Hurdles and Strategies in the Teaching of Algebra

Part 1

by Tony Barnard

"Why do they write $a^2+a^3=a^5$ and $\frac{24+5\sqrt{3}}{6}=4+5\sqrt{3}$ and ...?"

In talking about algebra in recent years with groups of teachers, various questions arose that were not always simple to answer. For example, there were questions like: "How would you respond to a pupil in an 'intermediate tier' group who asked what algebra was needed for?", or: "How do you get them to stop writing $(3x)^2=3x^2$?", and comments like: "Many of my pupils have particular difficulty with $1+\frac{a}{b}=\frac{b+a}{b}$", or: "Most would multiply the numerators in $\frac{x+y}{5} \times \frac{7}{2(x+y)}$, rather than cancel the $(x+y)$, to give $\frac{7}{5\times2}=\frac{7}{10}$". Often an interesting discussion followed in which individual teachers talked about what they had found helpful, but none of the ideas were ever recorded for subsequent circulation.

With a view to putting together a small collection of 'hurdles and strategies' in the teaching of this curriculum area, experienced mathematics teachers across the country were asked to submit lists of questions about the teaching of algebra in general, or algebraic manipulation in particular, for which they felt answers/suggestions would be of value. The responses, which included items that the teachers found challenging, as well as questions they had been asked by colleagues or had been discussed at a departmental meeting, were collated and sent round with the invitation to submit strategies, suggestions or comments that people had found to be practically useful. The responses to this second enquiry constituted a wealth of ideas based on the practice of real maths teachers in real classrooms. As one respondent later put it: "... many of the techniques and strategies for surmounting hurdles belong to an oral tradition of teachers – the images, metaphors and similes employed are not to be found in textbooks and, paradoxically, although algebra is often perceived (by students and teachers) as mechanical, the *teaching* of algebra is all but mechanical!".

In presenting the suggested strategies in this series of articles, one is aware that what works for one pupil might not always work for another. So much depends on the previous learning experiences of the pupil. Indeed, when problems are several layers deep, ideas for locating the various sources of difficulty and addressing these could fill a PhD thesis (and often have). In these cases, using one of the suggested strategies might seem like putting a plaster over a serious wound. There are also many misconceptions that can be partly caused by inappropriate previous teaching, or teaching not adequately matched to the pattern of development of the pupil. But teachers don't have time to complete a PhD thesis in each lesson and they can't change unfortunate learning experiences that have already occurred. They have to do something there and then. It is in this spirit (and with the above reservations) that the strategies are presented.

Underlying Causes of Pupil Errors

It can be argued that algebra starts when the things one is talking and thinking about have become mentally manipulable objects. At the heart of many errors is the failure to conceive the objects of manipulation (e.g. $3/4$, -7, $2x+5$, a^2, $\sqrt{r^2+1}$) as meaningful 'things' in their own right. The four features below can be thought of as variants on this theme.

Meanings attached to algebraic letters

There are theories that suggest a classification of pupils' interpretations of letters along the following lines:

(a) (i) ignored, (ii) given an arbitrary value, (iii) the name of a particular item;

(b) (i) specific number whose value is not known, (ii) generalized number from a certain range of possibilities, (iii) thing in its own right with the same properties as familiar numbers.

Some theories relate these different interpretations to stages of intellectual development (saying that those in (b) are not possible if the pupil is not 'cognitively ready'), while others are more optimistic about the role of teaching. Whichever view you take, what *is* clear is that if letters don't have meaning for a pupil, very little algebra is possible other than fragmentary successful instrumental rote learning of rules.

Fractions

Initially a pupil might think of a fraction such as $4/5$ only in relation to 4 of the parts of a shape which has been divided into 5 equal parts, or as the end result of the process of dividing 4 by 5. In order to operate successfully with algebraic fractions, it is essential to have an understanding of numerical fractions in which a fraction such as $^4/_5$ is seen as a single independent entity that can be manipulated arithmetically with the same lightness of touch as any whole number. It is also important to have a thorough understanding of the fact that $^4/_5$, $^8/_{10}$, $4 \times {}^1/_5$, $4 \div 5$, etc., are all the same 'thing'. Indeed, a semantic understanding of arithmetic generally is a necessary prerequisite for algebra to be meaningful.

Computations versus relationships

The computational world of arithmetic has several features that must be 'put in their place' as pupils progress into algebra. For example, pupils are sometimes so overwhelmed by the need for closure (the need to produce an answer that they accept as being a meaningful end result) that they might write, say, $2x$ instead of $2+x$, simply because they have a meaning for $2x$ (as two lots of x, whatever x is; e.g. two bags of sweets) whereas they do not have a meaning for $2+x$ (without knowing what x is; e.g. you can't add the number 2 to a bag). Another symptom of immersion in the computational world is seeing the equals sign as an instruction to do something (to produce an answer), rather than as a signifier of two things being the same. For algebra it is important to develop the feature of equality as a two-way relationship. Similarly, pupils' disregard for brackets is often rooted in a mind-set of sequential process, where brackets may be seen merely as a commentary on the process and not of the same importance as 'the answer'. Appreciating the role of brackets in an expression as a whole is part of the relational world of algebra.

Expressions as single items

If pupils are able to see an expression like $\sqrt{a^2+b^2}$ as one complete object, not only will they not feel the need to 'work it out' further (perhaps incorrectly replacing it with $a+b$), but also they will be able to move it around in an equation just as easily as they could move around a single letter or number. Stumbling blocks are often caused by the appearance of unsimplifiable expressions that have no meaning for pupils.

In the remainder of this article we will consider two general errors which are very common among pupils and which are particularly difficult to remedy. Items inside boxes are specific instances of these 'hurdles' submitted in the first stage of the project and comments attached to bullets are mostly 'strategies' submitted in the second stage.

Operating on One Piece of a Compound Term

Confusion between $(xy)^2$ and xy^2

Confusion between $(2x)^2$ and $2x^2$, $(3x)^2$ and $3x^2$, etc.

- We felt that confusion between $(2x)^2$ and $2x^2$ could be reduced by ensuring that the teacher always stresses the difference by using a pause in the correct place: $(2x)^2$ should always be referred to as "two x ... squared" with the 'two x' stressed to indicate togetherness and spoken at speed, before allowing a pause for the 'squared'. Whilst $2x^2$ would be referred to as "2 ... x squared", with an obvious verbal separation between the coefficient and the variable.
- Getting pupils to write it out in full, e.g. $(3x) \times (3x)$ for $(3x)^2$, often helps.
- Write out at length $(2x)^2=(2\times x)\times(2\times x)=4x^2$ every time during the introduction of this topic and repeat regularly.
- Use of brackets when not strictly needed: e.g. writing $2x^2$ as $2(x^2)$.
- Lots of practice with 'brackets' and 'order of operations'.
- Thorough reference to the BIDMAS convention (brackets, indices, ...) and a substitution exercise incorporating ax^2 and $(ax)^2$.
- Counterexamples with particular numbers. The easy thing here is for the pupil to see that $(3x)^2=3x^2$ is wrong, but this will not stick if the expresssions have no meaning for the pupil.

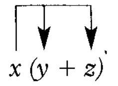
$\frac{1}{2}x + 2 = \frac{3}{4}x - \frac{3}{4} \rightarrow 2x+2=3x-3$ (forgetting to multiply the 'non-fraction' term)

- I would insist they replace the 2 by $\frac{2}{1}$. This reminds them that there is a denominator there, and reduces the number of slips.
- Put in extra steps: multiply both sides by 4, writing $4\left(\frac{1}{2}x + 2\right) = 4\left(\frac{3}{4}x - \frac{3}{4}\right)$.

$x(y+z)=xy+z$

$a+b=\frac{f}{c} \rightarrow a+bc=f$

$x+4$ times 3 equals $x+4\times3$

- Draw arrows: 

- When introducing brackets it was felt that concrete imagery was useful. Imagining the bracket as a container so that $3(a+b)$ means three containers each holding a letter a and a letter b, making a total of three as and three bs.
- Cuisenaire rods as an aid to explaining the use and expansion of brackets.
- Use more brackets.
- I try to get them to use brackets from day 1 to impress on them that it is the whole expression that is being multiplied. I prefer to have a plethora of brackets, rather than a dearth.
- Talk about the need to 'parcel' the $x+4$ and that we use ( ) for this.

  Write $x+4$ times 3 as $\begin{array}{r} x+4 \\ x+4 \\ x+4 \\ \hline 3x+12 \end{array}$

There are many possible explanations and contributory factors for errors of the kind, $(3x)^2 \rightarrow 3x^2$ and $x(y+z) \rightarrow xy+z$. One theory is that without a memorized pattern (pre-established frame of the form $x(y+z) = xy+xz$) that the input can match, a pupil is likely to resort to operating sequentially. So, although pupils may be easily convinced of this error by substituting in numerical values, they will continue to repeat it unless it is addressed at a deeper level. In order for the expression $(3x)^2$ to have meaning, pupils need to be able to think of $3x$ as a unified single entity. Similarly, in the case of $x(y+z)$, they need to be able to think of $y+z$ as a single entity. Some of the strategies above may help in achieving this perception.

$2x-x=2$

- This error is thought to be often induced by language, in which reading "two x, take away x" is unconsciously interpreted as "remove the x from 'two x' to leave 'two'". Pupils can be woken up from the spell of this false interpretation with something like, "I have two xylophones and I lose one xylophone ...". This may help to bring the meaning back to the 2x.

- Write the '1' back into the algebra: $2x-1x=1x$.

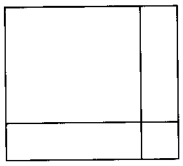
$$\boxed{\frac{3}{2x} \rightarrow 3 \times 2x^{-1} \rightarrow 6x^{-1} \rightarrow \frac{1}{6x}}$$

- Encourage students to think of $\frac{3}{2x}$ as $\frac{3}{2}\frac{1}{x}$. This separates the number and power (to the minus one) issues.

False Linearity

$$\boxed{\begin{array}{c}(x+y)^2 = x^2+y^2 \\ (x+4)^2 = x^2+16\end{array}}$$

- A graduated approach, first using a drawing

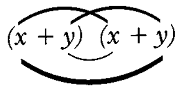


and if that fails then getting the student to construct a similar arrangement using Multilink cubes (thereby gradually increasing the number of senses involved).

- Multiplying out one or two brackets is introduced through finding areas of the corresponding rectangles.
- Get the pupils to write the brackets in full and show each step. They may find this tedious, but fewer make the obvious mistakes if they spend the time writing it out:

$$\begin{aligned}(x+4)^2 &= (x+4)(x+4) \\ &= x(x+4)+4(x+4) \\ &= x^2+4x+4x+16 \\ &= x^2+8x+16\end{aligned}$$

- I found that asking them to do a long multiplication by the usual algorithm (including using zeros as place keepers), and then using just the same format to multiply a pair of brackets together (with x^2-coefficient, x-coefficient and constant term in place of 'hundreds', 'tens' and 'units', etc.) seemed to work well. It was the 'known to the unknown' principle, capitalizing on and valuing their existing knowledge and emphasizing the isomorphism between arithmetic and algebra (well, this one, anyway).
- The 'face' method:



Makes a smiley face if you get it right!

- Two methods to raise pupil awareness of false linearity:
  - (i) A drip-feed effect of forcing pupils to confront such situations more frequently, through careful choice of examples and exercises (thereby raising awareness of the problem).
  - (ii) A lesson planned about a particular misconception. e.g. Investigate whether this is true: $(x+y)^2 = x^2+y^2$. If it isn't, what is? Or maybe a sheet of such misconceptions, complete with a few truths which the pupils could decide on the validity of by corroborating through the substitution of numbers (thereby reinforcing the need to check *and* a method for checking).

With something like $(x+y)^2$, we have an operation ('square it') acting on an expression consisting of two pieces. If the associated meaning, 'multiply $x+y$ by $x+y$' (in which $x+y$ is initially thought of as a single entity), is not firmly established in the mind of a pupil, it is likely that he/she will simply apply the operation to each of the pieces x and y in turn. Similar behaviour occurs frequently with operations such as $\sqrt{\ }$ ('take the square root'), ( )/( ) ('divide'), etc., where the interpretation of the operation as an instruction to do something overrides an understanding of what the expression means. As a first step in addressing this error, it can be useful to demonstrate the falsity by substituting in numbers (or asking the pupil to do so), but this on its own will not be long-lasting. They'll just say: "Oh yes, how dumb!", and then do the same thing again the next day. So recognition of the error should be followed by an attempt to establish what $(x+y)^2$ actually means, and this invariably involves getting to grips with a dual perception of $x+y$, first thinking of it as a single item and then unpacking it into its two components. Some of the suggested strategies may help with this (and *after* the understanding has been established, the smiley face method may be a handy way of keeping track).

$$\boxed{\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}}$$

- Much time is needed practising adding and subtracting numerical fractions and really getting the pupils to understand the principles behind the processes.
- There is some mileage in interchanging between number and algebra in a repeated way, each time presenting numeric and algebraic problems with *identical structure*,

  e.g. $\frac{2}{7}+\frac{3}{8}$ and $\frac{2}{x}+\frac{3}{y}$.

$$\boxed{\sqrt{a}+\sqrt{b}=\sqrt{a+b}}$$

- Show it doesn't work with numbers;

  e.g. $\sqrt{9}+\sqrt{16}=3+4=7$
  $\sqrt{9+16}=\sqrt{25}=5$

  This remedy may not be very long-lasting if the abstract expressions are not meaningful to the pupil.

- Explicitly relate to the concept of functionality, stressing that the square root function is not linear.

$$\boxed{\begin{array}{l}\sin(x+30)=\sin x+\sin 30 \\ 1/2 \sin 2x=\sin x \\ \frac{1}{2}+\frac{1}{x}=\frac{1}{5} \rightarrow 2+x=5\end{array}}$$

- Ask the students to think about what the consequences, in terms of the graphs of the functions, would be if these were true. With graphic calculators to hand this can be a more sophisticated and striking version of 'put in some numbers'.

- '$\sin(x+30)=\sin x+\sin 30$' is a teaching point: raise it (and shoot it down!) when introducing $\sin(A+B)$.

$$\boxed{2 \times \frac{a}{b} = \frac{2a}{2b}}$$

- Emphasize the similarity with multiplication of numerical fractions and encourage pupils to write the question as $\frac{2}{1} \times \frac{a}{b}$ initially.
- Use numerical examples where a and b are prime numbers; e.g. $1/3+1/5=8/15$. The algebra is the same, but the utensils are more concrete.

Further articles will consider pupil errors involving: 'confusion between operations' (e.g. $x^2=2x$, $a^2+a^2=a^4$), 'misapplied rules' (e.g. $(x+3)\ (x+7)=12 \Rightarrow x+3=12$ or $x+7=12$), 'minus' (e.g. $5-x=7 \rightarrow x=2$), 'rearranging formulae' (e.g. $T=2p+ph \rightarrow \frac{T}{h} = 2p+p$), inappropriate cancelling (e.g. $\frac{ax+b}{cx} = \frac{a+b}{c}$) and other all too familiar behaviours. There will also be summaries of responses regarding general principles for algebra teaching, techniques for motivating pupils in algebra and questions regarding teaching approaches for particular algebra topics.

Acknowledgements

I should like to thank the following people, all of whom have contributed to this project in one way or another: Steve Abbott, Teresa Alexander, Richard Ambrose, Miles Baillie, David Banham, John Bradshaw, David Crawford, Mary Eaton, Chris du Feu, Doug French, Martin Glazier, Trisha Haslam, David Holland, Steve Holmes, Gerry Leversha, Nick Lord, Heather Mendick, Paul Metcalf, Bernard Murphy, Stephen Nicholson, Chris Poole, Jenny Ramsden, Peter Ransom, Lesley Ravenscroft, Tony Robin, Steve Rout, Caroline Starkey, David Tall, Peter Thomas, David Waugh, Cathy Williamson, Sue Wilson.

Keywords: Algebra; Pupil Errors.

Author
Dr Tony Barnard, Department of Mathematics, King's College, Strand, London WC2R 2LS.
e-mail: tony.barnard@kcl.ac.uk