Addendum to Generalized forms, Chern-Simons and Einstein-Yang-Mills theory

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Abstract

This technical report is an updated addendum to [1] where polychains and a Stokes' theorem for generalized forms were introduced. Here these concepts are studied further with particular emphasis on type $N = 2$ generalized forms. The second Chern class and first Pontrjagin class and their variations are considered. An example related to the full and linearized Einstein vacuum field equations is also presented. This addendum includes review material in order to make it reasonably self contained.

1 Introduction

In [1] polychains (poly-chains) and a Stokes' theorem for generalized forms were introduced and used in the construction of generalized characteristic classes and actions for Einstein-Yang-Mills theory. Whereas an ordinary characteristic class may be obtained by integrating an ordinary differential form over an ordinary chain, generalized characteristic classes are obtained by integrating generalized forms over polychains. The latter are ordered sets of ordinary (real, singular) chains. The primary focus in this addendum is on type $N = 2$ generalized forms and its main purpose is to exhibit their use in the construction of generalized second Chern classes, generalized first Pontrjagin classes and Chern-Simons three-forms.

An outline of the properties of type N generalized forms and integration over type N polychains is given in section two. As in [1] two cases, corresponding to two classes of exterior derivatives, are carried along. In the third section these results are considered further for type $N = 2$ forms. Both of these sections contain discussions of generalized connections, the generalized second Chern class and first Pontrjagin class, and generalized Chern-Simons three forms.

The fourth section contains an example in which a class of type $N = 2$ $so(p, q)-$valued connections is exhibited. These connections and fields are first defined on a manifold M, of dimension $n \geqq p+q$ which is not assumed to be metric. Metric structures on the chains, if they exist, are defined by the pull-backs of the generalized connections and their group structure. There can be singularities in the metric geometry, or no metric geometry at all, on the polychains. These connections are flat on the polychains when either the full Einstein's vacuum field equations are satisfied or the linearized equations are satisfied, depending on which class of exterior deriviative is used.

The notation employed in this report is the same as in [1]. Bold-face Roman letters again denote generalized forms. Ordinary forms are again denoted by Greek letters. Where it is useful the degree of a form is indicated above it, for example $\overset{p}{\mathbf{a}}$. When the degree is obvious from the context it will not be indicated so explicitly. The exterior product of any two forms, α and β, is written $\alpha\beta$. As usual any ordinary form $\overset{q}{\alpha}$, with q either negative or greater than n is zero. Hence type N generalized forms, $\overset{p}{\mathbf{a}}$, with p less than minus N or greater than n are zero. This report develops results which can be found primarily in [1], [2] and [5]. References to other work on generalized forms can be found in those papers.

2 Review of generalized forms

A generalized p-form of type $N = 0$ is an ordinary differential p-form satisfying the standard exterior algebra and calculus. Any generalized p-form of type $N \geq 1$ can be uniquely expressed in terms of a basis of ordinary forms on M augmented by N linearly independent minus one-forms, $\{\mathbf{m}^i\}$ $(i = 1, 2, .., N)$. These latter quantities have the same algebraic properties as ordinary exterior p-forms, apart from p taking the value minus one in the usual formulae. In particular they are assumed to satisfy the ordinary distributive and associative laws of exterior algebra; the product rules, $\mathbf{m}^i\mathbf{m}^j = -\mathbf{m}^j\mathbf{m}^i$ and $\overset{p}{\alpha}\mathbf{m}^i = (-1)^p\mathbf{m}^i\overset{p}{\alpha}$, where $\overset{p}{\alpha}$ is any ordinary p-form; together with the condition of linear independence, $\mathbf{m}^1\mathbf{m}^2....\mathbf{m}^N \neq 0$. A generalized p-form of type $N \geq 1$, $\overset{p}{\mathbf{a}}_{(N)} \in \Lambda^p_{(N)}(M)$, is thus a geometrical object with a unique expansion of the form

$$\overset{p}{\mathbf{a}}_{(N)} = \overset{p}{\alpha} + \sum_{1 \leqq j \leqq N} \frac{1}{j!}\overset{p+j}{\alpha}_{i_1....i_j}\mathbf{m}^{i_1}.....\mathbf{m}^{i_j},$$

where $\overset{p}{\alpha}, \overset{p+j}{\alpha}_{i_1....i_j} = \overset{p+j}{\alpha}_{[i_1....i_j]}$ are, respectively, p-and $(p+j)$- ordinary forms; $-N \leqq p \leqq n$, $1 \leqq j \leqq N$ and $i_1,...i_j, ..., i_N$ range and sum over $1, 2, ..\ N$. It follows that generalized forms satisfy the usual distributive and associative laws of exterior algebra, together with the product rule

$$\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}} = (-1)^{pq}\overset{q}{\mathbf{b}}\overset{p}{\mathbf{a}}. \tag{1}$$

For each allowed value of p, $\Lambda^p_{(N)}(M)$ is a module over the ring of function on M and $\Lambda_{(N)}(M) = \oplus_{p=-N}^{p=n}\Lambda^p_{(N)}(M)$ is a graded algebra with the above exterior product $\Lambda^p_{(N)}(M) \times \Lambda^q_{(N)}(M) \rightarrow \Lambda^{p+q}_{(N)}(M)$.

Any p-form of type $N \geq 1$ can be expressed as a of a pair generalized forms of type $N - 1$, that is as

$$\overset{p}{\mathbf{a}}_{(N)} \equiv \overset{p}{\mathbf{a}}_{(N-1)} + \overset{p+1}{\mathbf{a}}_{(N-1)}\mathbf{m}^N, \tag{2}$$

and, when $N > 1$, each of the type $(N - 1)$ forms can be expressed in terms of an ordered pair of $(N - 2)$ forms, and so on. It follows that if

$\overset{q}{\mathbf{b}}_N = \overset{q}{\mathbf{b}}_{(N-1)} + \overset{q+1}{\mathbf{b}}_{(N-1)}\mathbf{m}^N$ is a q-form of type $N \geq 1$, then the exterior product of $\overset{p}{\mathbf{a}}_{(N)}$and $\overset{q}{\mathbf{b}}_{(N)}$is the p+q-form of type N given (recursively) by

$$\overset{p}{\mathbf{a}}_{(N)}\overset{q}{\mathbf{b}}_{(N)} = \overset{p}{\mathbf{a}}_{(N-1)}\overset{q}{\mathbf{b}}_{(N-1)} + [\overset{p}{\mathbf{a}}_{(N-1)}\overset{q+1}{\mathbf{b}}_{(N-1)} + (-1)^q\overset{p+1}{\mathbf{a}}_{(N-1)}\overset{q}{\mathbf{b}}_{(N-1)}]\mathbf{m}^N. \quad (3)$$

If φ is a smooth map $\varphi : M_1 \to M_2$, then the induced map of generalized forms, $\varphi^*_{(N)} : \Lambda^p_{(N)}(M_2) \to \Lambda^p_{(N)}(M_1)$, is the linear map defined (recursively) by using the standard pull-back map for ordinary forms

$$\varphi^*_{(N)}(\overset{p}{\mathbf{a}}) = \varphi^*_{(N-1)}(\overset{p}{\mathbf{a}}_{(N-1)}) + \varphi^*_{(N-1)}(\overset{p+1}{\mathbf{a}}_{(N-1)})\mathbf{m}^N, \quad (4)$$

with $\varphi^*_{(0)}$ being the ordinary map, φ^*, of ordinary forms; then $\varphi^*_{(N)}(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) = \varphi^*_{(N)}(\overset{p}{\mathbf{a}})\varphi^*_{(N)}(\overset{q}{\mathbf{b}})$. In future $\varphi^*_{(N)}$ will be written φ^*.

In the exterior calculus of generalized forms the exterior derivative operators $d : \Lambda^p_{(N)} \to \Lambda^{p+1}_{(N)}$ agree with the usual exterior derivative when they act on type $N = 0$ (ordinary) forms and the usual type of rules are also satisfied when $N > 0$, that is

$$\begin{aligned} d(\overset{p}{\mathbf{a}} + \overset{p}{\mathbf{b}}) &= d\overset{p}{\mathbf{a}} + d\overset{p}{\mathbf{b}},\ d\varphi(X) = X(\varphi), \\ d(\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}) &= (d\overset{p}{\mathbf{a}})\overset{q}{\mathbf{b}} + (-1)^p\overset{p}{\mathbf{a}}d\overset{q}{\mathbf{b}}, \\ d^2\overset{p}{\mathbf{a}} &= 0. \end{aligned} \quad (5)$$

where X is any vector field and φ any function on M. As is well known, when $N = 0$ the exterior derivative $d : \Lambda^p_{(0)}(M) \to \Lambda^{p+1}_{(0)}(M)$ satisfies these conditions and is unique [3], but when N is greater than zero d is not unique [2]. Here, as in [2] and [1] two cases will be considerd. These correspond to the choices of bases for which

$$d\mathbf{m}^i = \delta^i_1\epsilon, \quad (6)$$

$1 \leqq i \leqq N$. In the first case $\epsilon = 0$. In the second case ϵ is a non-zero constant. It can be convenient to scale the minus one-form $\mathbf{m}^1$ so that $\epsilon = 1$ but that will not be done here. These bases are not necessarily unique but will be assumed fixed in this paper.

It follows that the exterior derivative of a generalized $p-$form $\overset{p}{\mathbf{a}}_{(N)}$is the generalized $(p+1)-$form

$$d\overset{p}{\mathbf{a}}_{(N)} = d(\overset{p}{\mathbf{a}}_{(N-1)}) + d(\overset{p+1}{\mathbf{a}}_{(N-1)})\mathbf{m}^N, \tag{7}$$

when $N > 1$, and when $N = 1$

$$d\overset{p}{\mathbf{a}}_{(1)} = d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha} + d\overset{p+1}{\alpha}\mathbf{m}^1, \tag{8}$$

where $\overset{p}{\mathbf{a}}_{(1)} = \overset{p}{\alpha} + \overset{p+1}{\alpha}\mathbf{m}^1$. These exterior derivatives cleary satisfy $\varphi^*(d\overset{p}{\mathbf{a}}_{(N)}) = d(\varphi^*\overset{p}{\mathbf{a}}_{(N)})$. In the second case, where ϵ is non zero, a closed type $N > 0$ form is closed if and only if it is exact and the cohomology is trivial. In the first case a closed type $N > 0$ is closed (respectively exact) if and only if all the closed ordinary forms defining it are closed (respectively exact).

Integration of generalized forms is defined by using polychains [1]. The latter may be defined recursively. A $p-$polychain of type $N = 1$ in M is an ordered pair of type $N = 0$, or ordinary, (real, singular) chains in M

$$\mathbf{c}_p^{(1)} = (c_p, c_{p+1}), \tag{9}$$

where c_p is an ordinary $p-$chain and c_{p+1} is an ordinary $p+1-$chain. When $N > 1$ a $p-$polychain of type N in M is an ordered pair of type $N-1$ (real, singular) chains in M

$$\mathbf{c}_p^{(N)} = (\mathbf{c}_p^{(N-1)}, \mathbf{c}_{p+1}^{(N-1)}), \tag{10}$$

If the ordinary chains have respective boundaries $\partial c_p, \partial c_{p+1}$ the boundary of the polychain $\mathbf{c}_p^{(1)}$ is the $(p-1)-$polychain

$$\partial\mathbf{c}_p^{(1)} = (\partial c_p, \partial c_{p+1} + (-1)^p\epsilon c_p), \tag{11}$$

and when $N > 1$ the boundary is

$$\partial\mathbf{c}_p^{(N)} = (\partial\mathbf{c}_p^{(N-1)}, \partial\mathbf{c}_{p+1}^{(N-1)}). \tag{12}$$

When $N > 0$ the integral of a generalized form $\overset{p}{\mathbf{a}}_{(N)}$over a polychain $\mathbf{c}_p^{(N)}$ is

$$\int_{\mathbf{c}_p^{(N)}} \overset{p}{\mathbf{a}}_{(N)} = \int_{\mathbf{c}_p^{(N-1)}} \overset{p}{\mathbf{a}}_{(N-1)} + \int_{\mathbf{c}_{p+1}^{(N-1)}} \overset{p+1}{\mathbf{a}}_{(N-1)}. \tag{13}$$

Stokes' theorem for generalized forms and polychains is

$$\int_{\mathbf{c}_p^{(N)}} d\overset{p-1}{\mathbf{a}}_{(N)} = \int_{\partial\mathbf{c}_p^{(N)}} \overset{p-1}{\mathbf{a}}_{(N)}. \tag{14}$$

Lie groups of generalized zero-forms of type $N > 1$ can also be constructed iteratively from the $N = 1$ case. Let $\mathbf{G}_{(N-1)} = \{\overset{0}{\mathbf{g}}_{(N-1)}\}$, $N > 1$,be a Lie group of type $(N-1)$ generalized zero-forms, and let $\mathbf{H}_{(N-1)} = \{\overset{1}{\mathbf{g}}_{(N-1)}\}$ be an additive abelian Lie group of type $(N-1)$ generalized one-forms where there is an ad-action, as above, of $\mathbf{G}_{(N-1)}$ on $\mathbf{H}_{(N-1)}$. Then $\mathbf{G}_{(N)} = \{\overset{0}{\mathbf{g}}_{(N)}\}$, where

$$\overset{0}{\mathbf{g}}_{(N)} = (1_{(N-1)} + \overset{1}{\mathbf{g}}_{(N-1)}\mathbf{m}^N)\overset{0}{\mathbf{g}}_{(N-1)}, \tag{15}$$

with inverse $(\overset{0}{\mathbf{g}}_{(N)})^{-1} = (\overset{0}{\mathbf{g}}_{(N-1)})^{-1}(1_{(N-1)} - \overset{1}{\mathbf{g}}_{(N-1)}\mathbf{m}^N)$.

Type N generalized connections, with values in the Lie algebra $\mathfrak{g}$ of a matrix Lie group G, will be considered here. Hence $H = \mathfrak{g}$ and the connection one-forms, $\mathbf{A}_{(N)}$, are $\mathfrak{g}$-valued type N generalized one-forms. It will be assumed that matrix representations are being used so that the generalized connections (and their constituent ordinary forms) are square matrix-valued. The curvature 2-form is defined by the standard formula

$$\mathbf{F}_{(N)} = d\mathbf{A}_{(N)} + \frac{1}{2}[\mathbf{A}_{(N)}, \mathbf{A}_{(N)}] \tag{16}$$

where, as usual, the last term includes both the matrix and the exterior product. It is convenient to introduce a differential operator $D_{(N)}$ - the covariant exterior derivative defined by $\mathbf{A}_{(N)}$. The covariant exterior derivative of a type N generalized square matrix-valued p-form $\mathbf{P}_{(N)}$ is defined to be

$$D_{(N)}\mathbf{P}_{(N)} = d\mathbf{P}_{(N)} + [\mathbf{A}_{(N)}, \mathbf{P}_{(N)}], \tag{17}$$

and then

$$D^2_{(N)}\mathbf{P}_{(N)} = [\mathbf{F}_{(N)}, \mathbf{P}_{(N)}]. \tag{18}$$

The bracket of generalized (square matrix-valued) $p-$ and $q-$forms $\mathbf{P}_{(N)}$ and $\mathbf{Q}_{(N)}$ is given by

$$[\mathbf{P}_{(N)}, \mathbf{Q}_{(N)}] = \mathbf{P}_{(N)}\mathbf{Q}_{(N)} - (-1)^{pq}\mathbf{Q}_{(N)}\mathbf{P}_{(N)}.$$

As was noted above formulae can often be conveniently defined, recursively, in terms of formulae for forms of lower type.

Writing, for $N \geqq 1$

$$\mathbf{A}_{(N)} = \mathbf{A}_{(N-1)} + \mathbf{B}_{(N-1)}\mathbf{m}^N, \tag{19}$$

where $\mathbf{A}_{(N-1)}$ and $\mathbf{B}_{(N-1)}$ are respectively type $(N-1)$ Lie algebra valued one- and two-forms, it follows that

$$\mathbf{F}_{(N)} = \mathbf{F}_{(N-1)} + \epsilon\mathbf{B}_{(N-1)}\delta_1^N + D_{(N-1)}\mathbf{B}_{(N-1)}\mathbf{m}^N. \tag{20}$$

The generalized connection is flat when $\mathbf{F}_{(N)} = 0$ and therefore

$$\begin{aligned} \mathbf{F}_{(N-1)} + \epsilon\mathbf{B}_{(N-1)}\delta_1^N &= 0, \\ D_{(N-1)}\mathbf{B}_{(N-1)} &= 0. \end{aligned} \tag{21}$$

The generalized connection $\mathbf{A}_{(N)} = (\overset{0}{\mathbf{g}}_{(N)})^{-1}d\overset{0}{\mathbf{g}}_{(N)}$, where $\overset{0}{\mathbf{g}}_{(N)}$is a $\mathbf{G}_{(N)}$ - valued function, is flat. Under a generalized gauge transformation, [2],

$$\mathbf{A}_{(N)} \rightarrow (\overset{0}{\mathbf{g}}_{(N)})^{-1}d\overset{0}{\mathbf{g}}_{(N)} + (\overset{0}{\mathbf{g}}_{(N)})^{-1}\mathbf{A}_{(N)}\overset{0}{\mathbf{g}}_{(N)}, \tag{22}$$

the curvature transforms in the usual way

$$\mathbf{F}_{(N)} \rightarrow (\overset{0}{\mathbf{g}}_{(N)})^{-1}\mathbf{F}_{(N)}\overset{0}{\mathbf{g}}_{(N)}. \tag{23}$$

Henceforth it is assumed that, for any N, $Tr(\mathbf{F}_{(N)}) = 0$.

Generalized gauge transformations also preserve the generalized versions of various characteristic classes, such as those determined by the generalized Chern-Pontrjagin four-form

$$\mathbf{CP}_{(N)} = \frac{1}{8\pi^2}[Tr(\mathbf{F}_{(N)}\mathbf{F}_{(N)})], \tag{24}$$

which is equal to the exterior derivative of the generalized Chern-Simons three-form $\mathbf{CS}_{(N)}$

$$\mathbf{CS}_{(N)} = \frac{1}{8\pi^2}[Tr(\mathbf{A}_{(N)}\mathbf{F}_{(N)} - \frac{1}{3}\mathbf{A}_{(N)}\mathbf{A}_{(N)}\mathbf{A}_{(N)})]. \tag{25}$$

In the case of particular interest here, where $N > 1$, it is a straightforward matter to show thar

$$\mathbf{CP}_{(N)} = \mathbf{CP}_{(N-1)} + \frac{1}{4\pi^2}d[Tr(\mathbf{F}_{(N-1)}\mathbf{B}_{(N-1)})]\mathbf{m}^N, \tag{26}$$

and

$$\mathbf{CS}_{(N)} = \mathbf{CS}_{(N-1)} \tag{27}$$
$$+ \frac{1}{8\pi^2}\{Tr[\mathbf{F}_{(N-1)}\mathbf{B}_{(N-1)} - (D_{(N-1)}\mathbf{B}_{(N-1)})\mathbf{A}_{(N-1)} - \mathbf{A}_{(N-1)}\mathbf{A}_{(N-1)}\mathbf{B}_{(N-1)}]\}\mathbf{m}^N,$$

and so on. Hence if $\mathbf{c}_4^{(N)} = (\mathbf{c}_4^{(N-1)}, \mathbf{c}_5^{(N-1)})$,

$$\int_{\mathbf{c}_4^{(N)}} \mathbf{CP}_{(N)} = \int_{\mathbf{c}_4^{(N-1)}} \mathbf{CP}_{(N-1)} + \frac{1}{4\pi^2}\int_{\partial\mathbf{c}_5^{(N-1)}} [Tr(\mathbf{F}_{(N-1)}\mathbf{B}_{(N-1)})], \tag{28}$$

and if $\mathbf{c}_3^{(N)} = (\mathbf{c}_3^{(N-1)}, \mathbf{c}_4^{(N-1)})$,

$$\int_{\mathbf{c}_3^{(N)}} \mathbf{CS}_{(N)} = \int_{\mathbf{c}_3^{(N-1)}} \mathbf{CS}_{(N-1)} + \tag{29}$$
$$+ \frac{1}{8\pi^2}\int_{\mathbf{c}_4^{(N-1)}} Tr[\mathbf{F}_{(N-1)}\mathbf{B}_{(N-1)} - (D_{(N-1)}\mathbf{B}_{(N-1)})\mathbf{A}_{(N-1)} - \mathbf{A}_{(N-1)}\mathbf{A}_{(N-1)}\mathbf{B}_{(N-1)}].$$

Under the generalized gauge transformation above

$$\mathbf{CP}_{(N)} \rightarrow \mathbf{CP}_{(N)}, \tag{30}$$
$$\mathbf{CS}_{(N)} \rightarrow \mathbf{CS}_{(N)} - \frac{1}{8\pi^2}d\{Tr[(d\mathbf{g}_{(N)})(\mathbf{g}_{(N)})^{-1}\mathbf{A}]\} - \frac{1}{24\pi^2}Tr[(\mathbf{g}_{(N)}^{-1}d\mathbf{g}_{(N)})^3].$$

The last (generalized winding number) term is closed so that when $\mathbf{c}_3^{(N)} = \partial\widetilde{\mathbf{c}}_4^{(N)}$

$$\int_{\mathbf{c}_3^{(N)}} \mathbf{CS} \rightarrow \int_{\mathbf{c}_3^{(N)}} \mathbf{CS}. \tag{31}$$

Finally in this section it should be noted that results for global ordinary connections and their local connection one-form formulations may be generalized straightforwardly to apply to generalized connections. Briefly, and avoiding using the label (N) where possible, a global generalized connection may be defined by specifying for any open covering $\{U_i\}$ of M, transition functions $\mathbf{t}_{ij} : U_i \cap U_j \rightarrow G_{(N)}$ by $p \rightarrow \overset{0}{\mathbf{t}}_{ij}(p)$ satisfying $\overset{0}{\mathbf{t}}_{ii}(p) = 1$, $p \in U_i$, $\overset{0}{\mathbf{t}}_{ij}(p) = [\overset{0}{\mathbf{t}}_{ji}(p)]^{-1}$ $p \in U_i \cap U_j$, $\overset{0}{\mathbf{t}}_{ij}(p)\overset{0}{\mathbf{t}}_{jk}(p) = \overset{0}{\mathbf{t}}_{ik}(p)$ $p \in U_i \cap U_j \cap U_k$ and local generalized connection one-forms $\mathbf{A}_i$, on each U_i, related by $\mathbf{A}_j = (\mathbf{t}_{ij})^{-1}\mathbf{A}_i\mathbf{t}_{ij} + (\mathbf{t}_{ij})^{-1}d\mathbf{t}_{ij}$ on $U_i \cap U_j$. Transition functions $\{\mathbf{t}_{IJ}\}$ and $\{\widetilde{\mathbf{t}}_{IJ}\}$ are (gauge) equivalent when $\widetilde{\mathbf{t}}_{IJ} = (\mathbf{g}_I)^{-1}\mathbf{t}_{IJ}\mathbf{g}_J$ and $\mathbf{g}_I$ and $\mathbf{g}_J$ respectively determine generalized gauge transformations in U_I and U_J as in Eq.(22) above.

3 Type $N = 2$ generalized forms

This section expands on the results above in the case of type $N = 2$ generalized forms on an n dimensional manifold M. These are elements of the $\Lambda_{(2)}(M) = \oplus_{p=-2}^{p=n}\Lambda_{(2)}^{p}(M)$, where $\Lambda_{(2)}^{p}(M)$ denotes the type $N = 2$ generalized $p-$forms on M. A generalized $p-$form, $\overset{p}{\mathbf{a}} \in \Lambda_{(2)}^{p}(M)$, may be written out in terms of ordinary forms and minus one-forms as

$$\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha}_1\mathbf{m}^1 + \overset{p+1}{\alpha}_2\mathbf{m}^2 + \overset{p+2}{\alpha}\mathbf{m}^1\mathbf{m}^2, \tag{32}$$

where $\mathbf{m}^1$ and $\mathbf{m}^2$ are two linearly independent degree minus one$-$forms, with non-zero exterior product $\mathbf{m}^1\mathbf{m}^2$. The degrees of the ordinary differential forms $\overset{p}{\alpha}$, $\overset{p+1}{\alpha}_1$, $\overset{p+1}{\alpha}_2$, $\overset{p+2}{\alpha}$ are p, $p+1$ and $p+2$. For non-zero, type $N = 2$, generalized forms p can take integer values from -2 to n. If

$$\overset{q}{\mathbf{b}} = \overset{q}{\beta} + \overset{q+1}{\beta}_1\mathbf{m}^1 + \overset{q+1}{\beta}_2\mathbf{m}^2 + \overset{q+2}{\beta}\mathbf{m}^1\mathbf{m}^2 \tag{33}$$

is a generalized $q-$form of type $N = 2$, the exterior product, $\overset{p}{\mathbf{a}}\overset{q}{\mathbf{b}}$, is given by the degree $(p+q)$ generalized form $\overset{p+q}{\mathbf{c}} \in \Lambda_{(2)}^{p+q}(M)$ where

$$\begin{aligned}
\overset{p+q}{\mathbf{c}} &= \overset{p+q}{\gamma} + \overset{p+q+1}{\gamma_1}\mathbf{m}^1 + \overset{p+q+1}{\gamma_2}\mathbf{m}^2 + \overset{p+q+2}{\gamma}\mathbf{m}^1\mathbf{m}^2, \text{ where} \\
\overset{p+q}{\gamma} &= \overset{p}{\alpha}\overset{q}{\beta}, \\
\overset{p+q+1}{\gamma_1} &= \overset{p}{\alpha}\overset{q+1}{\beta}_1 + (-1)^q\overset{p+1}{\alpha}_1\overset{q}{\beta}, \quad \overset{p+q+1}{\gamma_2} = \overset{p}{\alpha}\overset{q+1}{\beta_2} + (-1)^q\overset{p+1}{\alpha_2}\overset{q}{\beta}, \\
\overset{p+q+2}{\gamma} &= \overset{p}{\alpha}\overset{q+2}{\beta} + (-1)^{q+1}\overset{p+1}{\alpha}_1\overset{q+1}{\beta_2} + (-1)^q\overset{p+1}{\alpha}_2\overset{q+1}{\beta}_1 + \overset{p+2}{\alpha}\overset{q}{\beta}.
\end{aligned} \tag{34}$$

As noted in the previous section there are two cases of exterior derivative to be considered

$$d\mathbf{m}^i = \delta_1^i\epsilon, \tag{35}$$

with i, j, k summing and ranging over one to two. In the first case $\epsilon = 0$ while in the second case ϵ is a fixed non-zero constant. It can be shown that the basis of minus one-forms can always be so chosen, [2]. The exterior derivative of a generalized $p-$form $\overset{p}{\mathbf{a}}_{(2)}$is the generalized $(p+1)-$form

$$d\overset{p}{\mathbf{a}}_{(2)} = [d\overset{p}{\alpha} + (-1)^{p+1}\epsilon\overset{p+1}{\alpha}_1] + d\overset{p+1}{\alpha}_1\mathbf{m}^1 + [d\overset{p+1}{\alpha}_2 + (-1)^p\epsilon\overset{p+2}{\alpha}]\mathbf{m}^2 + d\overset{p+2}{\alpha}\mathbf{m}^1\mathbf{m}^2. \tag{36}$$

A type $N=2$ form $\overset{p}{\mathbf{a}}$ is closed if and only if

$$
\begin{aligned}
d\overset{p}{\alpha} + (-1)^{p+1}\epsilon \overset{p+1}{\alpha}{}_1 &= 0, \\
d\overset{p+1}{\alpha}{}_1 &= 0, \\
d\overset{p+1}{\alpha}{}_2 + (-1)^{p}\epsilon \overset{p+2}{\alpha} &= 0, \\
d\overset{p+2}{\alpha} &= 0.
\end{aligned} \tag{37}
$$

Hence in case (i), where $\epsilon = 0$, $\overset{p}{\mathbf{a}}$ is closed if and only if all the ordinary forms defining it are closed. On the other hand in case (ii), where ϵ is non-zero, $\overset{p}{\mathbf{a}}$ is closed if and only if it is exact. In the latter case

$$
\begin{aligned}
\overset{p}{\mathbf{a}} &= \overset{p}{\alpha} + \epsilon^{-1}(-1)^p d\overset{p}{\alpha}\mathbf{m}^1 + \overset{p+1}{\alpha}{}_2\mathbf{m}^2 + \epsilon^{-1}(-1)^{p+1} d\overset{p+1}{\alpha}_2\mathbf{m}^1\mathbf{m}^2 \\
&= d[(-1)^p\epsilon^{-1}\overset{p}{\alpha}\mathbf{m}^1 + (-1)^{p+1}\epsilon^{-1}\overset{p+1}{\alpha}_2\mathbf{m}^1\mathbf{m}^2].
\end{aligned} \tag{38}
$$

A $p-$polychain of type $N=2$ in M is an ordered quadruple of ordinary (real, singular) chains in M

$$
\mathbf{c}_p^{(2)} = (c_p, c^1_{p+1}, c^2_{p+1}, c_{p+2}), \tag{39}
$$

where c_p is an ordinary $p-$chain, c^1_{p+1} and c^2_{p+1} are ordinary $p+1-$chains and c_{p+2} is an ordinary $p+2-$chain, [1]. The ordinary chains have respective boundaries $\partial c_p, \partial c^1_{p+1}, \partial c^2_{p+1}$ and ∂c_{p+2}. The boundary of the polychain $\mathbf{c}_p^{(2)}$ is the $(p-1)-$polychain

$$
\partial\mathbf{c}_p^{(2)} = (\partial c_p, \partial c^1_{p+1} + (-1)^p\epsilon c_p, \partial c^2_{p+1}, \partial c_{p+2} + (-1)^{p+1}\epsilon c^2_{p+1}), \tag{40}
$$

and $\partial^2\mathbf{c}_p^{(2)} = 0$. The integral of a generalized form $\overset{p}{\mathbf{a}}_{(2)}$ over a polychain $\mathbf{c}_p^{(2)}$ is

$$
\int_{\mathbf{c}_p^{(2)}} \overset{p}{\mathbf{a}}_{(2)} = \int_{c_p} \overset{p}{\alpha} + \int_{c^1_{p+1}} \overset{p+1}{\alpha}{}_1 + \int_{c^2_{p+1}} \overset{p+1}{\alpha}{}_2 + \int_{c_{p+2}} \overset{p+2}{\alpha}. \tag{41}
$$

When $N=2$ Stokes' theorem for generalized forms and polychains incorporates these expressions in the formula

$$
\int_{\mathbf{c}_p^{(2)}} d\overset{p-1}{\mathbf{a}}{}_{(2)} = \int_{\partial\mathbf{c}_p^{(2)}} \overset{p-1}{\mathbf{a}}{}_{(2)}. \tag{42}
$$

Now consider type $N-2$ connections, the generalized Chern-Pontrjagin four-forms and Chern-Simons integrals

Let M be a smooth manifold of dimension n greater than or equal to six. Consider now a $\mathfrak{g}-$valued type $N=2$ connection one-form $\mathbf{A}$ on M, where $\mathfrak{g}$ is the Lie algebra of a (unimodular) matrix Lie group G,

$$\mathbf{A} = \omega + \Theta\mathbf{m}^1 + \Sigma\mathbf{m}^2 + \Pi\mathbf{m}^1\mathbf{m}^2. \tag{43}$$

Here ω is a $\mathfrak{g}-$valued one-form, Θ and Σ are a pair of $\mathfrak{g}-$valued two-forms, and Π is a $\mathfrak{g}-$valued three-form and matrix representations are used.

The generalized curvature two-form is

$$\mathbf{F} = (\Omega + \epsilon\Theta) + D\Theta\mathbf{m}^1 + (D\Sigma - \epsilon\Pi)\mathbf{m}^2 + (D\Pi + \Theta\Sigma - \Sigma\Theta)\mathbf{m}^1\mathbf{m}^2, \tag{44}$$

where $\Omega = d\omega + \omega\omega$.

Here $D = D_{(0)}$ denotes the covariant exterior derivative with respect to ω.

Under a $G-$gauge transformation

$$\begin{aligned} &\mathbf{A}\longmapsto (\gamma)^{-1}d\gamma + \gamma^{-1}\mathbf{A}\gamma, \\ &\mathbf{F}\longmapsto(\gamma)^{-1}\mathbf{F}\gamma, \end{aligned} \tag{45}$$

where γ is a $G-$valued function on M. A global generalized connection, on a $G-$bundle over M can be constructed from these local expressions in the usual way, for instance by first constructing a coordinate bundle, [4].

The generalized connection is flat, that is the generalized curvature $\mathbf{F}=0$, if and only if

$$\Omega = -\epsilon\Theta,\ D\Theta = 0,\ D\Sigma = \epsilon\Pi,\ D\Pi + \Theta\Sigma - \Sigma\Theta = 0. \tag{46}$$

In case (i) where $\epsilon = 0$, $\mathbf{F}=0$ if and only if

$$\Omega = 0,\ D\Theta = 0,\ D\Sigma = 0,\ D\Pi + \Theta\Sigma - \Sigma\Theta = 0. \tag{47}$$

Then a $G-$gauge can be chosen so that, in a contractible open set,

$$\omega = 0,\ \Theta = d\vartheta,\ \Sigma = d\sigma,\ \Pi = \sigma d\vartheta - d\vartheta\sigma + d\pi,$$

where ϑ, σ and π are respectively two $\mathfrak{g}-$valued one-forms and a $\mathfrak{g}$-valued two-form. The latter three forms are not unique of course (with the freedom

$\vartheta \to \vartheta + d\overset{0}{\chi}_1$, $\sigma \to \sigma + d\overset{0}{\chi}_2$, $\pi \to \pi + d\vartheta\overset{0}{\chi}_2 - \overset{0}{\chi}_2 d\vartheta + d\overset{1}{\chi}$, where $\overset{0}{\chi}_1$and $\overset{0}{\chi}_2$ are arbitrary $\mathfrak{g}$-valued functions and $\overset{1}{\chi}$ is a $\mathfrak{g}$-valued one-form). Thus, when $\epsilon = 0$, the flat generalized potential can be written, in a general $G-$gauge, as

$$\mathbf{A} = (\gamma)^{-1} d\gamma + \gamma^{-1}[d\vartheta\mathbf{m}^1 + d\sigma\mathbf{m}^2 + (\sigma d\vartheta - d\vartheta\sigma + d\pi)\mathbf{m}^1\mathbf{m}^2]\gamma, \qquad (48)$$

where γ is a $G-$valued function on M.

In case (ii) where ϵ is non-zero, $\mathbf{F} = 0$, if and only if

$$\Theta = -\epsilon^{-1}\Omega, \Pi = \epsilon^{-1} D\Sigma, \qquad (49)$$

and hence

$$\mathbf{A} = \omega - \epsilon^{-1}\Omega\mathbf{m}^1 + \Sigma\mathbf{m}^2 + \epsilon^{-1} D\Sigma\mathbf{m}^1\mathbf{m}^2. \qquad (50)$$

As was noted in the previous section a generalized gauge transformation can be associated with the matrix group $G_{(2)}$ and its Lie algebra $\mathfrak{g}_{(2)}$. This is given by

$$\begin{aligned} \mathbf{A} &\longmapsto (\mathbf{g}_{(2)})^{-1} d\mathbf{g}_{(2)} + (\mathbf{g}_{(2)})^{-1}\mathbf{A}\mathbf{g}_{(2)} \\ \mathbf{F} &\longmapsto (\mathbf{g}_{(\mathbf{2})})^{-1}\mathbf{F}\mathbf{g}_{(2)}, \end{aligned} \qquad (51)$$

where $\mathbf{g}_{(2)}$ belongs to $\mathbf{G}_{(2)}$, the group of type $N = 2$ zero-forms on M. Any element of $\mathbf{G}_{(2)}$ has the form

$$\mathbf{g}_{(2)} = [1 + \overset{1}{\gamma}_1\mathbf{m}^1 + \overset{1}{\gamma}_2\mathbf{m}^2 + (\overset{2}{\gamma} + \overset{1}{\gamma}_2\overset{1}{\gamma}_1)\mathbf{m}^1\mathbf{m}^2]\overset{0}{\gamma}, \qquad (52)$$

where $\overset{0}{\gamma}$ is an ordinary G-valued zero-form, and $\overset{1}{\gamma}_1, \overset{1}{\gamma}_2, \overset{2}{\gamma}$ are two ordinary $\mathfrak{g}$-valued one-forms and a $\mathfrak{g}$-valued two-form. The curvature $\mathbf{F}$ is zero if and only if the connection one form is given by

$$\mathbf{A} = (\mathbf{g}_{(2)})^{-1} d\mathbf{g}_{(2)}, \qquad (53)$$

for some, non-unique, $\mathbf{g}_{(2)}$ belonging to $\mathbf{G}_{(2)}$.

In case (i) where $\epsilon = 0$, when $\mathbf{F} = 0$ a choice of $\mathbf{g}_{(2)}$, in a contractible open region in M, is

$$\mathbf{g}_{(2)} = [1 + \vartheta\mathbf{m}^1 + \sigma\mathbf{m}^2 + \{\widetilde{\pi} + \sigma\vartheta\}\mathbf{m}^1\mathbf{m}^2]\gamma. \qquad (54)$$

Here the $\mathfrak{g}-$valued two-form $\widetilde{\pi} = \pi - (\sigma\vartheta + \vartheta\sigma)$ and γ is a $G-$valued function as above.

In case (ii) where ϵ is non-zero, when $\mathbf{F} = 0$ a choice of $\mathbf{g}_{(2)}$ is

$$\mathbf{g}_{(2)} = [1 - \epsilon^{-1}\omega\mathbf{m}^1 + \epsilon^{-1}\Sigma\mathbf{m}^1\mathbf{m}^2]. \tag{55}$$

Next consider the generalized Chern-Pontrjagin forms and integrals.

Since $Tr(\mathbf{F}) = \mathbf{0}$, the four-forms corresponding to the generalized second Chern class or generalized first Pontrjagin class on M are equal to

$$\mathbf{CP}{=}kTr(\mathbf{FF}) \tag{56}$$

$$\begin{aligned}&= kTr\{\Omega\Omega + 2\epsilon\Omega\Theta + \epsilon^2\Theta\Theta+\\ &+ d(2\Omega\Theta + \epsilon\Theta\Theta)\mathbf{m}^1 + [2d(\Omega\Sigma) + 2\epsilon(\Theta D\Sigma - \Omega\Pi - \epsilon\Theta\Pi)]\mathbf{m}^2\\ &+ 2d(\Omega\Pi + \epsilon\Theta\Pi - \Theta D\Sigma)\mathbf{m}^{12}\},\end{aligned} \tag{57}$$

letting the constant $k = \frac{\kappa}{8\pi^2}$, (in the previous section $k = \frac{1}{8\pi^2}$). The generalized four-form, $\mathbf{CP}$, is of course invariant under both the gauge transformations and the generalized gauge transformations above.

Using the results of section two with $p = 4$, together with Stoke's theorem, the generalized second Chern-Pontrjagin class integral for a polychain $\mathbf{c}_4^{(2)} = (c_4, c_5^1, c_5^2, c_6)$ is given by the integral

$$\int_{\mathbf{c}_4^{(2)}} \mathbf{CP}{=}k\int_{\mathbf{c}_4^{(2)}} Tr(\mathbf{FF}) \tag{58}$$

$$\begin{aligned}&= k[\int_{c_4} Tr(\Omega\Omega + 2\epsilon\Omega\Theta + \epsilon^2\Theta\Theta]+\\ &+ \int_{\partial c_5^1} Tr(2\Omega\Theta + \epsilon\Theta\Theta) + 2\int_{\partial c_5^2} Tr(\Omega\Sigma) + 2\epsilon\int_{c_5^2} Tr(\Theta D\Sigma - \Omega\Pi - \epsilon\Theta\Pi)\\ &+ 2\int_{\partial c_6^2} Tr(\Omega\Pi + \epsilon\Theta\Pi - \Theta D\Sigma)].\end{aligned}$$

The generalized four-form, $\mathbf{CP}$, is the exterior derivative of a generalized Chern-Simons three-form, $\mathbf{CS}$, $\mathbf{CP} = d(\mathbf{CS})$, where

$$\mathbf{CS} = kTr(\mathbf{AF} - \frac{1}{3}\mathbf{AAA}) \tag{59}$$

$$\begin{aligned}&= kTr\{\omega\Omega - \frac{1}{3}\omega\omega\omega + \epsilon\omega\Theta + [2\Omega\Theta + \epsilon\Theta\Theta - d(\omega\Theta)]\mathbf{m}^1\\ &+ [2\Omega\Sigma + \epsilon\Sigma\Theta - d(\omega\Sigma) - \epsilon\omega\Pi]\mathbf{m}^2\\ &+ [\Sigma D\Theta - \Theta D\Sigma + 2\Pi\Omega + 2\epsilon\Pi\Theta - d(\omega\Pi)]\mathbf{m}^1\mathbf{m}^2\}.\end{aligned} \tag{60}$$

The generalized Chern-Simons integral for a polychain $\mathbf{c}_3^{(2)} = (c_3, c_4^1, c_4^2, c_5)$ is, after using Stoke's theorem, given by

$$
\begin{aligned}
\int_{\mathbf{c}_3^{(2)}} \mathbf{CS} = k\{ & \int_{c_3} Tr(\omega\Omega - \frac{1}{3}\omega\omega\omega + \epsilon\omega\Theta) + \\
& \int_{c_4^1} Tr(2\Omega\Theta + \epsilon\Theta\Theta) - \int_{\partial c_4^1} Tr(\omega\Theta) + \\
& + \int_{c_4^2} Tr(2\Omega\Sigma + \epsilon\Sigma\Theta - \epsilon\omega\Pi) - \int_{\partial c_4^2} Tr(\omega\Sigma) + \\
& + \int_{c_5} Tr(\Sigma D\Theta - \Theta D\Sigma + 2\Pi\Omega + 2\epsilon\Pi\Theta) - \int_{\partial c_5} Tr(\omega\Pi)\}.
\end{aligned} \tag{61}
$$

When the polychain $\mathbf{c}_3^{(2)}$ is the boundary of a polychain $\widetilde{\mathbf{c}}_4^{(2)} = (\widetilde{c}_4, \widetilde{c}_5^1, \widetilde{c}_5^2, \widetilde{c}_6)$, so that

$$
\begin{aligned}
\mathbf{c}_3^{(2)} = \partial\widetilde{\mathbf{c}}_4^{(2)} = (\partial\widetilde{c}_4, \partial\widetilde{c}_5^1 + \epsilon\widetilde{c}_4, \partial\widetilde{c}_5^2, \partial\widetilde{c}_6 - \epsilon\widetilde{c}_5^2), \text{ that is} \\
c_3 = \partial\widetilde{c}_4,\ c_4^1 = \partial\widetilde{c}_5^1 + \epsilon\widetilde{c}_4,\ c_4^2 = \partial\widetilde{c}_5^2,\ c_5 = \partial\widetilde{c}_6 - \epsilon\widetilde{c}_5^2,
\end{aligned} \tag{62}
$$

then

$$
\begin{aligned}
\int_{\partial\widetilde{\mathbf{c}}_4^{(2)}} \mathbf{CS} = k\{ & \int_{\partial\widetilde{c}_4} Tr(\omega\Omega - \frac{1}{3}\omega\omega\omega) + \int_{\partial\widetilde{c}_5^1 + \epsilon\widetilde{c}_4} Tr(2\Omega\Theta + \epsilon\Theta\Theta) + \\
& + \int_{\partial\widetilde{c}_5^2} Tr(2\Omega\Sigma + \epsilon\Sigma\Theta - \epsilon\omega\Pi) + \\
& \int_{\partial\widetilde{c}_6 - \epsilon\widetilde{c}_5^2} Tr(\Sigma D\Theta - \Theta D\Sigma + 2\Pi\Omega + 2\epsilon\Pi\Theta) + \int_{\epsilon\partial\widetilde{c}_5^2} Tr(\omega\Pi)\}.
\end{aligned} \tag{63}
$$

By Stoke's theorem for generalized forms

$$
\int_{\partial\widetilde{\mathbf{c}}_4^{(2)}} \mathbf{CS} = \int_{\widetilde{\mathbf{c}}_4^{(2)}} \mathbf{CP}. \tag{64}
$$

Under the $G-$gauge transformation

$$
\begin{aligned}
& \mathbf{CP} \rightarrow \mathbf{CP}, \\
& \mathbf{CS} \rightarrow \mathbf{CS} - kd\{Tr[(d\gamma)\gamma^{-1}\mathbf{A}]\} - \frac{k}{3} Tr[(\gamma^{-1}d\gamma)^3].
\end{aligned} \tag{65}
$$

The last (winding number) term is closed so when $\mathbf{c}_3^{(2)} = \partial\widetilde{\mathbf{c}}_4^{(2)}$

$$\int_{\mathbf{c}_3^{(2)}} \mathbf{CS} \to \int_{\mathbf{c}_3^{(2)}} \mathbf{CS}. \tag{66}$$

Again, when generalized gauge transformations, $\mathbf{G}_{(2)}-$gauge transformations, are considered the generalized winding number term is again closed and both of the above equations hold when γ is replaced by $\mathbf{g}_{(2)}$.

Next consider variations of these forms and invariants. Analogously to the result for ordinary forms the variation of a generalized Chern-Simons three-form is given by

$$\delta\mathbf{CS} = k2Tr[(\delta\mathbf{A})\mathbf{F}]+kd\{Tr[(\delta\mathbf{A})\mathbf{A}]\}. \tag{67}$$

and hence

$$\delta\int_{\mathbf{c}_3^{(2)}} \mathbf{CS} = k\{2\int_{\mathbf{c}_3^{(2)}} Tr[(\delta\mathbf{A})\mathbf{F}]+\int_{\partial\mathbf{c}_3^{(2)}} Tr[(\delta\mathbf{A})\mathbf{A}]\}. \tag{68}$$

For type $N = 2$ connections

$$\delta\mathbf{A} = \delta\omega + \delta\Theta\mathbf{m}^1 + \delta\Sigma\mathbf{m}^2 + \delta\Pi\mathbf{m}^1\mathbf{m}^2, \tag{69}$$

where $\delta\omega$, $\delta\Theta$, $\delta\Sigma$and $\delta\Pi$ denote variations of ordinary forms. The terms entering Eq.(67) are therefore the traces of

$$\begin{aligned}(\delta\mathbf{A})\mathbf{F} = {} & \delta\omega(\Omega + \epsilon\Theta) + [\delta\omega D\Theta + \delta\Theta(\Omega + \epsilon\Theta)]\mathbf{m}^1 \\ & + [\delta\omega(D\Sigma - \epsilon\Pi) + \delta\Sigma(\Omega + \epsilon\Theta)]\mathbf{m}^2 \\ & + [\delta\omega(D\Pi + \Theta\Sigma - \Sigma\Theta) + \delta\Theta(\epsilon\Pi - D\Sigma) + \\ & + \delta\Sigma D\Theta + \delta\Pi(\Omega + \epsilon\Theta)]\mathbf{m}^1\mathbf{m}^2,\end{aligned} \tag{70}$$

and

$$\begin{aligned}(\delta\mathbf{A})\mathbf{A} = {} & \delta\omega\omega + (\delta\omega\Theta - \delta\Theta\omega)\mathbf{m}^1 + (\delta\omega\Sigma - \delta\Sigma\omega)\mathbf{m}^2 \\ & + (\delta\omega\Pi + \delta\Pi\omega + \delta\Theta\Sigma - \delta\Sigma\Theta)\mathbf{m}^1\mathbf{m}^2.\end{aligned} \tag{71}$$

When, as in Eq.(62), $\mathbf{c}_3^{(2)} = \partial\widetilde{\mathbf{c}}_4^{(2)}$ it follows that

$$
\begin{aligned}
\delta\int_{\partial\mathbf{c}_4^{(2)}}\mathbf{CS} = kTr\{&\int_{\partial\widetilde{c}_4}\delta\omega(\Omega+\epsilon\Theta)+\int_{\partial\widetilde{c}_5^1+\epsilon\widetilde{c}_4}[\delta\omega D\Theta+\delta\Theta(\Omega+\epsilon\Theta)] \\
&+\int_{\partial\widetilde{c}_5^2}[\delta\omega(D\Sigma-\epsilon\Pi)+\delta\Sigma(\Omega+\epsilon\Theta)] \\
&+\int_{\partial\widetilde{c}_6-\epsilon\widetilde{c}_5^2}[\delta\omega(D\Pi+\Theta\Sigma-\Sigma\Theta)+\delta\Theta(\epsilon\Pi-D\Sigma)+ \\
&+\delta\Sigma D\Theta+\delta\Pi(\Omega+\epsilon\Theta)]\}.
\end{aligned} \tag{72}
$$

Expressions like this have been used in the Euler-Lagrange approach to generalized Chern-Simons formulations of field theories in the type $N = 1$ case [1]. More generally variational principles, with variations and Euler-Lagrange equations on different manifolds of different dimensions, arise naturally when action integrals of generalized fields are used.

Finally in this section it should be recalled, [1], that if $\mathbf{a}$ is a closed generalized zero-form and

$$\mathbf{a} = \overset{0}{\alpha} + \overset{1}{\alpha}_1\mathbf{m}^1 + \overset{1}{\alpha}_2\mathbf{m}^2 + \overset{2}{\alpha}\mathbf{m}^1\mathbf{m}^2, \tag{73}$$

then in the first case where $\epsilon = 0$ the ordinary forms $\overset{0}{\alpha}$, $\overset{1}{\alpha}_1$, $\overset{1}{\alpha}_2$ and $\overset{2}{\alpha}$ are all closed and in the second case where ϵ is non-zero

$$
\begin{aligned}
\mathbf{a} &= \alpha + \epsilon^{-1}d\alpha\mathbf{m}^1 + \beta\mathbf{m}^2 - \epsilon^{-1}(d\beta)\mathbf{m}^1\mathbf{m}^2 \\
&= \epsilon^{-1}d[\alpha\mathbf{m}^1 - \beta\mathbf{m}^1\mathbf{m}^2],
\end{aligned} \tag{74}
$$

where $\alpha = \overset{0}{\alpha}$ and $\beta = \overset{1}{\alpha}_2$. In both cases it follows that

$$\mathbf{aCP} = d(\mathbf{aCS}). \tag{75}$$

4 An so(p,q)-valued type N=2 connection

In this section generalized connections which have been introduced in an earlier paper, [5], will be discussed and then employed in the construction of generalized Chern-Simons forms.

Consider the $so(p,q)-$valued type $N=2$ generalized connection one-form $\mathbf{A}$ on an $n-$dimensional manifold M, where $n \geqq p+q$ and $so(p,q)$ is the Lie algebra of $SO(p,q)$,

$$\mathbf{A} = \omega + (\frac{\mu}{2}\Psi + \nu\Sigma)\mathbf{m}^1 + \Sigma\mathbf{m}^2. \tag{76}$$

Here μ and ν are constants, ω, Ψ and Σ are a $so(p,q)-$valued one-form and a pair of $so(p,q)-$valued two-forms respectively and $(p+q)\times(p+q)$ matrix representations are used. Using an index notation

$$\begin{gathered}\omega \leftrightarrow {\omega^a}_b,\ \Sigma \leftrightarrow \Sigma^a_{.b} = \theta^a\theta_b,\ \Psi \leftrightarrow \Psi^a_{.bcd}\Sigma^{cd}, \\ \mathbf{A}^a_b = {\omega^a}_b + (\frac{\mu}{2}\Psi^a_{.bcd}\Sigma^{cd} + \nu\Sigma^a_{.b})\mathbf{m}^1 + \Sigma^a_{.b}\mathbf{m}^2,\end{gathered} \tag{77}$$

and lower case Latin indices a,b,c,d range and sum over 1 to $(p{+}q) \leqq n$. The one-forms θ^a are ordinary forms on M and $\Psi^a_{.bcd}$ has the *algebraic* symmetries of a $SO(p,q)$ *Weyl conformal* tensor. Indices are raised and lowered by using the $SO(p,q)$ metric

$$(\eta_{ab}) = \begin{pmatrix} 1_{p\times p} & 0 \\ 0 & -1_{q\times q}\end{pmatrix}. \tag{78}$$

In this example it will be assumed again that the basis of minus one-forms is chosen so that $d\mathbf{m}^1 = \epsilon$ and $d\mathbf{m}^2 = 0$, where ϵ is a fixed real parameter. The curvature two-form is

$${\mathbf{F}^a}_b = ({\Omega^a}_b + \frac{\epsilon\mu}{2}{\Psi^a}_{bcd}\Sigma^{cd} + \epsilon\nu\Sigma^a_{.b}) + [D(\frac{\mu}{2}{\Psi^a}_{bcd}\Sigma^{cd} + \nu\Sigma^a_{.b})]\mathbf{m}^1 + D\Sigma^a_{.b}\mathbf{m}^2; \tag{79}$$

where D denotes the covariant exterior derivative with respect to ${\omega^a}_b$ and ${\Omega^a}_b = d{\omega^a}_b + {\omega^a}_c{\omega^c}_b$. Henceforth assume that $4 \leqq (p+q) \leqq n$.

In case (i) where $\epsilon = 0$ the curvature two-form ${\mathbf{F}^a}_b$ is zero if and only if

$${\Omega^a}_b = 0;\ \mu D({\Psi^a}_{bcd}\Sigma^{cd}) = 0;\ D\Sigma^a_{.b} = 0. \tag{80}$$

When (the pullbacks of) the one-forms $\{\theta^a\}$ are linearly independent on a $(p{+}q)-$ dimensional chain (or sub-manifold) S in M, ${\mathbf{F}^a}_b = 0$ if and only if $\omega^a_{.b}$ is the Levi-Civita connection of the flat $SO(p,q)$ metric $ds^2 = \eta_{ab}\theta^a \otimes \theta^b$ on S. In local coordinates $\{\xi^a\}$ on S for which $ds^2 = \eta_{ab}d\xi^a \otimes d\xi^b$ the generalized connection then takes the form

$$\mathbf{A}^a_b = (\gamma^{-1})^a_c d\gamma^c_b + (\gamma^{-1})^a_e[(\frac{\mu}{2}{\Psi^e}_{fcd}d\xi^c d\xi^d + \nu d\xi^e d\xi_f)\mathbf{m}^1 + d\xi^e d\xi_f\mathbf{m}^2]\gamma^f_b, \tag{81}$$

where (γ^a_b) is a $SO(p,q)-$valued function. In addition, if μ is non-zero, on S

$$\nabla_{[e}(\Psi^{ab}_{..cd]}) = 0, \tag{82}$$

where ∇ denote the the Levi-Civita covariant derivative. When S is topologically $\mathbb{R}^{p+q}$ this is equivalent to the Ricci tensor, linearized about flat space, vanishing. In particular, when $p+q = 4$, and the signature is Lorentzian, the metric on S is the flat Minkowski four-metric and $\Psi^a_{.bcd}$ are the components of a zero rest-mass spin two field in Minkowski space-time. The equivalence of the zero rest-mass spin two field equations in four dimensional Minkowski space-time and the linearized Einstein vacuum field equations was pointed out by Trautman [6], see also [7] .

In case (ii) where ϵ is a non-zero constant the curvature two-form $\mathbf{F}^a{}_b$ is zero if and only if

$$\Omega^a{}_b = -\frac{\epsilon\mu}{2}\Psi^a{}_{bcd}\Sigma^{cd} - \epsilon\nu\Sigma^a_{.b}, \text{ and } D\Sigma^a_{.b} = 0. \tag{83}$$

When the choice $\epsilon\mu = -1$ is made it follows that, for any $(p+q)-$ dimensional chain S in M on which (the pullbacks of) the one-forms $\{\theta^a\}$ are linearly independent, $\mathbf{F}^a{}_b = 0$ if and only if

$$\begin{aligned} d\omega^a_{.b} + \omega^a_{.c}\omega^c_{.b} &= \frac{1}{2}\Psi^a_{.bcd}\Sigma^{cd} + \lambda\theta^a\theta_b, \\ D(\theta^a\theta_b) &= 0, \end{aligned} \tag{84}$$

where $\lambda = -\epsilon\nu$. In other words $\omega^a_{.b}$ is the Levi-Civita connection of an Einstein $SO(p,q)$ metric, $ds^2 = \eta_{ab}\theta^a \otimes \theta^b$ on S, with non-zero cosmological constant if ν is non-zero.

To conclude this example the corresponding generalized Chern-Pontrjagin and Chern-Simons expressions will be exhibited.

The corresponding four-form $\mathbf{CP}$ on M is

$$\begin{aligned} \mathbf{CP} =& k\mathbf{F}^a{}_b\mathbf{F}^b{}_a \\ =& k\{\Omega^a{}_b\Omega^b{}_a + 2\epsilon\Omega^a{}_b(\frac{\mu}{2}\Psi^b{}_{acd}\Sigma^{cd} + \nu\Sigma^b_{.a}) + \epsilon^2\frac{\mu^2}{4}\Psi^a{}_{bcd}\Psi^b{}_{aef}\Sigma^{cd}\Sigma^{ef} \\ &+ d[2\Omega^a{}_b(\frac{\mu}{2}\Psi^b{}_{acd}\Sigma^{cd} + \nu\Sigma^b_{.a}) + \epsilon\frac{\mu^2}{4}\Psi^a{}_{bcd}\Psi^b{}_{aef}\Sigma^{cd}\Sigma^{ef}]\mathbf{m}^1 + 2d(\Omega^a{}_b\Sigma^b{}_a)\mathbf{m}^2\}. \end{aligned} \tag{85}$$

By using Stokes' theorem, the invariant on a polychain $\mathbf{c}_4^{(2)} = (c_4, c_5^1, c_5^2, c_6)$ can be written as

$$\int_{\mathbf{c}_4^{(2)}} \mathbf{CP} = k \int_{\mathbf{c}_4^{(2)}} \mathbf{F}^a{}_b \mathbf{F}^b{}_a \tag{86}$$
$$= k\{ \int_{c_4} [\Omega^a{}_b \Omega^b{}_a + 2\epsilon \Omega^a{}_b (\frac{\mu}{2} \Psi^b{}_{acd} \Sigma^{cd} + \nu \Sigma^b_{.a}) + \epsilon^2 \frac{\mu^2}{4} \Psi^a{}_{bcd} \Psi^b{}_{aef} \Sigma^{cd} \Sigma^{ef}] +$$
$$+ \int_{\partial c_5^1} [2\Omega^a{}_b (\frac{\mu}{2} \Psi^b{}_{acd} \Sigma^{cd} + \nu \Sigma^b_{.a}) + \epsilon \frac{\mu^2}{4} \Psi^a{}_{bcd} \Psi^b{}_{aef} \Sigma^{cd} \Sigma^{ef}] + 2 \int_{\partial c_5^2} [\Omega^a{}_b \Sigma^b{}_a] \}.$$

The generalized Chern-Simons three-form, **CS**, is

$$\mathbf{CS} = k(\mathbf{A}^a{}_b \mathbf{F}^b{}_a - \frac{1}{3} \mathbf{A}^a{}_b \mathbf{A}^b{}_c \mathbf{A}^c{}_a) \tag{87}$$
$$= k\{ [\omega^a{}_b \Omega^b{}_a - \frac{1}{3} \omega^a{}_b \omega^b{}_c \omega^c{}_a + \epsilon \omega^a{}_b (\frac{\mu}{2} \Psi^b{}_{acd} \Sigma^{cd} + \nu \Sigma^b_{.a})] +$$
$$+ [2\Omega^a{}_b (\frac{\mu}{2} \Psi^b{}_{acd} \Sigma^{cd} + \nu \Sigma^b_{.a}) + \epsilon \frac{\mu^2}{4} \Psi^a{}_{bcd} \Psi^b{}_{aef} \Sigma^{cd} \Sigma^{ef} - d(\omega^a{}_b \frac{\mu}{2} \Psi^b{}_{acd} \Sigma^{cd} + \nu \omega^a{}_b \Sigma^b_{.a})] \mathbf{m}^1$$
$$+ [2\Omega^a{}_b \Sigma^b{}_a - d(\omega^a{}_b \Sigma^b{}_a)] \mathbf{m}^2 \},$$

where the property $\Psi_{a[bcd]} = 0$ has been used.

If

$$\mathbf{c}_3^{(2)} = (c_3, c_4^1, c_4^2, c_5), \tag{88}$$

then, after using Stoke's theorem, the generalized Chern-Simons integral is

$$\int_{\mathbf{c}_3^{(2)}} \mathbf{CS} = k\{ \int_{c_3} [\omega^a{}_b \Omega^b{}_a - \frac{1}{3} \omega^a{}_b \omega^b{}_c \omega^c{}_a + \epsilon \omega^b{}_a (\frac{\mu}{2} \Psi^a{}_{bcd} \Sigma^{cd} + \nu \Sigma^a_{.b})] + \tag{89}$$
$$+ \int_{c_4^1} [2\Omega^b{}_a (\frac{\mu}{2} \Psi^a{}_{bcd} \Sigma^{cd} + \nu \Sigma^a_{.b}) + \epsilon \frac{\mu^2}{4} \Psi^a{}_{bcd} \Psi^b{}_{aef} \Sigma^{cd} \Sigma^{ef}]$$
$$- \int_{\partial c_4^1} [\omega^b{}_a (\frac{\mu}{2} \Psi^a{}_{bcd} \Sigma^{cd} + \nu \Sigma^a_{.b})] + \int_{c_4^2} (2\Omega^a{}_b \Sigma^b{}_a) - \int_{\partial c_4^2} (\omega^a{}_b \Sigma^b{}_a) \}.$$

When, as above, $\mathbf{c}_3^{(2)} = \partial \widetilde{\mathbf{c}}_4^{(2)} = (\partial \widetilde{c}_4, \partial \widetilde{c}_5^1 + \epsilon \widetilde{c}_4, \partial \widetilde{c}_5^2, \partial \widetilde{c}_6 - \epsilon \widetilde{c}_5^2)$ so that $\partial \widetilde{\mathbf{c}}_3^{(2)} = 0$, this integral reduces to

$$\int_{\mathbf{c}_3^{(2)}} \mathbf{CS} = k\{\int_{\partial\widetilde{c}_4} (\omega^a{}_b\Omega^b{}_a - \frac{1}{3}\omega^a{}_b\omega^b{}_c\omega^c{}_a) + \tag{90}$$
$$+ \int_{\partial\widetilde{c}_5^1 + \epsilon\widetilde{c}_4} [2\Omega^b{}_a(\frac{\mu}{2}\Psi^a{}_{bcd}\Sigma^{cd} + \nu\Sigma^a_{.b}) + \epsilon\frac{\mu^2}{4}\Psi^a{}_{bcd}\Psi^b{}_{aef}\Sigma^{cd}\Sigma^{ef}] +$$
$$+ \int_{\partial\widetilde{c}_5^2} (2\Omega^a{}_b\Sigma^b{}_a)\}.$$

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