# STRUCTURE IN MATHEMATICS AND MATHEMATICAL THINKING

*What counts as mathematics and why is precision considered to be so important? This article is an attempt to relate the notion of precision to notions of structure - in both mathematics and people's internalisation of mathematics. In the picture described, precision plays a fundamental role in the building of mental mathematical structures that have a flexibility through which large sections can be compressed and subsequently expanded without loss of detail. It is suggested that substantial engagement with mathematics that is rich in structure and lends itself to the experience of such mental operations, is a basic ingredient in mathematical development.*

## Right and wrong answers

The opening paragraph of the mathematics policy document of a certain primary school includes the assertion that children "should not see mathematics as an isolated school subject with right and wrong answers." Indeed such sentiments have been fairly widespread for several years, perhaps partly as a reaction to a lingering image of school maths as consisting of rows of sums with ticks and crosses.

In fact, however, one of the strongest characteristics of mathematics is that it *is* a precise subject with right and wrong answers. Estimating that the area of a sheet of paper 21 cm by 29 cm is roughly 600 $\text{cm}^2$ is not a matter of scientific relativism. $21 \times 29$ has an exact answer which, for many practical purposes, is close enough to 600, which is the exact answer to $20 \times 30$. It is important to distinguish between the *exactness* of mathematics and the *approximations* in which mathematics is used in everyday situations. Leaving children with the impression that mathematics is not an exact subject, and that what is really important about 'equality' is nothing more than a notion of 'similar enough' for some particular purpose, is an erosion of what should be a solid mental baseline from which they can then go on to make their estimations.

There are also verbal reasoning type situations and investigations in cross-curricular topics, with a mathematical component, which may have open-ended outcomes. Again, there is a distinction between the precise 'right and wrong answer' aspect of the mathematical component and the open-ended aspect of the context in which this is embedded. It is important for children to be clear about this distinction in order to maintain a solid framework within which to make their reasoned judgements in the open-ended situation.

## The hierarchical nature of mathematics

A major reason for the importance of the distinctions referred to in the previous paragraphs, has to do with the way that mathematics is structured. Mathematics has developed into an extensive hierarchy or network of concepts, each more abstract than, and dependent upon, those feeding into it. For example, in the terminology of Skemp (1971), a *pair of objects* would be a primary concept, *two* a secondary concept, *number* a tertiary concept, *addition* a fourth-degree concept, and so on. The sequence *arithmetic, fractions, algebra, calculus* is a chain of a similar nature further along in the hierarchy.

Sitting alongside this, and related to it, is the way that individuals develop mathematically via periods of *expansion* and *reconstruction* (or *assimilation* and *accommodation*, to use Piaget's terms for similar ideas). Periods of expansion are described as those in which the emphasis, on interacting with new ideas, is on making new connections within existing cognitive structures, while periods of reconstruction are those in which the emphasis is on loosening old connections and building new structures. The important issue, it would therefore seem, is to identify those aspects of mathematics that will best serve as building blocks and links in helping children to construct secure mathematical cognitive structures. Although many university mathematics lecturers might not describe it in these terms, it is this issue which lies at the root of their concerns, a perception that the mathematical mental frameworks in the minds of even the brightest incoming students with the highest A-level grades, are, in the sense of their being *structures* with hierarchical networks of linked concepts, becoming increasingly fragile and limited.

**Fractions and decimals**

To illustrate the ideas that might be worth bearing in mind when trying to identify suitable building blocks, let us consider the two standard ways of representing numbers which are not whole numbers, namely fractions and decimals. If you want to get three equal lengths of cloth from a 4-metre roll, you just cut off lengths of $1.333$ metres. For this, and many other practical purposes, $1.333$ is close enough to $\frac{4}{3}$. Judged purely in relation to size, or in relation to position on a number line, the two numbers $\frac{4}{3}$ and $1.333$ are indeed very similar. Judged purely in relation to weight, a person who weighs 12 stone is very similar to a person who weighs 12 stone and a bit, but, just as people have characteristics other than their weight, numbers have characteristics other than their size.

Other aspects such as the operations of addition and multiplication, the concepts of divisibility and factorisation, also play important roles in the hierarchical structure of mathematics at school level. Familiarity with the distinction between even and odd numbers may be of limited use in measuring out lengths of cloth, but it helps us to decide instantly that there can be no whole numbers $x$ and $y$ which satisfy the equation $6x+8y=13$. And if we look for solutions which are not necessarily whole numbers (and put $x=1$), it is the ratio 7:8 that is more intrinsic to the structure of the problem than is the magnitude of $0.875$.

When it comes to the building of mathematical cognitive structures, the characteristic of size is arguably even less central in relation to other characteristics of number. One thing that $\frac{7}{8}$ has got in this respect that $0.875$ has not got is the potential of being unpacked and thereby connected to other parts of the structure. $\frac{7}{8}$ carries with it, just beneath the surface, and ready if needed, the information that it is 7 divided by 8. While $0.875$ tells you nothing more than the amount of pizza you end up with if you are one of 8 people who wish to equally share 7 pizzas, $\frac{7}{8}$ also connects you with your pizza eating colleagues and with the actual process of the sharing.

Another thing $\frac{7}{8}$ has got that 0.875 has not got is a multi-representational character. As well as being 7 divided by 8, $\frac{7}{8}$ is also 7 times $\frac{1}{8}$. $\frac{7}{8}$ carries with it, simultaneously, the two separate processes of dividing 7 by 8, and of multiplying $\frac{1}{8}$ by 7. In short, 0.875 is just a pale shadow of $\frac{7}{8}$.

**Mental compression**

The amount that can be held in the mind and attended to at any one time is very limited and, in order to minimise the consequent constraints on thinking, various strategies are adopted for reducing the mental load of data to be considered. Such strategies include, for example, grouping items and considering them in sequence, labelling items and groups of items, writing things down in the form of abbreviated notes.

When the data being attended to are of the kind that are part of a hierarchical network spanning several layers of abstraction with various interconnections, the reduction in mental load can also be achieved by a compression which results in linked items becoming associated with a single entity. A collection of related items, which could be, say, processes, sentences, objects, properties, steps of logical deduction, become mentally compressed into one single entity which can then be easily manipulated using a minimum of thinking space and subsequently unpacked whenever needed. Such a compressed entity plays the role of an *operative* label. More than just saving mental space by being a shorthand *in place of* a collection of items, it actually *carries with it*, just beneath the surface, the structure of the collection and is operative in the sense that the live connection with this structure is able to guide the manipulation of the single compressed entity. This sort of phenomenon is discussed in Gray and Tall (1994) (*procepts*), Krutetskii (1976) (*curtailed structures*), Dubinsky (1991) (*encapsulation*), Sfard (1991) (*reification*).

In some instances the compressed entity might be associated with a natural visual label, such as in the encapsulation of the process 'times 3 and add 5' as the object '$3x+5$'. However in other instances, such as in the compression of a chain of steps of manipulation, the mental image, if there is one, may be more vague. For example, in seeing that a linear combination of a linear combination is a linear combination, the essence of the rearrangement of terms in the equation

$$a(rx+sy)+b(ux+vy)=(ar+bu)x+(as+bv)y,$$

might be conceived as a single 'unit of thought' without there being a clearly defined label.

To return to the case of the fraction $\frac{7}{8}$ and the processes of dividing 7 by 8 and of multiplying $\frac{1}{8}$ by 7, once someone has compressed these processes and the concept of $\frac{7}{8}$ as a single object, they are no longer weighed down by holding in their mind the complexity of the different representations. They can then operate arithmetically with

$\frac{7}{8}$ with no more difficulty than they would have with any integer and yet, at the same time, they have not lost awareness of the related processes.

An example which occurs as a step in various algebraic arguments, is the deduction that if $xa + yb = h$ and $d$ divides both $a$ and $b$, then $d$ must divide $h$. A student who has compressed '$d$ divides $a$', '$a = ds$ for some $d$' and '$a$ is a multiple of $d$' as a single entity, will make the deduction immediately. Other students, for whom these mathematically equivalent relationships between $d$ and $a$ are separate mental units, will proceed to write down "$a = ds,\ b = dt$" and substitute these into the equation "$xa + yb = h$". Because they are having to hold more in their minds, such students are more likely to lose the thread of the argument of which this deduction might have been a part, or even make a technical slip along the way and thereby get totally lost.

What makes mental compression such an important feature in mathematical thinking is the hierarchical nature of the way that mathematics is structured. In addition to promoting mental agility in moving between concepts, it gives strength to the links in the cognitive structures being built in the mind of the learner.

**Compression-rich mathematics**

While all areas of the mathematics curriculum require the development of abilities for reducing the mental load of given data, not all areas lend themselves equally to the development of the ability described in the last section. There are some topics, like arithmetic, fractions and algebra which spread into several layers of abstraction and provide considerable scope for mental compression. Such topics might be described as being *compression-rich*. On the other hand there are also topics, such as data-handling, approximation methods and combinatorics, where the mental demands are confined to a more narrow band in the hierarchical structure of mathematics, and where there is less scope for this particular feature of mathematical thinking. For example, finding the positive root of the quadratic equation $(f(x) =)\ x^2 + 2x - 4 = 0$ by observing successively that
$f(2) > 0 > f(1),\ f(1.3) > 0 > f(1.2),\ f(1.24) > 0 > f(1.23),\ f(1.237) > 0 > f(1.236)$,
etc., does not lend itself to compression in the same way as does the sequence of steps

$$x^2 + 2x = 4, \quad x^2 + 2x + 1 = 5, \quad (x+1)^2 = 5, \quad x + 1 = \sqrt{5}, \quad x = \sqrt{5} - 1.$$

Similarly, not all approaches to the same topic are equally compression-rich. For example, in trigonometry, $\sin 60^\circ$ can be obtained on a calculator as 0.866, or it can be related to the equilateral triangle in Fig.1 and seen as $\sqrt{3}/2$ .

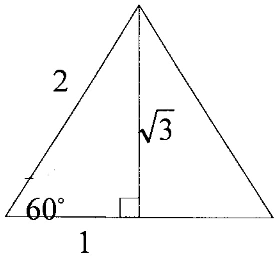


Fig.1: The 'canonical' equilateral triangle.

For someone who has compressed all the features connected with the 'canonical' equilateral triangle into a single mental entity, there is no difference in mental load between the two approaches. However, while the latter is full of potential links to related topics, the former is like an empty shell. When the connections with the structure of the mathematics are discarded, there is nothing left to distinguish 0.866 from any other decimal in its vicinity, and this number is weakened in its power to guide further thinking.

### Time-consuming activities

> In recent years English school mathematics has seen a marked shift of emphasis, introducing a number of time-consuming activities (investigations, problem-solving, data surveys, etc.) at the expense of core technique. In practice, many of these activities are poorly focused; moreover, inappropriate insistence on working within a context uses precious time and can often obscure the underlying mathematics. Such approaches, if well-directed, have value, but priorities must be agreed. At no stage has there been a serious debate as to which topics or skills are of primary, and which are of secondary, importance for students' subsequent progress. (LMS, 1995, p.9.)

It is all too easy to dismiss sentiments such as these as reflecting a concern for only those children who have the ability and interest to continue with mathematics-based studies in higher education. However, in relating this to a perception of devaluation of mathematical structure and shift of emphasis away from compression-rich mathematics, the suggestion is that many pupils who *could* benefit are being denied sufficient opportunities to experience and develop a form of mental operation that is central to mathematical thinking.

Just as there is no set of conditions that is known to be sufficient to ensure that someone will learn to swim, there is no set of conditions that is known to be *sufficient* to ensure that someone will mentally compress a collection of related mathematical items in the manner described. However, just as going into the water is certainly a necessary condition for learning how to swim, one might argue that a *necessary* condition for mental compression in mathematics is a substantial engagement with compression-rich mathematics.

There is a striking statistic that is not entirely unrelated to this. It comes from diagnostic tests of students entering the mathematics course at the University of Melbourne. The tests were given to 1518 students in 1989 and 1344 students in 1993, the latter group having experienced a programme of open-ended problem solving in the state of Victoria.

> They have been completing essentially the same mathematical skills test since 1989. The results show a steady decline in scores, the decline being generally attributable to loss of algebraic skill. ... In 1989, only 3% of students gave $x = a + b$ as the solution of $\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$. By 1993, 14% of students gave this answer to (essentially) the same question. This data ... does support the contention that the replacement of time spent on skills by time spent on open-ended problems and projects has been very seriously detrimental to basic mathematical skills. (Stacey, 1995, p.66.)

It would be interesting to compare the quality of this evidence with the quality of evidence for claims supporting positive learning gains in other abilities. Even if there *was* equally hard evidence on a similar scale for the latter, it would be necessary to question whether, in regard to their role in mathematical development, the gains were as important as the losses.

**Precision**

By manipulating mathematical objects, statements, processes, steps of logical deduction in a precise way, nothing is lost in moving between them. Items which are equivalent are *seen* to be equivalent and items which are not equivalent are *seen* to be not equivalent, so that the hierarchical structure of the mathematics is mirrored in the hierarchical structure being built in the mind of the learner.

Consider the diagram in Fig.1. Although 1.732050808 is what the calculator shows when you press 3 and then $\sqrt{\ }$, it is not true that 1.732050808 squared is equal to 3. Also measurement of the sides of a right angled triangle can never provide certainty that the square of the hypotenuse is exactly equal to the sum of the squares of the other two sides. These deficiencies are irrelevant for almost all practical purposes in measurement where one is only concerned with numerical size, but they are highly significant in regard to the construction of mental frameworks. Working with the expression $\sqrt{3}$, one is immediately connected with the fact that the square of this number is exactly 3, and this in turn enables a connection with the 'canonical' equilateral triangle via Pythagoras' Theorem and the equality $1^2 + \left(\sqrt{3}\right)^2 = 2^2$. This awareness of the fact that the square of the number is 3 and the accompanying mental picture of related mathematical structure is then able to guide further manipulation and help in the search for ideas during problem solving. However if $\sqrt{3}$ is replaced by a decimal approximation and Pythagoras' Theorem is replaced by an approximate relationship based on measurement, the links with the mathematical structure become less tight and the mental picture goes mathematically out of focus. All the interconnections in the picture, and the *compressibility* of these connections, are no longer active. In maintaining a tightness of links, precision plays an important role in the construction and manipulation of flexible mental structures which mirror structures in mathematics.

The attribute of precision also ensures that nothing is lost when a collection of mathematical items is mentally compressed into a single entity. When the compressed entity is unpacked, the original information, including all details and interconnections, is regained. For example, consider again the identity

$$a(rx+sy)+b(ux+vy)=(ar+bu)x+(as+bv)y$$

(or more generally $\sum_{i=1}^{m} a_i\left(\sum_{j=1}^{n} r_j x_{ij}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i r_j x_{ij}$ ), which can be thought of as saying that a linear combination of a linear combination is a linear combination. If in compressing the derivation of this identity we were left only with the thought "mixture of mixture is mixture", we would not, in the subsequent unpacking, have a feel for whether or not the final mixture of $x$ and $y$ could include non-linear terms like

$x^3$, $xy$ and $y^2$. But the phenomenon is more precise than that. It allows for the derivation to be compressed into the thought "linear combination of linear combination is linear combination" which retains *all* aspects of the mathematics.

There is also an aesthetic quality to the role of precision in regard to connections and compression. Consider, for example, the concept of an abstract group. This is not just some vague notion that has something roughly to do with symmetry. On the contrary, the concept of a group captures the essence of the notion of symmetry and is connected in a precise way to the concept of an equivalence relation, which itself is a precise abstract formulation of the notion of 'sameness' with respect to a given property. Moreover the properties defining a group are both sufficiently *general* to be satisfied by a large variety of relatively concrete mathematical objects, and also sufficiently *special* to have lots of powerful consequences in abstract levels. Thus a group is a precisely defined concept which sits at a major junction in the mathematical network of relations. Indeed, one of the most beautiful features of mathematics is the way it allows such precision at even the deepest levels of abstraction.

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Tony Barnard
King's College London