A potential-theoretical review of some exit problems of spectrally negative Lévy processes

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Summary. In this note we consider first exit problems of completely asymmetric (reflected) Lévy processes and present an alternative derivation of their Laplace transforms essentially based on potential theory of Markov processes.

Key words: Potential theory, first passage, Wiener-Hopf factorisation, Lévy processes

1 Introduction and main results

Let X be a spectrally negative Lévy process, i.e. a stochastic process with càdlàg paths without positive jumps that has stationary independent increments defined on some probability space $(\Omega, \mathcal{F}, P)$ that satisfies the usual conditions. By $(P_x, x \in \mathbb{R})$ we denote the family of measures under which the Lévy process X is translated over a constant, that is P_x denotes the measure P conditioned on $\{X_0 = x\}$. We exclude the case that X has monotone paths. By the absence of positive jumps, the moment generating function of X_t exists for all $\theta \geq 0$ and is given by

$$E[\mathrm{e}^{\theta X_t}] = \exp(t\ \psi(\theta)), \qquad \theta \geq 0,$$

for some function $\psi(\theta)$ which is well defined at least on the positive half axis, where it is convex with the property $\lim_{\theta\to\infty}\psi(\theta) = +\infty$. Let $\Phi(0)$ denote its largest root. On $[\Phi(0),\infty)$ the function ψ is strictly increasing and we denote its right-inverse function by $\Phi : [0,\infty) \to [\Phi(0),\infty)$. Denote by I and S the past infimum and supremum of X respectively, that is,

$$I_t = \inf_{0\leq s\leq t}(X_t \wedge 0), \qquad S_t = \sup_{0\leq s\leq t}(X_t \vee 0)$$

and write $Y = X - I$ for X reflected at its past infimum I. By T_a^-, T_a^+ we denote

$$T_a^- = \inf\{t \geq 0 : X_t < a\}, \qquad T_a^+ = \inf\{t \geq 0 : X_t > a\},$$

the first passage times of X into the sets $(-\infty, a)$ and (a, ∞), respectively. Similarly, we write

$$\tau_a^+ = \inf\{t \geq 0 : Y_t > a\}$$

for the first passage time of Y into the set (a, ∞). The following theorem gives the form of the Laplace transforms of these passage times:

Theorem 1 *(i) For $q > 0$, the q-potential measure of X*

$$U^q(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-qt} P(X_t \in \mathrm{d}x)\mathrm{d}t \tag{1}$$

is absolutely continuous with respect to the Lebesgue measure and a version of its density on $[0, \infty)$ is given by

$$u^q(x) = \Phi'(q) \exp(-\Phi(q)x). \tag{2}$$

(ii) For $q \geq 0$, there exists a continuous increasing function $W^{(q)} : [0, \infty) \to [0, \infty)$ with Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda x} W^{(q)}(x)\mathrm{d}x = (\psi(\lambda) - q)^{-1}, \qquad \lambda > \Phi(q)$$

and, denoting by u^q a version of $U^q(\mathrm{d}x)/\mathrm{d}x$, it holds that for $q > 0$

$$W^{(q)}(x) = \Phi'(q) \exp(\Phi(q)x) - u^q(-x) \text{ for a.e. } x \geq 0. \tag{3}$$

(iii) (Exit from a half-line) For $q \geq 0$, $x \leq a$ and $y \geq 0$ we have

$$E_x\left[\mathrm{e}^{-qT_a^+} I_{(T_a^+ < \infty)}\right] = \mathrm{e}^{\Phi(q)(x-a)}; \tag{4}$$

$$E_y\left[\mathrm{e}^{-qT_0^-} I_{(T_0^- < \infty)}\right] = Z^{(q)}(y) - q\Phi(q)^{-1} W^{(q)}(y), \tag{5}$$

where

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)\mathrm{d}y$$

and for $q = 0$, $q\Phi(q)^{-1}$ is understood in the limiting sense, $\lim_{q \downarrow 0} q\Phi(q)^{-1}$.

(iv) (Exit from a finite interval) For $x \in [0, a]$ and $q \geq 0$ we have

$$E_x\left[\mathrm{e}^{-qT_a^+} I_{(T_a^+ < T_0^-)}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}; \tag{6}$$

$$E_x\left[\mathrm{e}^{-qT_0^-} I_{(T_a^+ > T_0^-)}\right] = Z^{(q)}(x) - Z^{(q)}(a)\frac{W^{(q)}(x)}{W^{(q)}(a)}; \tag{7}$$

$$E_x[\mathrm{e}^{-q\tau_a^+}] = \frac{Z^{(q)}(x)}{Z^{(q)}(a)}. \tag{8}$$

Remark. The function $W^{(q)}$ is called the q-scale function of X in the literature. In particular, one calls $W = W^{(0)}$ the scale function of the Lévy process, in analogy with the theory of diffusions.

Remark. (Probabilistic derivation of formula (3)). Noting that, for $q > 0$, $\{\exp(\Phi(q)X_t - qt), t \geq 0\}$ is a martingale, we define the tilted measure $P^{\Phi(q)}$ by

$$P^{\Phi(q)}(A) = E[\exp(\Phi(q)X_t - qt)I_A], \qquad A \in \mathcal{F}_t.$$

Under the measure $P^{\Phi(q)}$ the process X is still a Lévy process and its characteristic exponent $\psi_{\Phi(q)}$ can be checked to be given by

$$\psi_{\Phi(q)}(\lambda) = \psi(\Phi(q) + \lambda) - \psi(\Phi(q)) = \psi(\Phi(q) + \lambda) - q.$$

We write $W_{\Phi(q)}$ for the scale function of X under $P^{\Phi(q)}$. Comparing Laplace transforms yields the identity

$$W^{(q)}(x) = \mathrm{e}^{\Phi(q)x} W_{\Phi(q)}(x).$$

Since $qu^q(x)\mathrm{d}x = P(X_{\eta(q)} \in \mathrm{d}x)$, where $\eta(q)$ denotes an independent exponential time, the strong Markov property yields for $x < 0$

$$\begin{aligned}
&q^{-1}P(X_{\eta(q)} \in \mathrm{d}x) \\
&= \int E[\mathrm{e}^{-qT_x^-}, X_{T_x^-} \in \mathrm{d}y]u^q(x-y) = \int E[\mathrm{e}^{-qT_x^-}, X_{T_x^-} \in \mathrm{d}y]\Phi'(q)\mathrm{e}^{\Phi(q)(y-x)} \\
&= \Phi'(q)\mathrm{e}^{-\Phi(q)x}E[\mathrm{e}^{-qT_x^- + \Phi(q)X_{T_x^-}}] = \Phi'(q)\mathrm{e}^{-\Phi(q)x}P^{\Phi(q)}(T_x^- < \infty) \\
&= \Phi'(q)\mathrm{e}^{-\Phi(q)x}\left(1 - \frac{W_{\Phi(q)}(-x)}{W_{\Phi(q)}(\infty)}\right) = \Phi'(q)\mathrm{e}^{-\Phi(q)x} - W^{(q)}(-x),
\end{aligned}$$

where in the first line we used the explicit form (2) and in the second line a change of measure. The third line follows by letting $a \to \infty$ and taking $q = 0$ in (6) and noting next that $W_{\Phi(q)}(\infty) = \lim_{x\to\infty} W_{\Phi(q)}(x)$ is equal to $1/\psi'(\Phi(q)) = \Phi'(q)$ by a Tauberian theorem applied to the Laplace-Stieltjes transform $\lambda/\psi_{\Phi(q)}(\lambda)$ of $W_{\Phi(q)}$.

The explicit form of the potential density given in Theorem 1 allows one to determine whether X is transient of recurrent. Let U^0 denote the potential measure of X, given by (1) with $q = 0$.

Definition. The process X is called *transient* if $U^0(K) < \infty$ for every compact set $K \subset \mathbb{R}$ and it is called *recurrent* if $U^0(B) = \infty$ for every open interval of the form $B = (-r, r)$, $r > 0$.

Corollary 1 *The process X is recurrent if*

$$\Phi'(0^+) := \lim_{q\downarrow 0} \frac{\Phi(q) - \Phi(0)}{q} = \infty.$$

Otherwise, $\Phi'(0^+) < \infty$ and X is transient, and the potential measure $U^0(\mathrm{d}x)$ of X is given by

$$U^0(\mathrm{d}x) = \left(\Phi'(0^+)\exp(-\Phi(0)x) - W(-x)I_{(x<0)}\right)\mathrm{d}x. \qquad (9)$$

Another consequence from Theorem 1 is the following result on the downward 'creeping' of X. The process X is said to creep across the level $x < 0$ if X first enters $(-\infty, x)$ continuously, that is if $X_{T_x^-} = x$. Recall that we excluded the case where X is a negative deterministic drift and denote by $\sigma^2 = 2\lim_{\lambda\to\infty}\lambda^{-2}\psi(\lambda)$ the Gaussian coefficient of X.

Corollary 2 *The process X creeps across $x < 0$ if and only if X has a nonzero Gaussian coefficient σ^2 and then*

$$P\left(X_{T_x^-} = x\right) = \frac{\sigma^2}{2}[W'(-x) - \Phi(0)W(-x)], \qquad x < 0. \qquad (10)$$

In the literature there exist already several proofs for the statements in Theorem 1 and Corollaries 1 and 2. The one-sided exit identities (4) – (5) were first studied by Zolotarev [17], although formulated in a different form. The existence and properties of the scale function were proved by Bingham [4] and Emery [8]. The well established identities (4) – (5) and (6) – (7) are related to the two-sided exit problem to which among others Takács [16], Rogers [14], Emery [8] and Bertoin [2] made significant contributions. In its current form it was first formulated by Bertoin [2, 3]. The given proofs rely on (complex-) analytic and combinatorial methods or invoke Itô-excursion theory applied to the excursions of X away from its supremum S. The identity (8) was first proved in [13] using a martingale argument. Recently, these identities received more attention in the literature and several short proofs were given. Kyprianou and Palmovski [12] gave proofs for the identities invoking the Kella-Whitt martingale [11] and Doney [7] used excursion theory to prove the identity (8).

Here we follow yet another approach which exploits the connection between potential analysis and Markov processes: We show that, essentially, potential theory allows us to give simple proofs of the above results. For a deeper analysis of the relationship between Markov processes on the one hand and potential analysis on the other hand, we refer the reader to the classical works by Blumenthal and Getoor [5] and Dellacherie and Meyer [6].

The rest of this note is organised as follows. In the next section, we state and prove a first hitting time identity for a certain class of continuous time Markov processes in terms of their potential density. In the third section, we then derive explicit expressions for the potential densities of X killed upon entering a negative half-line and of X reflected at its infimum and give the proofs of Theorem 1 and Corollaries 1 and 2.

2 Potential theory and first hitting

Denote by $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ a measurable space consisting of some interval $\mathcal{S}$ of the real line and its Borel sigma-algebra $\mathcal{B}(\mathcal{S})$ and fix $a \in \mathcal{S}$. Let Z be a continuous time strong Markov process with state space $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$ defined on a probability space $(\Omega, \mathcal{F}, P)$ that satisfies the usual conditions. In the sequel we restrict ourselves to processes Z which are *quasi-left continuous*, that is, if $(T_n, n \in \mathbb{N})$ is an increasing sequence of stopping times with $T = \lim_{n\to\infty} T_n$ almost surely, then $Z_T = \lim_{n\to\infty} Z_{T_n}$ almost surely on the event $\{T < \infty\}$. For $x \in \mathcal{S}$ we denote by $U^q(x, \cdot)$ the potential measure of Z

$$U^q(x, A) = \int_0^\infty \mathrm{e}^{-qt} P_x(Z_t \in A)\mathrm{d}t, \qquad A \in \mathcal{B}(\mathcal{S}),$$

where P_x denotes the measure P conditioned on $\{Z_0 = x\}$. Assume that for any $x \in \mathcal{S}$, $U^q(x, \cdot)$ restricted to some open interval containing a is absolutely continuous with respect to the Lebesgue measure with density $u^q(x, \cdot)$, say. Denote the first passage time of Z into the set A by

$$T'_A = \inf\{t \geq 0 : Z_t \in A\}, \qquad A \in \mathcal{B}(\mathcal{S}). \tag{11}$$

Below the Laplace transform of $T'_{\{a\}}$ is expressed in terms of known quantities. To formulate the result we define for any $\epsilon > 0$ the open sets $B_a(\epsilon) = (a - \epsilon, a + \epsilon)^2$ and $D_a(\epsilon) = \{(x, y) \in \mathbb{R}^2 : x < y, a - \epsilon < y < a + \epsilon\}$.

Proposition 1 *Let $x \in \mathcal{S}$ and $q \geq 0$.*

(i) If, for some $\epsilon_0 > 0$, u^q restricted to $B_a(\epsilon_0)$ is continuous, we have

$$E_x\left[\mathrm{e}^{-qT'_{\{a\}}} I_{(T'_{\{a\}}<\infty)}\right] = \frac{u^q(x, a)}{u^q(a, a)}, \tag{12}$$

provided $u^q(a, a) > 0$.

(ii) If Z has no positive jumps, for some $\epsilon_0 > 0$ u^q restricted to $D_a(\epsilon_0)$ is continuous and $u^q(a^-, a) = \lim_{\epsilon\downarrow 0} u^q(a - \epsilon, a) > 0$, the identity (12) holds for $x < a$ with $u^q(a, a)$ replaced by $u^q(a^-, a)$.

Proof. Write T'_ϵ as shorthand for $T'_{(a-\epsilon,a+\epsilon)}$. The strong Markov property of Z yields that for $\epsilon > 0$ $\frac{1}{\epsilon}U^q(x, (a - \epsilon, a + \epsilon))$ is equal to

$$\frac{1}{\epsilon}\int_{a-\epsilon}^{a+\epsilon} u^q(x, y)\mathrm{d}y = \int E_x\left[\mathrm{e}^{-qT'_\epsilon}; Z_{T'_\epsilon} \in \mathrm{d}z\right] \frac{1}{\epsilon}\int_{a-\epsilon}^{a+\epsilon} u^q(z, y)\mathrm{d}y. \tag{13}$$

If ϵ tends to zero, T'_ϵ increases to a stopping time, T say, with $T \leq T'_{\{a\}}$. By quasi-left continuity of Z we find that $Z_{T'_\epsilon}$ tends to $Z_T = a$ on $\{T < \infty\}$ almost surely and thus $T = T'_{\{a\}}$ on $\{T < \infty\}$. If we let ϵ tend to zero the measures $E_x[\mathrm{e}^{-qT'_\epsilon}; Z_{T'_\epsilon} \in \mathrm{d}z]$ vaguely converge to $E_x[\mathrm{e}^{-qT_{\{a\}}}]\delta_0$, where δ_0 denotes the unit mass in zero. Combined with the continuity of u^q in an open neighbourhood of (a, a) we end up with (12) if we let $\epsilon \downarrow 0$ in (13). In the second case, by the fact that Z has no positive jumps, (13) reduces to

$$\frac{1}{\epsilon}\int_{a-\epsilon}^{a+\epsilon} u^q(x, y)\mathrm{d}y = E_x[\mathrm{e}^{-qT'_\epsilon}]\frac{1}{\epsilon}\int_{a-\epsilon}^{a+\epsilon} u^q(a - \epsilon, y)\mathrm{d}y.$$

Letting again ϵ tend to zero, the assumed continuity of u^q leads to the required identity. □

Consider now the set $\{a, b\}$ for some $a, b \in \mathcal{S}$. A related question that arises then is: what is the probability that Z hits $\{a\}$ before $\{b\}$? To be more precise, can we find an expression for

$$t^q_{a,b}(x) = E_x\left[\mathrm{e}^{-qT_{\{a,b\}}} I_{(T_{\{a\}}<T_{\{b\}}, T_{\{a,b\}}<\infty)}\right], \qquad x \in [a, b],$$

in terms of known quantities? The answer in terms of the potential density u^q is given in the following result.

Corollary 3 *Let, for some $\epsilon_0 > 0$, u^q restricted to $B_a(\epsilon_0) \cup B_b(\epsilon_0)$ be continuous. Then we have for $q > 0$*

$$t^q_{a,b}(x) = \frac{u^q(x, a)u^q(b, b) - u^q(x, b)u^q(b, a)}{u^q(a, a)u^q(b, b) - u^q(a, b)u^q(b, a)}, \qquad x \in [a, b], \tag{14}$$

provided $u^q(a, a)u^q(b, b) > 0$. If $q = 0$, the identity (14) remains valid, where the right-hand side of (14) is to be understood in the limiting sense of $q \downarrow 0$ if $u^0(a, a)u^0(b, b) = u^0(a, b)u^0(b, a)$.

Proof. If $u^q(a, a) > 0$, the strong Markov property combined with Proposition 1 yields that for $q \geq 0$

$$u^q(x, a)/u^q(a, a) = t^q_{a,b}(x) + t^q_{b,a}(x)u^q(b, a)/u^q(a, a), \qquad x \in [a, b].$$

By interchanging the role of a and b, we can derive a similar second identity. For $q > 0$, this system of two equations is non-singular. Indeed, since $T_{\{a\}} > 0$ P_b-a.s., $E_b[\mathrm{e}^{-qT_{\{a\}}}] < 1$ and it follows from (12) that $u^q(b, a) < u^q(a, a)$. Interchanging a and b, we find that $u^q(a, b)u^q(b, a) < u^q(a, a)u^q(b, b)$. Solving this system finishes the proof for $q > 0$. Note that $t^q_{a,b}(x)$ increases to $t^0_{a,b}(x)$ if $q \downarrow 0$. Hence $t^0_{a,b}(x)$ is equal to the limit of $q \downarrow 0$ of the right-hand side of (14). If $u^0(a, a)u^0(b, b) \neq u^0(a, b)u^0(b, a)$ then this limit is given by (14) for $q = 0$ as the previously derived system is non-singular. □

Example. For a Brownian motion Z the potential density u^q is given by $u^q(x, a) = (2q)^{-1/2}\mathrm{e}^{-\sqrt{2q}|x-a|}$. By Corollary 3 and de l'Hôpital's rule we find back the well known identity

$$P_x(T_{\{a\}} < T_{\{b\}}) = (b - x)/(b - a), \qquad x \in [a, b].$$

3 Proofs of Theorem 1 and Corollaries 1 and 2

Let $\eta(q)$ denote an independent exponential random variable with parameter $q > 0$. We start recalling the following results which we will frequently use in the proof of Theorem 1:

Lemma 1 *Let X be a Lévy process.*

(i) For each fixed $t > 0$ $(S_t - X_t, S_t)$ has the same law as $(-I_t, X_t - I_t)$.

(ii) The processes $X - I$ and $S - X$ are strong Markov process.

(iii) For $q > 0$, $S_{\eta(q)} - X_{\eta(q)}$ is independent of $S_{\eta(q)}$.

Proof. (i) This result follows as consequence of the duality lemma (see e.g. Lemma II.2 and Proposition VI.3 in [2]).

(ii) This follows straightforwardly from the independence and stationarity of the increments of X. See e.g. [2, Prop. VI.1] for a proof.

(iii) The independence can for example be proved using Itô-excursion theory applied to the excursions of the Markov process $S - X$ away from zero, see Greenwood and Pitman [10]. □

Proof of Theorem 1(i) – (iii) We divide the proof in several steps.

1. (Absolute continuity of the potential measure U^q) By the strong Markov property of X and the spatial homogeneity we note that

$$P(S_{\eta(q)} > t + s) = P(S_{\eta(q)} > s)P(S_{\eta(q)} > t) \qquad \text{for all } t, s \geq 0, q > 0.$$

Hence we deduce that $S_{\eta(q)}$ is exponentially distributed with parameter $\lambda(q)$, say. Using then Lemma 1(iii), we get that

$$\begin{aligned} P(X_{\eta(q)} \in \mathrm{d}x) &= \int_{0\vee x}^{\infty} P(S_{\eta(q)} \in \mathrm{d}(z-x))P((S-X)_{\eta(q)} \in \mathrm{d}z) \\ &\leq \lambda(q)\mathrm{d}x. \end{aligned}$$

2. (Existence of the scale function $W^{(q)}$ for $q \geq 0$) Next we show, following Bingham [4], that, for $\theta > \Phi(q)$, the function $\theta/(\psi(\theta) - q)$ can be represented as a Laplace-Stieltjes transform. To be more precise, we prove that there exists a measure $\mathrm{d}W^{(q)}$ on $[0, \infty)$ such that

$$\int_0^\infty \theta \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = \int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(\mathrm{d}x) = \theta/(\psi(\theta) - q), \quad \theta > \Phi(q). \tag{15}$$

By the Lévy-Khintchine formula and by partial integration we have the following representation for ψ and $\theta \in [0, \infty)$

$$\begin{aligned} \psi(\theta) &= a\theta + \frac{\sigma^2}{2}\theta^2 + \int_0^\infty (\mathrm{e}^{-\theta x} - 1 + \theta x I_{(x<1)})\Lambda(\mathrm{d}x) \\ &= \theta\left(a' + \frac{\sigma^2}{2}\theta - \int_0^\infty (\mathrm{e}^{-\theta x} - I_{(x<1)})\Lambda((x,\infty))\mathrm{d}x\right), \end{aligned} \tag{16}$$

where a, σ are constants and $a' = a - \Lambda((1, \infty))$ with Λ a measure satisfying $\int_0^\infty (1 \wedge x^2)\Lambda(\mathrm{d}x) < \infty$. The measure Λ is related to the Lévy measure ν of X by $\nu(\mathrm{d}x) = \Lambda(-\mathrm{d}x)$. From the previous display, we see that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\psi(\theta) - q}{\theta} = \frac{\sigma^2}{2} + \int_0^\infty x\mathrm{e}^{-\theta x}\Lambda((x,\infty))\mathrm{d}x + \frac{q}{\theta^2}.$$

Hence $(\psi(\theta)-q)/\theta$ has derivatives that oscillate in sign and has thus a completely monotone derivative. Since also $1/\theta$ is completely monotone it follows (e.g. Feller [9, XIII.4, Criterion 2]) that $\theta/(\psi(\theta)-q)$ itself is completely monotone. By Feller [9, XII.4, Thm. 1a] a function on $[0,\infty)$ is completely monotone if and only if it can be represented as Laplace-Stieltjes transform of a measure. Partial integration yields then also the first identity of (15) and the claim is proved.

3. (Form of the potential density u^q) Denote by u^q a version of the density of U^q with respect to the Lebesgue measure. For $q>0$ the Fourier-transform of u^q, $\mathcal{F}u^q$, is given by

$$\mathcal{F}u^q(\xi)=\int \mathrm{e}^{\mathrm{i}x\xi}u^q(x)\mathrm{d}x=q^{-1}E[\mathrm{e}^{\mathrm{i}\xi X_{\eta(q)}}]=(q-\psi(i\xi))^{-1}.$$

Note that $\xi\mapsto\psi(\mathrm{i}\xi)$ is an analytic function in $\Im(\xi)<0$. By the independence from Lemma 1(iii) and the fact from part 1 above that $S_{\eta(q)}$ has an exponential distribution with mean $\lambda(q)^{-1}$, we see that for $\xi\in\mathbb{R}$

$$\begin{aligned} q(q-\psi(\mathrm{i}\xi))^{-1}&=E[\mathrm{e}^{i\xi X_{\eta(q)}}]=E[\mathrm{e}^{\mathrm{i}\xi S_{\eta(q)}}]E[\mathrm{e}^{-i\xi(S-X)_{\eta(q)}}]\\ &=\lambda(q)(\lambda(q)-\mathrm{i}\xi)^{-1}E[\mathrm{e}^{-i\xi(S-X)_{\eta(q)}}].\end{aligned}$$

Since $\xi\mapsto E[\mathrm{e}^{-i\xi(S-X)_{\eta(q)}}]$ can be analytically extended to $\Im(\xi)<0$, this identity remains valid for ξ in $\Im(\xi)<0$. In particular, we see that $(q-\psi(\mathrm{i}\xi))^{-1}$ is meromorphic in $\Im(\xi)<0$ with one pole in $\xi=-\mathrm{i}\lambda(q)$. Since $\psi(\lambda)=q$ has only one positive real root in $\lambda=\Phi(q)$, we deduce that $\lambda(q)=\Phi(q)$. The inversion formula for characteristic functions yields now that for $a>0$

$$U^q([0,a])=\lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{1-\mathrm{e}^{-\mathrm{i}\xi a}}{\mathrm{i}\xi}\frac{1}{q-\psi(\mathrm{i}\xi)}\mathrm{d}\xi=\frac{1-\mathrm{e}^{-\Phi(q)a}}{\Phi(q)\psi'(\Phi(q))},\tag{17}$$

where the second equality can be seen as follows. Let C_T be the (clockwise) contour in the complex plane that consists of the interval $[-T,T]$ on the real axis joined to the semi-circle R_T of radius T in the lower half of the complex plane and set $f(\xi)=\frac{1-\mathrm{e}^{-\mathrm{i}\xi a}}{2\pi\mathrm{i}\xi}(q-\psi(\mathrm{i}\xi))^{-1}$. Then by Cauchy's theorem,

$$\int_{C_T}f(t)\mathrm{d}t=2\pi\mathrm{i}\cdot\mathrm{Res}_{t=-\mathrm{i}\Phi(q)}f(t)=-2\pi\left(\frac{1-\mathrm{e}^{-\Phi(q)a}}{\Phi(q)}\right)(-\psi'(\Phi(q))^{-1}).$$

On the other hand, since we also have that

$$\int_{C_T}f(t)\mathrm{d}t=\int_{-T}^{T}f(t)\mathrm{d}t+\int_{R_T}f(t)\mathrm{d}t,$$

where, by Jordan's lemma, $\int_{R_T}f(t)\mathrm{d}t$ converges to zero as T tends to infinity, the result (17) follows.

Noting that $\psi'(\Phi(q))=\Phi'(q)^{-1}$ and differentiating (17) with respect to a we find that $u^q(a)=\Phi'(q)\exp(-\Phi(q)a)$ for $a>0$.

4. (An identity between Laplace transforms) Note that for $q, \lambda > 0$ with $q > \psi(\lambda)$ (or equivalently $\Phi(q) > \lambda$) one has that

$$\begin{aligned}(q-\psi(\lambda))^{-1} &= q^{-1}E[\mathrm{e}^{\lambda X_{\eta(q)}}] = \int_0^\infty \mathrm{e}^{\lambda x} u^q(x)\mathrm{d}x + \int_0^\infty \mathrm{e}^{-\lambda x} u^q(-x)\mathrm{d}x \\ &= \Phi'(q)/(\Phi(q)-\lambda) + \int_0^\infty \mathrm{e}^{-\lambda x} u^q(-x)\mathrm{d}x. \end{aligned} \tag{18}$$

By analytic continuation in λ, the identity (18) remains valid for $\Re(\lambda) > 0$ except for $\lambda = \Phi(q)$ and then by continuity for all λ with $\Re(\lambda) \geq 0$. Inverting the Laplace transforms in λ leads then to equation (3).

5. (Wiener-Hopf factorisation) Since a Lévy process is quasi left continuous (e.g. [2, Proposition I.7]) and satisfies the strong Markov property (e.g. [2, Proposition I.6]), we deduce from Proposition 1(ii) that

$$P(S_{\eta(q)} > x) = E[\mathrm{e}^{-qT_x^+} I_{(T_x^+<\infty)}] = u^q(x)/u^q(0^+) = \mathrm{e}^{-x\Phi(q)}. \tag{19}$$

Lemma 1(i),(iii) imply then that

$$\begin{aligned}E[\exp(\lambda I_{\eta(q)})] &= E[\mathrm{e}^{\lambda X_{\eta(q)}}]E[\mathrm{e}^{\lambda S_{\eta(q)}}]^{-1} \\ &= \frac{q}{q-\psi(\lambda)} \times \frac{\Phi(q)-\lambda}{\Phi(q)}.\end{aligned}$$

Using (15) to invert this transform we find that

$$P(-I_{\eta(q)} \in \mathrm{d}x) = \frac{q}{\Phi(q)} W^{(q)}(\mathrm{d}x) - qW^{(q)}(x)\mathrm{d}x, \qquad x \geq 0. \tag{20}$$

6. (The function $q \mapsto W^{(q)}(x)$ is analytic for $x \geq 0$) Following Bertoin [3], we invert the Laplace transform (term wise)

$$(\psi(\lambda)-q)^{-1} = \sum_{k\geq 0} q^k \psi(\lambda)^{-k-1}$$

to find the series expansion

$$W^{(q)}(x) = \sum_{k\geq 0} q^k W^{\star k+1}(x) \qquad x, q \geq 0, \tag{21}$$

where $W^{\star k}$ denotes the kth convolution power of W, $W^{\star k} = W\star\cdots\star W$. This series converges since

$$W^{\star k}(x) \leq W(x)^k x^k/k!, \qquad k \geq 1, x \geq 0$$

as W is increasing (recalling that $\mathrm{d}W = \mathrm{d}W^{(0)}$ is a nonnegative measure).

7. (Continuity of the function $x \mapsto W^{(q)}(x)$) The next step is to prove that $P(-I_{\eta(q)} \in \mathrm{d}x)$ has no atoms. Applying the strong Markov property we get that for $x, q > 0$

$$P[-I_{\eta(q)} = x] = E\left[\mathrm{e}^{-qT^-_{-x}} I_{(X_{T^-_{-x}} = -x, T^-_{-x} < \infty)}\right] P[I_{\eta(q)} = 0]. \tag{22}$$

If 0 is regular for $(-\infty, 0)$, the second factor of the right-hand side of (22) is zero, whereas if 0 is irregular for $(-\infty, 0)$, the paths of the infimum form step functions almost surely and the first factor of the right-hand side of (22) is zero. Combining with (20) we see that the measure $\mathrm{d}W^{(q)}$ has no atoms and thus $x \mapsto W^{(q)}(x)$ is continuous for $q > 0$. Since $W = W^{(0)}$ is increasing, a discontinuity of W at $a > 0$ would imply $\lim_{x \downarrow a} W(x) > \lim_{x \uparrow a} W(x)$. In view of the continuity of $x \mapsto W^{(q)}(x)$ for $q > 0$ on the one hand and the expansion (21) on the other hand, this would yield a contradiction. □

To prove the identities in Theorem 1(iv), we express the resolvents of the strong Markov processes $X^\dagger$ (X killed upon entering the negative half line $(-\infty, 0)$) and $Y = X - I$ (Lemma 1(ii)) in terms of the scale functions $W^{(q)}, Z^{(q)}$ and then invoke Proposition 1(ii).

Lemma 2 *For $x, y > 0$, we have*

$$\begin{aligned} P_x(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < T_0^-)/\mathrm{d}y & \\ = q\mathrm{e}^{-\Phi(q)y} W^{(q)}(x) - 1_{\{x>y\}} q W^{(q)}(x-y). & \quad (23) \\ P_x(Y_{\eta(q)} \in \mathrm{d}y)/\mathrm{d}y = \Phi(q)\mathrm{e}^{-\Phi(q)y} Z^{(q)}(x) - 1_{\{x>y\}} q W^{(q)}(x-y). & \quad (24) \end{aligned}$$

Proof. A proof of the first identity can be found e.g. in Bertoin [3] or Suprun [15] and of the second identity in [13]. In order to be self-contained we provide the proofs here.

(i) Invoking the identities (19) and (20) and the independence and duality (Lemma 1(iii,i)) and noting that $\eta(q) < T^-_{-x}$ iff $I_{\eta(q)} > -x$ we find that $q^{-1} P_x(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < T_0^-)$ is equal to

$$\begin{aligned} & q^{-1} \int_0^x P(-I_{\eta(q)} \in \mathrm{d}z) P((X-I)_{\eta(q)} \in \mathrm{d}(y - x + z)) \\ & = \int_{(x-y)\vee 0}^x \mathrm{e}^{-(y-x+z)\Phi(q)} W^{(q)}(\mathrm{d}x) - \int_{(x-y)\vee 0}^x \Phi(q) \mathrm{e}^{-(y-x+z)\Phi(q)} W^{(q)}(x) \mathrm{d}x. \end{aligned}$$

The identity (23) follows now by performing a partial integration on the first integral in the second line of above display.

(ii) The strong Markov property of Y at the stopping time $\tau_0 = \inf\{t \geq 0 : Y_t = 0\}$ implies that

$$\begin{aligned} P_x(Y_{\eta(q)} \in \mathrm{d}y) &= P_x(Y_{\eta(q)} \in \mathrm{d}y, \eta(q) < \tau_0) + E_x[\mathrm{e}^{-q\tau_0}] P_0(Y_{\eta(q)} \in \mathrm{d}y) \quad (25) \\ &= P_x(X_{\eta(q)} \in \mathrm{d}y, \eta(q) < T_0^-) + E_x[\mathrm{e}^{-qT_0^-}] P_0(Y_{\eta(q)} \in \mathrm{d}y), \end{aligned}$$

where in the second line we used that $(Y_t, t \leq \tau_0)$ has the same law as $(X_t, t \leq T_0^-)$. By integrating (20) we find the Laplace transform of T_0^- to be equal to

$$P(I_{\eta(q)} < -x) = E_x[\mathrm{e}^{-qT_0^-}] = Z^{(q)}(x) - q\Phi(q)^{-1} W^{(q)}(x), \qquad x > 0. \tag{26}$$

Substituting (26) and (23) into (25) and recalling that $Y_{\eta(q)}$ has an exponential distribution with mean $\Phi(q)^{-1}$, we end up with the required identity (24). □

Now we can finish the proof of Theorem 1.

Proof of Theorem 1(iv). Since a Lévy process is a quasi left continuous strong Markov process and $W^{(q)}$ is continuous (proved above in part 6), it follows by combining with Lemma 2 that the conditions of Proposition 1 are met for the Markov processes $X^\dagger$ and Y (Lemma 1(ii)). Taking for u^q in (12) the resolvents (23), (24) we find (6) and (8) respectively. Finally, the strong Markov property yields that

$$E_x[\mathrm{e}^{-qT_0^-}] = E_x\left[\mathrm{e}^{-qT_0^-} I_{(T_0^- < T_a^+)}\right] + E_x\left[\mathrm{e}^{-qT_a^-} I_{(T_0^- > T_a^+)}\right] E_a[\mathrm{e}^{-qT_0^-}]. \quad (27)$$

Inserting (26) and (6) in (27) completes the proof. □

Proof of Corollary 1. Since ψ is differentiable, convex and increasing on $(0,\infty)$, it follows that the right-derivative of ψ in $\Phi(0)$ is finite and non-negative and equal to $(\Phi'(0^+))^{-1}$. Thus $\lim_{q\downarrow 0}(\Phi(q)-\Phi(0))/q$ is positive and finite or equal to $+\infty$. Suppose first that the latter is the case. By the identity (2) it then follows, taking $q \downarrow 0$, that $U^0((-r,r))$ is infinite for $r>0$.

In the case $\Phi'(0^+) < \infty$, we show that the identity (9) holds true. The fact that $\psi(\cdot)$ is C^1 on $(0,\infty)$ in conjunction with the implicit function theorem applied to $\psi(\lambda) = q$ implies that Φ and Φ' are continuous. Combining with the continuity of $q \mapsto W^{(q)}$ we find that, for any compact set K, $U^0(K) = \lim_{q\downarrow 0}\int_K u^q(x)\mathrm{d}x$ by dominated convergence. On the other hand, monotone convergence implies that $U^0(K) = \lim_{q\downarrow 0} U^q(K)$. Using then the explicit formulas (2) and (3), equation (9) follows and the proof is finished. □

Proof of Corollary 2. Let $q>0$ and let $T'_{\{x\}}$ be as in (11). If X creeps across $x<0$, this implies that $T'_{\{x\}}$ is smaller than T_b^- for all $b<x$ by right-continuity of its sample paths. On the other hand, if $X_{T_x^-} < x$, X enters $(-\infty,x)$ by a jump and it follows that there exists an $\epsilon_0>0$ such that $T'_{\{x\}} > T_b^-$ for all $b\in(x-\epsilon_0,x)$. Thus $\{T'_{\{x\}} < T_b^-\}$ increases to $\{X_{T_x^-}=x\}$ as $b\uparrow x$ and we have

$$E\left[\mathrm{e}^{-qT'_{\{x\}}} I_{\left(X_{T_x^-}=x\right)}\right] = \lim_{b\uparrow x} E\left[\mathrm{e}^{-qT'_{\{x\}}} I_{\left(T'_{\{x\}}<T_b^-\right)}\right]. \quad (28)$$

Invoking Proposition 1 applied to the Markov process $\tilde{X}^\dagger$ (X killed upon entering $(-\infty,b)$) in conjunction with Lemma 2 we see that the right-hand side of (28) is equal to

$$\lim_{b\uparrow x}\frac{W^{(q)}(-b)-\mathrm{e}^{-\Phi(q)(x-b)}W^{(q)}(-x)}{W^{(q)}(x-b)} = \tfrac{\sigma^2}{2}(W_+^{(q)\prime}(-x)-\Phi(q)W^{(q)}(-x)), \quad (29)$$

where $W_+^{(q)\prime}$ is the right-derivative of $W^{(q)}$ (see e.g. Lemma 1 in [13] for a proof of the right-differentiability of $W^{(q)}(\cdot)$) and we used that $z/W^{(q)}(z)$ converges to $\sigma^2/2$ for $z \downarrow 0$ (e.g. Lemma 4 in [13]). Letting now $q \downarrow 0$ in (29) and using the differentiability of $W^{(q)}$ if $\sigma > 0$ (e.g. Lemma 1 in [13]) and continuity of the maps $q \mapsto W^{(q)}(x), W_+^{(q)\prime}(x)$, we end up with (10) and the proof is complete. □

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