Generalized forms and Einstein's equations

D. C. Robinson
Mathematics Department
King's College London
Strand, London WC2R 2LS
United Kingdom
email: david.c.robinson@kcl.ac.uk

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Abstract:

Generalized differential forms of different types are defined and their algebra and calculus are discussed. Complex generalized p-forms, a particular class of type two generalized forms, are considered in detail. It is shown that Einstein's vacuum field equations for Lorentzian four-metrics are satisfied if and only if a complex generalized 1-form on the bundle of two component spinors is closed. A similar result for half-flat and anti self-dual holomorphic four-metrics is also presented.

I. INTRODUCTION

Recently the algebra and calculus of generalized differential forms have been developed and examples of their physical application have been presented[1,2]. This paper contains a self-contained extension of this work and an application of it to Einstein's vacuum field equations. Earlier work concentrated on the presentation of generalized forms corresponding to ordered pairs of of ordinary p- and p+1- forms. In this paper, a broader framework is introduced which includes those forms as a particular case. This new framework is constructed by introducing the concept of generalized p-forms of type N (N a non-negative integer) on an n-dimensional manifold M. Ordinary p-forms become generalized forms of type $N = 0$, and the generalized forms corresponding to ordered pairs become generalized forms of type $N = 1$. Forms of type N can be represented by 2^N-tuples of ordinary differential forms, where $-N \leq p \leq n$, and satisfy an exterior algebra and calculus which is a direct generalization of that satisfied by type $N = 0$ and $N = 1$ forms. As in the case of type 1 forms, forms of negative degree are admitted when N is positive[3,4]. The new framework naturally extends the one presented for type $N = 1$ forms in Refs. 1 and 2. It encompasses the extension briefly mentioned in Ref. 1.

In Sec. II the algebra and calculus of generalized differential forms of type N are defined. The definitions are presented in a recursive fashion so that they are similar, in general form, to the definitions previously given for type $N = 1$ generalized forms. Three different representations of generalized forms and their algebra and calculus are presented, including a matrix representation. While each of these representations has its uses, the main representation of type N forms used throughout the paper will be in terms of 2^N-tuples. Local definitions of generalized connections are presented, and on manifolds with metrics the Hodge (star) operator and duality, the co-differential and Laplacian for type N forms are defined. These definitions are the standard ones for $N = 0$ (that is ordinary) forms[5], and agree with the ones given in previous papers for $N = 1$ forms[1,2]. Forms of type N can always be regarded as special cases of forms of type $\acute{N}$ where $N \leq \acute{N}$ but this may not always be the most efficient point of view. In Sec. III the basic algebra and calculus for the particular case of type $N = 2$ generalized forms are presented in more detail and in terms of ordered quadruples of ordinary forms. The formulae here include results needed in the next section. They also illustrate in detail the difference between results for $N = 2$

forms and results for $N = 1$ forms obtained previously. The fourth section is devoted to a discussion of Lorentzian four-metrics and the condition of Ricci flatness. First the representation of the metric geometry via the Cartan structure equations for ordinary forms, including a two component spinor version, is reviewed. Then, by using that formalism, the condition for a four-metric to be Ricci flat is formulated as the requirement that a generalized one-form, defined on the bundle of two component spinors over a four dimensional manifold, be closed. Although the focus is on Lorentzian four-metrics it is clear that a similar result hold for four-metrics of all signatures. Half-flat four-metrics are also considered. The condition for a metric to be half-flat is re-formulated as the requirement that a generalized 1-form to be closed. The holomorphic anti-self dual case is considered explicitly but Riemannian and ultra-hyperbolic four-metrics can be dealt with in a similar way. These results can be viewed as giving a geometrical interpretation of Ricci flatness for four-metrics. They also provide an illustration of the relationship between half-flat metrics and Ricci flat, but not necessarily half-flat, metrics.

Most of the considerations in this paper are local in nature. Emphasis is placed on the algebra, calculus and local geometry rather than the global geometry. The letters over the forms indicate the degrees of the forms and whenever these degrees are obvious they will be omitted. By standard convention, ordinary p-forms (that is of type $N = 0$) with p negative are zero. Where it is helpful a subscript will be used to denote the type of the form. Usually bold Latin letters will be used for generalized forms and normal Greek letters for ordinary forms.

II. GENERALIZED FORMS OF DIFFERENT TYPES

In this section generalized forms of different types will be defined and their exterior algebra and calculus will be discussed. The properties of generalized p-forms can be defined recursively (on N), using definitions which are formally similar to those for the special case, where $N = 1$, discussed in Refs. 1 and 2. Here, using the terminology of Ref. 1, the *left* exterior product and *left* generalized derivative will always be used. While a few examples are given in this section a more extensive collection is presented in Sec. III.

The module of generalized p-forms of type $N = 0$ is defined to be the module of ordinary p-forms on M, Λ_0^p, with the usual exterior product and exterior derivative. Then the module of generalized p-forms of type N, Λ_N^p, is defined as follows. For $N \geq 1$, a generalized p-form of type N, $\overset{p}{\mathbf{a}}_N \in \Lambda^p_N$,is defined to be ordered pairs of generalized p- and p+1-forms of type $N-1$

$$\overset{p}{\mathbf{a}}_N \equiv (\overset{p}{\mathbf{a}}_{N-1}, \overset{p+1}{\mathbf{a}}_{N-1}), \tag{1}$$

where N is any integer greater than or equal to one.

Hence forms of type $N = 1$ are ordered pairs of ordinary p- and p+1-forms forms, for example $\overset{p}{\mathbf{a}}_1 = (\overset{p}{\alpha}, \overset{p+1}{\alpha}) \in \Lambda^p_0 \times \Lambda^{p+1}_0$, as in Refs. 1 and 2. Forms of type $N = 2$ are ordered quadruples of ordinary p-, p+1-,p+1-and p+2-forms. For example, let $\overset{p}{\mathbf{a}}_2 = (\overset{p}{\mathbf{a}}_1, \overset{p+1}{\mathbf{a}}_1)$ where $\overset{p}{\mathbf{a}}_1 = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$, $\overset{p+1}{\mathbf{a}}_1 = (\overset{p+1}{\not\alpha}, \overset{p+2}{\alpha})$, then $\overset{p}{\mathbf{a}}_2 \in \Lambda^p_2$ is given by

$$\overset{p}{\mathbf{a}}_2 = (\overset{p}{\alpha}, \overset{p+1}{\alpha}, \overset{p+1}{\not\alpha}, \overset{p+2}{\alpha}) \in \Lambda^p_0 \times \Lambda^{p+1}_0 \times \Lambda^{p+1}_0 \times \Lambda^{p+2}_0. \tag{2}$$

More generally, a p-form of type N will correspond to an ordered set of 2^N ordinary q-forms, with $-N \leq p \leq q \leqq p + N$. Non-zero entries occur in the 2^N-tuple only when $0 \leq q \leq n$ since, as mentioned above, any ordinary form, $\overset{q}{\alpha}$,with q negative is zero. Forms of type N_1 are naturally included in forms of type N_2 when $N_1 < N_2$.

The exterior product is extended from ordinary forms to forms of type $N \geq 1$ as follows. Let $\overset{p}{\mathbf{a}}_N \equiv (\overset{p}{\mathbf{a}}_{N-1}, \overset{p+1}{\mathbf{a}}_{N-1})$ and $\overset{q}{\mathbf{b}}_N \equiv (\overset{q}{\mathbf{b}}_{N-1}, \overset{q+1}{\mathbf{b}}_{N-1})$ be a p-form and a q-form of type $N \geq 1$. Then the exterior product of $\overset{p}{\mathbf{a}}_N$and $\overset{q}{\mathbf{b}}_N$is the p+q-form of type N defined (recursively) by

$$\overset{p}{\mathbf{a}}_N \wedge \overset{q}{\mathbf{b}}_N = (\overset{p}{\mathbf{a}}_{N-1} \wedge \overset{q}{\mathbf{b}}_{N-1}, \overset{p}{\mathbf{a}}_{N-1} \wedge \overset{q+1}{\mathbf{b}}_{N-1} + (-1)^q \overset{p+1}{\mathbf{a}}_{N-1} \wedge \overset{q}{\mathbf{b}}_{N-1}). \tag{3}$$

The exterior product satisfies all the usual rules, in particular $\overset{p}{\mathbf{a}}_N \wedge \overset{q}{\mathbf{b}}_N = (-1)^{pq} \overset{p}{\mathbf{b}}_N \wedge \overset{q}{\mathbf{a}}_N$. Furthermore, it follows that when $p + q < -N$, the exterior product is zero.

The exterior derivative, $d : \Lambda^p_N \rightarrow \Lambda^{p+1}_N$, is defined in the usual way for $N = 0$ forms, and when $N \geq 1$ by

$$d\overset{p}{\mathbf{a}}_N = (d\overset{p}{\mathbf{a}}_{N-1} + (-1)^{p+1} k_N \overset{p+1}{\mathbf{a}}_{N-1}, d\overset{p+1}{\mathbf{a}}_{N-1}), \tag{4}$$

where k_N is constant.

When $d\overset{p}{\mathbf{a}}_N$is expressed in terms of ordinary forms it contains the constants $k_1, k_2, ...k_N$. These will all be assumed to be non-zero unless it is stated otherwise. This exterior derivative also satisfies all the usual rules, in particular $d^2 = 0$, and $d(\overset{p}{\mathbf{a}}_N \wedge \overset{q}{\mathbf{b}}_N) = d\overset{p}{\mathbf{a}}_N \wedge \overset{q}{\mathbf{b}}_N + (-1)^p \overset{p}{\mathbf{a}}_N \wedge d\overset{q}{\mathbf{b}}_N$.

The above representation of the algebra and calculus of generalized forms will be the main one used in this paper. However it is appropriate to note here two alternative representations which can be useful. In the first of these type N forms of degree minus one, ζ_1,....ζ_N, which are required to satisfy all the usual rules of exterior algebra and calculus, together with the two conditions

$$\zeta_1 \wedge .... \wedge \zeta_N \neq 0, \quad d\zeta_I = k_I, \; I = 1...N, \tag{5}$$

are introduced.[1] Then a generalized form of type N, $\overset{p}{\mathbf{a}}_N = (\overset{p}{\mathbf{a}}_{N-1}, \overset{p+1}{\mathbf{a}}_{N-1})$, can be identified with

$$\overset{p}{\mathbf{a}}_N = \overset{p}{\mathbf{a}}_{N-1} + \overset{p+1}{\mathbf{a}}_{N-1} \wedge \zeta_N, \tag{6}$$

and it follows that the exterior product and derivative agree with Eqs. (3) and (4). For example, it follows from Eq. (5) that

$$d\overset{p}{\mathbf{a}}_N = d\overset{p}{\mathbf{a}}_{N-1} + (-1)^{p+1} k_N \overset{p+1}{\mathbf{a}}_{N-1} + d\overset{p+1}{\mathbf{a}}_{N-1} \wedge \zeta_N. \tag{7}$$

The recursive use of this identification can be illustrated by using the example of an $N = 2$ form given above. In this case $\overset{p}{\mathbf{a}}_2 = (\overset{p}{\alpha}, \overset{p+1}{\alpha}, \overset{p+1}{\not\alpha}, \overset{p+2}{\alpha})$ is identified with $\overset{p}{\mathbf{a}}_2 = \overset{p}{\alpha} + \overset{p+1}{\alpha} \wedge \zeta_1 + \overset{p+1}{\not\alpha} \wedge \zeta_2 + \overset{p+2}{\alpha} \wedge \zeta_1 \wedge \zeta_2$. It should be noted that while the identification continues to be unambiguous when $N \geq 3$ the relationship between the ordering of the forms in the two types of representations is not as simple as it is in the $N \leq 2$ cases.

The second of the alternative representation is in terms of matrix valued forms and is again defined recursively. Here the generalized form $\overset{p}{\mathbf{a}}_N = (\overset{p}{\mathbf{a}}_{N-1}, \overset{p+1}{\mathbf{a}}_{N-1})$ is identified with a 2×2 matrix, $[\overset{p}{\mathbf{a}}_N]$, with entries that are forms of type $N - 1$,

$$[\overset{p}{\mathbf{a}}_N] = \begin{bmatrix} \overset{p}{\mathbf{a}}_{N-1} & \overset{p+1}{\mathbf{a}}_{N-1} \\ 0 & (-1)^p \overset{p}{\mathbf{a}}_{N-1} \end{bmatrix}. \tag{8}$$

Exterior multiplication of generalized forms $\overset{p}{\mathbf{a}}_N$and $\overset{q}{\mathbf{b}}_N$,as in Eq. (3), corresponds to matrix multiplication of $[\overset{p}{\mathbf{a}}_N]$ and $[\overset{q}{\mathbf{b}}_N]$. The matrix representation of the exterior derivative, given by Eq. (4), can be identified by using the 2×2 matrices

$$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, K_N = \begin{bmatrix} 0 & 0 \\ k_N & 0 \end{bmatrix},$$

and the bracket, $\{A, B\}_r$, of 2×2 matrices defined by

$$\{A, B\}_r = AB + (-1)^r BA.$$

Then the matrix corresponding to $d\overset{p}{\mathbf{a}}_N$is given by

$$[d\overset{p}{\mathbf{a}}_N] = Sd[\overset{p}{\mathbf{a}}_N] + \{K_N, [\overset{p}{\mathbf{a}}_N]\}_{p+1}. \tag{9}$$

Using the definitions recursively it follows that a generalized form of type N is identified with a $2^N \times 2^N$ matrix with entries which are ordinary forms. For example if $\overset{p}{\mathbf{a}}_2 = (\overset{p}{\alpha}, \overset{p+1}{\alpha}, \overset{p+1}{\not{\alpha}}, \overset{p+2}{\alpha})$, as above, then the 4×4 matrix representation of $\overset{p}{\mathbf{a}}_2$ is given by

$$[\overset{p}{\mathbf{a}}_2] = \begin{bmatrix} \overset{p}{\alpha} & \overset{p+1}{\alpha} & \overset{p+1}{\not{\alpha}} & \overset{p+2}{\alpha} \\ 0 & (-1)^p\overset{p}{\alpha} & 0 & (-1)^{p+1}\overset{p+1}{\not{\alpha}} \\ 0 & 0 & (-1)^p\overset{p}{\alpha} & (-1)^p\overset{p+1}{\alpha} \\ 0 & 0 & 0 & \overset{p}{\alpha} \end{bmatrix}. \tag{10}$$

The Poincaré lemma for generalized forms of type N$\geqq$ 1 can be obtained by straightforward calculation.

Theorem 1: Let $\overset{p}{\mathbf{a}}_N = (\overset{p}{\mathbf{a}}_{N-1}, \overset{p+1}{\mathbf{a}}_{N-1})$,with N$\geqq$ 1, be non-zero and closed, so that $d\overset{p}{\mathbf{a}}_N = 0$. Then

(a) d$\overset{-N}{\mathbf{a}}_N = 0$ if and only if $k_I = 0$ for all $I, 1 \leq I \leq N$, and $\overset{-N}{\mathbf{a}}_N = (0, ..., 0, c)$ where c is a non-zero constant;

(b) if p$\geq -N+1$, d$\overset{p}{\mathbf{a}}_N = 0$ if and only if *either* $\overset{p}{\mathbf{a}}_N$ is exact *or* $k_I = 0$ for all $I, 1 \leq I \leq N$, and all the ordinary forms in the 2^N-tuple are closed.

(c) When k_N is non-zero, d$\overset{p}{\mathbf{a}}_N = 0$ if and only if $\overset{p}{\mathbf{a}}_N$is exact.

Furthermore, $\overset{p+1}{\mathbf{a}}_{N-1} = (-1)^p k_N^{-1} d\overset{p}{\mathbf{a}}_{N-1}$and $\overset{p}{\mathbf{a}}_N = d\overset{p-1}{\mathbf{b}}_N$where $\overset{p-1}{\mathbf{b}}_N = (0, (-1)^p k_N^{-1}\overset{p}{\mathbf{a}}_{N-1}) + \overset{p-1}{\mathbf{c}}_N$, for any closed form $\overset{p-1}{\mathbf{c}}_N$.

Hence the de Rham cohomology determined by generalized forms is trivial unless *all* the constant k's are zero in which case the standard de Rham cohomology applies. These ideas can be straightforwardly extended to other differential operators. For example, suppose M is a complex manifold and ∂ and $\bar{\partial}$ are the usual Dolbeault differential operators on ordinary differential forms. These operators can be extended to operators acting on generalized forms by, for example, defining

$$\begin{gathered} \partial\zeta_I = l_I,\ \bar{\partial}\zeta_I = m_I \\ l_I + m_I = k_I,\ I = 1...N, \end{gathered} \tag{11}$$

where l_I and m_I are constants, and replacing d and k_N in Eq.(5) by ∂ and l_N (respectively $\bar{\partial}$ and m_N) in the obvious way. The nature of the Dolbeault cohomology determined by generalized forms, and the cohomology associated with the real operator $d^c = i(\bar{\partial} - \partial)$ will clearly be determined by the vanishing or non-vanishing of the constants m_I and $m_I - l_I$.

Let V be a vector field tangent to M. Then the inner product of V with a generalized p-form $\overset{p}{\mathbf{a}}_N \equiv (\overset{p}{\mathbf{a}}_{N-1}, \overset{p+1}{\mathbf{a}}_{N-1})$ is defined in the usual way for $N = 0$ forms. For $N > 0$ it is defined to be zero if p$= -N$ and if p$> -N$ it is defined to be the generalized (p-1)- form

$$V \rfloor \overset{p}{\mathbf{a}}_N \equiv (V \rfloor \overset{p}{\mathbf{a}}_{N-1}, V \rfloor \overset{p+1}{\mathbf{a}}_{N-1}). \tag{12}$$

The Lie derivative of $\overset{p}{\mathbf{a}}_N$is defined to be the p-form

$$L_V \overset{p}{\mathbf{a}}_N \equiv V \rfloor d\overset{p}{\mathbf{a}}_N + d(V \rfloor \overset{p}{\mathbf{a}}_N). \tag{13}$$

It follows from this definition that

$$L_V \overset{p}{\mathbf{a}}_N = (L_V \overset{p}{\mathbf{a}}_{N-1}, L_V \overset{p+1}{\mathbf{a}}_{N-1}). \tag{14}$$

Let G be a Lie group and let $\mathcal{G}$ the Lie algebra of G. Generalized $\mathcal{G}$-valued connection 1-forms and curvature 2-forms of type N are defined in a similar manner to ordinary connection 1-forms and curvature 2-forms.[1,6] However the ordinary forms are replaced by generalized type N forms. Here only the local definition of curvature is given. The appropriate geometrical setting for a global formulation needs further investigation. Let $\mathbf{A}_N$ be a generalized connection 1-form, of type N, with values in $\mathcal{G}$. The curvature 2-form of this connection is defined by the usual type of formula to be

$$\mathbf{F}_N = d\mathbf{A}_N + \frac{1}{2}[\mathbf{A}_N, \mathbf{A}_N]. \tag{15}$$

Hence if the commutation relations of the Lie algebra $\mathcal{G}$ are given by

$$[X_j, X_k] = C^i_{jk} X_i,$$

and if $\mathbf{A}_N = \mathbf{A}^i_N X_i$, where $\mathbf{A}^i_N$ are generalized 1-forms, then if $\mathbf{F}_N = \mathbf{F}^i_N X_i$,

$$\mathbf{F}^i_N = d\mathbf{A}^i_N + \frac{1}{2} C^i_{jk} \mathbf{A}^j_N \wedge \mathbf{A}^k_N. \tag{16}$$

A generalized connection, $\mathbf{A}_N$, determines a generalized covariant exterior derivative,$\mathbf{D}$. Let $\overset{p}{\mathbf{b}}_N$ be any $\mathcal{G}$-valued generalized p-form of type N, then

$$\mathbf{D}\overset{p}{\mathbf{b}}_N = d\overset{p}{\mathbf{b}}_N + [\mathbf{A}_N, \overset{p}{\mathbf{b}}_N]. \tag{17}$$

As an example of a generalized connection, consider *affine* generalized connections with structure group $G=$ IGL(n) and Lie algebra $\mathcal{G}$. The generators, X^b_a, of $\mathcal{G}$ satisfy the commutation relations

$$[X_a, X_b] = 0, [X^a_b, X_c] = \delta^a_c X_b, [X^a_b, X^c_d] = (\delta^a_d X^c_b - \delta^c_b X^a_d).$$

The generalized connection 1-form $\mathbf{A}_N$ and curvature 2-form $\mathbf{F}_N$ are given by

$$\mathbf{A}_N = \mathbf{e}^a_N X_a + \mathbf{\Gamma}^a_{Nb} X^b_a; \mathbf{F}_N = \mathbf{T}^a_N X_a + \mathbf{F}^a_{Nb} X^b_a, \tag{18}$$

where $\mathbf{e}^a_N$ is a moving co-frame of generalized 1-forms on M, and $\mathbf{\Gamma}^a_{Nb}$ and the pair $\mathbf{T}^a_N, \mathbf{F}^a_{Nb}$ are respectively generalized 1-forms and 2-forms on M. The lower case Latin indices range and sum over 1 to n. Computing the generalized curvature, as above, gives the Cartan structure equations. The first and second generalized Cartan structure equations are

$$\mathbf{T}^a_N = d\mathbf{e}^a_N - \mathbf{e}^b_N \wedge \mathbf{\Gamma}^a_{Nb}; \mathbf{F}^a_{Nb} = d\mathbf{\Gamma}^a_{Nb} + \mathbf{\Gamma}^a_{Nc} \wedge \mathbf{\Gamma}^c_{Nb}, \tag{19}$$

where $\mathbf{T}^a_N$ is the generalized torsion and $\mathbf{F}^a_{Nb}$ is the generalized curvature of the generalized affine connection $\mathbf{\Gamma}^a_{Nb}$.

Next consider an oriented manifold M with a metric g of signature (r,n-r) so that a Hodge star operator, duality, co-differential and Laplacian etc for generalized forms, $\overset{p}{\mathbf{a}}_N = (\overset{p}{\mathbf{a}}_{N-1}, \overset{p+1}{\mathbf{a}}_{N-1})$, of type N can be defined. The sign conventions of Ref. 2 are again used so the definitions and results below agree with the ones in that reference when $N = 0$ and $N = 1$. The definitions are again recursive in nature.

The (Hodge) star operator,$\star : \Lambda^p_N \rightarrow \Lambda^{n-p-N}_N$, and dual for generalized forms are defined as follows. For $N = 0$, $\star\overset{p}{\mathbf{a}}_0 \equiv *\overset{p}{\alpha}$, where $*$ denotes the usual Hodge star operator on ordinary forms, and for $N \geq 1$,

$$\overset{p}{\mathbf{a}}_N \mapsto \star\overset{p}{\mathbf{a}}_N \equiv ([-1]^{n+p+N} \star \overset{p+1}{\mathbf{a}}_{N-1}, \star\overset{p}{\mathbf{a}}_{N-1}). \tag{20}$$

This definition gives, as the dual to a type N generalized p-form, a generalized (n-p-N)-form. It follows that if $\overset{p}{s} = (-1)^{np+p+n-r}$, then

$$\star\star\overset{p}{\mathbf{a}}_N = \overset{p}{s}(-1)^{Np}\overset{p}{\mathbf{a}}_N \equiv \lambda^2_N\overset{p}{\mathbf{a}}_N, \tag{21}$$

where λ_N depends on N and, as usual, n, p and the signature of the metric. The possible eigenvalues of $\star$ are $\pm\lambda_N$, where in fact $\lambda_N = 1$ or i. This agrees with the standard result for ordinary forms and the previously obtained expression for N=1 forms. With the above sign conventions, a generalized p-form is said to be self-dual if $\star\overset{p}{\mathbf{a}}_N = \lambda_N\overset{p}{\mathbf{a}}_N$, and anti self-dual if $\star\overset{p}{\mathbf{a}}_N = -\lambda_N\overset{p}{\mathbf{a}}_N$. It is straightforward to see that necessary conditions for a generalized p-form to be either self-dual or anti self-dual are that when N is zero or an even integer, the dimension of M, n, must be be even, when N is odd n must be odd, and that $p = \frac{n-N}{2.}$. In fact $\star\overset{p}{\mathbf{a}}_N = \overset{+}{-}\lambda_N\overset{p}{\mathbf{a}}_N$ if and only if $\overset{p}{\mathbf{a}}_N = (\overset{\frac{n-N}{2}}{\mathbf{a}}_{N-1}, \pm\lambda^{-1}_N \star \overset{\frac{n-N}{2}}{\mathbf{a}}_{N-1})$.

Now

$$\overset{p}{\mathbf{a}}_N \wedge \star\overset{q}{\mathbf{b}}_N = ([-1]^{n+q+N}\overset{p}{\mathbf{a}}_{N-1} \wedge \star\overset{q+1}{\mathbf{b}}_{N-1}, \overset{p}{\mathbf{a}}_{N-1} \wedge \star\overset{q}{\mathbf{b}}_{N-1} + \overset{p+1}{\mathbf{a}}_{N-1} \wedge \star\overset{q+1}{\mathbf{b}}_{N-1}).$$

The simplest definition of a symmetric inner product of two generalized p-forms $\overset{p}{\mathbf{a}}_N$ and $\overset{p}{\mathbf{b}}_N$ is given by the usual expression for ordinary forms when N=0, $\left\langle \overset{p}{\alpha}, \overset{p}{\beta} \right\rangle$, and recursively, when N$\geq$ 1, by

$$\left\langle \overset{p}{\mathbf{a}}_N, \overset{p}{\mathbf{b}}_N \right\rangle \equiv \left\langle \overset{p}{\mathbf{a}}_{N-1}, \overset{p}{\mathbf{b}}_{N-1} \right\rangle + \left\langle \overset{p+1}{\mathbf{a}}_{N-1}, \overset{p+1}{\mathbf{b}}_{N-1} \right\rangle. \tag{22}$$

This inner product is positive definite for a Riemannian manifold and can be used in the construction of Lagrangians. (As was noted in Ref. 2, alternative definitions may also be useful.)

A co-differential operator $\delta : \Lambda_N^p \to \Lambda_N^{p-1}$, by $\overset{p}{\mathbf{a}}_N \mapsto \delta \overset{p}{\mathbf{a}}_N$ is defined recursively as follows. If $\overset{p}{\sigma} = (-1)^{np-r+1}$, then

$$\delta \overset{p}{\mathbf{a}}_N = (-1)^{N(p+1)} \overset{p}{\sigma} \star d \star \overset{p}{\mathbf{a}}_N. \tag{23}$$

This definition agrees with the definition for ordinary forms, and the previously presented definition for $N = 1$ forms, and implies that

$$\delta \overset{p}{\mathbf{a}}_N = (\delta \overset{p}{\mathbf{a}}_{N-1}, \delta \overset{p+1}{\mathbf{a}}_{N-1} + (-1)^p k_N \overset{p}{\mathbf{a}}_{N-1}). \tag{24}$$

From these definitions it follows that $\delta^2 = 0$,and $\delta \overset{-N}{\mathbf{a}}_N = 0$. If $\overset{p}{\mathbf{a}}_N$ is co-closed, that is $\delta \overset{p}{\mathbf{a}}_N = 0$, then in a result analogous to the Poincaré lemma above, it is co-exact, that is it is the co-differential of a generalized (p+1)-form.

Theorem 2: If $\overset{p}{\mathbf{a}}_N = (\overset{p}{\mathbf{a}}_{N-1}, \overset{p+1}{\mathbf{a}}_{N-1})$ and $\delta \overset{p}{\mathbf{a}}_N = 0$, then if $-N \leq p \leq n-1$, and k_N is non-zero, $\overset{p}{\mathbf{a}}_{N-1} = (-1)^{p+1} k_N^{-1} \delta \overset{p+1}{\mathbf{a}}_{N-1}$and $\overset{p}{\mathbf{a}}_N = \delta \overset{p+1}{\mathbf{b}}_N$,where $\overset{p+1}{\mathbf{b}}_N = ((-1)^{p+1} k_N^{-1} \overset{p+1}{\mathbf{a}}_{N-1}, 0) + \overset{p+1}{\mathbf{c}}_N$, for any $\overset{p+1}{\mathbf{c}}_N$ which is co-closed. Furthermore, $\delta \overset{n}{\mathbf{a}}_N = 0$ if and only if $\overset{n}{\mathbf{a}}_N = 0$. Any type N form, with p=-N, is both coclosed and co-exact.

When M is compact without boundary, the condition, $\langle d\mathbf{a}_N, \mathbf{b}_N \rangle = \langle \mathbf{a}_N, \delta \mathbf{b}_N \rangle$, for this co-differential operator on generalized forms to be the adjoint of d, holds.

A Laplacian for generalized forms, $\triangle : \Lambda_N^p \to \Lambda_N^p$, is defined to be $\triangle = d\delta + \delta d$.

Computation, with the choice of signs made in this paper, gives the simple expression, in agreement with the previously presented N=0 and N=1 cases,

$$\triangle \overset{p}{\mathbf{a}}_N = (\triangle \overset{p}{\mathbf{a}}_{N-1} + k_N^2 \overset{p}{\mathbf{a}}_{N-1}, \triangle \overset{p+1}{\mathbf{a}}_{N-1} + k_N^2 \overset{p+1}{\mathbf{a}}_{N-1}). \tag{25}$$

It follows from this that $\overset{p}{\mathbf{a}}_N$ is a harmonic generalized form, that is $\triangle \overset{p}{\mathbf{a}}_N = 0$, if and only if $\triangle \overset{p}{\mathbf{a}}_{N-1} + k_N^2 \overset{p}{\mathbf{a}}_{N-1} = 0$, and $\triangle \overset{p+1}{\mathbf{a}}_{N-1} + k_N^2 \overset{p+1}{\mathbf{a}}_{N-1} = 0$. That is, a generalized form is harmonic only when its constituent ordinary forms satisfy a Klein-Gordon type of equation with a "mass squared" term given by k_N^2.

The choices of signs in the above definitions have been made in order to make generalized forms eigenforms of the operator $\star\star$, to give a definition of δ which was simply related to the definition for ordinary forms and to ensure that the Laplacian on generalized forms was computable in terms of the Laplacian, not some other second order differential operator, acting on ordinary forms.

III. GENERALIZED FORMS OF TYPE N=2

In this section aspects of the algebra and calculus of generalized forms of type 2 will be considered in more detail and further applications of the results in Sec. II will be presented. Henceforth the generalized forms considered will be mainly of type $N = 2$, so the subscript $N = 2$ will be omitted in the remainder of the paper. As will be seen there is a loose analogy between going from real numbers to complex numbers to quaternions and going from ordinary forms to forms of type 1 and forms of type 2.

A. Basic algebra and calculus

As was noted in Sec. II, a generalized p-form of type 2, $\overset{p}{\mathbf{a}}$, (-2$\leq p \leq n$), is an ordered quadruple of ordinary p-, p+1-, p+1- and p+2- forms; that is since $\overset{p}{\mathbf{a}}$ is an ordered pair of the type 1 forms $\overset{p}{\mathbf{a}}_1 = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$ and $\overset{p+1}{\mathbf{a}}_1 = (\overset{p+1}{\not\alpha}, \overset{p+2}{\alpha})$

$$\overset{p}{\mathbf{a}} \equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha}, \overset{p+1}{\not\alpha}, \overset{p+2}{\alpha}).$$

A minus two-form is an ordered quadruple $\overset{-2}{\mathbf{a}} = (0, 0, 0, \overset{0}{\alpha})$, where $\overset{0}{\alpha}$ is a function on M, and a minus one-form is an ordered quadruple $\overset{-1}{\mathbf{a}} = (0, \overset{0}{\alpha}, \overset{0}{\not\alpha}, \overset{1}{\alpha})$, where $\overset{1}{\alpha}$ is an ordinary 1-form on M and $\overset{0}{\alpha}$, $\overset{0}{\not\alpha}$ are functions on M. A generalized p-form of type 2, given by a quadruple $(\overset{p}{\alpha}, 0, 0, 0)$, will be identified with the ordinary p-form $\overset{p}{\alpha}$. Consequently a function on M, $\overset{0}{\alpha}$, will be identified with the generalized 0-form $(\overset{0}{\alpha}, 0, 0, 0)$ while the quadruples $(0, \overset{0}{\alpha}, 0, 0), (0, 0, \overset{0}{\alpha}, 0)$ and $(0, 0, 0, \overset{0}{\alpha})$ respectively define two linearly independent generalized minus one-forms and a minus two-form. Just as an ordinary p-form $\overset{p}{\alpha}$ is naturally included in the generalized p-forms of type 2 as $(\overset{p}{\alpha}, 0, 0, 0)$, a generalized p-form of type 1, $(\overset{p}{\alpha}, \overset{p+1}{\alpha})$, can be naturally included in the generalized p-forms of type 2 as $(\overset{p}{\alpha}, \overset{p+1}{\alpha}, 0, 0)$.

If $\overset{p}{\mathbf{a}} \equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha}, \overset{p+1}{\not\alpha}, \overset{p+2}{\alpha})$ and $\overset{q}{\mathbf{b}} \equiv (\overset{q}{\beta}, \overset{q+1}{\beta}, \overset{q+1}{\not\beta}, \overset{q+2}{\beta})$, then the generalized exterior product and the generalized exterior derivative, d, defined in section two, are given in terms of ordinary forms by :

$$
\begin{aligned}
\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} &= \overset{p+q}{\mathbf{c}} \equiv (\overset{p+q}{\gamma}, \overset{p+q+1}{\gamma}, \overset{p+q+1}{\not\gamma}, \overset{p+q+2}{\gamma}),\\
\overset{p+q}{\gamma} &= \overset{p}{\alpha} \wedge \overset{q}{\beta},\\
\overset{p+q+1}{\gamma} &= \overset{p}{\alpha} \wedge \overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha} \wedge \overset{q}{\beta},\\
\overset{p+q+1}{\not\gamma} &= \overset{p}{\alpha} \wedge \overset{q+1}{\not\beta} + (-1)^q \overset{p+1}{\not\alpha} \wedge \overset{q}{\beta},\\
\overset{p+q+2}{\gamma} &= \overset{p}{\alpha} \wedge \overset{q+2}{\beta} + (-1)^{q+1} \overset{p+1}{\alpha} \wedge \overset{q+1}{\not\beta} + (-1)^q \overset{p+1}{\not\alpha} \wedge \overset{q+1}{\beta} + (-1)^q \overset{p+2}{\alpha} \wedge \overset{q}{\beta},
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
d\overset{p}{\mathbf{a}} &\equiv \overset{p+1}{\mathbf{c}} \equiv (\overset{p+1}{\gamma}, \overset{p+2}{\gamma}, \overset{p+2}{\not\gamma}, \overset{p+3}{\gamma}),\\
\overset{p+1}{\gamma} &= d\overset{p}{\alpha} + (-1)^{p+1} k_1 \overset{p+1}{\alpha} + (-1)^{p+1} k_2 \overset{p+1}{\not\alpha},\\
\overset{p+2}{\gamma} &= d\overset{p+1}{\alpha} + (-1)^{p+1} k_2 \overset{p+2}{\alpha},\\
\overset{p+2}{\not\gamma} &= d\overset{p+1}{\not\alpha} + (-1)^p k_1 \overset{p+2}{\alpha},\\
\overset{p+3}{\gamma} &= d\overset{p+2}{\alpha}.
\end{aligned}
\tag{27}
$$

The ordinary forms (and manifold) may be real or complex and a bar over ordinary forms denotes the usual complex conjugate.

The conjugate of $\overset{p}{\mathbf{a}}$, denoted $\overset{p}{\bar{\mathbf{a}}}$, is defined to be

$$
(\overset{p}{\bar{\alpha}}, \overset{p+1}{\bar{\not\alpha}}, \overset{p+1}{\bar{\alpha}}, -\overset{p+2}{\bar{\alpha}}). \tag{28}
$$

The generalized form is said to be self-conjugate, $\overset{p}{\mathbf{a}} = \overset{p}{\bar{\mathbf{a}}}$, when $\overset{p}{\alpha}$ is real, $\overset{p+1}{\not\alpha}$ is the complex conjugate of $\overset{p+1}{\alpha}$ and $\overset{p+2}{\alpha}$ is a purely imaginary ordinary 1-form.

(Hence if $\overset{p}{\mathbf{a}} = \overset{p}{\alpha} + \overset{p+1}{\alpha} \wedge \zeta + \overset{p+1}{\not\alpha} \wedge \bar{\zeta} + \overset{p+2}{\alpha} \wedge \zeta \wedge \bar{\zeta}$, with $d\zeta = k$, $d\bar{\zeta} = \bar{k}$ the conjugate is just the obvious complex conjugate.)

The Poincare lemma of Sec. II can be written in the following way for type 2 forms.

Proposition 1: If $\overset{p}{\mathbf{a}} = (\overset{p}{\alpha}, \overset{p+1}{\alpha}, \overset{p+1}{\not\alpha}, \overset{p+2}{\alpha})$ is closed, so that $d\overset{p}{\mathbf{a}} = 0$, then

(a) $(-1)^p d\overset{p}{\alpha} = k_1 \overset{p+1}{\alpha} + k_2 \overset{p+1}{\not\alpha}$,

(b) $\overset{p}{\mathbf{a}}$ is exact, and $\overset{p}{\mathbf{a}} = d\overset{p-1}{\mathbf{b}} = d\overset{p-1}{\mathbf{c}}$, where $\overset{p-1}{\mathbf{b}} = (-1)^p k_2^{-1}(0, 0, \overset{p}{\alpha}, \overset{p+1}{\alpha})$,and $\overset{p-1}{\mathbf{c}} = (-1)^p k_1^{-1}(0, \overset{p}{\alpha}, 0, -\overset{p+1}{\not{\alpha}})$.

The second equality in (b) illustrates a consequence of the freedom to add the exterior derivative of a complex generalized (p-2)-form to $\overset{p-1}{\mathbf{b}}$ or $\overset{p-1}{\mathbf{c}}$. If $\overset{p}{\mathbf{a}}$ is self-conjugate then $\overset{p-1}{\mathbf{b}}$ is the conjugate of $\overset{p-1}{\mathbf{c}}$.

B. Matrix Lie groups

Next consider Lie groups and Lie algebras, for simplicity matrix Lie groups. In the present context, and following Ref. 1, it is natural to associate with a Lie group G the semi-direct product of G and Lie algebra-valued forms (viewed as the direct product of additive abelian groups). Define the (associated) Lie group $\mathbf{G}$ by

$$\mathbf{G} = \{\mathbf{a} \mid \mathbf{a} = \alpha(1, \overset{1}{A}, \overset{1}{\not{A}}, \overset{2}{A})\}, \tag{29}$$

$$\alpha(1, \overset{1}{A}, \overset{1}{\not{A}}, \overset{2}{A}) \equiv (\alpha, 0, 0, 0) \wedge (1, \overset{1}{A}, \overset{1}{\not{A}}, \overset{2}{A}) = (\alpha, \alpha A, \alpha \overset{1}{\not{A}}, \alpha \overset{2}{A}),$$

where $\mathbf{a}$ is a complex generalized 0-form, α belongs to the Lie group G, with identity 1, and $\overset{1}{A}, \overset{1}{\not{A}}, \overset{2}{A}$ are ordinary forms with values in the (matrix) Lie algebra of G (or more generally H, where G is a sub-group of H).

The product of two elements of $\mathbf{G}$, $\mathbf{a} = \alpha(1, \overset{1}{A}, \overset{1}{\not{A}}, \overset{2}{A}))$ and $\mathbf{b} = \beta(1, \overset{1}{B}, \overset{1}{\not{B}}, \overset{2}{B})$ is given by the above rules for exterior multiplication, and is $\mathbf{c} = \mathbf{a} \wedge \mathbf{b} = \alpha\beta(1, \overset{1}{C}, \overset{1}{\not{C}}, \overset{2}{C})$, where

$$\begin{aligned}
\overset{1}{C} &= \overset{1}{B} + \beta^{-1}\overset{1}{A}\beta, \\
\overset{1}{\not{C}} &= \overset{1}{\not{B}} + \beta^{-1} \overset{1}{\not{A}} \beta, \\
\overset{2}{C} &= \overset{2}{B} + \beta^{-1}\overset{2}{A}\beta + \overset{2}{L},
\end{aligned} \tag{30}$$

where

$$\overset{2}{L} = \beta^{-1} \overset{1}{\not{A}} \beta \wedge \overset{1}{B} - \beta^{-1}\overset{1}{A}\beta \wedge \overset{1}{\not{B}}.$$

Here, in order to ensure that the ordinary forms take values in the Lie algebra of G, it is henceforth assumed that $\overset{1}{\not{A}} = c\overset{1}{A}$, $\overset{1}{\not{B}} = c\overset{1}{B}$, where c equals one if necessary. Hence in the following, $\overset{2}{L} = 0$.

The inverse of $\mathbf{a}$ is $\mathbf{a}^{-1} = \alpha^{-1}(1, -\alpha\overset{1}{A}\alpha^{-1}, -c\alpha\overset{1}{A}\alpha^{-1}, -\alpha\overset{2}{A}\alpha^{-1})$ and the identity is $(1,0,0,0)$. Left fundamental 1-forms, denoted $\mathbf{l}$, are formally defined by

$$
\begin{aligned}
&\mathbf{l} = \mathbf{a}^{-1} \wedge d\mathbf{a} = (\overset{1}{\lambda}, \overset{2}{\lambda}, \overset{2}{\not\lambda}, \overset{3}{\lambda}),\\
&\overset{1}{\lambda} = \alpha^{-1}d\alpha - (k_1 + ck_2)\overset{1}{A},\\
&\overset{2}{\lambda} = d\overset{1}{A} - (k_1 + ck_2)\overset{1}{A} \wedge \overset{1}{A} - k_2\overset{2}{A} + \alpha^{-1}d\alpha \wedge \overset{1}{A} + \overset{1}{A} \wedge \alpha^{-1}d\alpha, \qquad (31)\\
&\overset{2}{\not\lambda} = c(d\overset{1}{A} - (k_1 + ck_2)\overset{1}{A} \wedge \overset{1}{A}) + k_1\overset{2}{A} + c(\alpha^{-1}d\alpha \wedge \overset{1}{A} + \overset{1}{A}\wedge\alpha^{-1}d\alpha) + dc \wedge \overset{1}{A},\\
&\overset{3}{\lambda} = d\overset{2}{A} + (k_1 + ck_2)(\overset{2}{A} \wedge \overset{1}{A} - \overset{1}{A} \wedge \overset{2}{A}) + \alpha^{-1}d\alpha \wedge \overset{2}{A} - \overset{2}{A} \wedge \alpha^{-1}d\alpha + dc \wedge \overset{1}{A} \wedge \overset{1}{A},
\end{aligned}
$$

and $\mathbf{l}$ satisfies the Maurer-Cartan equation

$$d\mathbf{l} + \mathbf{l} \wedge \mathbf{l} = 0. \qquad (32)$$

In the special case where $\alpha = 1, k_1 + ck_2 = -1$, and c is constant,

$$\mathbf{l} = \mathbf{a}^{-1} \wedge d\mathbf{a} = (\overset{1}{A}, F - k_2\overset{2}{A}, c[F + c^{-1}k_1\overset{2}{A}], D\overset{2}{A}), \qquad (33)$$

where here,

$$
\begin{aligned}
F &\equiv d\overset{1}{A} + \overset{1}{A} \wedge \overset{1}{A},\\
D\overset{2}{A} &\equiv d\overset{2}{A} + \overset{1}{A} \wedge \overset{2}{A} - \overset{2}{A} \wedge \overset{1}{A}. \qquad (34)
\end{aligned}
$$

C. Connections

The following is an outline of the basic formulae for type 2 connections and curvature. As in Sec.II, a discussion of connections in terms of bundles will be avoided here by working locally with type $N = 2$ forms on M. Let the generalized connection and curvature forms on M be given by $\mathbf{A}^i = (\overset{1}{\alpha}{}^i, \overset{2}{\alpha}{}^i, \overset{2}{\not\alpha}{}^i, \overset{3}{\alpha}{}^i), \mathbf{F}^i = (\overset{2}{F}{}^i, \overset{3}{F}{}^i, \overset{3}{\not F}{}^i, \overset{4}{F}{}^i)$. Then it follows from Eqs. (18) and

(19) that

$$
\begin{aligned}
\overset{2}{F}{}^{i} &= d\overset{1}{\alpha}{}^{i} + \frac{1}{2}C^{i}_{jk}\overset{1}{\alpha}{}^{j} \wedge \overset{1}{\alpha}{}^{k} + k_1\overset{2}{\alpha}{}^{i} + k_2 \overset{2}{\not{\alpha}}{}^{i}, \\
\overset{3}{F}{}^{i} &= D\overset{2}{\alpha}{}^{i} + k_2\overset{3}{\alpha}{}^{i}, \\
\overset{3}{\not{F}}{}^{i} &= D \overset{2}{\not{\alpha}}{}^{i} - k_1\overset{3}{\alpha}{}^{i}, \\
\overset{4}{F}{}^{i} &= D\overset{3}{\alpha}{}^{i} + C^{i}_{jk}\overset{2}{\alpha}{}^{j} \wedge \overset{2}{\not{\alpha}}{}^{k} .
\end{aligned}
\tag{35}
$$

Here D denotes the (formal) covariant exterior derivative of a $\mathcal{G}$-valued ordinary differential form with respect to the ordinary $\mathcal{G}$-valued 1-form $\overset{1}{\alpha}{}^{i}$;

$$
D\overset{p}{\beta}{}^{i} \equiv d\overset{p}{\beta}{}^{i} + C^{i}_{jk}\overset{1}{\alpha}{}^{j} \wedge \overset{p}{\beta}{}^{k}.
$$

Generalized gauge transformations, following Ref.1, are determined by generalized 0-forms on M with values in the Lie group **G,** as above. The gauge transformations determined by**, a**, an element of **G,** as in Eq. (29), are given by the standard formulae

$$
\begin{aligned}
\mathbf{A} &\to (\mathbf{a}^{-1})d\mathbf{a} + (\mathbf{a}^{-1})\mathbf{A}\mathbf{a}, \\
\mathbf{F} &\to (\mathbf{a}^{-1})\mathbf{F}\mathbf{a}.
\end{aligned}
\tag{36}
$$

It should be noted, for example, that although it appears in the above expressions as if $\overset{1}{\alpha}$ in the above equations can be regarded as a connection 1-form, it does not necessarily transform, under generalized gauge transformations in the same way as an ordinary connection transforms under ordinary gauge transformations. Consequently the appropriate non-local geometrical formulation and application requires further investigation, possibly along lines similar to those referred to and discussed in, for example, Ref. 7.

Any ordinary connection $\overset{1}{\alpha}$ can determine flat (zero curvature) generalized connections. In this flat case it follows from Eq. (35) that the curvature of the connection $\overset{1}{\alpha}$ is given by $f^{i} = -(k_1\overset{2}{\alpha}{}^{i} + k_2 \overset{2}{\not{\alpha}}{}^{i})$. Further reference to Eq. (35), in the flat case, shows that the 2-forms $\overset{2}{\alpha}{}^{i}$ (respectively $\overset{2}{\not{\alpha}}{}^{i} = -\ k_2^{-1}(f^{i} - k_1\overset{2}{\alpha}{}^{i})$) determine the 3-forms $\overset{3}{\alpha}{}^{i}$ which automatically satisfies the fourth equation. Eq. (33) gives an alternative representation of a flat generalized connection.

D. Metric geometries, the co-differential and Laplacian

When M has a metric the dual of $\overset{p}{\mathbf{a}}$ is the n-p-2 form given by

$$\star\overset{p}{\mathbf{a}} \equiv (*\overset{p+2}{\alpha}, (-1)^{n+p}*\overset{p+1}{\not{\alpha}}, (-1)^{n+p+1}*\overset{p+1}{\alpha}, *\overset{p}{\alpha}) \tag{37}$$

If the dimension of M is even type 2 forms may be self-dual, or anti-self dual, when $p = \frac{1}{2}(n-2)$. Such forms are given by

$$\begin{aligned}\overset{p}{\mathbf{a}} &\equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha}, \pm\lambda^{-1}*\overset{p+1}{\alpha}, \pm\lambda^{-1}*\overset{p}{\alpha}),\\ \lambda^2 &= sgn(\det g).\end{aligned}$$

For example, $N=2$ self/anti self-dual forms on four manifolds are determined by a pair of 1-forms and 2-forms (or any 1-form of type $N=1$).

The co-differential is given by

$$\delta\overset{p}{\mathbf{a}} = (\delta\overset{p}{\alpha}, \delta\overset{p+1}{\alpha}+(-1)^p k_1\overset{p}{\alpha}, \delta\overset{p+1}{\not{\alpha}}+(-1)^p k_2\overset{p}{\alpha}, \delta\overset{p+2}{\alpha}+(-1)^{p+1}k_1\overset{p+1}{\not{\alpha}}+(-1)^p k_2\overset{p+1}{\alpha}). \tag{38}$$

In the case of generalized forms of type 2, Theorem 2 implies the following.

Proposition 2: When k_2 is non-zero and the generalized form $\overset{p}{\mathbf{a}}$ is co-closed it must have the form

$$\overset{p}{\mathbf{a}} = ((-1)^{p+1}k_2^{-1}\delta\overset{p+1}{\not{\alpha}}, (-1)^{p+1}k_2^{-1}\delta\overset{p+2}{\alpha} + k_2^{-1}k_1\overset{p+1}{\not{\alpha}}, \overset{p+1}{\not{\alpha}}, \overset{p+2}{\alpha}) \tag{39}$$

and $\overset{p}{\mathbf{a}} = \delta\overset{p+1}{\mathbf{c}}$ where $\overset{p+1}{\mathbf{c}}$ may be chosen to be

$$\overset{p+1}{\mathbf{c}} = ((-1)^{p+1}k_2^{-1}\overset{p+1}{\not{\alpha}}, (-1)^{p+1}k_2^{-1}\overset{p+2}{\alpha}, 0, 0). \tag{40}$$

From Eq. (22), the inner product is given by

$$\left\langle \overset{p}{\mathbf{a}}, \overset{p}{\mathbf{b}} \right\rangle = \left\langle \overset{p}{\alpha}, \overset{p}{\beta} \right\rangle + \left\langle \overset{p+1}{\alpha}, \overset{p+1}{\beta} \right\rangle + \left\langle \overset{p+1}{\not{\alpha}}, \overset{p+1}{\not{\beta}} \right\rangle + \left\langle \overset{p+2}{\alpha}, \overset{p+2}{\beta} \right\rangle.$$

The Laplacian of $\overset{p}{\mathbf{a}}$ is given by

$$\Delta\overset{p}{\mathbf{a}} = (\Delta\overset{p}{\alpha} + c\overset{p}{\alpha}, \Delta\overset{p+1}{\alpha} + c\overset{p+1}{\alpha}, \Delta\overset{p+1}{\not{\alpha}} + c\overset{p+1}{\not{\alpha}}, \Delta\overset{p+2}{\alpha} + c\overset{p+2}{\alpha}), \tag{41}$$

where $c = k_1^2 + k_2^2$. Hence, unlike the case for $N = 1$ forms in Ref. 2, the "mass" term c can be zero even if both (complex) k_1 and k_2 are non-zero.

IV. LORENTZIAN METRICS AND EINSTEIN'S EQUATIONS

The aim of this section is to consider Einstein's vacuum field equations on a four-manifold M. In order to fix notation, a standard formulation, which includes the use of 2-component spinors, of the Cartan moving frame approach to four-metrics will be reviewed. Next a complex generalized one form , **E**, will be constructed on S, the total space of the bundle of two-component spinors over a four- manifold M with four-metric g. It will be shown that **E** is closed if and only if g is Ricci flat. Primary attention will be paid to Lorentzian four-metrics although similar results apply straightforwardly to other signatures and holomorphic four-metrics. Once again everything is local.

First, the Cartan approach to metrics can be summarized as follows. Let θ^a be a basis of ordinary 1-forms, a Cartan co-frame for g, so that the line element for g is given by

$$ds^2 = \eta_{ab}\theta^a \otimes \theta^b \tag{42}$$

where, for 4-metrics

$$\eta_{ab} = \begin{bmatrix} 0 & \epsilon_{AB} \\ -\epsilon_{AB} & 0 \end{bmatrix}, \text{ and } \epsilon_{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

so that

$$ds^2 = \theta^1 \otimes \theta^4 + \theta^4 \otimes \theta^1 - \theta^2 \otimes \theta^3 - \theta^3 \otimes \theta^2. \tag{43}$$

In this section lower case Latin indices sum and range over 1-4. Upper case Latin indices sum and range over 0-1 and, as will be shown below, will be able to be interpreted as two-component spinor indices. Conventions include the standard two component spinor conventions[8–10].

The orientation is such that, in the case of Lorentzian four-metrics, where θ^1 and θ^4 are real and θ^2 is the complex conjugate of θ^3, the volume element is given by, $V = i\theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4$, and the structure group is S0(1,3) which is isomorphic to SL(2,C)/$\mathbb{Z}_2$.

The Cartan structure equations are given by

$$\begin{aligned} D\theta^a &\equiv d\theta^a - \theta^b \wedge \omega^a_b = 0, \\ \omega_{ab} + \omega_{ba} &= 0, \\ d\omega^a_b + \omega^a_c \wedge \omega^c_b &= \Omega^a_b = -\frac{1}{2} R^a_{bcd}\theta^c \wedge \theta^d, \end{aligned} \tag{44}$$

where ω^a_b denotes the Levi-Civita connection 1-form, and R^a_{bcd} are the components of its curvature 2-form Ω^a_b. Here, for any ordinary form the covariant exterior derivative is given by

$$D\alpha^a = d\alpha^a + \omega^a_{b\wedge}\alpha^b, \tag{45}$$

and the second covariant exterior derivative satisfies

$$D^2\alpha^a = \alpha^b \wedge \Omega^a_{b.}$$

The connection and curvature forms, which take values in the Lie algebra of the structure group, can be written as the sum of their self-dual and anti-self-dual parts on the algebra indices, ${}^+\omega^a_b, {}^-\omega^a_b, {}^+\Omega^a_b, {}^-\Omega^a_b$ respectively. Here, ${}^{*+}\Omega^a_b = i{}^+\Omega^a_b$, ${}^{*-}\Omega^a_b = -i{}^-\Omega^a_b$. In 4×4 matrix form

$$ {}^+\omega^a_b = \left[\begin{array}{cc} \bar{\omega}^{0\prime}_{0\prime}1 & \bar{\omega}^{0\prime}_{1\prime}1 \\ \bar{\omega}^{1\prime}_{0\prime}1 & -\bar{\omega}^{0\prime}_{0\prime}1 \end{array}\right], {}^-\omega^a_b = \left[\begin{array}{cc} \omega^A_B & 0 \\ 0 & \omega^A_B \end{array}\right], \tag{46}$$

where 1 denotes the unit 2×2 matrix, $\bar{\omega}^{0\prime}_{0\prime}, \bar{\omega}^{0\prime}_{1\prime}$, $\bar{\omega}^{1\prime}_{0\prime}$ denote the independent components of ${}^+\omega^a_b$, the trace of the 2×2 matrix $\left(\omega^A_B\right)$ is zero and its entries are the complex conjugates of $\bar{\omega}^{A'}_{B'}$. Other self-dual and anti self-dual objects can be written similarly, for instance,

$$\begin{aligned} {}^-\Omega^a_b &= \left[\begin{array}{cc} \Omega^A_B & 0 \\ 0 & \Omega^A_B \end{array}\right], \\ \Omega^A_B &= d\omega^A_B + \omega^A_C \wedge \omega^C_B. \end{aligned} \tag{47}$$

In the case of Lorentzian four-metrics the self-dual connection and curvature are the complex conjugates of the anti-self dual connection and curvature and take (complex conjugate) values in the Lie algebras sl(2,C)$_R$ and sl(2,C)$_L$.

The two-component spinor approach to the Cartan equations for 4-metrics can be summarized, using notation which is compatible with the above, as follows. The line element, given by Eqs. (42) and (43), can be written

$$ds^2 = \epsilon_{AB}\epsilon_{A'B'}\theta^{AA'} \otimes \theta^{BB'}, \tag{48}$$

where the co-frame is represented by a 2×2 matrix $\theta^{AA'}$,

$$\theta^{AA'} = \left[\begin{array}{cc} \theta^{00'} & \theta^{01'} \\ \theta^{10'} & \theta^{11'} \end{array}\right] = \left[\begin{array}{cc} \theta^1 & \theta^3 \\ \theta^2 & \theta^4 \end{array}\right]. \tag{49}$$

For Lorentzian four-metrics this is a Hermitian matrix valued 1-form. By using the correspondences

$$
\begin{aligned}
&\theta^{AA'} \leftrightarrow \theta^{a},\\
&\delta^A_B \bar{\omega}^{A'}_{B'} \leftrightarrow^{+} \bar{\omega}^a_b, \delta^A_B \bar{\Omega}^{A'}_{B'} \leftrightarrow^{+} \bar{\Omega}^a_b,\\
&\delta^{A'}_{B'} \omega^A_B \leftrightarrow^{-} \omega^a_b, \delta^{A'}_{B'} \Omega^A_B \leftrightarrow^{-} \Omega^a_b,
\end{aligned} \tag{50}
$$

the Cartan structure equations, Eqs. (44), can be seen to take the spinorial form

$$
\begin{aligned}
D\theta^{AA'} &\equiv d\theta^{AA'} - \theta^{AB'} \wedge \omega^A_B - \theta^{BA'} \wedge \varpi^{A'}_{B'} = 0,\\
\Omega^A_B &\equiv d\omega^A_B + \omega^A_C \wedge \omega^C_B,\\
\bar{\Omega}^{A'}_{B'} &\equiv d\bar{\omega}^{A'}_{B'} + \bar{\omega}^{A'}_{C'} \wedge \bar{\omega}^{C'}_{B'}.
\end{aligned} \tag{51}
$$

The anti self-dual and self-dual components of the Lorentzian Levi-Civita spin connection are given, respectively, by ω^A_B and $\bar{\omega}^{A'}_{B'}$, in agreement with Eq. (46). Unprimed upper case Latin indices and primed upper case Latin indices represent, respectively, transformation properties under $SL(2,C)_L$ and $SL(2,C)_R$. The components of the curvature 2-forms are given by

$$
\begin{aligned}
\Omega^A_B &= \Psi^A_{BCD}\Sigma^{CD} + 2\Lambda\Sigma^A_B + \Phi^A_{BC'D'}\Sigma^{C'D'},\\
\bar{\Omega}^{A'}_{B'} &= \bar{\Psi}^{A'}_{B'C'D'}\bar{\Sigma}^{C'D'} + 2\Lambda\bar{\Sigma}^{A'}_{B'} + \Phi^{A'}_{B'CD}\Sigma^{CD},
\end{aligned} \tag{52}
$$

where $\Sigma^A_B = 1/2\theta^A_{A'} \wedge \theta^{A'}_B$ and $\bar{\Sigma}^{A'}_{B'} = 1/2\theta^{A'}_A \wedge \theta^A_{B'}$. The anti self-dual and self-dual components of the Weyl spinor are given, respectively, by the totally symmetric, complex conjugate spinors Ψ_{ABCD} and $\bar{\Psi}_{A'B'C'D'}$; and $-2\Phi^A_{BC'D'}$ and 24Λ correspond respectively to the trace free part of the Ricci tensor and the Ricci scalar.

Einstein's field equations, with cosmological constant λ, are given by

$$
G_{ab} = -8\pi T_{ab} - \lambda g_{ab}, \tag{53}
$$

and the spinor form of the Einstein tensor $G^{AA'}_{BB'} = -6(1/3\Phi^{AA'}_{BB'} + \Lambda\delta^A_B\delta^{A'}_{B'})$, is given by the 3-form equation

$$
(1/3\Phi^{AA'}_{BB'} + \Lambda\delta^A_B\delta^{A'}_{B'})\theta^{BC'} \wedge \theta^{CB'} \wedge \theta_{CC'} = \Omega^A_B \wedge \theta^{BA'} = -\bar{\Omega}^{A'}_{B'} \wedge \theta^{AB'}. \tag{54}
$$

Hence the metric is Ricci-flat if and only if

$$
\Omega^A_B \wedge \theta^{BA'} = \bar{\Omega}^{A'}_{B'} \wedge \theta^{AB'} = 0. \tag{55}
$$

Consider now the two-component spinor bundle over M with fibre coordinates π_A and define on the total space, S, the *self conjugate* generalized 1-form $\mathbf{E}$ given by the quadruple

$$\mathbf{E} =(\pi_A\bar{\pi}_{A'}\theta^{AA'}, -k^{-1}\pi_A D\bar{\pi}_{A'}\wedge\theta^{AA'}, -\bar{k}^{-1}D\pi_A\wedge\bar{\pi}_{A'}\theta^{AA'}, (k\bar{k})^{-1}D\pi_A\wedge D\bar{\pi}_{A'}\wedge\theta^{AA'}), \tag{56}$$

where $\theta^{AA'}$ is a Hermitian matrix valued 1-form on M, and ω^A_B and $\bar{\omega}^{A'}_{B'}$ are complex conjugate sl(2,C) valued connection 1-forms on M. Here, for example, D denotes a covariant exterior derivative, for example,

$$D\pi_A \equiv d\pi_A - \pi_B\omega^B_A, \tag{57}$$

and the choice $k_1 = k$, $k_2 = \bar{k}$ has been made.

By using the results above and the identities for the second derivative, such as

$$D^2\pi_A = -\pi_B\Omega^B_A, \tag{58}$$

it is a straightforward matter to show that

$$\begin{aligned}
d\mathbf{E} &= (\overset{2}{\varepsilon}, \overset{3}{\varepsilon}, \overset{3}{\not{\varepsilon}}, \overset{4}{\varepsilon})\\
\overset{2}{\varepsilon} &= \pi_A\bar{\pi}_{A'}D\theta^{AA'},\\
\overset{3}{\varepsilon} &= k^{-1}[\pi_A\bar{\pi}_{A'}\bar{\Omega}^{A'}_{B'}\wedge\theta^{AB'} + \pi_A D\bar{\pi}_{A'}\wedge D\theta^{AA'}],\\
\overset{3}{\not{\varepsilon}} &= \bar{k}^{-1}[\pi_A\bar{\pi}_{A'}\Omega^A_B\wedge\theta^{BA'} + D\pi_A\wedge\bar{\pi}_{A'}D\theta^{AA'}],\\
\overset{4}{\varepsilon} &= (k\bar{k})^{-1}[D\pi_A\wedge D\bar{\pi}_{A'}\wedge D\theta^{AA'} - \pi_A D\bar{\pi}_{A'}\wedge\Omega^A_B\wedge\theta^{BA'} + D\pi_A\wedge\bar{\pi}_{A'}\bar{\Omega}^{A'}_{B'}\wedge\theta^{AB'}].
\end{aligned} \tag{59}$$

This leads immediately to the following theorem.

Theorem 3: The self-conjugate generalized 1-form, $\mathbf{E}$, on S, is closed if and only if

$$D\theta^{AA'} = 0, \tag{60}$$
$$\Omega^A_B\wedge\theta^{BA'} = \bar{\Omega}^{A'}_{B'}\wedge\theta^{AB'} = 0.$$

Consequently, when the four real 1-forms defined by $\theta^{AA'}$ are linearly independent, and hence define a Lorentzian four-metric on M (as in Eq. (48)), the connection ω^a_b is the torsion free Levi-Civita connection, and the metric is Ricci flat, if and only if the complex generalized 1-form $\mathbf{E}$ is closed.

The one form $\mathbf{E}$, which is closed and hence exact when Einstein's vacuum field equations are satisfied, is not unique. For example $\mathbf{F} \wedge \mathbf{E}$, where $\mathbf{F}$ is a zero form which is either closed or satisfies the condition $\mathbf{E} = d\mathbf{F}$, is also closed when Einstein's equations are satisfied. The generalized 1-form $\mathbf{E}$ incorporates both the Witten-Nester 2-form and the Sparling 3-form which play an important role in the discussion of quantities such as energy in general relativity. For a review of the latter and references to higher dimensions see, for example, Ref. 9. The generalized 1-form $\mathbf{E}$ presented here effectively encodes the conditions that the 1-forms $\theta^{AA'}$ must satisfy in order to determine a Ricci flat, Lorentzian, four-metric. By using Theorem 1, the following corollaries may be obtained. The first is a gauge non-invariant form, on M, of the gauge invariant result in the theorem.

Corollary 1: Einstein's vacuum field equations are satisfied if and only if

$$\mathbf{E}^{AA'} \equiv (\theta^{AA'}, -k^{-1}\theta^{AB'} \wedge \bar{\omega}^{A'}_{B'}, -\bar{k}^{-1}\theta^{BA'} \wedge \omega^{A}_{B}, (k\bar{k})^{-1}\theta^{BB'} \wedge \omega^{A}_{B} \wedge \bar{\omega}^{A'}_{B'}), \tag{61}$$

is closed. In non-spinorial notation

$$\begin{aligned}
\mathbf{E}^a &= (\theta^a, -k^{-1}\theta^b \wedge^+ \omega^a_b, -\bar{k}^{-1}\theta^b \wedge^- \omega^a_b, (k\bar{k})^{-1}\theta^c \wedge^- \omega^a_b \wedge^+ \omega^b_c),\\
d\mathbf{E}^a &= (\overset{2}{\epsilon}{}^a, \overset{3}{\epsilon}{}^a, \overset{3}{\not{\epsilon}}{}^a, \overset{4}{\epsilon}{}^a),\\
\overset{2}{\epsilon}{}^a &= D\theta^a, \overset{3}{\epsilon}{}^a = k^{-1}[\theta^b \wedge^+ \Omega^a_b - D\theta^b \wedge^+ \omega^a_b],\\
\overset{3}{\not{\epsilon}}{}^a &= \bar{k}^{-1}[\theta^b \wedge^- \Omega^a_b - D\theta^b \wedge^- \omega^a_b],\\
\overset{4}{\epsilon}{}^a &= (k\bar{k})^{-1}[{}^+\omega^a_c \wedge^- \Omega^c_b \wedge \theta^b -{}^- \omega^a_c \wedge^+ \Omega^c_b \wedge \theta^b +{}^- \omega^a_c \wedge^+ \omega^c_b \wedge D\theta^b].
\end{aligned} \tag{62}$$

Corollary 2: (a) $\mathbf{E}$ is closed if and only if

$$\begin{aligned}
&\mathbf{E} = d\mathbf{F} = d\bar{\mathbf{F}}, \text{ where}\\
&\mathbf{F} = -(\bar{k})^{-1}(0, 0, \pi_A\bar{\pi}_{A'}\theta^{AA'}, -k^{-1}\pi_A D\bar{\pi}_{A'} \wedge \theta^{AA'}),\\
&\bar{\mathbf{F}} = -(k)^{-1}(0, \pi_A\bar{\pi}_{A'}\theta^{AA'}, 0, \bar{k}^{-1}\pi_{A'} D\pi_A \wedge \theta^{AA'}).
\end{aligned} \tag{63}$$

(b) If $\mathbf{E}$ is closed then $\mathbf{F} - \bar{\mathbf{F}}$ is closed.
(c) Since

$$\mathbf{F} - \bar{\mathbf{F}} = (k\bar{k})^{-1}(0, \bar{k}\pi_A\bar{\pi}_{A'}\theta^{AA'}, -k\pi_A\bar{\pi}_{A'}\theta^{AA'}, D(\pi_A\bar{\pi}_{A'}) \wedge \theta^{AA'}), \tag{64}$$

$\mathbf{F}-\bar{\mathbf{F}}$ is closed if and only if $D\theta^{AA'}=0$, that is the connection is torsion free.

In (a) use is made of the equation for the exterior derivative of the complex generalized zero-form $\mathbf{F}$,

$$d\mathbf{F}=\mathbf{E}-(k\bar{k})^{-1}(0,0,k\pi_A\bar{\pi}_{A'}D\theta^{AA'},\pi_A\bar{\pi}_{A'}\bar{\Omega}^{B'}_{A'}\wedge\theta^{AA'}+\pi_A D\bar{\pi}_{A'}\wedge D\theta^{AA'}).$$

(An unprimed connection 1-form ω^A_B does not appear in $\mathbf{F}$, but $\mathbf{E}=d\mathbf{F}$ implies that the connection 1-form ω^A_B appearing in the last equation is the anti-self dual part of the Levi-Civita connection defined by $\theta^{AA'}$.)

It is a straightforward matter to construct, in a similar way, 1-forms which are closed if and only if a 4- metric is half-flat. It suffices to demonstrate this here in the case of anti-self dual half-flat holomorphic four-metrics.[9–11]

Theorem 4: Let g be a holomorphic four-metric on a four-manifold M given by $ds^2=\epsilon_{AB}\epsilon_{A'B'}\theta^{AA'}\otimes\theta^{BB'}$, and let $\mu_{A'}$ be any 2-component spinor field with constant components. Consider the generalized 1-form on the spin bundle over M given by

$$\mathbf{E}=(\pi_A\mu_{A'}\theta^{AA'},0,-\bar{k}^{-1}D\pi_A\wedge\mu_{A'}\theta^{AA'},0), \tag{65}$$

where π_A are fibre coordinates on the bundle. Then $\mathbf{E}$ is closed if and only if $d\theta^{AA'}-\theta^{BA'}\wedge\omega^A_B=0$; that is $\mathbf{E}$ is closed if and only if the self-dual part of the curvature of g is zero.

The formulation of this corollary emphasizes certain similarities and differences between the requirement that a metric be half-flat on the one hand and Ricci flat, but not necessarily half flat, on the other. However it can clearly be more economically expressed in terms of the closure of a generalized 1-form of type $N=1$ given by $(\pi_A\mu_{A'}\theta^{AA'},-k_1^{-1}D\pi_A\wedge\mu_{A'}\theta^{AA'})$.

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