Generating Einstein-scalar solutions

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Abstract: In this comment on a recent paper by Dunajski a method of generating solutions of the Einstein-scalar field equations from Einstein metrics is presented. Two spherically symmetric examples are presented.

In a recent paper,[1], Maciej Dunajski used the standard Kaluza-Klein reduction ansatz from five to four dimensions to construct new four-metrics which satisfy the Einstein-Maxwell-Dilaton field equations from Einstein four-metrics. The aim of this comment is to point out a way in which Lorentzian solutions of the Einstein-scalar equations can be similarly constructed from any known Lorentzian Einstein four-metric which admits a non-null hypersurface orthogonal Killing vector field. The construction follows the same main lines as Dunajski's but differs in some details. It leads to Einstein-scalar four metrics which do not necessarily possess continuous isometries. The zero rest-mass scalar fields are minimally coupled in the standard (non-phantom) way and the cosmological constant is zero.

First let $ds^2_{(4)}$ be the line element of a Lorentzian four-metric on a four dimensional manifold M, with signature (3,1), which admits a hyper-surface orthogonal, non-null killing vector field K. It is assumed to be Einstein so that it satisfies the equations

$$G_{\alpha\beta} = -\Lambda g_{\alpha\beta}. \tag{1}$$

In coordinates adapted to K the line element takes the form

$$ds^2_{(4)} = g_{ij}dx^i dx^j + V(dx^4)^2, \tag{2}$$

where $K = \partial/\partial x^4$and $\partial/\partial x^4(g_{ij}) = \partial/\partial x^4 V = 0$. Two cases need to be considered.

(i) The function $V<0$, the signature of g_{ij} is Euclidean and K is time-like.

(ii) The function $V > 0$,the signature of g_{ij} is (2,1) and K is space-like.

In the first case when V is replaced by $-V$ the resulting Riemannian metric also satisfies Equation (1) and it this metric that is employed in the following construction. To emphasize this let $|V| = U^2$. Now it is known that any Einstein four-metric on M can be used to construct a Ricci flat five-metric on $M \times R$, [2]. Here the Ricci- flat five-metric can be written as

$$ds^2_{(5)} = \epsilon(dx^0)^2 + (ax^0 + b)^2 g_{ij}dx^i dx^j + (ax^0 + b)^2 U^2(dx^4)^2. \tag{3}$$

Here $|\epsilon| = 1$ and a and b are constants with

$$a^2 = \epsilon\Lambda/3. \tag{4}$$

Consequently ϵ and a non-zero Λ must have the same sign. The five-metric is Lorentzian, as is chosen to be the case here, when $\epsilon = -1$ in case (i) and $\epsilon = 1$ in case (ii).

Now, following Dunajski, this line element, with hypersurface orthogonal Killing vector field $\partial/\partial x^4$, is identified with the standard form used in the Kaluza-Klein reduction, that is with the metric form

$$ds^2_{(5)} = \exp(-2\Phi/\sqrt{3})ds^2 + \exp(4\Phi/\sqrt{3})(dx^4)^2. \tag{5}$$

This leads to the identification of a new four-metric

$$ds^2 = \epsilon \left|U(ax^0 + b)\right| (dx^0)^2 + \left|U(ax^0 + b)^3\right| g_{ij}dx^i dx^j, \tag{6}$$

and a scalar field

$$\Phi = \frac{\sqrt{3}}{2} \ln \left|U(ax^0 + b)\right|. \tag{7}$$

With this identification it follows automatically that the new Lorentzian four-metric and scalar field satisfy the Einstein-scalar field equations, where the zero rest-mass scalar field is conventionally and minimally coupled. An Einstein-scalar solution obtained in this way, and the original Einstein metric are both contained in a Ricci flat five-metric and are extracted from it via different fibrations of the five manifold.

When Λ is non-zero the final four-metric does not necessarily admit a Killing vector field. Henceforth this is the case that will be considered and it will then simplify the formulae if the new coordinate $y = \ln|ax^0 + b|$ is introduced. It then follows that the five-metric is given by

$$ds^2_{(5)} = \exp(2y)[\frac{3}{\Lambda}dy^2 + g_{ij}dx^i dx^j + U^2(dx^4)^2], \tag{8}$$

the new four-metric is

$$ds^2 = |U| \exp(3y)[\frac{3}{\Lambda}dy^2 + g_{ij}dx^i dx^j], \tag{9}$$

and the scalar field is

$$\Phi = \frac{\sqrt{3}}{2}(y + \ln|U|). \tag{10}$$

Simple illustrative applications of this procedure are to the Kottler and Narai metrics with non-zero Λ. First consider the Kottler metric, [3], in the static patch

$$ds^2_{(4)} = -U^2 dt^2 + U^{-2}dr^2 + r^2 d\Omega^2, \tag{11}$$

where

$$U^2 = (1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}) > 0, \tag{12}$$
$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2.$$

These coordinates are adapted to the two hypersurface orthogonal Killing vector fields $\partial/\partial t$ and $\partial/\partial\varphi$. It will suffice here to exhibit the procedure using case (i) above with $x^4 = t$. In this case the starting point is chosen to be the Schwarzschild-anti-de Sitter family with $\Lambda < 0$. From the above results, and writing $y = T$, it follows immediately that the spherically symmetric Einstein-scalar solutions generated from this family are given by

$$ds^2 = \left|1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right|^{1/2} \exp(3T)[\frac{3}{\Lambda}dT^2 + \left|1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right|^{-1} dr^2 + r^2 d\Omega^2], \tag{13}$$

and

$$\Phi = \frac{\sqrt{3}}{2}[T + \frac{1}{2}\ln\left|1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right|]. \tag{14}$$

Next take as starting Einstein four-metrics the Narai family, [4], in the form [5]

$$ds^2_{(4)} = \Lambda^{-1}[\cosh^2 t dr^2 + d\Omega^2 - dt^2]. \tag{15}$$

with $\Lambda > 0$. These coordinates are adapted to the Killing vector fields $\partial/\partial r$ and $\partial/\partial\varphi$. By applying the general procedure as in case (ii) above, with $x^4 = r$ and $y = R$, the spherically symmetric Einstein-scalar solutions

$$ds^2 = \frac{\exp(3R)\cosh t}{\Lambda^{3/2}}[-dt^2 + 3dR^2 + d\Omega^2], \tag{16}$$

$$\Phi = \frac{\sqrt{3}}{2}[R + \ln(\frac{\cosh t}{\Lambda^{1/2}})], \tag{17}$$

are generated from the Nariai metrics.

The solution generating method discussed in this paper is an addition to earlier methods of generating Einstein-scalar solutions from known solutions. Examples of such methods can be found in reference [6].

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