Starting points and end points

Mathematics is not just a collection of numerical and spatial facts and methods. There are also features of structure, coherence and order that are intrinsic to the subject. Within the school mathematics curriculum, an area particularly rich in opportunities for seeing how things hang together logically is geometry. Indeed, much of what geometry has to contribute to a school education has to do with *connecting* geometrical facts in short chains of logical reasoning and *locating* these facts and links within a wider hierarchical framework. But how feasible is it for pupils to engage with the idea of a deductive system in geometry? I claim that, with a carefully chosen set of starting points, it is both very feasible and potentially of great value for a wide range of pupils.

No axiomatics for school pupils

There are two features of axiomatics that are not appropriate for school pupils. The first relates to the way that, at a certain level in university mathematics, the deductive side of geometry is emancipated from its roots in physical experience. As Hilbert put it,

> "One must be able to say at all times - instead of points, straight lines and planes - tables, chairs and beer mugs."

Here one is able to use different axiom systems without reference to intuitions of physical space on which they are based. For example, one can work with a geometry in which 'lines' are great circles on the surface of a sphere and 'points' are pairs of antipodal points on the sphere. (*Aside:* Given two distinct 'points' in this geometry, there is precisely one 'line' that passes through both. However it is also true that given two distinct 'lines', there is precisely one 'point' that lies on both. So in this geometry there are no parallel 'lines'!) Although thinking at this level is fundamental for research mathematicians, it is neither necessary nor appropriate for school pupils who are beginning to engage with systems of logical deduction in mathematics.

The second feature of 'axiomatics' not appropriate at school level relates to the desire for a *minimal* set of axioms. Looking at the mathematics that is appropriate as part of an education for school pupils, one finds a large number of geometric facts. An intriguing quality of this collection of facts is that it is imbued with lots and lots of logical connections, such as the those between the facts that (a) the angle in a semicircle is a right angle, (b) angles in the same segment of a circle are equal and (c) the angle at the centre of a circle is twice the angle at the circumference. These logical connections are an intrinsic part of geometry and a study of the subject that does not recognise this (to an extent that will depend on the ability of the pupil) is incomplete. Moreover, it is global networks of connections, as well as individual links, that are intrinsic to the subject. However, once we start looking at logical connections, we are into the realm of 'starting points' and 'end points'. The starting points in geometry are particularly problematic, partly on account of the relationship between the ideal mathematical universe and the physical world in which we live. At least, with arithmetic, you do not have integers as concrete objects physically confronting you wherever you go.

Deciding what to take as 'self-evident'

So the question is, 'What should the starting points be?' There is a whole spectrum of possibilities. At one end, there would be a minimal set of 'agreed facts' ('agreed' in the sense that the population in question find these 'assumptions'/'agreed facts' compatible with their counterparts in the concrete world). But such a set is not unique. So there is still an element of choice as to which minimal set of axioms to work from. At the other end, there would be the complete collection of isolated 'facts', devoid of any logical connections. Given that working from a *minimal* set of 'agreed facts' (the absolutist position, let us say) is not feasible or desirable within school mathematics, the totally relativist position is sometimes advocated. One often hears statements like, "If in proving a result in school geometry you use other results that have not themselves been proved, then you're no better off in knowing if the result is true". However, this is missing an important point. It amounts to saying that the only feasible position is the collection of isolated facts and, in doing so, is having no regard to the logical connections which are intrinsic to geometry, and are the very essence of mathematics. It is undermining the educational value of the interconnections and coherence within the body of geometric material and is ignoring its explanatory role in developing understanding. It is like confining pupils to be spectators at a display, rather than allowing them to go behind the scenes and get a feeling for why, in the context of the 'surrounding mathematics', a relationship must hold, or a theorem must be true, and what makes it tick. So one has to go for something in between, namely a small, but not minimal, set of 'agreed facts'. (A reason for wanting the set of assumptions to be small, in addition to the aesthetic reason, is that the fewer the assumptions there are, the less chance there is of arriving at contradictory conclusions.)

It is also important that the 'facts' in this "small set of agreed facts" are *explicitly* 'agreed'. The absolutist/relativist dichotomy enters here too. Although it may not be helpful to make explicit *every* assumption, this does not mean that there has to be complete anarchy on this point. A perceptive teacher should be able to make a judgment, depending on the teaching situation, as to which assumptions (within the small set he/she is implicitly working from) merit explicit discussion, and which do not.

Distinguishing between what is assumed and what is deduced

One reason for being explicit about assumptions is that it enables us to distinguish between that which is assumed and that which is deduced. If asked to give a reason for some geometrical fact, it is not good enough to simply say, "It's obvious". These words cover a whole range of responses, such as "It feels right to me" (*subjectivity*), "I can't think of any reason why it's not true" (*apparent consistency*), "It follows from some other fact and I needn't tell you what this is or what the steps of deduction are, as these are easy enough for you to see for yourself" (*vagueness*). In contrast, what is needed is clarity and agreement about the starting points from which the conclusion follows (even if all that this consists of is empirical support).

One often hears in geometrical explanations the phrase, "by symmetry". However, if the understandings associated with this phrase are not made explicit, it may not be clear whether the phrase is an allusion to precise mathematical results or a vague description just one step removed from the phrase, "it's obvious". In seeing that a certain line is a line of symmetry of some geometrical figure and in deducing various consequences from this, there are steps of logical deduction involving equal distances and perpendicular lines. If pupils are to be more than spectators, these steps of deduction, together with their starting points and endpoints, should not remain hidden. If we think of a logical deduction as being like a journey from a fact A to a fact B, then when we simply state, "by symmetry", as the reason for fact B being true, there is no way of knowing where the journey is starting from.

For example, it may seem intuitively obvious that the opposite sides of a rectangle are equal, but how could we explain this fact to someone? After all, if something in mathematics is obvious we should be able to explain mathematically *why* it is obvious. Starting from the facts that a rectangle has opposite sides parallel and all angles equal to 90°, one could either work with the reflection in the perpendicular bisector of one of the sides, or put in a diagonal and use congruent triangles. The first of these approaches is relatively long if all the necessary steps are included and, unlike the second, does not work for other, more general, parallelograms. Again, how are we to explain the 'obvious fact' that a chord of a circle is bisected by a diameter perpendicular to it? It might sound neat to say that the midpoint of the chord is invariant under the reflection in the diameter and must therefore be *on* the diameter, but this assumes that the reflection interchanges the endpoints of the chord. One can of course prove this latter fact by assuming that the chord and circle are both invariant under the reflection but this, taken together with the assumptions used in proving the equality of opposite sides of a parallelogram and other basic geometric results, is just moving around from one 'intuitive' assumption to another without any sense of direction.

Limitations of the transformations approach

The concept of symmetry is tied in with the idea of a transformation. Transformations, such as reflections and rotations, operate on the whole plane (or on the whole of three-dimensional space). A *symmetry* of a two-dimensional figure is a distance-preserving transformation of the plane in which the figure lies, whose effect on the figure is to produce an identical looking figure in the same position. In other words, although individual points of the figure may be moved, the points they are moved to are always points that were occupied by the figure before the mapping was effected. When we say that a two-dimensional figure has line symmetry, with respect to a particular line, we mean that a reflection of the plane in that line is a symmetry of the figure. Similarly, a two-dimensional figure has rotational symmetry, about a point, if a rotation of the plane about that point is a symmetry of the figure. When we make such a statement, or derive consequences from it, we are talking about equal lengths and equal angles, and this invariably involves pairs of congruent triangles.

In order to be explicit about the starting assumptions surrounding the term, 'symmetry', and its role in logical deductions, one would have to either (a) produce a suitable 'small set of agreed facts' about transformations, from which other properties of transformations could be deduced, or (b) deduce the relevant properties of transformations from 'agreed facts' about angles, lengths and congruent triangles. In principle, any collection of 'agreed facts' which brought order to the network of assumptions and deductions, and which led to the necessary properties of transformations and symmetry, would do, but it is the small collection involving congruent triangles and parallels that, in the long run, is the simplest and most appropriate for doing the job within 11-16 school mathematics. In the absence of (a) or (b), saying, "by symmetry", is like saying, "abracadabra", without explaining how the trick works. Put another way, if the properties of symmetry are not derived from clearly stated starting assumptions, the only place for a school pupil to seek meaning for the phrase, "by symmetry", is in the empirical/experiential mode. Thus an undue emphasis on transformation geometry in secondary school (at this level, almost of necessity lacking in a firm logical base) is unhelpful to the aim of moving pupils from merely *perceiving* spatial things onto *understanding logical relationships* between spatial things.

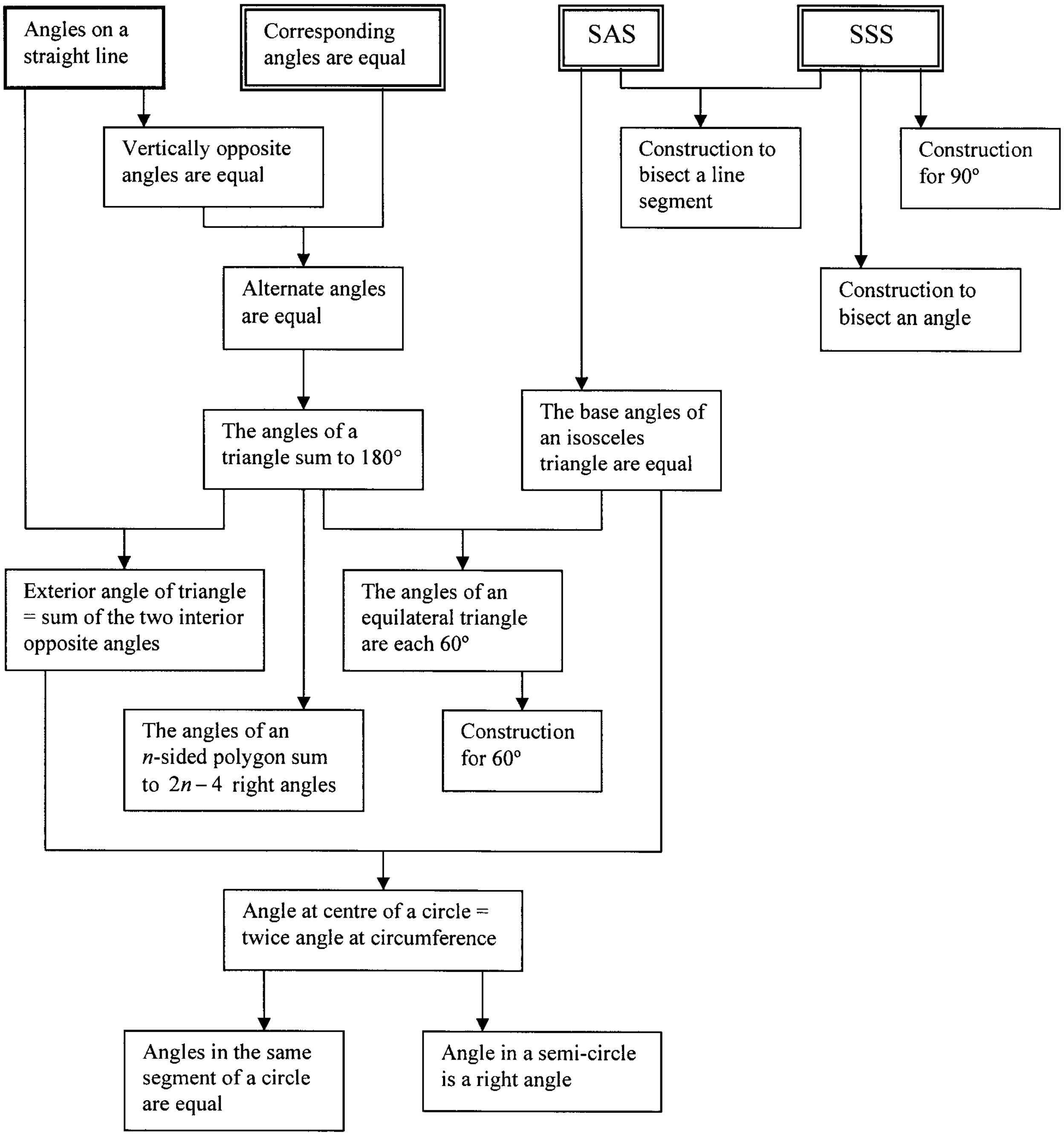
Another example of incompatibility (at school level) in perspective between the experiential and deductive aspects of a concept in transformation geometry occurs when trying to provide an accessible mathematical definition for the idea of 'amount of rotational symmetry' of a geometrical figure. The 'order of rotational symmetry' of a geometrical figure is usually defined as the largest integer n for which a rotation (about a suitable centre) of $360/n$ degrees is a symmetry of the figure. As far as the school curriculum goes, the main role of this concept is as part of a language for distinguishing certain kinds of shapes. However, from a mathematical point of view, three unfortunate consequences of the above definition are (i) it gives no insight into where the concept is located within the wider picture (order of a symmetry as a transformation, order of an element in a group, groups of transformations, *etc.*), (ii) it fails to address the question of whether a figure can have rotational symmetries about more than one centre (a bounded figure can't, but it is not entirely trivial to see why), and (iii) it does not generalise to also include other types of symmetries. The concept of the order of rotational symmetry of a geometrical figure, like many other aspects of transformation geometry, is one that lies at the surface of a body of interesting mathematics in the deductive domain, but whose accessibility for most school pupils is limited to the experiential descriptive domain. The gap between the intuitive idea, 'amount of rotational symmetry', and a usable mathematical formulation of this linking in with a wider deductive framework, is simply too big to allow a smooth progression.

Getting started

So what starting assumptions should one make explicit? This would depend on the level of the pupils. For pupils at a level appropriate for introducing proofs of results such as 'the angle sum of a triangle is 180°', appropriate assumptions to make explicit could be: 'Corresponding angles are equal', all four of SAS, SSS, AAS, RHS and the equivalence of certain properties of similar triangles. There is a lot of redundancy here, but at this level, and given the relative complications (for an average beginning secondary school pupil) of deducing SSS and AAS from SAS, the above would be minimal enough to serve the purpose outlined earlier. It would also be appropriate to assume Playfair's Axiom, or something equivalent to it (maybe just the existence and uniqueness of a line through a point parallel to a given line), but it might not, at this initial stage, be helpful to make this assumption explicit. With a sixth form class, perhaps a class working for Further Mathematics, one could say something more about the whole issue of there being choice in the set of 'initial assumptions' (things taken to be self-evident) and also talk about axioms for parallels, triangles on a sphere, *etc.* The suggestion then is that the set of 'near minimal' initial assumptions should be chosen on pedagogical grounds, rather than on mathematical grounds. The choice should be honest to the mathematics, but need not be subservient to what might be best from the point of view of strict formal logic.

Moving on

The flow chart of logical reasoning below illustrates how one might proceed from such a set of initial assumptions. In the bold-lined box, we take as a definition (which includes certain unstated assumptions) that angles at a point sum to 360° and that angles on a straight line sum to180°. In the double-lined boxes, we take as explicit assumptions, that corresponding angles are equal (when a transversal intersects a pair of parallel lines), SAS (the 'two sides and included angle' condition for congruent triangles) and SSS (the 'three sides' condition). The flow chart indicates where there are possible links (not necessarily the shortest) between the following consequences:


Angles on a straight line
Corresponding angles are equal
SAS
SSS
Vertically opposite angles are equal
Construction to bisect a line segment
Construction for 90°
Alternate angles are equal
Construction to bisect an angle
The angles of a triangle sum to 180°
The base angles of an isosceles triangle are equal
Exterior angle of triangle = sum of the two interior opposite angles
The angles of an equilateral triangle are each 60°
The angles of an n-sided polygon sum to 2n – 4 right angles
Construction for 60°
Angle at centre of a circle = twice angle at circumference
Angles in the same segment of a circle are equal
Angle in a semi-circle is a right angle

This network of connections could, within the bounds of 11-16 school mathematics, be extended to include also results on parallelograms, tangents to circles, similar triangles, *etc.*, and to do this we would also explicitly assume ASA (the 'two angles and a side' condition for congruent triangles), RHS (the 'right angle, hypotenuse and side' condition) and that triangles with equal angles have proportional sides, and conversely (similar triangles).

The claim is that the approach via congruent triangles and parallels does allow an explicit statement of relatively accessible assumptions, so that pupils can get a genuine insight into the world of stating what you are assuming, doing a mathematical argument and coming to logical a conclusion.

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