A thousand million leagues

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Some time ago a friend, who was organising a chess league, rang me up and asked "How many different ways are there of arranging a round-robin league tournament?" Thinking of his phone bill I told him I'd ring him back. Which was just as well, because the problem was much more difficult than I had realised. Indeed it is unsolved in general. In the language of graph theory it is the problem of finding the number of one-factorisations of a complete graph. These and related objects have been the focus of considerable study both in combinatorics and recreational mathematics and, although the enumeration question remains unanswered in general, it is, even for small numbers of players, remarkably full of interest.

Counting league schedules

We shall suppose that the number of players is even, so that each player plays in every round. Think of a *player* as one of the integers 1, 2, 3, ... , 2n and a *game* as an unordered pair of players. Then if P is the set of players and G is the set of all possible games, a *round* is a partition of P into n games (thought of as being played simultaneously) and a *league* is a partition of G into $2n-1$ rounds (the order in which the rounds are played to be regarded as immaterial). Let R denote the set of all possible rounds and L the set of all possible leagues.

It is easy enough to count the numbers of possible games and rounds. Indeed we have

$$|G| = {}_{2n}C_2 = n(2n-1) \text{ and } |R| = (2n-1)(2n-3) \ldots 5.3.1 .$$

However the difficult problem is to find $|L|$, the number of leagues. For $n = 1$ (2 players) there are of course just one game, one round and one league. For $n = 2$ (4 players) there are 6 possible games, 3 possible rounds, and only one league:

12	34
13	24
14	23

It is when $n = 3$ (six players) that things begin to get more interesting. For $n = 3$ there are 15 possible games and 15 possible rounds (apart from n = 1 there are no other values of n for which $|G| = |R|$), and we shall see that there is a striking duality between P and L in which "players in a game" corresponds to "leagues having a common round". First let us determine all the leagues with 6 players. Taking the first round as that which includes the game {1, 2}, the first round can be fixed in three ways:

12 34 56   12 35 46   12 36 45

Having fixed the first round, there are then two possibilities for the remaining four rounds:

12 34 56		12 35 46		12 36 45	
13 25 46	13 26 45	13 24 56	13 26 45	13 24 56	13 25 46
14 26 35	14 25 36	14 25 36	14 23 56	14 26 35	14 23 56
15 24 36	15 23 46	15 26 34	15 24 36	15 23 46	15 26 34
16 23 45	16 24 35	16 23 45	16 25 34	16 25 34	16 24 35
A	*B*	*C*	*D*	*E*	*F*

So the total number of leagues is 6. Moreover, in addition to the fact that $|L| = |P|$, it will be noticed that,

(i) any two leagues have precisely one round in common, and
(ii) every round is in precisely two leagues.

Furthermore, any two distinct rounds either share precisely one game or share precisely one league, but not both. This one-to-one correspondence between league-pairs and rounds can be expressed in Table 1.

	A	*B*	*C*	*D*	*E*	*F*
A		12 34 56	16 23 45	15 24 36	14 26 35	13 25 46
B	12 34 56		14 25 36	13 26 45	15 23 46	16 24 35
C	16 23 45	14 25 36		12 35 46	13 24 56	15 26 34
D	15 24 36	13 26 45	12 35 46		16 25 34	14 23 56
E	14 26 35	15 23 46	13 24 56	16 25 34		12 36 45
F	13 25 46	16 24 35	15 26 34	14 23 56	12 36 45	

TABLE 1.

This remarkable property of the number 6 seems first to have been noticed by Sylvester 150 years ago [8], [9], (and see also [2] and [3]). It has the group theoretic interpretation that S_6, the symmetric group on 6 symbols, has outer automorphisms. For no other value of n does S_n have an outer automorphism.

Relabelling the players

Let us say that two leagues are *equivalent* if one can be obtained from the other by relabelling the players. In this sense all the leagues with 6 players are equivalent, since B, C, D, E and F can be obtained from A by the respective permutations (56), (45), (456), (654), (64).

In order to count the number of leagues in general, one might hope to partition L into its equivalence classes and count the number of leagues in each class. Now it may happen, for a given league l, that relabelling the players produces a league that is actually the same as l though the rounds may be written in a different order. Denoting the number of such permutations by $a(l)$, it can be seen that the number of different leagues in the equivalence class of l is equal to $(2n)!/a(l)$.

Lucas leagues

Perhaps the simplest method of arranging a league schedule is that given by Lucas in 1883 [6]. This can be described by saying that its rounds are obtained by rotating the "clock dial" as in the example in the figure below which is for a league of 8 players and gives the first three rounds of the league

12	38	47	56
13	24	58	67
14	26	35	78
15	28	37	46
16	23	48	57
17	25	34	68
18	27	36	45

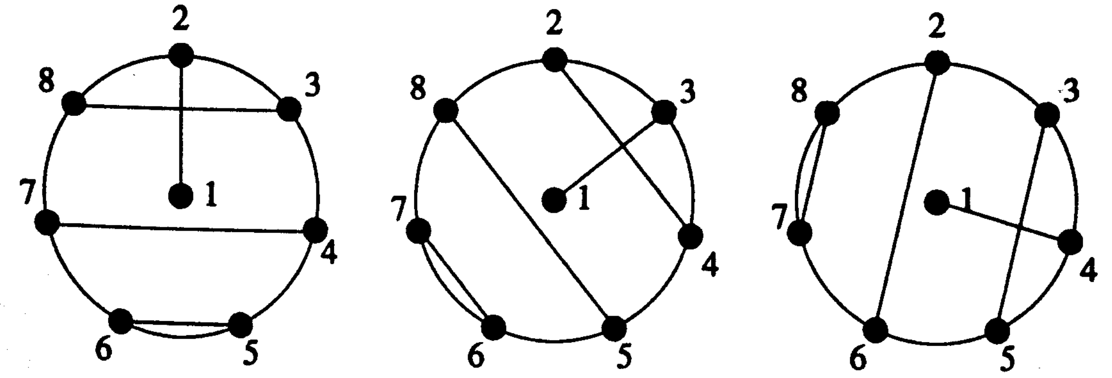


FIGURE 1. The first three rounds of a Lucas league for 8 players.

Such a collection of rounds does constitute a league, because no two of the 7 directions of the "hour hand" can be parallel.

In general, consider the clock dial in figure 2. The players $P = \{1, 2, 3, ..., 2n\}$ can be placed at the marked points on the diagram in any arrangement. Accordingly the construction gives a whole equivalence class of *Lucas leagues*. Let $\pi = (2\ 3\ 4\ ...\ 2n)$, the permutation of P for

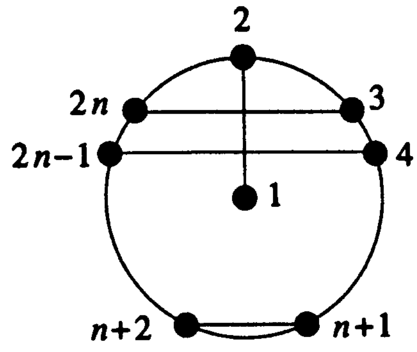


FIGURE 2.

which $1 \to 1$, $2 \to 3$, $3 \to 4$, ... , $2n \to 2$. For each of the integers s in the range $1 < s < 2n-1$ and having no common factor with $2n-1$, let θ_s denote the permutation of P given by $\theta_s(1) = 1$; and for $i=2, \ldots, 2n$, $\theta_s(\mathrm{i})$ = the remainder obtained when si is divided by $2n-1$. Then it can be shown that, for $n > 4$, the permutations

$$\pi^t \theta_s \ (t = 1, 2, \ldots, 2n-1;\ \text{values of } s \text{ as above})$$

are all distinct and are precisely those permutations of P which leave fixed the Lucas league l given by figure 2. It follows that

$$\mathrm{a}(l) = (2n-1)\,\phi(2n-1).$$

where $\phi(m)$ is the number of positive integers no larger than m which are coprime to m, the Euler function. Hence the number of leagues in this class is

$$\frac{(2n)!}{(2n-1)\,\phi(2n-1)}\,.$$

League schedules for 8 players

For $n = 4$, there are 28 games, 105 possible rounds and every league consists of 7 rounds with 4 games in each.

Given a league l, let $\mathrm{r}(l)$ denote the number of divisions of the 8 players into two halves such that, for each half, the 6 mutual games between the 4 players involved occur in three rounds: 2, 2, 2. Also let $\mathrm{s}(l)$ denote the number of divisions of the 8 players into two halves such that, for each half, the 6 mutual games between the 4 players involved occur in four rounds: 2, 2, 2, 1. It is clear that if two leagues l and l' are equivalent, then $\mathrm{r}(l) = \mathrm{r}(l')$ and $\mathrm{s}(l) = \mathrm{s}(l')$.

Table 2 shows six leagues together with their values of r and s. As the six pairs of (r, s) values are all different, it follows that no two of the leagues in the table are equivalent.

Verification of the values listed for r and s is aided by Table 3, which shows the relevant divisions of the 8 players into two halves, together with the rounds which contain 2 of the 6 mutual games between the 4 players in each half.

A

12	34	56	78
13	25	47	68
14	26	38	57
15	28	37	46
16	27	35	48
17	24	36	58
18	23	45	67
$r=0$			$s=0$

B

12	34	56	78
13	24	57	68
14	25	38	67
15	27	36	48
16	28	37	45
17	23	46	58
18	26	35	47
$r=0$			$s=3$

C

12	34	56	78
13	24	57	68
14	23	58	67
15	26	38	47
16	28	37	45
17	25	36	48
18	27	35	46
$r=1$			$s=4$

D

12	34	56	78
13	24	57	68
14	23	58	67
15	27	38	46
16	28	37	45
17	25	36	48
18	26	35	47
$r=1$			$s=6$

E

12	34	56	78
13	24	57	68
14	23	58	67
15	26	37	48
16	25	38	47
17	28	36	45
18	27	35	46
$r=3$			$s=4$

F

12	34	56	78
13	24	57	68
14	23	58	67
15	26	37	48
16	25	38	47
17	28	35	46
18	27	36	45
$r=7$			$s=0$

TABLE 2.

A	$r=0$	$s=0$	Lucas league	
B	$r=0$		$s=3$	
			1234/5678	1,4
			1467/2358	3,6
			1268/3457	5,7
C	$r=1$		$s=4$	
	1234/5678	1,2,3	1256/3478	1,4
			1278/3456	1,7
			1538/2467	4,7
			1637/2458	5,6
D	$r=1$		$s=6$	
	1234/5678	1,2,3	1527/3468	4,6
			1538/2467	4,7
			1546/2358	4,5
			1628/3457	5,7
			1637/2458	5,6
			1748/2356	6,7
E	$r=3$		$s=4$	
	1234/5678	1,2,3	1357/2468	2,4
	1256/3478	1,4,5	1368/2457	2,5
	1278/3456	1,6,7	1458/2367	3,4
			1467/2358	3,5
F	$r=7$		$s=0$	
	1234/5678	1,2,3		
	1256/3478	1,4,5		
	1278/3456	1,6,7		
	1357/2468	2,4,6		
	1368/2457	2,5,7		
	1458/2367	3,4,7		
	1467/2358	3,5,6		

TABLE 3.

It can also be shown that every league with 8 players is equivalent to one of those in Table 2. So to determine the total number of leagues with 8 players, it remains to find the numbers in each of the six equivalence classes.

For the Lucas league A this number is, from the formula in the last section, equal to 8! / (7.6) = 960.

An interesting feature of the league F is that if a game {i,j} is thought of as a transposition (i j) and a round is accordingly thought of as a product of four disjoint transpositions, then the seven rounds of league F are the non-identity elements of the group

$$\Gamma = \{1, a, b, c, bc, ca, ab, abc\}$$

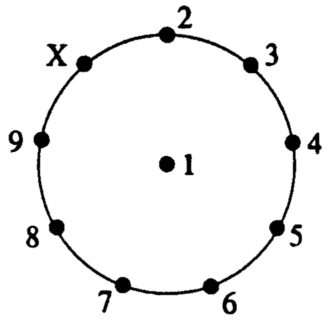
where $a = (12)(34)(56)(78)$, $b = (13)(24)(57)(68)$, $c = (15)(26)(37)(48)$. As $a^2 = b^2 = c^2 = 1$, Γ can be regarded as a 3-dimensional vector space over the field of two elements $\mathbf{F}_2$, and this observation can be used to show that the number of relabellings of the players which do not change league F is equal to 8 times the number of different invertible 3×3 matrices over $\mathbf{F}_2$; namely 1344. Therefore the number of leagues in the equivalence class of F is 8!/1344 = 30.

The numbers of leagues in the equivalence classes of leagues B, C, D and E are 1680, 2520, 420 and 630 respectively. Details can be found in [4] and [10]. Hence the total number of leagues with 8 players is 6240.

League schedules for 10 players

For $n = 5$, there are 45 games, 945 possible rounds and every league consists of 9 rounds with 5 games in each. Table 4 exhibits three such leagues. (Player 10 is denoted by the symbol X).

League A is the Lucas league corresponding to the diagram shown below. It turns out that there are 396 equivalence classes of leagues with 10 players, and the numbers in each class, found with the help of a computer, are given in [5]. The smallest of the 396 classes is that of league C, while at the other end of the scale 298 of the classes consist of leagues where, as for league B, each of the 10! ways of relabelling the players gives a different league in the same class.



League A is the Lucas league corresponding to this diagram.

	A					B					C				
	12	3X	49	58	67	12	34	56	78	9X	12	34	56	78	9X
	13	24	5X	69	78	13	24	57	69	8X	13	24	57	69	8X
	14	26	35	7X	89	14	25	37	6X	89	14	23	5X	68	79
	15	28	37	46	9X	15	23	49	68	7X	15	26	37	4X	89
League	16	2X	39	48	57	16	28	35	4X	79	16	25	39	48	7X
	17	23	4X	59	68	17	26	3X	48	59	17	28	35	49	6X
	18	25	34	6X	79	18	29	36	47	5X	18	27	3X	46	59
	19	27	36	45	8X	19	2X	38	45	67	19	2X	36	47	58
	1X	29	38	47	56	1X	27	39	46	58	1X	29	38	45	67
No. of leagues in class	$\frac{10!}{9\phi(9)} = 67\,200$					$10! = 3\,628\,800$					8400				

TABLE 4.

Anyone who has ever tried to arrange a schedule for a league tournament, without using one of the standard constructions, may find it surprising that there are so many different possibilities. In fact, adding up the numbers of leagues in each of the equivalence classes gives a grand total of 1 225 566 720 different ways of scheduling a league with 10 players.

Leagues with more than 12 players

For 12 or more players the exact number of leagues is not known, but it is known that the number of leagues grows very rapidly as n increases.

Straightforward estimation and use of Stirling's formula, show that the total number of leagues with $2n$ players cannot increase with n any faster than

$$(2n)^{2n^2}.$$

On the other hand, using Latin squares to construct leagues, and a recently proved result about doubly stochastic matrices, it can be shown that the number of equivalence classes of leagues with $2n$ players increases with n at least as fast as this expression. Therefore both the number of leagues and the number of equivalence classes tend to infinity at this rate. Further details can be found in [1].

Other problems of interest concern league schedules satisfying certain conditions, such as those which arise from venue considerations. There are also unsolved problems concerning sets of rounds that have no common game and do not form part of any league. See for example [11] and [12]. A general survey and extensive bibliography is given in [7].

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