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Multiplication Modules

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Let R be a commutative ring with identity. An R-module M is said to be *distributive* if the lattice of submodules of M is distributive. We shall call an R-module M a *multiplication module* if every submodule of M is of the form IM, for some ideal I of R. We show that distributive modules are characterised as modules for which every finitely generated submodule is a multiplication module.

Let M be a finitely generated distributive R-module. Then M itself is a multiplication module, so that every submodule of M is of the form IM for some ideal I of R. If M is Noetherian we can say more about the choice of I. Indeed we then have that every submodule of M, which is locally non-zero at every maximal ideal of R, is of the form IM where I is a unique product of maximal ideals of R. If on the other hand M is Artinian, it is known that M must be cyclic [4]. However, it turns out that every finitely generated Artinian multiplication module is cyclic.

Throughout the term "ring" will mean "commutative ring with identity." If R is a ring and X is a submodule of an R-module M, the ideal $\{r \in R \mid rM \subseteq X\}$ will be denoted by $(X : M)$. Thus $(0 : M)$ is the annihilator of M, Ann (M).

1. We begin by noting the following simple analogue of Nakayama's Lemma.

PROPOSITION 1. *Let M be a multiplication module over a ring R and let I be an ideal of R contained in the Jacobson radical of R. Then $M = IM$ implies $M = 0$.*

Proof. Given $x \in M$, there is an ideal E of R such that $Rx = EM$. Thus $Rx = EM = EIM = IEM = Ix$. Therefore $x = ax$, for some $a \in I$. But $1 - a$ is a unit in R. Hence $x = 0$.

LEMMA 2. *Let R be a ring.*

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(i) *Let S be a multiplicatively closed subset of R. If an R-module M is a multiplication module, then the $S^{-1}R$-module $S^{-1}M$ is a multiplication module.*

(ii) *A finitely generated R-module M is a multiplication module if, and only if, the R_P-module M_P is a multiplication module for all prime/maximal ideals P of R.*

Proof. Straightforward. (For (ii) observe that, if X is a submodule of M, then $X = IM$ for some ideal I of R if, and only if, $X = (X : M)M$.)

LEMMA 3. *Let R be a ring and let M be an R-module whose annihilator is contained in only finitely many maximal ideals $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ of R. If $M_{\mathfrak{M}_i}$ is a cyclic $R_{\mathfrak{M}_i}$-module for $1, \ldots, n$, then M is a cyclic R-module.*

Proof. Take $x_1, \ldots, x_n \in M$ such that $M_{\mathfrak{M}_i} = (Rx_i)_{\mathfrak{M}_i}$ for $i = 1, \ldots, n$, and let X be the submodule of M generated by $x_1, \ldots, x_n$. Then for $i = 1, \ldots, n$, $M_{\mathfrak{M}_i} = (Rx_i)_{\mathfrak{M}_i} \subseteq X_{\mathfrak{M}_i} \subseteq M_{\mathfrak{M}_i}$; so that $X_{\mathfrak{M}_i} = M_{\mathfrak{M}_i}$. Now choose $a_i \in (\bigcap_{j \neq i} \mathfrak{M}_j) - \mathfrak{M}_i$, for $i = 1, \ldots, n$, and let Z be the cyclic submodule of M generated by $a_1x_1 + \cdots + a_nx_n$. Using the standard techniques of localising and Nakayama's Lemma, $M = Z$ readily follows.

Every cyclic module is a multiplication module, for if X is a submodule of a cyclic R-module $Z = Rz$, $z \in Z$, then $x \in X$ implies $x = rz$ for some $r \in (X:Z)$, so that $X = (X:Z)Z$. Over a ring with only finitely many maximal ideals the converse is also true.

PROPOSITION 4. *Let R be a semi-local ring. Then an R-module is a multiplication module if, and only if, it is cyclic.*

Proof. In view of Lemmas 2 and 3, it is enough to show that every multiplication module over a local ring is cyclic.

Let R be a local ring with maximal ideal $\mathfrak{M}$ and let M be a non-zero multiplication module over R. By Proposition 1, we can choose an element $x \in M - \mathfrak{M}M$. Then $Rx = IM$, where I is an ideal of R and $I \not\subseteq \mathfrak{M}$. Therefore $I = R$ and so $M = Rx$.

Let us say that a module M over a ring R is "locally cyclic" if $M_{\mathfrak{M}}$ is a cyclic $R_{\mathfrak{M}}$-module for all maximal ideals $\mathfrak{M}$ of R. Lemma 2 and Proposition 4 together give

PROPOSITION 5. *A finitely generated module is a multiplication module if, and only if, it is locally cyclic.*

2. Turning now to distributive modules, the following criterion follows easily from [2, Lemma 6.2] and [3, 2.3, Corollary 1].

PROPOSITION 6. *A module is distributive if, and only if, every finitely generated submodule is locally cyclic.*

From Propositions 5 and 6, we have

PROPOSITION 7. *A module is distributive if, and only if, every finitely generated submodule is a multiplication module.*

It is not the case that every finitely generated multiplication module is distributive, for as a (Noetherian) counterexample we may take $M = R =$ polynomial ring $k[x, y]$, k a field. Thus the following gives a generalisation and alternative proof of the Proposition of [4], in the commutative case.

PROPOSITION 8. *Every finitely generated Artinian multiplication module is cyclic.*

Proof. If M is a finitely generated Artinian module over a ring R, then $R/\mathrm{Ann}(M)$ is isomorphic to a submodule of the direct sum of finitely many copies of M, and is therefore an Artinian ring. Thus $\mathrm{Ann}(M)$ is contained in only finitely many maximal ideals of R. Hence the result follows from Lemma 3 and Proposition 5.

3. Let M be a module over a ring R. A submodule X of M will be called a *distributive submodule* if the following equivalent conditions are satisfied:

$$(Y + Z) \cap X = (Y \cap X) + (Z \cap X), \qquad \text{for all submodules } Y, Z \text{ of } M;$$

$$(Y \cap Z) + X = (Y + X) \cap (Z + X), \qquad \text{for all submodules } Y, Z \text{ of } M.$$

Thus M is a distributive module if every submodule of M is a distributive submodule.

A submodule X of M will be called a *supporting submodule* if $X_{\mathfrak{M}} \neq 0$ for all maximal ideals $\mathfrak{M}$ of R for which $M_{\mathfrak{M}} \neq 0$.

PROPOSITION 9. *Let M be a Noetherian module over a ring R. Then every distributive and supporting submodule of M is uniquely of the form $(\mathfrak{M}_1^{k_1} \cdots \mathfrak{M}_n^{k_n})M$, where $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ are maximal ideals of R belonging to the support of M and $k_1, \ldots, k_n$ are positive integers.*

Proof. We first prove the existence of the expression in the local case. Let R be a local ring with maximal ideal $\mathfrak{M}$ and let M be a Noetherian module over R. We have to show that every distributive non-trivial submodule of M is of the form $\mathfrak{M}^k M$, where k is a positive integer.

Let X be a distributive submodule of M. It is easily verified that $X + \mathfrak{M}M/\mathfrak{M}M$ is then a distributive submodule of the $R/\mathfrak{M}$-module $M/\mathfrak{M}M$. But a module over a field has no non-trivial distributive submodules. (For any non-trivial submodule X, the definition fails if we take Y to be a complement of X and Z to be the submodule generated by $x+y$, where x and y are non-zero elements of X and Y, respectively.) Therefore either $X + \mathfrak{M}M = M$, in which case $X = M$, by Nakayama's Lemma; or $X \subseteq \mathfrak{M}M$, in which case the argument can be repeated to show that either $X = \mathfrak{M}M$ or $X \subseteq \mathfrak{M}^2M$. It follows therefore that either $X = \mathfrak{M}^kM$, for some positive integer k, or $X = \bigcap_{n=1}^{\infty} \mathfrak{M}^nM = 0$, by Krull's Intersection Theorem.

Consider now the general case. Let M be a Noetherian module over a ring R and let X be a distributive and supporting proper submodule of M. Then X has a normal primary decomposition $X = Q_1 \cap \cdots \cap Q_n$. (Details can be found in [1].) Let $\mathfrak{M}$ be a maximal ideal of R containing the radical of Q_1. Then $(Q_1)_{\mathfrak{M}} \neq M_{\mathfrak{M}}$. Therefore $X_{\mathfrak{M}} \neq M_{\mathfrak{M}}$ and $M_{\mathfrak{M}} \neq 0$. Hence also $X_{\mathfrak{M}} \neq 0$, since X is a supporting submodule of M. But $X_{\mathfrak{M}}$ is a distributive submodule of the $R_{\mathfrak{M}}$-module $M_{\mathfrak{M}}$. Therefore by the local case, $X_{\mathfrak{M}} = \mathfrak{M}^k_{\mathfrak{M}}M_{\mathfrak{M}}$ for some $k \geqslant 1$. Let P_i denote the radical of Q_i, $i = 1,\ldots, n$, and suppose that $P_1,\ldots, P_t \subseteq \mathfrak{M}$; and $P_{t+1},\ldots, P_n \nsubseteq \mathfrak{M}$. Then $X_{\mathfrak{M}} = (Q_1)_{\mathfrak{M}} \cap \cdots \cap (Q_t)_{\mathfrak{M}}$ is a normal primary decomposition of $X_{\mathfrak{M}}$ in $M_{\mathfrak{M}}$. But $X_{\mathfrak{M}} = \mathfrak{M}^k_{\mathfrak{M}}M_{\mathfrak{M}}$ is an $\mathfrak{M}_{\mathfrak{M}}$-primary submodule of $M_{\mathfrak{M}}$. Therefore $(P_1)_{\mathfrak{M}} = \cdots = (P_t)_{\mathfrak{M}} = \mathfrak{M}_{\mathfrak{M}}$, by uniqueness of radicals. Hence $P_1 = \cdots = P_t = \mathfrak{M}$, and therefore $t = 1$. Thus Q_1 and $\mathfrak{M}^kM$ are $\mathfrak{M}$-primary submodules of M with $(Q_1)_{\mathfrak{M}} = (\mathfrak{M}^kM)_{\mathfrak{M}}$. Therefore $Q_1 = \mathfrak{M}^kM$.

Similarly there are maximal ideals $\mathfrak{M}_i$ of R and integers $k_i \geqslant 1$ such that $Q_i = \mathfrak{M}_i^{k_i}M$, $i = 1,\ldots, n$. The $\mathfrak{M}_i$ are distinct because $\mathfrak{M}_i \supseteq P_i$ for $i = 1,\ldots, n$; and $\mathfrak{M}_i \nsupseteq P_j$ for $i \neq j$. Hence the $\mathfrak{M}_i^{k_i}$ are pairwise coprime. Therefore $X = \mathfrak{M}_1^{k_1}M \cap \cdots \cap \mathfrak{M}_n^{k_n}M = (\mathfrak{M}_1^{k_1} \cdots \mathfrak{M}_n^{k_n})\, M$.

The uniqueness of the expression follows by localising and observing that, if R is a local ring with maximal ideal $\mathfrak{M}$ and M is a Noetherian R-module, then $\mathfrak{M}^kM = \mathfrak{M}^hM \neq 0$ implies $k = h$, by Nakayama's Lemma.

COROLLARY. *Let M be a Noetherian distributive module over a ring R. Then every submodule of M which is locally non-zero at every maximal ideal of R, is of the form IM where I is a unique product of maximal ideals of R.*

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