When are examples enough to give a proof?

When is an Example Not an Example?

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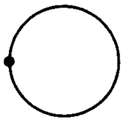
I knew a schoolteacher once who used to teach the fact that the angle sum of a triangle is 180 degrees by getting each pupil to draw a triangle, measure the angles and add them up. Depending on accuracies of measurement and addition, this usually gave 30 or so answers of about 180. When I remarked one day that this exercise did not constitute a proof, the teacher replied, 'Well how many triangles do you have to do before it *does* count as a proof?' You may like to ponder on this story while trying the following investigations.

FOR each of the following statements, see if it is true for $n = 1, 2, 3, 4, 5, \ldots$ (go as far as you think is worthwhile) and decide if it is true for all values of n. In each case the first two values have been done to help you get started.

(1) The sum of the first n odd numbers is n^2.

$$1 = 1^2,\ 1 + 3 = 2^2 \ldots$$

(2) Given n points on the circumference of a circle, if every chord joining two of the points is drawn, the circle is divided into exactly 2^{n-1} regions.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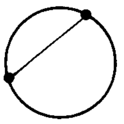
2 regions

...

(3) The number of different arrangements of noughts and crosses in a row of n such symbols is 2^n.

$n = 1$ (0, X),
$n = 2$ (00, 0X, X0, XX) ...

(4) When $x^n - 1$ is factorised as a product of irreducible polynomials with integer coefficients, no integer other than 1, 0 or -1 occurs as a coefficient in any of the factors.

$$x^1 - 1 = x - 1,$$
$$x^2 - 1 = (x-1)(x+1) \ldots$$

(5) $n^2 + n + 41$ is prime.

The values for $n = 1$ and $n = 2$ are the primes 43 and 47, respectively.

(6) $\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24) = 2^{n-1}$.

The values of

$$n^4 - 6n^3 + 23n^2 - 18n + 24$$

for $n = 1$ and $n = 2$ are 24 and 48, respectively.

(7) Let P be a positive integer whose decimal representation is

$$a_n\ a_{n-1}\ \ldots\ a_2\ a_1\ a_0,$$

i.e.

$$P = a_n\,10^n + a_{n-1}\,10^{n-1} + \ldots + a_1\,10 + a_0$$

($a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are the 'digits' of P). If $a_0 = 1$ and $a_1 = a_2 = \ldots = a_{n-1} = a_n = 3$, then P is prime.

31 and 331 are prime.

(8) With the same notation as in (7), if $a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0$ is divisible by 3 then P is divisible by 3.

$10a_1 + a_0$ and $a_1 + a_0$ differ by $9a_1$, which is a multiple of 3. Similarly $100a_2 + 10a_1 + a_0$ and $a_2 + a_1 + a_0$ differ by $3(33a_2 + 3a_1)$.

(9) $(a+b)^{2n+1} = a^{2n+1} + b^{2n+1} + (2n+1)ab(a+b)\,(a^2+ab+b^2)^{n-1}$.

$$(a+b)^3 = a^3 + b^3 + 3\,ab(a+b),$$
$$(a+b)^5 = a^5 + b^5 + 5ab(a+b)\,(a^2+ab+b^2)$$

etc.

(10) For $n \geq 3$,

$$n! - (n-1)! + (n-2)! - (n-3)! + \ldots$$

is prime.

$$3! - 2! + 1! = 5,$$
$$4! - 3! + 2! - 1! = 19 \ldots$$

(11) For $n \geq 2$, it is possible to place $2n$ pawns on an $n \times n$ chessboard so that no straight line (horizontal, vertical or in any other direction) contains more than two pawns.

X	X
X	X

	X	X
X		X
X	X	

...

(12) $1 + 61n^2$ is not a perfect square.

The values for $n = 1$ and $n = 2$ are 62 and 245, respectively, neither of which is the square of an integer.

Some Answers

(1) *True* Perhaps the simplest way to prove this statement is to write

$$S = 1 + 3 + 5 + \ldots + (2n-3) + (2n-1)$$

Then also

$$S = (2n-1) + (2n-3) + (2n-5) + \ldots + 3 + 1$$

So

$$2S = 2n + 2n + 2n + \ldots + 2n + 2n \quad (n \text{ times})$$
$$= n.2n = 2n^2$$

Therefore

$$S = n^2$$

(2) *False* The statement is true for n = 1, 2, 3, 4, 5 but for n = 6 there are 31 regions, not 32. This seeming prank of nature can be explained as follows. Whereas 2^{n-1} is the sum of *all* the n binomial coefficients

$$\binom{n-1}{r} \quad 0 \leq r < -1$$

(which can be seen by putting x = 1 in the formula:

$$(1+x)^{n-1} = 1 + \binom{n-1}{1} x + \binom{n-1}{2} x^2 + \ldots + \binom{n-1}{n-2} x^{n-2} + x^{n-1})$$

it can be shown that the number of regions of the circle is only the sum of the first five; namely

$$1 + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-1}{4}$$

For $n \leq 5$, there *are* no more than five such terms!

(3) *True* The statement is easily proved by induction, since extending the row of symbols by one place multiplies the total number of arrangements by the number of possibilities for the additional symbol, in this case giving $2^n \times 2 = 2^{n-1}$. Alternatively, the total number of arrangements is the sum

$$\sum_{r=0}^{n} \text{(number of arrangements with } r \text{ crosses)}$$

$$= \sum_{r=0}^{n} \binom{n}{r}$$

$$= (1+1)^n$$

by the binomial theorem,

$$= 2^n$$

(On the other hand, the comparison of these two ways of counting the number of arrangements can be viewed as an independent proof of the formula

$$1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + 1 = 2^n)$$

(4) *False* The statement is true for all values of n up to 104; for example,

$$\begin{aligned} x^{100}-1 = {} & (x-1)(x+1)(x^2+1) \\ & (x^4+x^3+x^2+x+1) \\ & (x^4-x^3+x^2-x+1) \\ & (x^8-x^6+x^4-x^2+1) \\ & (x^{20}+x^{15}+x^{10}+x^5+1) \\ & (x^{20}-x^{15}+x^{10}-x^5+1) \\ & (x^{40}-x^{30}+x^{20}-x^{10}+1) \end{aligned}$$

However, one of the irreducible factors of $x^{105}-1$ is

$$\begin{aligned} & x^{48}+x^{47}+x^{46}-x^{43}-x^{42}-2x^{41} \\ & -x^{40}-x^{39}+x^{36}+x^{35}+x^{34}+x^{33} \\ & +x^{32}+x^{31}-x^{28}-x^{26}-x^{24}-x^{22} \\ & -x^{20}+x^{17}+x^{16}+x^{15}+x^{14}+x^{13} \\ & +x^{12}-x^9-x^8-2x^7-x^6-x^5+x^2 \\ & +x+1 \end{aligned}$$

The reason for no counterexample being reached until this point is connected with the fact that 105 is the smallest positive integer which is divisible by more than two distinct odd primes.

(5) *False* The values for n = 40 and n = 41 are clearly not prime ($40^2+40+41 = 41^2$ and $41^2+41+41 = 41.43$), but n^2+n+1 *is* prime for all n up to 39. In fact it is prime for all integer values of n in the range $-40 \leq n \leq 39$.

(6) *False* The statement is true for n = 1, 2, 3, 4, 5 but false for all $n \geq 6$. This can be explained in the same way as (2) above, since

$$\tfrac{1}{24}(n^4-6n^3+23n^2-18n+24) = 1 + \binom{n}{2} + \binom{n}{4}$$

(7) *False* The statement is true for n = 1, 2, 3, 4, 5, 6, 7, but

$$333{,}333{,}331 = 17 \times 19{,}607{,}843$$

(8) *True* Since $10^n - 1 = 99 \ldots 9$ (n digits), it is clear that

$$P - (a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0)$$

is a multiple of 3. Therefore if

$$(a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0)$$

is divisible by 3, then so is P.

(9) *False* The statement is true for n = 1, 2, 3. However it is false for n = 4, as can be seen by putting $a = b = 1$, when

$$(a+b)^9 = 512$$

and

$$a^9+b^9+9ab\ (a+b)(a^2+ab+b^2)^3 = 488$$

(10) *False* The values for n = 3, 4, 5, 6, 7, 8 are, respectively, 5, 19, 101, 619, 4421, 35899 (all prime), but the value for n = 9 is 326,981 = 79 × 4,139.

(11) *Unknown* The statement is true for $2 \leq n \leq 26$. For example, a solution for a standard 8 × 8 chessboard is

	X		X				
					X		X
		X		X			
X						X	
X						X	
		X		X			
					X		X
	X		X				

However, it is *conjectured* that for large n it is not possible to place $2n$ pawns on an $n \times n$ chessboard so that no straight line contains more than two. (One of the difficulties when n is large is that the number of possible directions for straight lines is also large.)

(12) *False* $1 + 61n^2$ is not a perfect square for any value of n less than two hundred million, but

$$1 + 61\,(226{,}153{,}980)^2 = (1{,}766{,}319{,}049)^2$$

In fact there are *infinitely* many integer values of n for which $1 + 61n^2$ is a perfect square. This is a special case of a more general fact that if d is a positive integer, with no repeated prime factor, then there are infinitely many integer values of n for which $1 + dn^2$ is a perfect square. What makes d = 61 so fascinating is the relatively large size of the smallest solution.

The Moral of the Story

Considering special cases is often an excellent way of suggesting ideas for solving a problem and getting a feel for the mathematics involved. A well chosen example may contain all the essential features and exhibit these very clearly. However, one should never regard the verification of a general mathematical fact in a number of special cases, as a complete reason for believing the fact.

This is not to deny the importance of developing a mathematical intuition. Indeed the real 'action' of mathematics takes place in the interplay between intuition and logic. But the moral of the story is that, although examples are the food of intuition, not all examples are equally nutritious and, furthermore, one can never be sure of the truth of a statement if one's trust in it depends only on examples. In adopting an 'experimental' approach we must not forget that, in mathematics, it is logical reasoning which is needed to actually establish the results.

Further Reading

Guy, R.K. (1988), 'The Strong Law of Small Numbers', *American Mathematical Monthly*, No. 95, pp. 697–712.

Guy, R.K. (1990), 'The Second Strong Law of Small Numbers', *Mathematics Magazine*, No. 63, pp. 3–20.

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