The Impact of 'Meaning' on Students' Ability to Negate Statements

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This paper reports on a study to investigate students' capabilities for handling logical structures in mathematics, in particular in negating statements involving quantifiers. Undergraduates, both at early and later stages of a university course, were asked to negate a variety of statements set in everyday and mathematical contexts. It was found that, even after two years at university, one in three students could not negate apparently simple statements. Comparison of the performances of the two groups showed that the ways in which they differed reflected characteristics of the parallel transitions in the nature of the mathematics encountered and in the intellectual development of the students.

Introduction

Mathematical discourse at university is permeated with structures of the form "Suppose A is not true. This is the same as saying that B is true". Consideration of equivalent ways of expressing the falsity of a given statement, such as "for all $x > 0$, $a < x$" or "p divides ab implies p divides a or p divides b", occurs abundantly in both exposition and construction of mathematical proofs. Thus the ability to negate statements correctly is fundamental to meaningful mathematical communication at this level. Students who have difficulty with such structures may willingly accept, learn and reproduce instances of these in a mathematical argument, but they will be missing the point of such an argument in that it will have contributed little to their overall understanding of what is going on in the mathematics.

In an attempt to gain insight into the difficulties students have with 'negations', lists of statements of the following kinds were drawn up.

1. x satisfies P, for all x in X.
2. x satisfies P, for some x in X.
3. x and y satisfy P.
4. x satisfies P and Q, for all x in X.
5. A implies B.
6. There exists x in X such that $S(x,y)$ is true for all y in Y.
7. Given x in X, there exists y in Y such that $S(x,z)$ is true for all z in Z (the 'limit' definition structure).

These statements were set both in everyday contexts and mathematical contexts, and students were tested on their ability to negate them. The students were drawn from two groups: students in the first term of their first year, and a mixed group of second and third year students who had completed at least one year of formal mathematics.

The most notable finding was perhaps the sheer number of wrong answers, even with what many lecturers would regard as "just common sense". Thus for statements 2, 3 and 4, generally less than half of the first year students tested gave the correct answer. For statement 6, the number of correct answers was less than 1 in 4. The performance of the second and third year students was markedly better: generally 2 in 3 correct for each of statements 2, 3 and 4, and just under half correct for statement 6. However, the prevalence of such errors among students engaging with the more advanced mathematics of an undergraduate course was still far from ideal.

Subsequent interviews with students and consideration of the most common incorrect answers suggest that among the underlying causes of difficulty in performing negations are the following:

- logical structure,
- lexical representation (language, symbols),
- contextual influences,
- level of abstraction,
- degree of complexity.

It will be argued that ability to cope with these difficulties is related to progress in the transition from a descriptive view of mathematics, grounded in a practical domain in which objects and meanings of words are the dominant constructs, to one of definition and deduction, grounded in a theoretical domain in which symbols and words themselves are predominant. This aspect of mathematical ability is discussed in (Tall, 1994).

The test

Six lecturers in the mathematics department of a UK university were asked to run the test with their classes in the first term of the academic year. The total numbers of students involved were 78 from the first year and a further 78 from the second/third years. Before distributing the papers, the lecturers gave an explanation/reminder of the meaning of the word 'negation', following a prepared briefing sheet of notes and examples. Each student was then given a paper containing the following three sets of questions (figures 1, 2, 3).

For each of the following statements, circle the letter beside the statement below it which is its negation.

1.1 All people living in Cheltenham watch 'Neighbours'.
- A. No people living in Cheltenham watch 'Neighbours'.
- B. Some people living in Cheltenham watch 'Neighbours'.
- C. All people living in Cheltenham don't watch 'Neighbours'.
- D. Some people living in Cheltenham don't watch 'Neighbours'.

1.2 Some students stay awake at lunchtime.
- A. All students stay awake at lunchtime.
- B. Some students fall asleep at lunchtime.
- C. No students fall asleep at lunchtime.
- D. All students fall asleep at lunchtime.

1.3 Linford Christie and Sally Gunnell can run fast.
- A. Linford Christie and Sally Gunnell cannot run fast.
- B. Neither Linford Christie nor Sally Gunnell can run fast.
- C. Either Linford Christie or Sally Gunnell or both can run fast.
- D. Either Linford Christie or Sally Gunnell or both cannot run fast.

1.4 Long John Silver always has a briefcase and an umbrella.
- A. Long John Silver is sometimes either without a briefcase or without an umbrella or without both.
- B. Long John Silver is always either without a briefcase or without an umbrella or without both.
- C. Long John Silver is sometimes without a briefcase and without an umbrella.
- D. Long John Silver is always without a briefcase and without an umbrella.

1.5 What goes up must come down.
- A. What goes down must come up.
- B. What goes up must stay up.
- C. If something doesn't go up, it needn't come down.
- D. If something goes up, it needn't come down.

1.6 There is a station on the London Underground whose name contains no letters of the word 'MACKEREL'.
- A. There is a station on the London Underground whose name contains some letters of the word 'MACKEREL'.
- B. There is a station on the London Underground whose name contains all the letters of the word 'MACKEREL'.
- C. There is no station on the London Underground whose name contains all the letters of the word 'MACKEREL'.
- D. For any station on the London Underground, there is a letter of the word 'MACKEREL' which is not in the name of the station.
- E. For any station on the London Underground, there is a letter of the word 'MACKEREL' which is also in the name of the station.

1.7 For any lecture room, there is a time of day such that all students able to attend lectures at that time can fit into the room.
- A. There is a lecture room such that, for any time of day, there are students able to attend lectures at that time who cannot fit into the room.
- B. There is a lecture room such that, for any time of day, all students able to attend lectures at that time can fit into the room.
- C. For any lecture room and any time of day, there are students able to attend lectures at that time who cannot fit into the room.
- D. For any lecture room, there is a time of day for which there are students able to attend lectures at that time who cannot fit into the room.

Figure 1 : Negating statements in everyday contexts

For each of the following statements, circle the letter beside the statement below it which is its negation.

2.1 For all integers a, $a^2 \geq 0$.

A. There does not exist an integer a satisfying $a^2 \geq 0$.
B. $a^2 < 0$ for all integers a.
C. There exists an integer a such that $a^2 < 0$.
D. There exists an integer a such that $a^2 \geq 0$.

2.2 There exists a real number x such that $\log(x) = -1$.

A. There exists a real number x such that $\log(x) \neq -1$.
B. There does not exist a real number x such that $\log(x) \neq -1$.
C. $\text{Log}(x) = -1$ for all real numbers x.
D. $\text{Log}(x) \neq -1$ for all real numbers x.

2.3 $\text{Sin}(x) > 0{\cdot}1$ and $\cos(y) < 0{\cdot}9$.

A. $\text{Sin}(x) \leq 0{\cdot}1$ and $\cos(y) < 0{\cdot}9$.
B. $\text{Sin}(x) \leq 0{\cdot}1$ and $\cos(y) \geq 0{\cdot}9$.
C. $\text{Sin}(x) \leq 0{\cdot}1$ or $\cos(y) \geq 0{\cdot}9$.
D. $\text{Sin}(x) > 0{\cdot}1$ or $\cos(y) < 0{\cdot}9$.

2.4 For all $x \in X$, $x^2 \geq 1$ and $x^3 \leq 8$.

A. Given $x \in X$, either $x^2 < 1$ or $x^3 > 8$.
B. There exists $x \in X$ such that either $x^2 < 1$ or $x^3 > 8$.
C. There exists $x \in X$ such that $x^2 < 1$ and $x^3 > 8$.
D. For all $x \in X$, $x^2 < 1$ and $x^3 > 8$.

2.5 If $u > 7$, then $v = 3$.

A. If $u \leq 7$, then $v \neq 3$.
B. If $u > 7$, then $v \neq 3$.
C. '$u > 7$' does not imply '$v = 3$'.
D. '$u \leq 7$' does not imply '$v = 3$'.

2.6 There exists a positive integer m such that $m + n \geq 5$ for all positive integers n.

A. Given any positive integer m, there exists a positive integer n such that $m + n < 5$.
B. Given any positive integer m, there exists a positive integer n such that $m + n \geq 5$.
C. There exist positive integers m and n such that $m + n < 5$.
D. There does not exist a positive integer m such that $m + n < 5$ for all positive integers n.
E. There exists a positive integer m such that $m + n < 5$ for all positive integers n.

2.7 Given a prime number p, there exists an integer x such that $pa < x$ for all positive integers a.

A. There exists a prime number p such that, for any integer x, there is a positive integer a satisfying $pa < x$.
B. There exists a prime number p such that, for any integer x, there is a positive integer a satisfying $pa \geq x$.
C. Given a prime number p and an integer x, there exists a positive integer a such that $pa \geq x$.
D. Given a prime number p, there exists an integer x such that $pa \geq x$ for some positive integer a.

Figure 2 : Negating statements in mathematical contexts

For each of the following statements, write its negation in the space below it.

3.1 All people living in Neasden have black hair.

3.2 Some TV programmes are good.

3.3 Kylie Minogue and the Loch Ness Monster can sing.

3.4 Donald Duck always wears glasses and a hat.

3.5 Where there's a will, there's a way.

3.6 There is a tree in England whose number of leaves is not equal to the number of words in any book.

3.7 For any textbook, there is a price above which the number of students who can afford the book is less than the number of copies in the bookshop.

Figure 3 : Formulating the negation of statements

Responses of the students

In each of the boxes in the tables below, the upper italic figure relates to the first year students and the lower figure relates to the second and third year students.

	1	2	3	4	5	6	7
Section 1	*58*	*46*	*44*	*37*	*60*	*24*	*32*
	81	76	65	60	68	49	50
Section 2	*53*	*50*	*53*	*42*	*32*	*18*	*40*
	73	65	67	65	50	42	44
Section 3	*60*	*62*	*35*	*31*	*33*	*24*	*12*
	82	79	69	54	55	32	29

Table 1 : Percentage of students giving correct response to each section

	1.1	1.2	1.3	1.4	1.5.	1.6	1.7	2.1	2.2	2.3	2.4	2.5	2.6	2.7
A	*7*	*10*	*15*	***29***	*8*	*25*	***25***	*18*	*15*	*2*	*10*	*15*	***14***	*4*
	5	5	6	**47**	6	11	**39**	9	13	3	4	8	**33**	7
B	*11*	*27*	*26*	*7*	*11*	*16*	*9*	*12*	*17*	*26*	***33***	*29*	*7*	***31***
	4	10	17	8	8	7	3	10	11	17	**51**	18	4	**34**
C	*15*	*4*	*2*	*16*	*12*	*10*	*12*	***41***	*7*	***41***	*12*	***25***	*8*	*15*
	5	3	3	11	9	9	18	**57**	2	**52**	11	**39**	8	8
D	***45***	***36***	***34***	*26*	***47***	*6*	*27*	*7*	***39***	*9*	*20*	*9*	*26*	*21*
	63	**59**	**51**	10	**53**	6	6	2	**51**	4	10	12	13	14
E						***19***							*17*	
						38							14	

Table 2 : Number of students choosing each option (N=78)
(correct responses in **bold**)

The most common underlying error was that of negating a single part of the statement which had, for the student, a dominating presence. For the statements set in everyday contexts, this point of focus was often the section of the main verb. For example, in 1.2, by far the most common error was to solely convert "stay awake" to "fall asleep". However, in 2.2, where the section corresponding to "stay awake" was the less tangible "$\log(x) = -1$", the errors were more evenly distributed between solely converting "$\log(x) = -1$" to "$\log(x) \neq -1$" and solely converting "There exists" to "There does not exist". Similarly, in 1.6, 53% of the first year students and 23% of the second year students solely converted "contains no letters" to either "contains some letters" or "contains all letters", whereas in 2.6 the logically corresponding errors, C and E, were exceeded in popularity by the error of choosing D, the statement which converted "There exists" to "There does not exist" as well as "$m + n \geq 5$" to "$m + n < 5$". This behaviour was also widespread in section 3 where the students had to construct their own statement. For the negation of 3.6, 21% of the first year students changed only "not equal" to "equal", and even a very high proportion of the 'correct' answers consisted merely of the replacement of "a" by "no" after "There is".

Where students operated on one component of the statement with no relation to the others, and this was not a negation of the main verb, it was usually a transposition of two quantifiers. On being asked why they were focussing on just one part of the statement, typical student responses were "I was going for something a bit different", "I just want to make it not true ... minimum statement to make it false". This is the kind of behaviour that might be expected from students operating in the unifocal (Case, 1985), or unistructural (Biggs and Collis, 1982), mode of a developmental stage.

A possible explanation could be related to the opposing needs for coming to a conclusion, and for conclusions to be consistent. Students operating at a higher level of sophistication, for whom consistency was a factor of relative concern, were less likely to jump to hasty conclusions.As might be expected, the students' difficulties were greater with those statements which were more complex logically, such as statements 6 and 7 which were longer and had more than one quantifier. With a short term working memory of limited capacity, successful operation with these statements may require a chunking strategy and/or use of symbolic notation to mentally compress the components. As one student put it, "I think there was too much in that one"! However there were also complexities not related to logical structure. For example, "stay awake at lunchtime" in 1.2 was more complex linguistically than "are good" in 3.2. This variation was likely to be less significant to students more proficient in abstract reasoning, and could partially explain the different relative performances of the two groups at 1.2 and 3.2. For the first year students the percentage of correct answers for 3.2 was 35% greater than that for 1.2, while for the second and third year students the corresponding figure was only 4%.

Contextual influences

It will be noticed that the increases in success rates of the second and third year students over the first year students for the first four statements of each section were greater for sections 1 and 3, where the statements were set in everyday contexts, than they were for section 2, where they were set in mathematical contexts. (The increase for 3.2, which was slightly less than that for 2.2, may be related to the remarks of the previous paragraph.)

A possible explanation for these phenomena may lie in the role played by truth value. Students with less facility in abstract reasoning are generally less able to throw off the 'real world' true/false dimension when contemplating a given statement. For example, they are more comfortable writing down a statement they know to be true than one which they know to be false. A student comment on 1.5 and 3.5 was "I found them hard because they were phrases that you knew". For such students, more grounded in the practical than in the theoretical domain, the truth or falsity of a statement was a matter of relative importance and probably had a greater influence on their perfomances at negating statements than it did for students with a greater facility in abstract reasoning. Furthermore, this differential effect was likely to be greater with statements set in concrete everyday contexts than with more abstract statements where, for students whose abstract thought was more fragile, the true/false dimension had less immediacy.

Relative difficulties with statements set in everyday contexts and those set in mathematical contexts with concise symbols were also reflected in the following contrasting student remarks. While discussing her difficulty with 2.7, one student said, "It is harder with numbers than with the worded sentences because you've got the mathematical language as well, that you have to be thinking of. At the same time you have to think what $pa < x$ actually is, rather than in the common sense case." On the other hand a second student, whose best performance was on sheet 2, said, "(There was) less to keep in mind".

There is one final statistic which, though not surprising, does have its merits. Five lecturers were also given the 21 statements. While the percentage of students who gave correct answers in all 21 cases was 1%, the percentage of lecturers who achieved this was 100% !

Conclusion

Although the statements were chosen to have the same logical structure from section to section, there was no significant correlation of logical structure in the students' responses. The error patterns that did emerge arose rather from factors such as (a) complexity, (b) single, or unrelated multiple, operations, and (c) links with meaning via dominant phrases and truth value. For students at an early stage of development in detached theoretical thinking, the various components of a statement were likely to have attached weightings of importance, or presence, derived from a network of associations and meanings in their base of experience. They were less able to shake off logically irrelevant associations than students who had progressed further in the transition to the stage where it is the weightless words themselves which are the dominant feature.

References

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