

No-Arbitrage Pricing of Weather Derivatives in the Presence of a Liquid Swap Market

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Abstract

We consider the problem of pricing weather derivatives based on linear temperature indices. Anticipating the development of a liquid weather swap market, we address the issue of pricing weather derivative options using weather swaps as hedging instruments. Our analysis starts by considering stochastic dynamics that are appropriate for the modelling of actively traded swaps, which is an issue that involves weather forecast considerations. It then proceeds to an adaptation of the continuous-time mathematical finance theory that ensures the absence of arbitrage opportunities within the weather derivative market, and provides formulae for the no-arbitrage prices of weather derivative options. The results derived include the modifications of the Black and Scholes and Black formulae that are appropriate for weather derivative pricing.

1 Introduction

Weather derivatives are financial contracts with a payoff determined by measurements of the weather. Their economic purpose is to transfer weather risk from one counterparty of the contract to the other: by trading weather derivatives, an entity can insure themselves against weather conditions that might affect them adversely. Weather derivatives can be written on any weather variable. However, the vast majority of currently traded contracts are based on temperature, and we consider only temperature based contracts in this article. For such contracts, measurements of the temperature are aggregated into an index over a specified period of time known as the *contract period*. For instance, one class of commonly traded contracts considers the daily average temperature as the weather variable, and defines the

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contract index to be the sum of daily average temperatures between May and September. This index is linear in terms of daily average temperatures. Another commonly traded index is based on the sum over the contract period of the number of *heating degree days*, which are defined on a daily basis as $(18 - T)^+$, where T is the daily average temperature measured in °C. Heating degree days are *effectively* linear in daily average temperature, and equal to $18 - T$, when there is only a negligible probability of the temperature actually reaching 18°C. For most of the commonly traded contracts that are based on heating degree days this is a reasonable approximation. We therefore consider only indices that are linear in daily average temperature in our study here.

The question of how to price weather derivatives has attracted considerable interest in recent years, and has been considered by several authors. The interested reader can consult the book by Jewson and Brix [JB05] for a comprehensive exposition of all the meteorological, statistical, financial and mathematical issues that arise in the pricing and risk management of weather derivatives. At present, a number of rather different methods has been suggested. For example, actuarial methods based on stochastic models for the daily average temperature have been proposed by Benth [B03], Benth and Šaltytė-Benth [BS05], Brody, Syroka and Zervos [BSZ02], Caballero, Jewson and Brix [CJB02], Dischel [D98], Dornier and Queruel [DQ00], Jewson and Caballero [JC03], and Torro, Meneu and Valor [TMV01], while actuarial methods based on direct modelling of the distribution of the settlement index have been discussed by Brix, Jewson and Ziehmman [BJZ02] and Jewson and Brix [JB00]. Furthermore, adaptations of a number of standard incomplete financial markets methodologies have been proposed by Davis [D01] who considers hedging with gas contracts, Geman [G99] who considers hedging with electricity contracts, and Jewson [J02] who considers hedging with other weather contracts. The purpose of this article is to develop the arbitrage pricing ideas proposed by Jewson [J02], and Jewson and Zervos [JZ03] by extending them to a more general case, by consolidating the underlying mathematical theory, and by deriving a range of pricing results, including ones of an explicit nature.

At present, the weather derivatives market is still at a relatively early stage of development, and provides a prime example of an incomplete market. The current lack of liquidity indicates that the pricing of *any* weather derivative has to be addressed by means of actuarial methods or incomplete market techniques such as the ones discussed briefly above. However, the weather derivatives market is becoming increasingly liquid, and, in view of the important role that weather derivatives could play in modern economies, we can expect that the market will become more liquid in the near future. At present, certain contracts are much more actively traded than others. The most liquid ones are the linear swaps on the Chicago Mercantile Exchange that are traded several times every day. On the other hand, option contracts are traded less frequently. Assuming that this trend continues, we are faced with the problem of pricing a weather derivative option in the presence of a liquid weather swap market. In this scenario, the issue of pricing a weather swap becomes irrelevant because the market provides the swap's price. This market price takes into account weather forecasts as well as other issues such as imbalances in supply and demand. The

problem of pricing a weather derivative option on the index of actively traded swaps can then be considered within a complete market context. As a result of such a development, the various methods for weather option pricing, such as the ones mentioned above, become less important. However, they still provide a range of valuable techniques within a quantitative analyst's toolbox.

The objective of this paper is to develop theory that can be used to price European style weather derivatives in the presence of a market of actively traded swaps with the same underlying weather index. The paper is organised as follows. In Section 2, we describe the structure of weather swaps. We then derive stochastic dynamics for the swap price under the assumption that the swap market is a liquid one, and by applying the so-called efficient forecast hypothesis. Section 3 is concerned with deriving stochastic dynamics for portfolios composed of swap contracts. Sufficient conditions for the absence of arbitrage are presented in Section 4. Finally, a range of results concerning the pricing of weather derivative options in the presence of a liquid swap market is developed in Section 5. These results include the analogues of the Black and Scholes and Black formulae that are appropriate for the weather derivative markets.

2 Weather swaps based on temperature indices

A *weather swap* is a contract that is exchanged for no premium: the only associated cash flow is the payoff that is exchanged at the end of the contract. In practice, this is not always true because some exchange traded contracts may involve daily margin or performance bond payments. However, we do not consider such possibilities here. Two counterparties can enter into a weather swap agreement either before or during the contract period. The market strike level of weather swaps, which is defined below, fluctuates in time, in response to the arrival of new information about the likely weather during the contract period and in response to changes in supply and demand in the market. From about 15 days before the start of the contract period weather forecasts start to give a useful indication of the likely weather during some part of the contract period, and weather forecasts that suggest deviations from the expected weather may cause changes in market prices. From the start of the contract period, actual measurements of the weather during the contract period start becoming available, and these have a direct effect on prices. To simplify the exposition of our results, we restrict our attention to *linear* weather swaps and to fluctuations in the swap price during the contract period itself.

Now, to fix ideas, we consider swap contracts the *settlement index* I of which is the sum of daily average temperatures over the period $[0, T]$, where T is a constant. Trading one swap at time $t \in [0, T]$ is characterised by its *strike* S_t , and involves the exchange of a contract that presents the holder with a payoff equal to $I - S_t$ at time T . For values of the index I above the strike S_t , the buyer of the swap pays the seller, and vice versa. In practice, the "price" of the swap is quoted in terms of the level of the strike, which has the same units as

the settlement index.

Given the structure of a swap contract such as the one described above, it is reasonable to assume that

$$S_T \equiv \lim_{t \rightarrow T} S_t = I, \quad (1)$$

with probability one. Indeed, such a condition is reminiscent of the standard assumption in the interest rate theory that the price of the T -maturity discount bond converges to one unit of currency as time t converges to T . With respect to (1), it follows that one swap contracted at time $t \in [0, T]$ presents its holder with a payoff equal to $S_T - S_t$ at time T .

Thus, we are faced with the problem of determining an appropriate model for the swap strike process S in the context of the assumption that the swap market considered is liquid. To this end, we start by assuming that the swaps are traded in a market in which everybody uses swaps to hedge weather risk, and there are equal volumes of hedgers on either side, creating equal levels of supply and demand for the swap contract under consideration. In this situation there is no reason for the swap strike to be either above or below the expectation of the settlement index, which gives rise to the relationship

$$S_t = E[I | \mathcal{F}_t] \stackrel{(1)}{=} E[S_T | \mathcal{F}_t], \quad t \in [0, T]. \quad (2)$$

Here, expectations are computed with respect to the natural, or “market” probability measure, and \mathcal{F}_t denotes all information available to the market participants at time t , including measurements of temperature up to time t , and weather forecasts available at time t . We call the value value of S_t given by (2) the *fair strike* of the swap contract at time t .

To understand better the martingale character of the fair strike given by (2), we observe the following. The fair strike changes during the period of the weather swap contract because of the arrival of new weather forecasts and the gradual replacement of forecasts with observed temperatures. For brevity we refer to both of these effects as forecast changes. If the changes in weather forecasts were predictable, then they would have been incorporated into previous weather forecasts. We can therefore assume that they are unpredictable, so that their evolution can be modelled by a martingale. This conclusion has been called *the efficient forecast hypothesis*, because it is analogous to the efficient market hypothesis of economics that is used to justify the randomness of movements in share prices.

To proceed further, we address the issue of determining the dynamics of the martingale defined by (2). Temperature and temperature forecasts are normally distributed to a good approximation, and so, we can assume that forecast changes are normally distributed as well. Furthermore, since the indices that we consider here are linear, the fair strike, which we have defined as the expected index, is a linear combination of expected (i.e., forecast) temperatures. Since changes in forecast temperatures are independent and normally distributed, changes in the index are also normal random variables. This leads us to postulate that the fair strike satisfies

$$dS_t = \sigma_t dW_t. \quad (3)$$

Here, the “volatility” process σ is time-dependent. Indeed, at times long before the start of a weather contract there are no weather forecasts that are relevant to the contract period. Estimates of the fair strike made at such times are based solely on historical meteorological data, which do not change from day to day. The volatility should therefore be zero at such times. Nearer to the start of the weather contract, the volatility ramps up rapidly as more and more days of forecasts become relevant to the contract period. Finally, near the end of the contract period, as the number of remaining days reduces, the volatility reduces down to zero (recall that the index is the sum of daily average temperatures over several months). However, it is a reasonable approximation to assume that σ is deterministic. Further discussion of the derivation of (3), along with estimates for the size of σ for a real case, are given in Jewson [J03].

One significant extension of the model developed above arises if we relax the assumption that the swap market is balanced in supply and demand. The assumption that there is an asymmetry in the levels of supply and demand results in moving the market swap strike away from the swap’s fair strike, which leads to the introduction of a drift term in (3), so that the actual swap strike should be modelled by the stochastic equation

$$dS_t = \mu_t dt + \sigma_t dW, \quad (4)$$

with σ being as in (3), and for an appropriate choice of μ . In practice, we expect that the process μ should be stochastic because variations in supply and demand can arise for entirely unpredictable reasons. Otherwise, the rationale underlying (1) remains valid.

3 Portfolios in the swap market

We build the model of the market that we are going to study on a probability space (Ω, \mathcal{F}, P) supporting a standard, one-dimensional Brownian motion W . We denote by (\mathcal{F}_t) the usual augmentation of the natural filtration of W by the P -negligible sets in \mathcal{F} .

We consider swap contracts that are written on the sum of daily average temperatures over the swap’s period $[0, T]$, where $T > 0$ is a given constant. With regard to the arguments developed in the previous section, we model the swap’s market *strike* by

$$dS_t = \mu_t dt + \sigma_t dW_t, \quad S_0 = s \in \mathbb{R}. \quad (5)$$

The following assumption ensures that the process S is well-defined.

Assumption 1 The processes μ and σ are (\mathcal{F}_t) -progressively measurable, and

$$\int_0^T [|\mu_u| + \sigma_u^2] du < \infty, \quad P\text{-a.s.}$$

Now, consider an agent that enters swap contracts at discrete times $0 = t_0 < t_1 < \dots < t_n < T$. If the agent enters N_{t_j} contracts at time t_j , then the agent receives at time T a total amount of

$$\begin{aligned} X_T &:= \sum_{j=0}^n N_{t_j} (S_T - S_{t_j}) \\ &= \sum_{j=0}^n \left(\sum_{k=0}^j N_{t_k} \right) (S_{t_{j+1}} - S_{t_j}), \end{aligned}$$

where, we have defined $t_{n+1} := T$. If we define Π_{t_j} to be the total number of contracts that the agent has entered by time t_j , i.e., $\Pi_{t_j} := \sum_{k=0}^j N_{t_k}$, then the agent receives at time T the total payoff

$$X_T = \sum_{j=0}^n \Pi_{t_j} (S_{t_{j+1}} - S_{t_j}).$$

We now make the standard assumption of continuous time finance that the agent can enter swap contracts continuously. With obvious modifications of notation, we define the stochastic process X by

$$\begin{aligned} X_t &:= \int_0^t \Pi_u dS_u \\ &= \int_0^t \mu_u \Pi_u du + \int_0^t \sigma_u \Pi_u dW_u, \quad t \in [0, T], \end{aligned} \tag{6}$$

so that the agent faces a cash exchange of X_T at time T .

The conditions imposed on the trading strategies by the following definition ensure that the associated process X is well defined, and will be part of the assumptions that will ensure absence of arbitrage in the swap market.

Definition 1 A trading strategy Π is *admissible* if it is an (\mathcal{F}_t) -progressively measurable process satisfying

$$\int_0^T [|\mu_u \Pi_u| + \sigma_u^2 \Pi_u^2] du < \infty, \quad P\text{-a.s.},$$

and there exists a constant $K \geq 0$ such that the process X defined as in (6) satisfies $X_t \geq -K$, for all $t \in [0, T]$, P -a.s..

4 Conditions for absence of arbitrage

We now establish conditions on the model of the swap market formulated in the previous sections that ensure that there are no arbitrage opportunities. Specifically, we want to rule out the possibility that there exists a trading strategy that can result in a risk-free profit.

Definition 2 A trading strategy Π presents arbitrage opportunities if, at time T , it results to a payoff $X_T = X_T(\Pi)$ such that $X_T \geq 0$, P -a.s., and, for some event $A \in \mathcal{F}_T$ with $P(A) > 0$, $X_T(\omega) > 0$, for all $\omega \in A$.

The following assumption provides a sufficient condition for no admissible trading strategy to present arbitrage opportunities.

Assumption 2 There exists a process λ such that

$$\mu_t = \sigma_t \lambda_t, \quad t \in [0, T], \quad (7)$$

and $\int_0^T \lambda_u^2 du < \infty$, P -a.s.. Also, the process L defined by

$$L_t = \exp \left(-\frac{1}{2} \int_0^t \lambda_u^2 du - \int_0^t \lambda_u dW_u \right), \quad t \in [0, T], \quad (8)$$

is a martingale.

Proposition 1 *Suppose that Assumptions 1 and 2 hold. Then the market model considered allows for no arbitrage opportunities.*

Proof. We argue by contradiction. To this end, suppose that there exists an admissible trading strategy Π that yields a payoff $X_T = X_T(\Pi)$ such that

$$\begin{aligned} X_T &\geq 0, \text{ } P\text{-a.s., and} \\ \text{for some event } A &\in \mathcal{F}_T \text{ with } P(A) > 0, X_T(\omega) > 0, \text{ for all } \omega \in A. \end{aligned} \quad (9)$$

The assumption that the process L defined as in (8) is a martingale implies that we can define a new probability measure P^λ on (Ω, \mathcal{F}_T) such that $dP^\lambda/dP = L_T$, and, with regard to Girsanov's theorem, the process W^λ defined by

$$W_t^\lambda = W_t + \int_0^t \lambda_u du, \quad t \in [0, T], \quad (10)$$

is an $(\mathcal{F}_t, P^\lambda)$ -Brownian motion. With regard to (6),

$$X_t = \int_0^t \sigma_u \Pi_u dW_u^\lambda, \quad t \in [0, T].$$

It follows that X is an $(\mathcal{F}_t, P^\lambda)$ -local martingale. Since Π is admissible, the process X is bounded from below by a constant, and therefore, it is an $(\mathcal{F}_t, P^\lambda)$ -supermartingale. We conclude that $0 \geq E[X_T]$, which contradicts (9), and the proof is complete. \square

5 Derivatives pricing and hedging

We now consider the issue of derivative pricing.

Definition 3 A *contingent claim* is any contract that requires a premium and yields a payoff Z_T at time T , where Z_T is any given \mathcal{F}_T -measurable random variable such that $Z_T \geq 0$, P -a.s..

For instance, a contingent claim can yield a payoff $Z_T = f(S_T)$, where f is a given deterministic function, e.g., $f(s) = (s - K)^+$. Recall from Section 2 that S_T identifies with the value I of the swap's underlying index. Therefore, such a contingent claim corresponds to a European style weather derivative that is written on the swap's index. The payoff Z_T can also be path-dependent, i.e., it can depend on the entire sample path of the process S . Since S_t , for $t < T$, is a price and not a weather index, such a payoff structure cannot be identified with a weather derivative any more. However, it can be identified with a derivative in the swap market, which presents an interesting possible development.

To proceed further, we assume that agents in the market have access to a money market account that pays a rate of interest $r \geq 0$, assumed for simplicity to be constant (however, see Remark 3 below). Also, without loss of generality, we assume that a contingent claim can be traded for an appropriate premium at any time $t \in [0, T]$. The pricing idea considers the following two scenarios:

Scenario I An agent pays a premium \tilde{Z}_t at time $t < T$ to acquire a given contingent claim that pays Z_T at time T .

Scenario II At time $t < T$, an agent deposits an amount equal to Z_t in the money market account, and then starts managing a trading strategy Π that yields a payoff $X_T = X_T(\Pi)$ as in (6) at time T .

Now, given a time $t \in [0, T)$ suppose that we can find an \mathcal{F}_t -measurable random variable Z_t and a trading strategy Π^* such that

$$\Pi_u^* = 0, \quad \text{for } u < t, \quad \text{and} \quad Z_T = e^{r(T-t)} Z_t + X_T(\Pi^*), \quad P\text{-a.s.} \quad (11)$$

Then, since the investment strategies associated with the two scenarios yield identical payoffs at time T , they should require identical endowments at time t , i.e., we should have $\tilde{Z}_t = Z_t$. Indeed, if this is not the case, then it is totally straightforward to construct an investment strategy that yields arbitrage. We are thus faced with the following definition.

Definition 4 Fix any a contingent claim with a payoff Z_T at time T . Given a time $t \in [0, T)$, if there exists a \mathcal{F}_t -measurable random variable Z_t and an admissible trading strategy Π^* such that (11) is true, then Z_t is the *no-arbitrage* price of the claim at time t , and we call Π^* *hedging* strategy.

To derive expressions for the no-arbitrage price of a contingent claim, we need the following additional assumption.

Assumption 3 The process σ satisfies $\int_0^T \sigma_t^{-2} dt < \infty$, P -a.s..

Proposition 2 Fix any contingent claim that yields a payoff Z_T at time T , and let L be the martingale defined by (8). Suppose that all of the Assumptions 1, 2 and 3 hold, and $E[L_T Z_T] < \infty$. The no-arbitrage price of the claim is then given by

$$Z_t = e^{-r(T-t)} L_t^{-1} E[L_T Z_T | \mathcal{F}_t], \quad \text{for all } t \in [0, T]. \quad (12)$$

Proof. Consider the martingale M defined by

$$M_t = E[L_T Z_T | \mathcal{F}_t], \quad t \in [0, T]. \quad (13)$$

With reference to the martingale representation theorem, there exists a process K such that $\int_0^T K_u^2 du < \infty$, P -a.s., and

$$M_t = E[L_T Z_T] + \int_0^t K_u dW_u, \quad t \in [0, T]. \quad (14)$$

Using Itô's formula, we can check that

$$dL_t^{-1} = \lambda_t^2 L_t^{-1} dt + \lambda_t L_t^{-1} dW_t,$$

which, combined with (14) and the integration by parts formula, implies

$$L_t^{-1} M_t = E[L_T Z_T] + \int_0^t L_u^{-1} [M_u \lambda_u + K_u] \lambda_u du + \int_0^t L_u^{-1} [M_u \lambda_u + K_u] dW_u.$$

However, this equation and the observation $L_T^{-1} M_T = Z_T$, which follows from (13), imply

$$Z_T = L_t^{-1} E[L_T Z_T | \mathcal{F}_t] + \int_t^T L_u^{-1} [M_u \lambda_u + K_u] \lambda_u du + \int_t^T L_u^{-1} [M_u \lambda_u + K_u] dW_u. \quad (15)$$

Now, consider the trading strategy Π^* defined by

$$\Pi_u^* = L_u^{-1} [M_u \lambda_u + K_u] \sigma_u^{-1} \mathbf{1}_{\{t \leq u \leq T\}}, \quad (16)$$

and note that our assumptions imply that it is admissible. With regard to (6) and (7), this strategy is associated with the payoff

$$X_T(\Pi^*) = \int_t^T L_u^{-1} [M_u \lambda_u + K_u] \lambda_u du + \int_t^T L_u^{-1} [M_u \lambda_u + K_u] dW_u. \quad (17)$$

From (15) and (17), it follows that

$$Z_T = L_t^{-1} E [L_T Z_T | \mathcal{F}_t] + X_T(\Pi^*).$$

With reference to (11) we can conclude that the no-arbitrage price of the claim is given by (12), and the proof is complete. \square

Now, consider the change of probability measure defined in the proof of Proposition 1. It is straightforward to check that the dynamics of the swap price process S given by (5) satisfy

$$dS_t = \sigma_t dW_t^\lambda, \quad t \in [0, T], \quad (18)$$

where W^λ is the $(\mathcal{F}_t, P^\lambda)$ -Brownian motion defined by (10), and the no-arbitrage price of a contingent claim as in Proposition 2 is given by

$$Z_t = e^{-r(T-t)} E^\lambda [Z_T | \mathcal{F}_t], \quad t \in [0, T]. \quad (19)$$

In this formula, E^λ denotes expectation computed by means of the probability measure P^λ .

Remark 1 Note that P^λ corresponds to the so-called risk neutral probability measure of mathematical finance in the context of our problem. However, it is worth noting that the process λ is not the so-called *market price of risk* that appears in the standard theory of financial markets. With regard to its definition in (7) and the analysis of Section 2, it provides a measure of the changes in the balance of the swap's supply and demand, which are modelled by μ , relative to changes in the expected weather forecast, which are modelled by σ .

To refine the results of Proposition 2, we need to impose some additional structure on our model. To this end, suppose that the swap price process S is given by

$$dS_t = \mu_t dt + \sigma(t, S_t) dW_t, \quad t \in [0, T], \quad (20)$$

for some stochastic process μ and some deterministic function $\sigma : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, consistent with the requirements of Assumption 1. Also, suppose that the contingent claim's payoff is given by $Z_T = f(S_T)$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a given deterministic function. With regard to (18), (19), and the Feynman-Kac formula, we expect that the no-arbitrage price of the contingent claim at time t identifies with $e^{-r(T-t)} v(t, S_t)$, where v is the solution of the PDE

$$v_t(t, s) + \frac{1}{2} \sigma^2(t, s) v_{ss}(t, s) = 0, \quad (t, s) \in [0, T] \times \mathbb{R}, \quad (21)$$

with the boundary condition

$$v(T, s) = f(s), \quad s \in \mathbb{R}. \quad (22)$$

In view of this observation, the main contribution of the following result is the expression for the hedging strategy Π^* .

Proposition 3 *Suppose that the swap price satisfies (20), and that (21)–(22) has a $C^{1,2}$ solution. Then, the no-arbitrage price at time $t \in [0, T]$ of the contingent claim that yields a payoff equal to $f(S_T)$ at time T is given by*

$$Z_t = e^{-r(T-t)}v(t, S_t), \quad (23)$$

and the associated hedging strategy is given by

$$\Pi_u^* = v_s(u, S_u)\mathbf{1}_{\{t \leq u \leq T\}}. \quad (24)$$

Proof. Using Itô's formula, we calculate

$$\begin{aligned} v(T, S_T) &= v(t, S_t) + \int_t^T \left[v_t(u, S_u) + \frac{1}{2}\sigma^2(u, S_u)v_{ss}(u, S_u) + \mu_u v_s(u, S_u) \right] du \\ &\quad + \int_t^T \sigma(u, S_u)v_s(u, S_u) dW_u. \end{aligned} \quad (25)$$

Since v satisfies (21)–(22), it follows that

$$f(S_T) = v(t, S_t) + \int_t^T \mu_u v_s(u, S_u) du + \int_t^T \sigma(u, S_u)v_s(u, S_u) dW_u. \quad (26)$$

Also, with regard to (6), the trading strategy Π^* defined by (24) is associated with

$$X_T(\Pi^*) = \int_t^T \mu_u v_s(u, S_u) du + \int_t^T \sigma(u, S_u)v_s(u, S_u) dW_u. \quad (27)$$

Now, a straightforward comparison of (26) and (27) shows that $f(S_T) = v(t, S_t) + X_T(\Pi^*)$, which, with regard to the discussion associated with (11), establishes the result. \square

With a view to a further refinement of our results, suppose that the volatility structure of the swap price is deterministic. In particular, suppose that

$$\sigma : [0, T] \rightarrow (0, \infty) \text{ is a deterministic function such that } \Sigma_{0T} < \infty, \quad (28)$$

where $\Sigma_{tT} := \int_t^T \sigma_u^2 du$, for $t \in [0, T]$. In the light of Proposition 5, it is straightforward to establish the following result.

Proposition 4 *Suppose that the volatility of the swap price satisfies (28). The no-arbitrage price at time $t \in [0, T]$ of the contingent claim that yields $f(S_T)$ at time T is given by*

$$Z_t = e^{-r(T-t)} \frac{1}{\sqrt{2\pi\Sigma_{tT}}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{(y - S_t)^2}{2\Sigma_{tT}}\right) dy, \quad (29)$$

and the associated hedging strategy is given by

$$\Pi_u^* = \frac{1}{\sqrt{2\pi\Sigma_{uT}}} \int_{-\infty}^{\infty} (y - S_u) f(y) \exp\left(-\frac{(y - S_u)^2}{2\Sigma_{uT}}\right) dy, \quad u \in [t, T]. \quad (30)$$

Note that if the contingent claim considered has the structure of a European call option, i.e., if $f(s) = (s - K)^+$ for some constant K , and if the swap volatility σ is assumed to be constant, then (29) presents the analogue of the Black and Scholes and Black formulae for the pricing of weather derivative call options.

Remark 2 Note that the formulae for weather derivative option prices and the associating hedging portfolios provided by Proposition 4 do not involve the processes μ or λ . This implies that considerations relative to changes of the underlying swap's balance of supply and demand are of no relevance in the context of pricing and hedging. Indeed, the volatility structure σ and the market swap strike process S provide a sufficient statistic for the problem of pricing weather derivative options with structures satisfying the assumptions of Proposition 4. This is a property that parallels the fact that investor risk preferences, which are reflected by a choice of a market price of risk, do not influence option prices in the Black and Scholes theory of financial markets.

Remark 3 It is worth noting that the money market account considered at the beginning of this section appears only in the arguments related to (11). Moreover, as long as our analysis is concerned, the balance of this account changes at the time when a contingent claim is written and at the claim's maturity date T only. Therefore, throughout our analysis, we could have assumed stochastic interest rates defined on an appropriate extension of our underlying probability space (plainly, we can assume that interest rate processes are independent of weather forecast processes). In particular, the pricing formulas given by (12), (23) and (29) would all be valid if we replaced the discounting term $\exp(-r(T - t))$ by the price B_{tT} at time t of the T -maturity discount bond.

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