})^i_k d\Lambda^k_j + (\Lambda^{-1})^i_k\nu^k_l\Lambda^l_j + \mu_k\Lambda^k_j\Upsilon^i,\\
\widetilde{\rho}^i &= \Delta\{(\Lambda^{-1})^i_j\rho^j + d\Upsilon^i + \Upsilon^i\nu^j_j + [(\Lambda^{-1})^i_k d\Lambda^k_j + (\Lambda^{-1})^i_k\nu^k_l\Lambda^l_j]\Upsilon^j\\
&\quad + \mu_k\Lambda^k_j\Upsilon^i\Upsilon^j\},
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\Theta}^i &\equiv d\widetilde{\mu}^i + \widetilde{\mu}^j\widetilde{\nu}^i_j = (\Lambda^{-1})^i_j\Theta^j, \qquad (17)\\
\widetilde{\Phi}^i_j &\equiv d\widetilde{\nu}^i_j + \widetilde{\nu}^i_k\widetilde{\nu}^k_j - \widetilde{\rho}^i\widetilde{\mu}_j = (\Lambda^{-1})^i_k\Phi^k_l\Lambda^l_j - \Upsilon^i\Theta_k\Lambda^k_j,\\
\widetilde{\Psi}^i &\equiv d\widetilde{\rho}^i + \widetilde{\nu}^i_j\widetilde{\rho}^j - \widetilde{\rho}^i\widetilde{\nu}^j_j = \Delta[(\Lambda^{-1})^i_j\Psi^j + \Lambda^k_j\Theta_k\Upsilon^i\Upsilon^j + (\Lambda^{-1})^i_k\Phi^k_l\Lambda^l_j\Upsilon^j - \Upsilon^i\Phi^j_j].
\end{aligned}
$$

It follows from Eqs.(12)-(17), and the change of basis equations, that it is possible to construct bases of minus one-forms $\widetilde{\mathbf{m}}^i$ with simple "canonical" exterior derivatives. There are two distinct cases where such bases can be constructed in a straightforward way from the above equations. When the dimension of the manifold M is less than three the following calculations can be simplified but the results are the same.

Case (i) Consider exterior derivatives with $\mu^i = 0$ in a contractible domain $U \subseteq M$. Then a new (canonical) basis can be chosen in U so that $d\widetilde{\mathbf{m}}^i = 0$.

This follows from the observation that when $\mu^i = 0$ the solutions of Eq.(13) can be written in the form

$$
\begin{aligned}
\nu^i_j &= (\lambda^{-1})^i_k d\lambda^k_j,\\
\rho^i &= (\det\lambda)(\lambda^{-1})^i_j d\zeta^j, \qquad (18)
\end{aligned}
$$

where $\det\lambda$ is the determinant of the invertible matrix with entries λ^i_j, and ζ^i are one-forms. Then it follows from Eq.(16) that a canonical basis $\widetilde{\mathbf{m}}^i$, satisfying $d\widetilde{\mathbf{m}}^i = 0$, is given by Eq.(6) with $\Lambda^i_j = (\lambda^{-1})^i_j$, and $\Upsilon^i = -(\det\lambda)\zeta^i$.

Such a canonical basis is not unique. If $\widetilde{\mathbf{m}}^i = (\widetilde{\mathbf{m}}^1, \widetilde{\mathbf{m}}^2)$ is a basis satisfying $d\widetilde{\mathbf{m}}^i = 0$, then so is any basis $\widehat{\mathbf{m}}^i = (\widehat{\mathbf{m}}^1, \widehat{\mathbf{m}}^2)$, where

$$\begin{aligned} \widehat{\mathbf{m}}^i &= (\widetilde{\Lambda}^{-1})^i_j \widetilde{\mathbf{m}}^j + \widetilde{\Upsilon}^i \widetilde{\mathbf{m}}^1 \widetilde{\mathbf{m}}^2, \\ d\widetilde{\Lambda}^i_j &= d\widetilde{\Upsilon}^i = 0. \end{aligned} \tag{19}$$

Case (ii) Consider exterior derivatives with μ^i non-zero in a (not necessarily contractible) domain $U \subseteq M$. In this case, by introducing $\sigma^i_j \equiv \rho^i \mu_j$, the differential ideal given by Eq.(13) can be re-written as

$$\begin{aligned} \Theta^i &= d\mu^i + \mu^j \nu^i_j = 0, \\ \Phi^i_j &= d\nu^i_j + \nu^i_k \nu^k_j - \sigma^i{}_j = 0, \\ \mu_j \Psi^i + \Theta_j \rho^i &= d\sigma^i_j + \nu^i_k \sigma^k_j - \nu^k_j \sigma^i_k \\ &= -[d\Phi^i_j + \nu^i_k \Phi^k_j - \nu^k_j \Phi^i_k] = 0. \end{aligned} \tag{20}$$

Consequently when the second equation is satisfied so is the third. In fact with this notation the second equation has the form of the Cartan equation relating a connection and its curvature and the third equation has the form of the corresponding Bianchi identity.

In this case a new (canonical) basis can be chosen so that $d\widetilde{\mathbf{m}}^i = \delta^i_1$ in U . To see this first assume, without loss of generality, that μ^1 is non-zero. Then it follows directly from Eq. (16) that Λ^i_j, as in Eq.(6), can be chosen so that $\widetilde{\mu}^i = \delta^i_1$. With such a choice,and the choice $\Upsilon^i = -\Delta^{-1}[(\Lambda^{-1})^i_k d\Lambda^k_2 + (\Lambda^{-1})^i_k \nu^k_l \Lambda^l_2]$, it follows from Eqs.(16) and (20), together with the transformed version of Eq.(20), that $\widetilde{\nu}^i_j = 0$, $\widetilde{\rho}^i = 0$ and therefore $d\widetilde{\mathbf{m}}^i = \delta^i_1$.

This canonical basis is not unique. If $\widetilde{\mathbf{m}}^i = (\widetilde{\mathbf{m}}^1, \widetilde{\mathbf{m}}^2)$ is a basis satisfying $d\widetilde{\mathbf{m}}^i = \delta^i_1$, then so are the bases $(\widehat{\mathbf{m}}^1, \widehat{\mathbf{m}}^2)$, where

$$\begin{aligned} \widehat{\mathbf{m}}^1 &= \widetilde{\mathbf{m}}^1 + d(\pi \widetilde{\mathbf{m}}^1 \widetilde{\mathbf{m}}^2), \\ \widehat{\mathbf{m}}^2 &= d(\tau \widetilde{\mathbf{m}}^1 \widetilde{\mathbf{m}}^2), \end{aligned} \tag{21}$$

where π and τ are functions and τ is non-zero. Such mappings of canonical bases into canonical bases form a group, and if $\overset{p}{\mathbf{a}} = \overset{p}{\widetilde{\alpha}} + \overset{p+1}{\widetilde{\alpha}}{}_i \widetilde{\mathbf{m}}^i + \overset{p+2}{\widetilde{\alpha}} \widetilde{\mathbf{m}}^1 \widetilde{\mathbf{m}}^2 =$

$\overset{p}{\widehat{\alpha}} + \overset{p+1}{\widehat{\alpha}}_i \widehat{\mathbf{m}}^i + \overset{p+2}{\widehat{\alpha}} \widehat{\mathbf{m}}^1 \widehat{\mathbf{m}}^2$, then

$$\begin{aligned}
\overset{p}{\widetilde{\alpha}} &= \overset{p}{\widehat{\alpha}}, \quad \overset{p+1}{\widetilde{\alpha}}_1 = \overset{p+1}{\widehat{\alpha}}_1, \\
\overset{p+1}{\widetilde{\alpha}}_2 &= \tau \overset{p+1}{\widehat{\alpha}}_2 + \pi \overset{p+1}{\widehat{\alpha}}_1 \\
\overset{p+2}{\widetilde{\alpha}} &= \tau \overset{p+2}{\widehat{\alpha}} + \overset{p+1}{\widehat{\alpha}}_1 d\pi + \overset{p+1}{\widehat{\alpha}}_2 d\tau.
\end{aligned} \tag{22}$$

In the canonical bases the two exterior derivatives of a type $N = 2$ generalized p-form are given by

$$\begin{aligned}
d\overset{p}{\mathbf{a}} &= d\overset{p}{\alpha} + (-1)^{p+1}\epsilon \overset{p+1}{\alpha}_1 + d\overset{p+1}{\alpha}_1 \mathbf{m}^1 \\
&\quad + [d\overset{p+1}{\alpha}_2 + (-1)^p \epsilon \overset{p+2}{\alpha}]\mathbf{m}^2 + d\overset{p+2}{\alpha}\mathbf{m}^1\mathbf{m}^2,
\end{aligned} \tag{23}$$

where $\epsilon = 0$ in case (i) and $\epsilon = 1$ in case (ii). Then $\overset{p}{\mathbf{a}}$ is closed if and only if

$$\begin{aligned}
d\overset{p}{\alpha} + (-1)^{p+1}\epsilon \overset{p+1}{\alpha}_1 &= 0, \\
d\overset{p+1}{\alpha}_1 &= 0, \\
d\overset{p+1}{\alpha}_2 + (-1)^p \epsilon \overset{p+2}{\alpha} &= 0, \\
d\overset{p+2}{\alpha} &= 0.
\end{aligned} \tag{24}$$

Hence in case (i), where $\epsilon = 0$, $\overset{p}{\mathbf{a}}$ is closed if and only if all the ordinary forms defining it are closed. On the other hand in case (ii), where $\epsilon = 1$, $\overset{p}{\mathbf{a}}$ is closed if and only if it is exact. In this case

$$\begin{aligned}
\overset{p}{\mathbf{a}} &= \overset{p}{\alpha} + (-1)^p d\overset{p}{\alpha}\mathbf{m}^1 + \overset{p+1}{\alpha}_2 \mathbf{m}^2 + (-1)^{p+1} d\overset{p+1}{\alpha_2}\mathbf{m}^1\mathbf{m}^2 \\
&= d[(-1)^p \overset{p}{\alpha}\mathbf{m}^1 + (-1)^{p+1}\overset{p+1}{\alpha_2}\mathbf{m}^1\mathbf{m}^2],
\end{aligned} \tag{25}$$

if and only if $d\overset{p}{\mathbf{a}} = 0$. It can be seen from Eqs.(25) that, in case (ii), an ordinary closed form is always exact when viewed as a $N \geq 1$ form.

Conclusions in the two corresponding cases for type $N = 1$ forms can be read off from these results. In case (i) a canonical minus one-form $\mathbf{m}$ which satisfies $d\mathbf{m} = 0$ can always be constructed in a contractible region. It is not unique, if $\mathbf{m}$ is such a canonical minus one-form so is $\Lambda^{-1}\mathbf{m}$, where Λ is any non-zero constant. In case (ii) a canonical minus one-form $\mathbf{m}$ which satisfies $d\mathbf{m} = 1$ can always be constructed and this one form is unique. One consequence of this is that, in this case, the definitions of Lie derivative, duality, co-differential and Laplacian for generalized forms, on manifolds with metrics, made in previous papers, [7], [8], are uniquely defined relative to the canonical basis. This contrasts with the situation in case (i) or whenever $N > 1$, where those definitions are dependent on the choice of canonical basis. The method of calculation used in this example can be applied when N is greater than two but the computations become increasingly lengthy. A more efficient calculation for general N would be much more satisfactory, even though it might not give results that are qualitatively different from the $N = 2$ case.

Henceforth in this paper two exterior derivatives will be carried along together and used in calculations for all $N > 1$. The two derivatives are those which admit *canonical* bases of minus one-forms, that is bases for which

$$d\mathbf{m}^i = \epsilon \delta^i_1, \tag{26}$$

where $\epsilon = 0$ in the first case and $\epsilon = 1$ in the second case. Consequently for a type $N \geq 1$ form given by Eq.(2), the exterior derivatives considered are given by

$$d\overset{p}{\mathbf{a}}_{(N)} = d\overset{p}{\mathbf{a}}_{(N-1)} + (1)^{p+1}\epsilon\delta^N_1 \overset{p+1}{\mathbf{a}}_{(N-1)} + d\overset{p+1}{\mathbf{a}}_{(N-1)}\mathbf{m}^N. \tag{27}$$

In previous papers it was assumed, implicitly or explicitly, that the exterior derivative was such that a basis existed for which the only non vanishing ordinary forms in Eq.(11) were the zero-forms μ^i and that these were non-zero constants. When this is the case it is always possible to transform to a new basis of minus one forms satisfying $d\mathbf{m}^i = \delta^i_1$, $i = 1...N$. Therefore, in the terminology being used in this paper, the second type of exterior derivative and canonical basis was being used in earlier work.

Finally in this section it should be noted that analytic functions can be extended naturally, by using their power series expansions, to define functions of generalized zero-forms.

Example 2.3:

Let f be an analytic function of r real variables and let $\mathbf{a}^\mu = \alpha^\mu + \beta^\mu \mathbf{m}$, $\mu = 1..r$, be r type $N = 1$ generalized zero-forms. Then, by using the Maclaurin expansion for f the generalized zero-form $\mathbf{f}(\mathbf{a}^\mu)$ can be computed to be

$$\mathbf{f}(\mathbf{a}^\mu) = f(\alpha^\mu) + \frac{\partial f(\alpha^\mu)}{\partial \alpha^\nu}\beta^\nu \mathbf{m}. \tag{28}$$

3 Lie Groups and matrix-valued generalized forms

In this section a fuller discussion of groups of matrix-valued generalized forms than has been given in previous papers, such as [8], will be presented. It will always be assumed that matrix representations of groups, on a vector space, are being used which make the matrix operations in the calculations well-defined; in particular the identity will be the appropriate unit matrix. This approach means that obvious homomorphisms and isomorphisms need not be stated explicitly all the time.

First recall the general case of groups of type $N = 1$ generalized forms. For simplicity the starting point is taken to be a matrix Lie group. Let G be a matrix Lie group and let H an (additive) abelian Lie group. Let $G_{(0)} = \{\overset{0}{\mathbf{g}}_{(0)}\}$ be the space of G-valued zero-forms belonging to $\Lambda^0_{(0)}(M)$. This is a group under multiplication with identity written $1_{(0)}$. Let $H_{(0)} = \{\overset{1}{\mathbf{g}}_{(0)}\}$ be the additive abelian group of H-valued one-forms $\in \Lambda^1_{(0)}(M)$. Let there be an ad-action of of $G_{(0)}$ on $H_{(0)}$, that is a homomorphism $\Phi : G_{(0)} \rightarrow aut(H_{(o)})$ with $\Phi(\overset{0}{\mathbf{g}}_{(0)}) : \overset{1}{\mathbf{g}}_{(0)} \longrightarrow \overset{0}{\mathbf{g}}_{(0)}\overset{1}{\mathbf{g}}_{(0)}(\overset{0}{\mathbf{g}}_{(0)})^{-1}$. Then the set of type $N = 1$ matrix-valued generalized zero-forms, $G_{(1)} = \{\overset{0}{\mathbf{g}}_{(1)}\}$, where

$$\overset{0}{\mathbf{g}}_{(1)} = (1_{(0)} + \overset{1}{\mathbf{g}}_{(0)}\mathbf{m})\overset{0}{\mathbf{g}}_{(0)} \tag{29}$$

is a group under exterior multiplication. If $\overset{0}{\mathbf{f}}_{(1)} = (1_{(0)} + \overset{1}{\mathbf{f}}_{(0)}\mathbf{m})\overset{0}{\mathbf{f}}_{(0)} \in G_{(1)}$ the product is

$$\overset{0}{\mathbf{g}}_{(1)}\overset{0}{\mathbf{f}}_{(1)} = (1_{(0)} + [\overset{1}{\mathbf{g}}_{(0)} + \overset{0}{\mathbf{g}}_{(0)}\overset{1}{\mathbf{f}}_{(0)}(\overset{0}{\mathbf{g}}_{(0)})^{-1}]\mathbf{m})\overset{0}{\mathbf{g}}_{(0)}\overset{0}{\mathbf{f}}_{(0)}, \tag{30}$$

and the inverse, with identity $1_{(1)} = 1_{(0)}$, is given by

$$(\overset{0}{\mathbf{g}}_{(1)})^{-1} = (1_{(0)} - [(\overset{0}{\mathbf{g}}_{(0)})^{-1}\overset{1}{\mathbf{g}}_{(0)}\overset{0}{\mathbf{g}}_{(0)}]\mathbf{m})(\overset{0}{\mathbf{g}}_{(0)})^{-1}. \tag{31}$$

The group $G_{(1)}$ is isomorphic to the semi-direct product of $G_{(0)}$ and $H_{(0)}$. The Lie algebra $\mathfrak{g}_{(1)}$ of $G_{(1)}$ is given by $\{\overset{0}{l}_{(1)}\}$, where $\overset{0}{l}_{(1)} = \overset{0}{\lambda} + \overset{1}{\lambda}\mathbf{m}$ and $\overset{0}{\lambda}$ and $\overset{1}{\lambda}$ respectively take values in the Lie algebras of G and H.

Lie groups of generalized zero-forms of type $N > 1$ can be constructed iteratively (cf. Eq.(2)) from the $N = 1$ case as follows. Let $G_{(N-1)} = \{\overset{0}{\mathbf{g}}_{(N-1)}\}$, $N > 1$,be a Lie group of type $(N-1)$ generalized zero-forms, and let $H_{(N-1)} = \{\overset{1}{\mathbf{g}}_{(N-1)}\}$ be an additive abelian Lie group of type $(N-1)$ generalized one-forms where there is an ad-action, as above, of $G_{(N-1)}$ on $H_{(N-1)}$. Then $G_{(N)} = \{\overset{0}{\mathbf{g}}_{(N)}\}$, where

$$\overset{0}{\mathbf{g}}_{(N)} = (1_{(N-1)} + \overset{1}{\mathbf{g}}_{(N-1)}\mathbf{m}^N)\overset{0}{\mathbf{g}}_{(N-1)} \tag{32}$$

is a Lie group of type N forms. The product rule and inverses are given by the same formulae as in Eqs.(30)-(31) with the subscripts (1) and (0) respectively replaced by (N) and $(N-1)$. In the applications in this paper it will always be the case that $H_{(N)}$ is isomorphic to the Lie algebra $\mathfrak{g}$ of G, regarded as an additive abelian group.

Example 3.1:

Consider $G_{(2)} = \{\overset{0}{\mathbf{g}}_{(2)}\}$, where $\overset{0}{\mathbf{g}}_{(2)} = (1_{(1)} + \overset{1}{\mathbf{g}}_{(1)}\mathbf{m}^2)\overset{0}{\mathbf{g}}_{(1)}$, $\overset{0}{\mathbf{g}}_{(1)} = (1 + \overset{1}{\gamma}_1\mathbf{m}^1)\overset{0}{\gamma}$ and $\overset{1}{\mathbf{g}}_{(1)} = \overset{1}{\gamma}_2 + \overset{2}{\gamma}\mathbf{m}^1$. Here $\overset{0}{\gamma}$ is an ordinary G-valued zero form belonging to $G_{(0)}$, and $\overset{1}{\gamma}_1, \overset{1}{\gamma}_2, \overset{2}{\gamma}$ are, respectively, ordinary H-valued one-forms and two-forms. Furthermore $G_{(1)} = \{\overset{0}{\mathbf{g}}_{(1)}\}$ and $H_{(1)} = \{\overset{1}{\mathbf{g}}_{(1)}\}$ and $G_{(1)}$ acts on $H_{(1)}$ by $\overset{1}{\mathbf{g}}_{(1)} \rightarrow \overset{0}{\mathbf{g}}_{(1)}\overset{1}{\mathbf{g}}_{(1)}(\overset{0}{\mathbf{g}}_{(1)})^{-1}$. Written out more fully, with the identity written as 1,

$$\begin{aligned}
\overset{0}{\mathbf{g}}_{(2)} &= [1 + \overset{1}{\gamma}_1\mathbf{m}^1 + \overset{1}{\gamma}_2\mathbf{m}^2 + (\overset{2}{\gamma} + \overset{1}{\gamma}_2\overset{1}{\gamma}_1)\mathbf{m}^1\mathbf{m}^2]\overset{0}{\gamma}, && (33)\\
(\overset{0}{\mathbf{g}}_{(2)})^{-1} &= [1 - (\overset{0}{\gamma})^{-1}\overset{1}{\gamma}_1\overset{0}{\gamma}\mathbf{m}^1 - (\overset{0}{\gamma})^{-1}\overset{1}{\gamma}_2\overset{0}{\gamma}\mathbf{m}^2 \\
&\quad - (\overset{0}{\gamma})^{-1}(\overset{2}{\gamma} + \overset{1}{\gamma}_1\overset{1}{\gamma}_2)\overset{0}{\gamma}\mathbf{m}^1\mathbf{m}^2](\overset{0}{\gamma})^{-1};
\end{aligned}$$

$$\overset{0}{\mathbf{g}}_{(1)}\overset{1}{\mathbf{g}}_{(1)}(\overset{0}{\mathbf{g}}_{(1)})^{-1} = \overset{0}{\gamma}\overset{1}{\gamma}_2(\overset{0}{\gamma})^{-1} + \{(\overset{0}{\gamma}\overset{2}{\gamma}(\overset{0}{\gamma})^{-1} - [\overset{1}{\gamma}_1\overset{0}{\gamma}\overset{1}{\gamma}_2(\overset{0}{\gamma})^{-1} + \overset{0}{\gamma}\overset{1}{\gamma}_2(\overset{0}{\gamma})^{-1}\overset{1}{\gamma}_1]\}\mathbf{m}^1. \tag{34}$$

Note that, in Eq.(34), $\{\overset{1}{\gamma}_1[\overset{0}{\gamma}\overset{1}{\gamma}_2(\overset{0}{\gamma})^{-1}] + [\overset{0}{\gamma}\overset{1}{\gamma}_2(\overset{0}{\gamma})^{-1}]\overset{1}{\gamma}_1\}$ must be an H-valued two-form as is the case in the applications, where H is always a Lie-algebra regarded also as an additive abelian group.

Under a change of canonical basis, $\mathbf{m}^1 = \widetilde{\mathbf{m}}^1 + \pi\widetilde{\mathbf{m}}^2 + d\pi\widetilde{\mathbf{m}}^1\widetilde{\mathbf{m}}^2$, and $\mathbf{m}^2 = \tau\widetilde{\mathbf{m}}^2 + d\tau\widetilde{\mathbf{m}}^1\widetilde{\mathbf{m}}^2$ as in Eq.(21), it follows that $(\overset{0}{\gamma}, \overset{1}{\gamma}_1, \overset{1}{\gamma}_2, \overset{2}{\gamma}) \rightarrow (\overset{0}{\widetilde{\gamma}}, \overset{1}{\widetilde{\gamma}}_1, \overset{1}{\widetilde{\gamma}}_2, \overset{2}{\widetilde{\gamma}})$

where

$$\overset{0}{\tilde{\gamma}} = \overset{0}{\gamma}, \overset{1}{\tilde{\gamma}}_1 = \overset{1}{\gamma}_1, \overset{1}{\tilde{\gamma}}_2 = \pi\overset{1}{\gamma}_1 + \tau\overset{1}{\gamma}_2, \overset{2}{\tilde{\gamma}} = \tau\overset{2}{\gamma} - \pi\overset{1}{\gamma}_1\overset{1}{\gamma}_1 + \overset{1}{\gamma}_1 d\pi + \overset{1}{\gamma}_2 d\tau. \tag{35}$$

Example 3.2:

Let V be a real vector space of dimension $s = p + q$ equipped with a metric η of signature (p, q). First consider matrix-valued generalized forms which preserve the metric, that is forms $\overset{0}{\mathbf{g}}_{(N)} = (1_{(N-1)} + \overset{1}{\mathbf{g}}_{(N-1)}\mathbf{m}^N)\overset{0}{\mathbf{g}}_{(N-1)}$, with matrix entries belonging to $\Lambda^0_{(N)}(V)$, which satisfy

$$\overset{0}{\mathbf{g}}{}^T_{(N)}\eta\overset{0}{\mathbf{g}}_{(N)} = \eta, \tag{36}$$

where, for each N the metric is considered to be a matrix-valued type N zero-form (by a small abuse of notation written η) and the superscript T denotes the matrix transpose, $\overset{0}{\mathbf{g}}{}^T_{(N)} = \overset{0}{\mathbf{g}}{}^T_{(N-1)} + \overset{0}{\mathbf{g}}{}^T_{(N-1)}\overset{1}{\mathbf{g}}{}^T_{(N-1)}\mathbf{m}^N$. This condition holds if and only if

$$\begin{aligned} \overset{0}{\mathbf{g}}{}^T_{(N-1)}\eta\overset{0}{\mathbf{g}}_{(N-1)} &= \eta, \\ \overset{1}{\mathbf{g}}{}^T_{(N-1)}\eta + \eta\overset{1}{\mathbf{g}}_{(N-1)} &= 0. \end{aligned} \tag{37}$$

If $\overset{0}{\mathbf{X}}_{(N)}$and $\overset{0}{\mathbf{Y}}_{(N)}$are vector-valued generalized zero-forms, the bilinear form $\overset{0}{\mathbf{X}}{}^T_{(N)}\eta\overset{0}{\mathbf{Y}}_{(N)}$is preserved under the transformations $\overset{0}{\mathbf{X}}_{(N)} \mapsto \overset{0}{\mathbf{g}}_{(N)}\overset{0}{\mathbf{X}}_{(N)}$, $\overset{0}{\mathbf{Y}}_{(N)} \mapsto \overset{0}{\mathbf{g}}_{(N)}\overset{0}{\mathbf{Y}}_{(N)}$.

Example 3.3:

In the case of type $N = 1$ generalized forms, $\overset{0}{\mathbf{g}}_{(1)} = (1_{(0)} + \overset{1}{\mathbf{g}}_{(0)}\mathbf{m}^1)\overset{0}{\mathbf{g}}_{(0)}$, written here as $\overset{0}{\mathbf{g}}_{(1)} = (1 + \overset{1}{\gamma})\mathbf{m}\overset{0}{\gamma}$, the metric preserving conditions given by Eqs.(36-37) hold if and only if

$$\begin{aligned} \overset{0}{\gamma}{}^T\eta\overset{0}{\gamma} &= \eta, \\ \overset{1}{\gamma}{}^T\eta + \overset{1}{\gamma}\eta &= 0, \end{aligned} \tag{38}$$

that is if and only if the matrix $\overset{0}{\gamma}$ takes values in $SO(p, q)$ and the matrix-valued one-forms $\overset{1}{\gamma}$ take values in $so(p, q)$, the Lie algebra of $SO(p, q)$.

If $\overset{0}{\mathbf{X}}_{(1)} = \overset{0}{\xi} + \overset{1}{\xi}\mathbf{m}$ and $\overset{0}{\mathbf{Y}}_{(1)} = \overset{0}{\varsigma} + \overset{1}{\varsigma}\mathbf{m}$, where $\overset{0}{\xi}$, $\overset{1}{\xi}$, $\overset{0}{\varsigma}$ and $\overset{1}{\varsigma}$ are vector-valued ordinary forms, then

$$\overset{0}{\mathbf{X}}{}^T_{(1)}\eta_{(1)}\overset{0}{\mathbf{Y}}_{(1)} = \overset{0}{\xi}{}^T\eta\overset{0}{\varsigma} + (\overset{0}{\xi}{}^T\eta\overset{1}{\varsigma} + \overset{1}{\xi}{}^T\eta\overset{0}{\varsigma})\mathbf{m}. \tag{39}$$

Under the transformations $\overset{0}{\mathbf{X}}_{(1)} \mapsto \overset{0}{\mathbf{g}}_{(1)}\overset{0}{\mathbf{X}}_{(1)}$, $\overset{0}{\mathbf{Y}}_{(1)} \mapsto \overset{0}{\mathbf{g}}_{(1)}\overset{0}{\mathbf{Y}}_{(1)}$

$$\overset{0}{\xi} \mapsto \overset{0}{\gamma}\overset{0}{\xi} \text{ and } \overset{1}{\xi} \mapsto \overset{0}{\gamma}\overset{1}{\xi} + \overset{1}{\gamma}\overset{0}{\gamma}\overset{0}{\xi}, \tag{40}$$

and similarly for $\overset{0}{\varsigma}$ and $\overset{1}{\varsigma}$. Since $\overset{0}{\mathbf{X}}{}^T_{(1)}\eta_{(1)}\overset{0}{\mathbf{Y}}_{(1)}$is preserved under these transformations so are both $\overset{0}{\xi}{}^T\eta\overset{0}{\varsigma}$ and $(\overset{0}{\xi}{}^T\eta\overset{1}{\varsigma} + \overset{1}{\xi}{}^T\eta\overset{0}{\varsigma})$. The first transformation in Eq.(40) is the usual $SO(p, q)$ transformation, but the second is a generalization of the usual transformation induced on vector valued one-forms. Hence, when $p = 1$ and $q = 3$, the groups and transformations are generalizations of the usual Minkowski space-time Lorentz group and Lorentz transformations.

These results can be easily extended to forms of higher type. When Eq.(32) is used, repeatedly, to express type $N > 2$ forms, $\overset{0}{\mathbf{g}}_{(N)}$, in terms of their expansions in ordinary forms and the basis minus one-forms it is a straightforward matter to see that the degree zero ordinary form is $SO(p, q)$-valued and the higher degree ordinary forms are $so(p, q)$-valued. For example when $N = 2$ and $\overset{0}{\mathbf{g}}_{(2)}$is expanded in the form given by Eq.(33), $\overset{0}{\gamma}$ is $SO(p, q)$-valued and $\overset{1}{\gamma}_1, \overset{1}{\gamma}_2, \overset{2}{\gamma}$ are each $so(p, q)$-valued.

It is clear that these results extend straightforwardly to the case where G is a symplectic or unitary group. Again in these cases H is the corresponding Lie algebra. The formalism also extends in the obvious way to other groups such $ISO(p, q)$ and to affine, conformal, projective and other transformations.

4 Local generalized connections

The local theory of type N generalized connections, with values in the Lie algebra $\mathfrak{g}$ of a matrix Lie group G, will be discussed and some examples will be given. Here $H = \mathfrak{g}$ and the connection one-forms, $\mathbf{A}_{(N)}$, are $\mathfrak{g}$-valued type N generalized one-forms. It will be assumed that matrix representations are being used so that the generalized connections (and their constituent ordinary forms) are square matrix-valued. The curvature 2-form is defined by the standard formula

$$\mathbf{F}_{(N)}= \mathbf{dA}_{(N)} + \mathbf{A}_{(N)}\mathbf{A}_{(N)}, \tag{41}$$

where, as usual, the last term includes both the matrix and the exterior product. It is convenient to introduce a differential operator $D_{(N)}$ - the covariant exterior derivative defined by $\mathbf{A}_{(N)}$. The covariant exterior derivative of a type N generalized square matrix-valued p-form $\mathbf{P}_{(N)}$ is defined to be

$$D_{(N)}\mathbf{P}_{(N)} = d\mathbf{P}_{(N)} + \mathbf{A}_{(N)}\mathbf{P}_{(N)} + (-1)^{p+1}\mathbf{P}_{(N)}\mathbf{A}_{(N)}. \tag{42}$$

As in previous sections formulae for type $N \geq 1$ forms can often be conveniently constructed, iteratively, from formulae for forms of lower type. Writing

$$\mathbf{A}_{(N)} = \overset{1}{\mathbf{A}}_{(N-1)} + \overset{2}{\mathbf{A}}_{(N-1)}\mathbf{m}^N, \tag{43}$$

it follows that

$$\mathbf{F}_{(N)} = \overset{2}{\mathbf{F}}_{(N-1)} + \epsilon\overset{2}{\mathbf{A}}_{(N-1)}\delta_1^N + D_{(N-1)}\overset{2}{\mathbf{A}}_{(N-1)}\mathbf{m}^N. \tag{44}$$

The generalized connection is flat when $\mathbf{F}_{(N)} = 0$. The generalized connection $\mathbf{A}_{(N)} = (\overset{0}{\mathbf{g}}_{(N)})^{-1}d\overset{0}{\mathbf{g}}_{(N)}$, where $\overset{0}{\mathbf{g}}_{(N)}$is a $G_{(N)}$ - valued function, is flat. Under a "generalized gauge transformation"

$$\mathbf{A}_{(N)} \rightarrow (\overset{0}{\mathbf{g}}_{(N)})^{-1}d\overset{0}{\mathbf{g}}_{(N)} + (\overset{0}{\mathbf{g}}_{(N)})^{-1}\mathbf{A}_{(N)}\overset{0}{\mathbf{g}}_{(N)}, \tag{45}$$

the curvature transforms in the usual way

$$\mathbf{F}_{(N)} \rightarrow (\overset{0}{\mathbf{g}}_{(N)})^{-1}\mathbf{F}_{(N)}\overset{0}{\mathbf{g}}_{(N)}, \tag{46}$$

and the condition of flatness is preserved. These generalized gauge transformations also preserve the generalized versions of various characteristic classes, for instance the generalized second Chern class which is defined by

$${}_2\mathbf{C}_{(N)} = \frac{1}{8\pi^2}[Tr(\mathbf{F}_{(N)}\mathbf{F}_{(N)}) - Tr(\mathbf{F}_{(N)})Tr(\mathbf{F}_{(N)})], \tag{47}$$

and is equal to the exterior derivative of the generalized Chern-Simons three-form ${}_{CS}\mathbf{C}_{(N)}$ where

$${}_{CS}\mathbf{C}_{(N)} = \frac{1}{8\pi^2}[Tr(\mathbf{A}_{(N)}\mathbf{F}_{(N)} - \frac{1}{3}\mathbf{A}_{(N)}\mathbf{A}_{(N)}\mathbf{A}_{(N)} - \mathbf{A}_{(N)}d\mathbf{A}_{(N)})]. \tag{48}$$

The definitions of the second Chern class and Chern-Simons three-form presented here are formal analogues of the usual definition in terms of ordinary forms. They were introduced and used in references [3], [4] using type $N = 1$ connections and curvatures. Other generalized characteristic classes, such as the Pontrjagin classes, can also be constructed by replacing ordinary forms with generalized forms in the usual definitions. In an example below the generalized Euler class in four dimensions will introduced and used.

Next examples which exhibit various features of generalized connections will be exhibited. It suffices to discuss type $N = 1$ connections here. Many further examples, including type $N = 2$ connections, are given in references [6],[9] and also in [3], [4].

Example 4.1:

Let a type $N = 1$ connection, $\mathbf{A} = \overset{1}{\alpha} + \overset{2}{\alpha}\mathbf{m}$, be flat so that $d\overset{1}{\alpha} + \overset{1}{\alpha}\overset{1}{\alpha} + \epsilon\overset{2}{\alpha} = 0$ and $D\overset{2}{\alpha} = 0$, where D is the covariant exterior derivative with respect to $\overset{1}{\alpha}$.

Then in case (i) where $\epsilon = 0$, it is a straightforward matter to show that on a contractible neighbourhood U there exist ordinary G-valued zero- and $\mathfrak{g}$-valued one-forms μ and ν such that $\mathbf{A} = \mu^{-1}d\mu + \mu^{-1}(d\nu)\mu\mathbf{m}$. Furthermore, $\mathbf{A} = (\overset{0}{\mathbf{h}})^{-1}d\overset{0}{\mathbf{h}}$ where $\overset{0}{\mathbf{h}} = (1 + \nu\mathbf{m})\mu$. More generally, $\mathbf{A} = (\overset{0}{\mathbf{g}}\overset{0}{\mathbf{h}})^{-1}d(\overset{0}{\mathbf{g}}\overset{0}{\mathbf{h}})$ for any closed zero-form $\overset{0}{\mathbf{g}}$ in $G_{(1)}$. When $\epsilon = 0$ such a closed form can always be written as $\overset{0}{\mathbf{g}} = [1 + \beta\mathbf{m}]\gamma$ for some constant G-valued zero-form γ and closed $\mathfrak{g}$-valued one-form β.

In case (ii) where $\epsilon = 1$, the flat connection is always of the form $\mathbf{A} = \overset{1}{\alpha} - (d\overset{1}{\alpha} + \overset{1}{\alpha}\overset{1}{\alpha})\mathbf{m}$, and $\mathbf{A} = (\overset{0}{\mathbf{h}})^{-1}d\overset{0}{\mathbf{h}}$ where $\overset{0}{\mathbf{h}} = 1 - \overset{1}{\alpha}\mathbf{m}$. More generally $\mathbf{A} = (\overset{0}{\mathbf{g}}\overset{0}{\mathbf{h}})^{-1}d(\overset{0}{\mathbf{g}}\overset{0}{\mathbf{h}})$ for any closed zero-form $\overset{0}{\mathbf{g}}$ in $G_{(1)}$. When $\epsilon = 1$ such a closed form can always be written as $\overset{0}{\mathbf{g}} = [1 + (d\gamma)\gamma^{-1}\mathbf{m}]\gamma$ for some G-valued zero-form γ.

In the next example the case where $G = ISO(p, q)$ and affine generalized connections are considered. In particular it is shown how to recover the Cartan structure equations for a metric from a flat generalized connection. This is an extension and re-formulation of an earlier calculation in reference [6].

Example 4.2:

On an open subset of an n dimensional manifold M an element of the group $ISO(p, q)_{(1)}$ can be represented by $(n+1) \times (n+1)$ matrix-valued type

$N=1$ generalized zero-form,

$$\overset{0}{\mathbf{g}}_{(1)} = \begin{pmatrix} \mathbf{g}^a_b & \mathbf{g}^a \\ 0 & 1 \end{pmatrix}. \tag{49}$$

Here the Latin indices range and sum over $1...n$ and the $N=1$ generalized zero-forms $\mathbf{g}^a_b$ are the (a,b) entries in a $n\times n$ representation of $SO(p,q)_{(1)}$. The type $N=1$ generalized affine connections are represented by $(n+1,n+1)$ matrix valued generalized one-forms

$$\mathbf{A}_{(1)}= \begin{pmatrix} \mathbf{A}^a_b & \mathbf{A}^a \\ 0 & 0 \end{pmatrix}, \tag{50}$$

with values in the Lie algebra of $ISO(p,q)_{(1)}$. When $\mathbf{A}^a_b = \omega^a_b - \Omega^a_b\mathbf{m}$ and $\mathbf{A}^a = \theta^a - \Theta^a\mathbf{m}$ the curvature of $\mathbf{A}_{(1)}$ is given by

$$\mathbf{F}_{(1)} = \begin{pmatrix} \mathbf{F}^a_b & \mathbf{F}^a \\ 0 & 0 \end{pmatrix}, \tag{51}$$

where

$$\begin{aligned} \mathbf{F}^a_b &= d\omega^a_b + \omega^a_c\omega^c_b - \epsilon\Omega^a_b - D\Omega^a_b\mathbf{m}, \\ \mathbf{F}^a &= D\theta^a - \epsilon\Theta^a + (\Omega^a_b\theta^b - D\Theta^a)\mathbf{m}, \end{aligned}$$

and D is the covariant exterior derivative with respect to ω^a_b.

When $\epsilon = 0$ the generalized connection is flat if and only if

$$\begin{aligned} d\omega^a_b + \omega^a_c\omega^c_b = 0,\ D\theta^a = 0, \\ D\Omega^a_b = 0, \Omega^a_b\theta^b - D\Theta^a = 0. \end{aligned} \tag{52}$$

If the n ordinary one-forms θ^a are linearly independent so that they can form an orthonormal basis for a metric of signature (p,q), $ds^2 = \eta_{ab}\theta^a\otimes\theta^b$, then this metric, with metric connection ω^a_b, is flat. Locally $\theta^a = (\gamma^{-1})^a_b dx^b$ and $\omega^a_b = (\gamma^{-1})^a_c d\gamma^c_b$, where $\gamma^c_a\gamma^d_b\eta_{cd} = \eta_{ab}$. It then follows from the other flatness conditions that there are $so(p,q)$-valued one forms μ^a_b and one-forms ν^a such that $\Omega^a_b = (\gamma^{-1})^a_c(d\mu^c_d)\gamma^d_b$ and $\Theta^a = (\gamma^a_c)^{-1}[\mu^c_b dx^b + d\nu^c]$. Hence, when $\epsilon = 0$, the flat connection one-form $\mathbf{A}_{(1)}$ is locally given in this gauge by

$$\mathbf{A}_{(1)}= \begin{pmatrix} (\gamma^{-1})^a_c d\gamma^c_b & (\gamma^{-1})^a_b dx^b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} (\gamma^{-1})^a_c(d\mu^c_d)\gamma^d_b & (\gamma^a_c)^{-1}[\mu^c_b dx^b + d\nu^c] \\ 0 & 0 \end{pmatrix}\mathbf{m}. \tag{53}$$

When $\epsilon = 1$ the generalized connection is flat if and only if

$$\begin{aligned} d\omega^a_b + \omega^a_c\omega^c_b - \Omega^a_b &= 0, \\ D\theta^a - \Theta^a &= 0. \end{aligned} \tag{54}$$

Then $\mathbf{A}_{(1)} = (\overset{0}{\mathbf{g}}\overset{0}{\mathbf{h}})^{-1}d(\overset{0}{\mathbf{g}}\overset{0}{\mathbf{h}})$ where, directly from Example 4.1, $\overset{0}{\mathbf{g}}$ is any closed zero-form in $IS(p,q)_{(1)}$, and

$$\overset{0}{\mathbf{h}} = \begin{pmatrix} \delta^a_b - \omega^a_b\mathbf{m} & -\theta^a\mathbf{m} \\ 0 & 1 \end{pmatrix}. \tag{55}$$

When the one-forms θ^a are linearly independent, so that they can form an orthonormal basis for the metric $\eta_{ab}\theta^a\otimes\theta^b$, Eqs.(54) are the Cartan structure equations for the $SO(p,q)$ metric. The metric connection ω^a_b has torsion Θ^a and curvature Ω^a_b. The equations, $\Omega^a_b\theta^b - D\Theta^a = 0$ and $D\Omega^a_b = 0$, which must also hold when the connection is flat are just the Bianchi identities. It should be noted that the Cartan structure equations, and Einstein's gravitational field equations can also be expressed as the flatness of type $SO(p,q)_{(2)}$ connections, [9].

In four dimensions the Cartan structure equations for a metric connection can also be obtained from another flat connection by using spinor representations. In the next example a brief outline of the Lorentzian case will be given, using the two component spinor conventions and notation of references [10] and [11].

Example 4.3:

Consider a four dimensional manifold M and a generalized connection represented by a complex 4×4 matrix-valued generalized one-form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}^A_B & \mathbf{A}^A_{B'} \\ 0 & \mathbf{A}^{A'}_{B'} \end{pmatrix} = \begin{pmatrix} \omega^A_B - \Omega^A_B\mathbf{m} & \theta^A_{B'} - \Theta^A_{B'}\mathbf{m} \\ 0 & \omega^{A'}_{B'} - \Omega^{A'}_{B'}\mathbf{m} \end{pmatrix}, \tag{56}$$

where $\theta^A_{B'}$, ω^A_B, $\omega^{A'}_{B'}$and $\Theta^A_{B'}$, Ω^A_B, $\Omega^{A'}_{B'}$ are respectively one-forms and two-forms. Let ω^A_B, Ω^A_B take values in the Lie algebra $sl(2,C)$ and let $\omega^{A'}_{B'}$, $\Omega^{A'}_{B'}$ be their complex conjugates. The operator D, whenever it is used, is the relevant covariant exterior derivative with respect to ω^A_B and $\omega^{A'}_{B'}$. The curvature $\mathbf{F}$=$d\mathbf{A} + \mathbf{A}\mathbf{A}$ is equal to

$$\begin{pmatrix} F^A_B - \epsilon\Omega^A_B - D\Omega^A_B\mathbf{m}, & D\theta^A_{B'} - \epsilon\Theta^A_{B'} + (\Omega^A_C\theta^C_{B'} - \theta^A_{C'}\Omega^{C'}_{B'} - D\Theta^A_{B'})\mathbf{m} \\ 0 & F^{A'}_{B'} - \epsilon\Omega^{A'}_{B'} - D\Omega^{A'}_{B'}\mathbf{m} \end{pmatrix}, \tag{57}$$

where $F^A_B = d\omega^A_B + \omega^A_C\omega^C_B$ and $F^{A'}_{B'} = d\omega^{A'}_{B'} + \omega^{A'}_{C'}\omega^{C'}_{B'}$. Henceforth in this example let $\epsilon = 1$. Consider the case where the one-forms $\theta^{AA'}$ form a null co-frame for a Lorentzian metric, $\epsilon_{AB}\epsilon_{A'B'}\theta^{AA'} \otimes \theta^{BB'}$. Then the generalized curvature $\mathbf{F}$ is zero if and only if the Cartan structure equations for the metric are satisfied, with metric connection $\omega^a_b \leftrightarrow (\omega^A_B\delta^{A'}_{B'} + \omega^{A'}_{B'}\delta^A_B)$. The torsion of this connection is $\Theta^a \leftrightarrow \Theta^{AA'}$ and its curvature two-form $\Omega^a_b \leftrightarrow (\Omega^A_B\delta^{A'}_{B'} + \Omega^{A'}_{B'}\delta^A_B)$. Now let $\mathbf{V} = \begin{pmatrix} \mathbf{V}^A \\ \mathbf{V}^{A'} \end{pmatrix}$ be a generalized 4×1 matrix-valued p-form. Its covariant exterior derivative, with respect to $\mathbf{A}$, is

$$\begin{aligned}\mathbf{DV} &= (d\mathbf{V} + \mathbf{AV}) \\ &= \begin{pmatrix} d\mathbf{V}^A + \mathbf{A}^A_B\mathbf{V}^B + \mathbf{A}^A_{B'}\mathbf{V}^{B'} \\ d\mathbf{V}^{A'} + \mathbf{A}^{A'}_{B'}\mathbf{V}^{B'} \end{pmatrix} \equiv \begin{pmatrix} \mathbf{DV}^A \\ \mathbf{DV}^{A'} \end{pmatrix},\end{aligned} \tag{58}$$

When $\mathbf{V}^A = \alpha^A + \beta^A\mathbf{m}$ and $\mathbf{V}^{A'} = \xi^{A'} + \zeta^{A'}\mathbf{m}$, where $\alpha^A, \xi^{A'}$ and $\beta^A, \zeta^{A'}$ are respectively spinor-valued ordinary p-forms and $(p+1)$-forms

$$\begin{aligned}\mathbf{DV}^A &= D\alpha^A + \theta^A_{B'}\xi^{B'} + (-1)^{p+1}\beta^A + \\ &\quad + [D\beta^A + \theta^A_{B'}\zeta^{B'} + (-1)^{p+1}\Theta^A_{B'}\xi^{B'} + (-1)^{p+1}\Omega^A_B\alpha^B]\mathbf{m}, \\ \mathbf{DV}^{A'} &= D\xi^{A'} + (-1)^{p+1}\zeta^{A'} + [D\zeta^{A'} + (-1)^{p+1}\Omega^{A'}_{B'}\xi^{B'}]\mathbf{m}.\end{aligned} \tag{59}$$

Now consider the case where the generalized curvature two-form $\mathbf{F}$ is zero and where $\mathbf{V}$ is the generalized *three*-form obtained by choosing

$$\begin{aligned}\alpha^A &= \rho_{A'}\eta^{AA'}, \beta^A = (2-\mu)\sigma^A\upsilon, \\ \xi^{A'} &= \sigma_A\eta^{AA'}, \zeta^{A'} = -\mu\rho^{A'}\upsilon.\end{aligned} \tag{60}$$

Here μ is a real constant, υ is the non-zero volume four-form and $\eta^{AA'} = \frac{i}{3}\theta^{AB'}\theta^{BA'}\theta_{BB'}$ is the basis of three-forms dual to $\theta^{AA'}$. Then

$$\mathbf{DV} = \begin{pmatrix} D^{AA'}\rho_{A'} - \mu\sigma^A + \rho_{A'}[\Theta_B{}^{A'AB} + \Theta_{B'}{}^{A.A'B'}] \\ D^{AA'}\sigma_A - \mu\rho^{A'} + \sigma_A[\Theta_B{}^{A'AB} + \Theta_{B'}{}^{A.A'B'}] \end{pmatrix}\upsilon, \tag{61}$$

where $D_{AA'}$ denotes the covariant derivative determined by the metric connection and the torsion has been expanded in terms of its components as

$$\Theta^{AA'} = \frac{1}{2}(\Theta^{AA'BC}\theta_{BC'}\theta^{C'}_C + \Theta^{AA'B'C'}\theta_{B'C}\theta^C_{C'}), \tag{62}$$

The complex conjugate components $\Theta^{AA'BC}$ and $\Theta^{AA'B'C'}$ are symmetric in their last two indices. The vanishing of the covariant exterior derivative gives a generalization of the Dirac equation. When the torsion is zero the covariant derivative is determined by the Levi Civita (spin) connection and the equation $\mathbf{DV} = 0$ holds if and only if the four spinor zero-form $\psi = (\sigma_A, \rho_{A'})$ satisfies the classical Dirac equation

$$D^{AA'}\rho_{A'} = \mu\sigma^A;\ D^{AA'}\sigma_A = \mu\rho^{A'}. \tag{63}$$

Similar results hold for split and Euclidean signature metrics. It is straightforward to see how to write down the associated gauge groups by appropriately modifying the discussion of $ISO(p,q)_{(1)}$ in the previous example.

The next example illustrates the use of type $N = 1$ forms in four dimensions, and the generalized Euler class, to construct a Lagrangian which has the Einstein vacuum field equations with non-zero cosmological constant as Euler-Lagrange equations. The idea of using generalized characteristic classes to construct Lagrangians was introduced in references [3] and [4] where Lagrangians for various field theories, including Einstein's vacuum field equations with a non-zero cosmological constant and Yang-Mills fields, were constructed by using the generalized second Chern class. The type of notation employed in Example 4.2 will be used again.

Example 4.4:

The generalized Euler class in four dimensions is defined, by analogy with the standard definition, to be

$$\mathbf{E}_{(N)} = \frac{1}{32\pi^2}\varepsilon_{abcd}\mathbf{F}^{ab}_{(N)}\mathbf{F}^{cd}_{(N)}, \tag{64}$$

where the Latin indices sum and range over 1 to 4, ε_{abcd} is the totally antisymmetric Levi-Civita symbol, and $\mathbf{F}^{ab}_{(N)}$ is the generalized curvature of a type N generalized connection with $G = SO(r,s)$ and $r + s = 4$. This is also invariant under generalized gauge transformations. In four dimensions only the first two terms in the expansion of the expression for the connection contribute to $\mathbf{E}_{(N)}$ and ${}_2\mathbf{C}_{(N)}$. In this sense, $\mathbf{E}_{(N)} = \mathbf{E}_{(1)}$ and ${}_2\mathbf{C}_{(N)} ={}_2 \mathbf{C}_{(1)}$ for all $N \geq 1$. Therefore here only type $N = 1$ connections $\mathbf{A}^a_b$, with generalized curvature two-form $\mathbf{F}^a_b$, will be used. In the remainder of this example, and the next, the choice $d\mathbf{m} = 1$ will be made. Choosing

$$\mathbf{A}^a_b = \omega^a_b + k\theta^a\theta_b\mathbf{m}, \tag{65}$$

where k is a non-zero constant, it follows that

$$\mathbf{F}^a_b = \Omega^a_b + k\theta^a\theta_b + kD(\theta^a\theta_b)\mathbf{m}. \tag{66}$$

When $\mathbf{F}^a_b = 0$ the metric $\eta_{ab}\theta^a \otimes \theta^b$ has a torsion free metric connection ω^a_b with constant Riemannian curvature. The generalized Euler class is given by

$$\begin{aligned} \mathbf{E}_{(1)} &= E_{(0)} + L, \\ L &= \frac{k}{16\pi^2}(\varepsilon_{abcd}\Omega^{ab}\theta^c\theta^d + \frac{k}{2}\varepsilon_{abcd}\theta^a\theta^b\theta^c\theta^d). \end{aligned} \tag{67}$$

Here $E_{(0)}$ is the ordinary Euler class, $E_{(0)} = \frac{1}{32\pi^2}(\varepsilon_{abcd}\Omega^{ab}\Omega^{cd})$. Consequently $\mathbf{E}_{(1)}$ is in fact an ordinary four form, the sum of the topological term $E_{(0)}$ and the term L. The latter is essentially the usual first order Lagrangian four-form for Einstein's vacuum equations with non-zero cosmological constant. This result suggests that $\mathbf{E}_{(1)}$ is a natural Lagrangian four-form for four dimensional gravity when the cosmological constant is assumed to be non-zero. Examples of recent investigations employing Lagrangians in four dimensions which include a cosmological constant and a Gauss-Bonnet term can be found in references [12] and [13].

Example 4.5:

In four dimensions, the generalized second Chern class, ${}_2\mathbf{C}_{(1)} \equiv \mathbf{C}$, and generalized first Pontrjagin class, ${}_1\mathbf{P}_{(1)} \equiv \mathbf{P}$, corresponding to a type $N = 1$ (zero trace) generalized connection $\mathbf{A} = \alpha + \beta\mathbf{m}$, with curvature $\mathbf{F} = \Omega + \beta + D\beta\mathbf{m}$, have the form

$$\frac{\kappa}{8\pi^2}Tr(\Omega + \beta)^2 = \frac{\kappa}{8\pi^2}Tr(\Omega\Omega + 2\Omega\beta + \beta\beta), \tag{68}$$

where $\Omega = d\alpha + \alpha\alpha$, and $\kappa = 1$ for $\mathbf{C}$ and $\kappa = -1$ for $\mathbf{P}$. When α and β are respectively given by the $so(r,s)$-valued forms ω^a_b and $(a\theta^a\theta_b + \frac{b}{2}\epsilon^a_{bcd}\theta^c\theta^d)$, where a and b are constants and $r + s = 4$, the generalized first Pontrjagin class takes the form

$$P = -\frac{1}{8\pi^2}(\Omega^a_b\Omega^b_a + 2a\Omega^a_b\theta^b\theta_a + b\Omega^a_b\varepsilon^b_{acd}\theta^c\theta^d - ab\varepsilon_{abcd}\theta^a\theta^b\theta^c\theta^d). \tag{69}$$

The first term in this ordinary four-form is the ordinary first Pontrjagin class, ${}_1P_{(o)}$, corresponding to the connection one-form α. Altogether $\mathbf{P}$, with appropriate choices of a and non-zero b, is also a first order Lagrangian four-form for Einstein's vacuum equations. It includes the topological term ${}_1P_{(0)}$ and the choice $a = 0$ corresponds to the zero cosmological constant case.

Finally it should noted that when a fixed metric with signature (r, s) is assumed given on a four dimensional manifold M, and the two-form β is chosen to be $\beta = a\Omega + b * \Omega$, where $*$ denotes the Hodge dual, then the generalized second Chern class is the ordinary four-form

$$\mathbf{C} = \frac{1}{8\pi^2}\{[(1+a)^2 + (-1)^r b^2]Tr(\Omega\Omega) + 2b[1+a]Tr(\Omega * \Omega)\}. \tag{70}$$

This is essentially just the sum of the ordinary second Chern class and the usual Lagrangian four-form for the source-free Yang-Mills equations on M. This Lagrangian is a small generalization of the Yang-Mills Lagrangian computed in [4]. As a particular example consider the case where the metric has Euclidean signature, $b = (1 + a)$ and the gauge group is $SU(2$. Then the Lagrangian corresponding to $\mathbf{C}$ is non-negative and attains its minimum of zero when the generalized curvature

$$\mathbf{F} = (1 + a)[(\Omega + *\Omega) + D(*\Omega)\mathbf{m}] \tag{71}$$

is zero; that is when the ordinary curvature two-form Ω is anti-self dual.

5 Summary

The theory and formalism of generalized forms has been developed more fully than previously. In particular bases of minus one-form have been explored and used. The introduction and use of canonical bases has enabled a number of results to be expressed with increased generality. Previous studies of different representations of generalized forms, matrix groups of generalized forms and generalized connections have been extended, and a number of examples have been presented. In particular the Cartan structure equations and Lagrangians for relativistic field theories have been re-formulated within the context of the theory of generalized connections. Topics for future investigation include the global formulation of generalize connections and the development of further applications to physically interesting systems in different dimensions.

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6 Appendix: Representations of generalized forms

For calculational purposes the rules given in the previous sections suffice. However conceptually it is useful to be able to represent generalized forms and their exterior products and derivatives purely in terms of ordinary forms. In this appendix two such representations will be exhibited. First a representation of generalized forms in terms of matrix-valued forms - matrices with entries taking values in ordinary forms - will be recalled. The exterior product of generalized forms is represented by the matrix product of the matrix-valued forms and the exterior derivative is represented by the action of nilpotent differential operators, $d_{(N)}$, on the type N matrix-valued forms. Second an interesting representation of the exterior derivative of a generalized p-form on M will be exhibited in terms of the ordinary exterior derivative of an ordinary $(p+N)$-form, on an $n+N$ dimensional manifold $M \times I^N$, where locally $I \subseteq R$ can be taken to be an interval on the real line. This latter representation of the exterior derivative was first noted by Paul-Andi Nagy in the case of $N=1$ forms. Here this representation will be explicitly constructed for $N=1$ and $N=2$ forms and it will be shown how to generalize these constructions to forms of higher type.

In the first representation of a generalized form $\overset{p}{\mathbf{a}}_{(N)}$, introduced in [8], generalized forms of type $N>1$ are identified with $2^N \times 2^N$ matrix-valued forms $[\overset{p}{\mathbf{a}}_{(N)}] \in \overset{p}{M}_{(N)}$. The expressions for these matrices are found by exploiting Eq.(2) and the definition

$$[\overset{p}{\mathbf{a}}_{(N)}] = \begin{pmatrix} [\overset{p}{\mathbf{a}}_{(N-1)}] & [\overset{p+1}{\mathbf{a}}_{(N-1)}] \\ 0 & (-1)^p[\overset{p}{\mathbf{a}}_{(N-1)}] \end{pmatrix}. \tag{72}$$

The matrix representing the exterior product $\overset{p}{\mathbf{a}}_{(N)}\overset{q}{\mathbf{b}}_{(N)}$, denoted by $[\overset{p}{\mathbf{a}}_{(N)}\overset{q}{\mathbf{b}}_{(N)}]$, is equal to the matrix product $[\overset{p}{\mathbf{a}}_{(N)}][\overset{q}{\mathbf{b}}_{(N)}]$. The matrix representation of the exterior derivative of $\overset{p}{\mathbf{a}}_{(N)}$, $[d\overset{p}{\mathbf{a}}_{(N)}]$, is equal to $d_{(N)}[\overset{p}{\mathbf{a}}_{(N)}]$, where the nilpotent operator $d_{(N)} : \overset{p}{M}_{(N)} \to \overset{p+1}{M}_{(N)}$, is given by $d_{(0)} = d$, and for $N > 0$

$$d_{(N)}[\overset{p}{\mathbf{a}}_{(N)}] = S_{(N)} d[\overset{p}{\mathbf{a}}_{(N)}] + \{K_{(N)}, [\overset{p}{\mathbf{a}}_{(N)}]\}_{p+1}. \tag{73}$$

Here the bracket of $2^N \times 2^N$ matrices A and B is defined by $\{A, B\}_r = AB + (-1)^r BA$, and the $2^N \times 2^N$ constant matrices $S_{(N)}$ and $K_{(N)}$ satisfy $(S_{(N)})^2 = 1_{2^N \times 2^N}$, $(K_{(N)})^2 = 0$ and $K_{(N)} S_{(N)} + S_{(N)} K_{(N)} = 0$. The nilpotent operator $d_{(N)}$ must also satisfy the usual graded Leibniz rule when acting on products. The following example contains these matrices when $N = 1$ and $N = 2$ and corrects small errors in the discussion in reference [8].

Example A.1:

The matrix representation, $[d\overset{p}{\mathbf{a}}_{(2)}]$, of the exterior derivative of $\overset{p}{\mathbf{a}}_{(2)}$is equal to $d_{(2)}[\overset{p}{\mathbf{a}}_{(2)}]$ where, since $d\mathbf{m}^i = \epsilon \delta_1^i$,

$$\begin{aligned} d_{(2)}[\overset{p}{\mathbf{a}}_{(2)}] &= S_{(2)} d[\overset{p}{\mathbf{a}}_{(2)}] + \{K_{(2)}, [\overset{p}{\mathbf{a}}_{(2)}]\}_{p+1}; \\ S_{(2)} &= \begin{pmatrix} S_{(1)} & 0 \\ 0 & -S_{(1)} \end{pmatrix}, K_{(2)} = \begin{pmatrix} K_{(1)} & 0 \\ 0 & -K_{(1)} \end{pmatrix}, \\ S_{(1)} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, K_{(1)} = \begin{pmatrix} 0 & 0 \\ \epsilon & 0 \end{pmatrix}. \end{aligned} \tag{74}$$

The operator $d_{(2)}$ is nilpotent and satisfies the graded Leibniz rule

$$d_{(2)}[\overset{p}{\mathbf{a}}_{(2)} \overset{q}{\mathbf{b}}_{(2)}] = d_{(2)}[\overset{p}{\mathbf{a}}_{(2)}][\overset{q}{\mathbf{b}}_{(2)}] + (-1)^p [\overset{p}{\mathbf{a}}_{(2)}] d_{(2)}[\overset{q}{\mathbf{b}}_{(2)}]. \tag{75}$$

It is a straightforward matter to compute the constant matrices $K_{(N)}$ and $S_{(N)}$, and hence $d_{(N)}$, when $N > 2$. From Eq.(72) it follows that the 4×4 matrix representations of the type $N = 2$ canonical basis minus one-forms $\{m^i\}$ are

$$\begin{aligned} [\mathbf{m}^1] &= \begin{pmatrix} [\mathbf{m}] & 0_{2\times 2} \\ 0_{2\times 2} & -[\mathbf{m}] \end{pmatrix}, \text{ where } [\mathbf{m}] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \\ [\mathbf{m}^2] &= \begin{pmatrix} 0_{2\times 2} & 1_{2\times 2} \\ 0_{2\times 2} & 0_{2\times 2} \end{pmatrix}, \end{aligned} \tag{76}$$

and $d_{(2)}[\mathbf{m}^1] = \epsilon 1_{4\times 4}$, $d_{(2)}[\mathbf{m}^2] = 0$.

Example A.2

Let $\mathbf{A}$ be a type $N = 1$ ($s \times s$ matrix-valued) connection one-form with curvature two-form $\mathbf{F} = d\mathbf{A} + \mathbf{A}\mathbf{A}$. Then in the representation of generalized forms being considered here the standard formulae relating Lie algebra valued ordinary forms are replaced by analogous formulae which include ordinary forms of higher degree. For example here the connection matrix $[\mathbf{A}]$ is a $2s \times 2s$ matrix with entries ordinary one- and two forms (as in Eq.(72) but with $s \times s$ matrix valued ordinary forms inserted in the formula). The corresponding curvature matrix, $[\mathbf{F}]$, is

$$[\mathbf{F}] = d_{(1)}[\mathbf{A}] + [\mathbf{A}][\mathbf{A}], \tag{77}$$

where

$$d_{(1)}[\mathbf{A}] = Sd[\mathbf{A}] + K[\mathbf{A}] + [\mathbf{A}]K, \tag{78}$$

and the $2s \times 2s$ constant matrices S and K are given by

$$S = \begin{pmatrix} 1_{s\times s} & 0 \\ 0 & -1_{s\times s} \end{pmatrix}, \; K = \begin{pmatrix} 0 & 0 \\ \epsilon 1_{s\times s} & 0 \end{pmatrix}. \tag{79}$$

This expression for the curvature generalizes the standard formula to include ordinary forms of different degrees and can clearly be extended to the cases where $N > 1$.

In this first representation the exterior product of generalized forms is represented by the ordinary exterior product, albeit applied to matrix-valued forms, while the ordinary exterior derivative d is replaced by the differential operator $d_{(N)}$. In the second representation of type $N \geq 1$ generalized forms to be discussed in this appendix, the ordinary exterior derivative is retained, albeit applied to ordinary $(p+N)$-forms defined on manifolds of dimensions $n+N$. While the exterior product of generalized p-forms is not reproduced by the ordinary exterior products of these $(p+N)$-forms it is still useful to have this representation of the exterior derivative in, for example, the exploration of integrals and Stokes theorem for generalized forms.

Consider first generalized forms of type $N = 1$. Let $\overset{p}{\mathbf{a}}_{(1)} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m}$ be a type $N = 1$ generalized p-form on M. Then on $M \times I$ where I is an interval on the line with coordinate y, define the ordinary $(p+1)$-form

$$\overset{p+1}{\alpha}_{(1)} \equiv \overset{p}{\alpha} dy + y^{\epsilon} \overset{p+1}{\alpha}. \tag{80}$$

Comparing the exterior derivatives of $\overset{p}{\mathbf{a}}_{(1)}$and $\alpha_{(1)}$,

$$\begin{aligned} d\overset{p}{\mathbf{a}}_{(1)} &= [d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}] + d\overset{p+1}{\alpha}\mathbf{m}, \\ d\overset{p+1}{\alpha}_{(1)} &= [d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}]dy + y^{\epsilon} d\overset{p+1}{\alpha}, \end{aligned} \tag{81}$$

it is clear that the association of $\overset{p}{\mathbf{a}}_{(1)}$ with $\overset{p+1}{\alpha}_{(1)}$ is identical to the association of $d\overset{p}{\mathbf{a}}_{(1)}$with $d\overset{p+1}{\alpha}_{(1)}$. This is the observation of Paul-Andi Nagy. From this correspondence it follows that $\overset{p}{\mathbf{a}}_{(1)}$is closed if and only if $\overset{p+1}{\alpha}_{(1)}$is closed. When $\epsilon = 1$, $\overset{p+1}{\alpha}_{(1)}$ is closed if and only if it is exact; if $d\overset{p}{\mathbf{a}}_{(1)} = 0$ then $d\overset{p+1}{\alpha}_{(1)} = 0$ and it follows that $\overset{p}{\mathbf{a}}_{(1)} = (-1)^p d(\overset{p}{\alpha}\mathbf{m})$ and $\overset{p+1}{\alpha}_{(1)} = (-1)^p d(y\overset{p}{\alpha})$.

Example A.3

A Beltrami vector field on Euclidean three-space, $\mathbb{E}^3$, is a vector field $\overrightarrow{\alpha}$ which is parallel or anti-parallel to its curl, that is $curl\overrightarrow{\alpha} = \sigma\overrightarrow{\alpha}$, where σ is a non-zero function. This condition can be re-expressed in terms of the one-form α, corresponding to $\overrightarrow{\alpha}$ via the usual metric isomorphism, as the condition that $d\alpha = \sigma * \alpha$, where $*$ denotes the Hodge dual. This latter equality can be used to define Beltrami vector fields on any 3-manifold, with metric, M^3. Beltrami vector fields on $\mathbb{E}^3$ and other three dimensional manifolds arise in many physically interesting contexts such as fluid dynamics, magnetohydrodynamics and plasma physics, see for example, [14], [15]. On M^3 consider the type $N = 1$ generalized one-form $\mathbf{a} = \alpha - \sigma * \alpha\mathbf{m}$ with $d\mathbf{m} = 1$. This generalized form is closed (and therefore exact) if and only if α defines a Beltrami vector field $\overrightarrow{\alpha}$. Hence $\overrightarrow{\alpha}$ is a Beltrami vector field on M^3 if and only if on $M^3 \times I$ the two-form $\overset{2}{\alpha}_{(1)} \equiv \alpha dy - y\sigma * \alpha$ is closed. When the metric has Euclidean signature, and I does not contain $y = 0$, the non-zero form $\overset{2}{\alpha}_{(1)}$is of maximal rank. In this case any non-zero Beltrami vector field on M^3 defines a symplectic structure on $M^3 \times I$.

A straightforward extension of the definition is suggested by this formulation of the Beltrami condition in terms of generalized forms. For an n dimensional manifold M with metric, the dual, $*\alpha$, of a p-form α is a $(n-p)$-form. Consider therefore the generalized p-form on M, given by $\mathbf{a} = \alpha - \sigma * \alpha\mathbf{m}$, where σ is a non-zero $(2p+1-n)-$form . The generalized p-form $\mathbf{a}$ is closed if and only if α satisfies the "generalized Beltrami condition", $d\alpha + (-1)^p\sigma * \alpha = 0$. When $n = 3$ the two possible choices are $p = 1$ and $p = 2$. In the second case, the latter condition, expressed in terms of dual pseudo-vector fields $\overrightarrow{\alpha}$ and $\overrightarrow{\sigma}$ takes the form $div(\overrightarrow{\alpha}) + \langle\overrightarrow{\alpha}, \overrightarrow{\sigma}\rangle = 0$, where $\langle\overrightarrow{\alpha}, \overrightarrow{\sigma}\rangle$ denotes the metric inner product. This is a restriction on $\overrightarrow{\alpha}$ only when $\overrightarrow{\sigma}$ is assumed given.

Next consider forms of type $N = 2$. Let $\overset{p}{\mathbf{a}}_{(2)} = \overset{p}{\mathbf{a}}_{(1)} + \overset{p+1}{\mathbf{a}}_{(1)}\mathbf{m}^2$ where $\overset{p}{\mathbf{a}}_{(1)} = \overset{p}{\alpha} + \overset{p+1}{\alpha}_1\mathbf{m}^1$ and $\overset{p+1}{\mathbf{a}}_{(1)} = \overset{p+1}{\alpha}_2 + \overset{p+2}{\alpha}\mathbf{m}^1$ so that $\overset{p}{\mathbf{a}}_{(2)} = \overset{p}{\alpha} + \overset{p+1}{\alpha}_i\mathbf{m}^i + \overset{p+2}{\alpha}\mathbf{m}^1\mathbf{m}^2$.

Then on $M \times I^2$, with coordinates y^1, y^2 on I^2, define the ordinary $(p+2)$-form

$$\begin{aligned}
\overset{p+2}{\alpha}_{(2)} &\equiv \overset{p+1}{\alpha}_{(1)} dy^2 + \overset{p+2}{\alpha}_{(1)} \equiv (\overset{p}{\alpha} dy^1 + (y^1)^{\epsilon} \overset{p+1}{\alpha}_1) dy^2 + (\overset{p+1}{\alpha}_2 dy^1 + (y^1)^{\epsilon} \overset{p+2}{\alpha}) \\
&= \overset{p}{\alpha} dy^1 dy^2 + (y^1)^{\epsilon} \overset{p+1}{\alpha}_1 dy^2 + \overset{p+1}{\alpha}_2 dy^1 + (y^1)^{\epsilon} \overset{p+2}{\alpha}. \qquad (82)
\end{aligned}$$

Here Eq.(2) and the above constructions for $N = 1$ forms have been used to construct $\overset{p+2}{\alpha}_{(2)}$. Comparing the exterior derivatives of $\overset{p}{\mathbf{a}}$, and $\alpha_{(2)}$

$$\begin{aligned}
d\overset{p}{\mathbf{a}} &= [d\overset{p}{\alpha} + (-1)^{p+1} \epsilon \overset{p+1}{\alpha}_1] + d\overset{p+1}{\alpha}_1 \mathbf{m}^1 + \\
&\quad + [d\overset{p}{\alpha}_2 + (-1)^{p} \epsilon \overset{p+2}{\alpha}] \mathbf{m}^2 + d\overset{p+2}{\alpha} \mathbf{m}^1 \mathbf{m}^2, \qquad (83) \\
d\overset{p+2}{\alpha}_{(2)} &= [d\overset{p}{\alpha} + (-1)^{p+1} \epsilon \overset{p+1}{\alpha}_1] dy^1 dy^2 + (y^1)^{\epsilon} d\overset{p+1}{\alpha}_1 dy^2 + \\
&\quad + [d\overset{p+1}{\alpha}_2 + (-1)^{p} \epsilon \overset{p+2}{\alpha}] dy^1 + (y^1)^{\epsilon} d\overset{p+2}{\alpha},
\end{aligned}$$

it is clear that the association of $\overset{p}{\mathbf{a}}$ with $\overset{p+2}{\alpha}_{(2)}$ is identical to the association of $d\overset{p}{\mathbf{a}}$ with $d\overset{p+2}{\alpha}_{(2)}$. In addition, $\overset{p}{\mathbf{a}}$ is closed if and only if $\overset{p+2}{\alpha}_{(2)}$ is closed. When $\epsilon = 1$ both $\overset{p}{\mathbf{a}}$ and $\overset{p+2}{\alpha}_{(2)}$ are closed if and only if they are exact; when they are closed they are equal to the exact forms $\overset{p}{\mathbf{a}} = (-1)^p d[\overset{p}{\alpha} \mathbf{m}^1 - \overset{p+1}{\alpha}_2 \mathbf{m}^1 \mathbf{m}^2]$ and $\alpha_{(2)} = (-1)^p d[y^1 \overset{p}{\alpha} dy^2 - y^1 \overset{p+1}{\alpha}_2]$.

For generalized forms of type $N > 2$ the analogous correspondence is between generalized forms $\overset{p}{\mathbf{a}}_{(N)} = \overset{p}{\mathbf{a}}_{(N-1)} + \overset{p+1}{\mathbf{a}}_{(N-1)} \mathbf{m}^N$ on M and ordinary $(p+N)-$forms on $M \times I^N$. With coordinates $y^1, ..., y^N$ on I^N, $\overset{p}{\mathbf{a}}_{(N)}$corresponds to $\overset{p+N}{\alpha}_{(N)} = \overset{p+N-1}{\alpha}_{(N-1)} dy^N + \overset{p+N}{\alpha}_{(N-1)}$,where $\overset{p}{\mathbf{a}}_{(N-1)}$and $\overset{p+1}{\mathbf{a}}_{(N-1)}$respectively correspond to $\overset{p+N-1}{\alpha}_{(N-1)}$and $\overset{p+N}{\alpha}_{(N-1)}$on $M \times I^{N-1}$. It should be noted that this representation depends on the choice of canonical basis.

The exterior product of two type N generalized forms, say a p- and a q-form, does not correspond to the exterior product of the related $(p+N)$- and $(q+N)$- ordinary forms. The product of the two generalized forms is a $(p+q)$-generalized form of type N and so corresponds to a $(p+q+N)$-ordinary form. In fact the exterior product of the generalized forms can be viewed as defining a new product for ordinary forms belonging to the subset $\{\overset{p+N}{\alpha}_{(N)}\}$.

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