

DETECTION OF DISORDER BEFORE AN OBSERVABLE EVENT

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ABSTRACT. We consider a continuously observable process, which behaves like a standard Brownian motion up to a random time τ_1 , and as a Brownian motion with a known drift after τ_1 . At a stopping-time τ_2 that happens after the time τ_1 , an observable event occurs. We address the problem of detecting the change in the system's behaviour prior to the occurrence of the observable event. In particular, our formulation takes into account the information provided by the non-occurrence of the observable event, and where it is favourable to "raise the alarm" before this event. We show that this problem can be reduced to a one dimensional optimal stopping problem, to which we derive an explicit solution.

1. INTRODUCTION

We consider a continuously monitored random system and an observable event that is preceded by a change in the system's behaviour. Think of a continuously monitored noisy system that might crash or break down (observable event), where the crash is preceded by a change in the observable system's behaviour. Our aim is to detect the change in the system's behaviour and to anticipate the observable event. Examples of possible applications might be found in areas such as geology (e.g. volcanic activity), manufacturing and engineering (e.g. crash of electronic or mechanical components), finance (e.g. insider trading prior to press release) or medicine (detection of critical deceases prior to fatal observable symptoms). More precisely, we assume that the observable process is a standard Brownian motion up to the time τ_1 , which we assume has a known distribution of the exponential type. After this time, the observable process behaves like a Brownian motion with an a priori known drift. The observable event coincides with the stopping time τ_2 , which happens at an exponentially distributed time after τ_1 . We can model this by a jump of the observable process at time τ_2 , which makes τ_2 a stopping time with respect to the natural filtration of the observation process. This formulation is a variation of Shiryaev's classical Wiener disorder problem [23],[24] (see also [22],[25],[27],[28]). Independently of Shiryaev, Roberts [21] formulated and solved a similar problem. The reader is referred to Shiryaev [33] for a historical survey on quickest detection problems and Shiryaev [31] for their applications to technical analysis of financial data. For Poisson disorder problems,

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the reader is referred to Galchuk and Rozovsky [6], Davis [5], Peskir and Shiryaev [15],[16], Bayraktar et al. [2], and the references therein.

There is also a growing amount of literature studying discrete time versions of disorder problems. These sequential analysis formulations are often based on the cumulative sum procedure of Page [13],[14] (see also Pollak [17] and Moustakides [12]). Shiryaev [26] consider several criteria for optimality, and Poor [19] consider a criteria with exponential penalty for delay (Beibel [3] consider a similar version in continuous time). Pollak and Siegmund [18], Yakir [34] and Shiryaev [30] compare the cumulative sum procedure of Page with that of Shiryaev and Roberts.

The classical problem formulated by Shiryaev [23],[24] and subsequent variations, aims at detecting the unknown time at which an observable process changes from being a standard Brownian motion to a Brownian motion with drift. These formulations have in common that once the change has occurred, the observable process remains a Brownian motion with drift until infinity, and that the dynamics under partial information follows a one-dimensional Itô process. Notable exceptions are Gapeev and Peskir [8], which consider similar formulations, but on a finite horizon (see Gapeev and Peskir [7] for the sequential version of this problem), and Vellekoop and Clark [4] which consider a more general formulation. In the latter case, the dynamics under partial information is non-Markovian, and only asymptotic results are feasible. Our formulation with an observable time, is one of the few formulations that leads to dynamics which differs from the dynamics derived by Shiryaev [23],[24], yet is of the form of a one-dimensional Itô diffusion. The reason the dynamics in our case differs from the classical one is that the non-occurrence of the observable event provides information affecting the estimate of the time τ_1 .

Our criterion for optimality is a variation of the classical criteria of Shiryaev [23],[24], which consists of minimising the probability of “false alarm” plus a multiple of the expected delay in “sounding the alarm”. We adopt an analysis similar to that of Shiryaev and reduce the problem to a one-dimensional optimal stopping problem. The solution to this optimal stopping problem is characterised by a set of variational inequalities, which involves a one-dimensional differential equation. We prove that is equation is of the general confluent type, which enables us to completely solve the optimal stopping problem. The solution can be expressed in terms of known functions, one of them being Kummer’s U-function. This provides a solution to the disorder problem, for which we prove that there exists a unique $\pi^* \in (0, 1)$, such that the optimal time to “intervene”, or “raise the alarm”, is the first time $t \geq 0$ that either the observable event occurs or

$$(1.1) \quad \frac{p}{1-p} e^{\theta X_t - \frac{1}{2}\theta^2 t + (\lambda_1 - \lambda_2)t} + \lambda_1 \int_0^t e^{\theta(X_t - X_u) - \frac{1}{2}\theta^2(t-u) + (\lambda_1 - \lambda_2)(t-u)} du \geq \frac{\pi^*}{1 - \pi^*}.$$

In formula (1.1), X denotes the observable process, θ the drift of the observable process after time τ_1 , p the probability that τ_1 happens at time zero, and λ_1 and λ_2 parameters describing the distribution of τ_1 and τ_2 , respectively. The “threshold” number π^* is determined by an explicit equation only involving known functions.

2. FORMULATION OF THE PROBLEMS

Let $(\Omega, \mathcal{F}, \mathbf{F}, P)$ be a complete probability space satisfying the usual conditions, supporting a standard Brownian motion W and two independent random variables τ_1 and η with distributions

$$P(\tau_1 > t) = (1 - p) e^{-\lambda_1 t} \quad \text{and} \quad P(\eta > t) = e^{-\lambda_2 t}, \quad \lambda_1, \lambda_2 \in (0, \infty),$$

where both τ_1 and η are independent of W . Further, we assume that the filtration \mathbf{F} coincides with $\{\mathcal{F}_t\}_{0 \leq t < \infty}$, where $\mathcal{F}_t := \sigma(X_s + N_s; 0 \leq s \leq t)$, i.e. the filtration generated $X + N$. The continuous process X and the single jump process N are given by

$$(2.1) \quad X_t := W_t + \int_0^t \theta \mathbf{1}_{[\tau_1, \tau_2)}(u) du \quad \text{and} \quad N_t := \mathbf{1}_{[\tau_2, \infty)}(t).$$

Notice that both X and N are \mathbf{F} -adapted, since X is a continuous process and N is a pure jump process. The number θ appearing in the definition of X is assumed to be a known deterministic real constant. Further, τ_2 appearing in the definition of X and N , denotes the time of the observable event. We assume that

$$(2.2) \quad \tau_2 := \tau_1 + \eta.$$

Hence, the observed process X behaves like a standard Brownian motion up to time τ_1 . Between τ_1 and the observable time τ_2 , it behaves like a Brownian motion with drift θ . At time τ_2 , the process $X + N$ has an observable jump, after which X once again behaves like a standard Brownian motion. While continuously observing $X + N$, the aim is to detect the point τ_1 as accurately as possible, preferably before the occurrence of the observable event τ_2 .

Definition 2.1. Consider the problem of minimising the weighted average of the probability of “false alarm”, the expected delay in “raising the alarm”, and the probability of “failure to raise the alarm”. More specifically, the problem is to minimise the performance function

$$(2.3) \quad J(T; c_1, c_2) := P(T < \tau_1) + c_1 \mathbb{E}[(T - \tau_1)^+] + c_2 P(\tau_2 \leq T), \quad \text{for } c_1, c_2 \in (0, \infty),$$

over all \mathbf{F} -stopping times T .

We would like to remark that for $c_2 = 0$ this optimisation criterion coincides with Shiryaev’s classical criterion [23].

Introduce the \mathbf{F} -predictable process

$$(2.4) \quad \Pi_t := P(\tau_1 \leq t \mid \mathcal{F}_{t-}) = \mathbb{E}[\mathbf{1}_{[\tau_1, \infty)}(t) \mid \mathcal{F}_{t-}],$$

where $\mathbb{E}[\mathbf{1}_{[\tau_1, \infty)}(t) \mid \mathcal{F}_{t-}]$ denotes the predictable projection of the process $\mathbf{1}_{[\tau_1, \infty)}(\cdot)$. The predictable projection of $\mathbf{1}_{[\tau_1, \infty)}(\cdot)$ evaluated at a predictable finite stopping time t , is equal to $\lim_{n \rightarrow \infty} \mathbb{E}[\mathbf{1}_{[\tau_1, \infty)}(t) \mid \mathcal{F}_{t_n}]$, P-a.s., where $\{t_n\}_{n=1}^{\infty}$ is a sequence of predictable stopping times announcing t (see, e.g., Theorem 2.28 in Jacod and Shiryaev [9]). It follows from the results provided in Section 4 that Π provides a sufficient statistic for the optimal policy

of the disorder problem. In particular, according to Proposition 4.1, $\Pi_t = Y_t$, for $t \leq \tau_2$, where Y is a one-dimensional, time-homogeneous diffusion satisfying

$$(2.5) \quad dY_t = b(Y_t) dt + \sigma(Y_t) dB(t), \quad 0 \leq t \leq \infty,$$

where $b, \sigma : [0, 1] \rightarrow \mathbb{R}$ are given by

$$(2.6) \quad b(y) = (1 - y)(\lambda_1 - \lambda_2 y) \quad \text{and} \quad \sigma(y) = \theta y(1 - y).$$

The process B appearing in equation (2.5) is a standard \mathbf{F} -Brownian motion, often referred to as the innovation process. It is related to X by

$$B_t = X_t + \int_0^t \theta \Pi_s \mathbf{1}_{[0, \tau_2)}(s) ds, \quad 0 \leq t < \infty.$$

Note that this deviates from the dynamics in the classical Wiener disorder case (see e.g., Shiryaev [23], [24]). However, if we let λ tend to 0 (which corresponds to τ_2 being infinite almost surely), we recover the partial information dynamics for the classical case.

Observe that $\sigma(0) = \sigma(1) = b(1) = 0$ and $b(0) = \lambda_1 > 0$. Therefore, if $Y_0 = y \in [0, 1]$, it follows that Y never leaves $[0, 1]$. From Theorem 2.9 in Karatzas and Shreve [11], it follows that equation (2.5) has a unique strong solution. Further we note that since $P(\tau_2 < \infty) = 1$, it follows from Feller's test for explosions (see, e.g., Theorem 5.29 in Karatzas and Shreve [11]) that if $\Pi_0 = \pi \in [0, 1)$, then $P(\Pi_t \in [0, 1)) = 1$, for every $t \leq \tau_2$.

Definition 2.2 (Optimal stopping problem). Let V be the value function corresponding to the optimal stopping problem given by

$$(2.7) \quad V(\pi; \kappa) := \inf_{T \in \mathcal{S}} \mathbb{E}^\pi \left[\int_0^{T \wedge \tau_2} (\Pi_t - \kappa) dt \right],$$

for initial conditions $\pi \in [0, 1]$ and parameter $\kappa \in (0, 1)$.

The next result provides a relationship between the disorder problem (2.3) and optimal stopping problem (2.7).

Lemma 2.3. *Let Π be given by (2.4), with initial value $\Pi_0 = \pi \in [0, 1)$, and V be given by (2.7). The problem formulated in Definition 2.1 is related to V by*

$$(2.8) \quad \inf_{T \in \mathcal{S}} J(T; c_1, c_2) = 1 + (\lambda_1 + c_1 + c_2 \lambda_2) V \left(\pi; \frac{\lambda_1}{\lambda_1 + c_1 + c_2 \lambda_2} \right).$$

Proof. First, observe that since τ_2 is an \mathbf{F} -stopping time, an optimal stopping rule T for the disorder problem must satisfy $T = T \wedge \tau_2$. Now, for any $f \in C^2(\mathbb{R})$ and $T \in \mathcal{S}$, we calculate that

$$\mathbb{E}^\pi \left[\int_0^{T \wedge \tau_2} \sigma^2(\Pi_t) (v'(\Pi_t))^2 dt \right] \leq \frac{1}{4} \left(\max_{0 \leq \pi \leq 1} |v'(\pi)| \right)^2 \mathbb{E}^\pi [T \wedge \tau_2] < \infty,$$

since Π takes values in $[0, 1]$. By Itô's formula (see, e.g., Theorem 32 in Protter [20]), Proposition 4.1 and Proposition 4.2, we calculate that

$$(2.9) \quad \mathbb{E}^\pi [\mathbf{1}_{[0, \tau_1)}(T \wedge \tau_2)] = \mathbb{E}^\pi [\Pi_{T \wedge \tau_2} + \Delta \Pi_{T \wedge \tau_2}] = \mathbb{E}^\pi \left[\int_0^{T \wedge \tau_2} \lambda_1 (1 - \Pi_t) dt \right].$$

Identity (2.8) then follows from equation (2.9), Proposition 4.2, and the calculation

$$\mathbb{E}[(T \wedge \tau_2 - \tau_1)^+] = \mathbb{E}\left[\int_0^{T \wedge \tau_2} \mathbf{1}_{[\tau_1, \infty)}(t) dt\right] = \mathbb{E}\left[\int_0^{T \wedge \tau_2} \Pi_t dt\right].$$

□

Remark 2.4. From the remark by Shiryaev [32], it follows that

$$(2.10) \quad \inf_{T \in \mathcal{S}} R(T; c) = \mathbb{E}[\tau_1] + (2 + c\lambda_2)V\left(\pi; \frac{1}{2 + c\lambda_2}\right),$$

where R denotes the criterion given by

$$(2.11) \quad R(T; c) := \mathbb{E}[|T - \tau_1|] + cP(\tau_2 \leq T), \quad \text{for } c \in (0, \infty).$$

3. THE SOLUTION TO THE OPTIMAL STOPPING PROBLEM

Having established the relation between the disorder problem(2.3) and the optimal stopping problem (2.7), we proceed to find the solution to the stopping problem. The analysis is based on the principle of smooth fit, and is similar to the original analysis of Shiryaev (see e.g. [33]). With reference to the general theory of optimal stopping, the Hamilton-Jacobi-Bellman equation corresponding problem (2.7) is of the form

$$(3.1) \quad (\mathcal{L}v)(\pi) + \pi - \kappa = 0, \quad \text{for } \pi \in [0, \pi^*],$$

$$(3.2) \quad (\mathcal{L}v)(\pi) + \pi - \kappa > 0, \quad \text{for } \pi \in (\pi^*, 1],$$

$$(3.3) \quad v(\pi) < 0, \quad \text{for } \pi \in [0, \pi^*),$$

$$(3.4) \quad v(\pi) = 0, \quad \text{for } \pi \in [\pi^*, 1],$$

for some $\pi^* \in (0, 1)$, where \mathcal{L} denotes the operator given by

$$(\mathcal{L}v)(\pi) := \frac{1}{2}\theta^2\pi^2(1 - \pi)^2v''(\pi) + (1 - \pi)(\lambda_1 - \lambda_2\pi)v'(\pi) - \lambda_2\pi v(\pi).$$

In view of the results in, e.g., Shiryaev [33], we look for solutions to (3.1)–(3.4) that are of class $C^1([0, 1]) \cap C^2([0, 1] \setminus \{\pi^*\})$, for some $\pi^* \in (0, 1)$.

Define the function $g : [0, 1] \rightarrow \mathbb{R}$ by

$$(3.5) \quad g(y) := \frac{1}{\lambda_1\lambda_2} \left((\lambda_1 + \lambda_2)\kappa - \lambda_1 - \lambda_2\kappa y \right).$$

From the calculation $(\mathcal{L}g)(y) = y - \kappa$, we verify that if v is a solution to (3.1)–(3.4), then $u = v + g$ is a solution to (3.6)–(3.9). Moreover, if u is a solution to (3.6)–(3.9), then $v = u - g$ is a solution to (3.1)–(3.4). So, if we manage to find a solution u of class $C^1([0, 1]) \cap C^2([0, 1] \setminus \{y^*\})$ satisfying

$$(3.6) \quad (\mathcal{L}u)(y) = 0, \quad \text{for } y \in [0, y^*],$$

$$(3.7) \quad (\mathcal{L}u)(y) > 0, \quad \text{for } y \in (y^*, 1],$$

$$(3.8) \quad u(y) - g(y) < 0, \quad \text{for } y \in [0, y^*),$$

$$(3.9) \quad u(y) - g(y) = 0, \quad \text{for } y \in [y^*, 1],$$

for some $y^* \in [0, 1)$, then $v = u - g$ belongs to $C^1([0, 1]) \cap C^2([0, 1] \setminus \{\pi^*\})$ and satisfies (3.1)–(3.4), with $\pi^* = y^*$.

In view of (3.6), we proceed by addressing the question of possible solutions to

$$(3.10) \quad \frac{1}{2}\theta^2 y^2(1-y)^2 u''(y) + (1-y)(\lambda_1 - \lambda_2 y)u'(y) - \lambda_2 y u(y) = 0, \quad y \in [0, 1).$$

For $y \in (0, 1)$, we can write eq. (3.10) as

$$(3.11) \quad u''(y) + \frac{2(\lambda_1 - \lambda_2 y)}{\theta^2 y^2(1-y)} u'(y) - \frac{2}{\theta^2 y(1-y)^2} u(y) = 0.$$

Let $h : (0, 1) \rightarrow \mathbb{R}$ and $f : (0, 1) \rightarrow \mathbb{R}$ be the functions defined by

$$h(y) := \frac{2\lambda_1(1-y)}{\theta^2 y} \quad \text{and} \quad f(y) := \xi \ln\left(\frac{1-y}{y}\right),$$

where the constant ξ is given by

$$(3.12) \quad \xi := \frac{1+\gamma}{2} + \frac{\lambda_1 - \lambda_2}{\theta^2},$$

$$(3.13) \quad \gamma := \frac{1}{\theta^2} \left(4\theta^2(\lambda_1 + \lambda_2) + 4(\lambda_1 - \lambda_2)^2 + \theta^4 \right)^{1/2}.$$

Moreover, define the constants

$$(3.14) \quad a := 1 + \frac{\gamma}{2} + \frac{\lambda_1 - \lambda_2}{\theta^2} = \frac{1}{2} + \xi,$$

$$(3.15) \quad b := 1 + \gamma.$$

Now, with $A = -1$, straightforward, but lengthy calculations show that

$$\frac{2(\lambda_1 - \lambda_2 y)}{\theta^2 y^2(1-y)} = \frac{2A}{y} + 2f'(y) + \frac{bh'(y)}{h(y)} - h'(y) - \frac{h''(y)}{h'(y)},$$

and

$$\begin{aligned} -\frac{2}{\theta^2 y(1-y)^2} &= \left(\frac{bh'(y)}{h(y)} - h'(y) - \frac{h''(y)}{h'(y)} \right) \left(\frac{A}{y} + f'(y) \right) \\ &\quad + \frac{A(A-1)}{y^2} + \frac{2Af'(y)}{y} + f''(y) + (f'(y))^2 - \frac{a(h'(y))^2}{h(y)}. \end{aligned}$$

According to Chapter 13 in Abramowitz and Stegun [1], these calculations show that equation (3.11) is of the general confluent type, which has two independent solutions

$$(3.16) \quad \phi(y) := y^{-A} e^{-f(y)} M(a, b, h(y)) = y^{1-\xi} (1-y)^\xi M(a, b, h(y))$$

and

$$(3.17) \quad \psi(y) := y^{-A} e^{-f(y)} U(a, b, h(y)) = y^{1-\xi} (1-y)^\xi U(a, b, h(y)),$$

where M and U denote the Kummer M-function and the Kummer U-function, respectively. However, we are looking for a solution u of (3.10) for all $y \in [0, 1)$, not only for $y \in (0, 1)$.

A solution u of (3.11), for $y \in (0, 1)$, is a solution to (3.10) if and only if u satisfies $u'(0+) := \lim_{y \downarrow 0} u'(y) = 0$. According to Chapter 13 in Abramowitz and Stegun [1],

$$M'(a, b, x) = \frac{a}{b} M(a+1, b+1, x) \quad \text{and} \quad \lim_{x \rightarrow \infty} M(a, b, x) e^{-x} x^{b-a} = C, \quad a, b > 0,$$

for some constant $C \in \mathbb{R}$. Together with the observation $\lim_{y \downarrow 0} h(y) = \infty$ and the properties of U listed in Chapter 13 in Abramowitz and Stegun [1], it follows that u is a solution to (3.10) if and only if $u(y) = C \psi(y)$, for some constant $C \in \mathbb{R}$.

A function u of the form $u(y) = C \psi(y)$ is of class $C^1([0, 1]) \cap C^2([0, 1] \setminus \{y^*\})$, for some $y^* \in [0, 1)$, and a solution to (3.6) and (3.9), if and only if C and $y^* \in [0, 1)$ satisfy

$$(3.18) \quad C \psi(y^*) = g(y^*),$$

$$(3.19) \quad C \psi'(y^*) = g'(y^*).$$

Equations (3.18)–(3.19) has a solution if and only if there exists a $y^* \in [0, 1)$ which satisfies $F(y^*) = 0$, where

$$(3.20) \quad F(y) := g(y) \psi'(y) - g'(y) \psi(y).$$

Moreover, if such a y^* exists, the constant C appearing in (3.18)–(3.19), is given by $C = g(y^*)/\psi(y^*)$. Further, $F(0+) = 0$. However, in view of the calculation,

$$(3.21) \quad (\mathcal{L}0)(\pi) + \pi - \kappa < 0, \quad \text{for } x < \kappa,$$

we are looking for a $\pi^* \in [\kappa, 1)$.

We are now ready to state the solution to the optimal stopping problem formulated in Definition 2.2.

Theorem 3.1. *Let F be given by (3.20), and let π^* be the unique solution to $F(\pi) = 0$, for $\pi \in (\kappa, 1)$. Further, let ψ and g be the functions given by (3.17) and (3.5), respectively. Then*

$$(3.22) \quad v(\pi) := \begin{cases} \frac{g(\pi^*)}{\psi(\pi^*)} \psi(\pi) - g(\pi) & , \text{ for } \pi \in [0, \pi^*), \\ 0 & , \text{ for } \pi \in [\pi^*, 1], \end{cases}$$

is the unique function of class $C^1([0, 1]) \cap C^2([0, 1] \setminus \{\pi^*\})$, for some $\pi^* \in (0, 1)$, that satisfies (3.1)–(3.4). Moreover, v identifies with the value function V given by (2.7), and the stopping time

$$(3.23) \quad T^* := \inf \{t \geq 0; \Pi_t \geq \pi^*\},$$

provides an optimal stopping rule for (2.7).

Proof. We will start by proving that the equation $F(\pi) = 0$ has a unique solution, for $\pi \in (\kappa, 1)$. The proof will be based on the proof of Theorem 3.1 in Johnson and Zervos [10]. Since ψ satisfies equation (3.10), and $F'(y) = g(y) \psi''(y) - g''(y) \psi(y)$, we calculate that

$$F'(y) = -\frac{2b(y)}{\sigma^2(y)} F(y) - \frac{2\psi(y)}{\sigma^2(y)} (\mathcal{L}g)(y).$$

Combining this equation with the observation $(\mathcal{L}g)(y) = y - \kappa$, and the properties of the scale function, denote by S (see, e.g., Karatzas and Shreve [11]), we obtain

$$(3.24) \quad \left(\frac{F(y)}{S'(y)} \right)' = -\frac{2\psi(y)}{\sigma^2(y)S'(y)}(y - \kappa),$$

where

$$S'(y) = \exp\left(-2 \int_{x_0}^y \frac{b(u) du}{\sigma^2(u)}\right), \quad y \in (0, 1).$$

From (3.24), we conclude that F/S' is strictly increasing on $(0, \kappa)$, and strictly decreasing on $(\kappa, 1]$. This observation together with the continuity of g , the property $\psi(0+) = \psi'(0+) = 0$, and $S'(0+) = \infty$, imply that

$$(3.25) \quad 0 = \frac{F(0+)}{S'(0+)} \leq \frac{F(y)}{S'(y)} < \frac{F(\kappa)}{S'(\kappa)},$$

for all $y \in (0, \kappa)$. We conclude that $F(\kappa) > 0$. Furthermore, this observation and the monotonicity of F/S' , implies that (3.20) has a unique solution $\pi^* \in (\kappa, 1)$ if $\liminf_{y \uparrow 1} F(y) < 0$. To prove that this is indeed the case, observe that

$$(3.26) \quad F(y) = g^2(y) \left(\frac{\psi(y)}{g(y)} \right)', \quad \text{for } y \in (0, 1) \setminus \{g(0)\}.$$

Suppose, that $\liminf_{y \uparrow 1} F(y) \geq 0$. In view of (3.26), this assumption implies that

$$\liminf_{y \uparrow 1} \frac{\psi(y)}{g(y)} > -\infty.$$

However, since $\psi(1-) := \lim_{y \uparrow 1} \psi(y) = +\infty$ and $g(1) = \frac{\kappa-1}{\lambda_2} < 0$, we see that this is a contradiction. Hence, $\liminf_{y \uparrow 1} F(y) < 0$, and we conclude that (3.20) has a unique solution $\pi^* \in (\kappa, 1)$.

We continue by proving that v given by (3.22) is the unique solution to (3.1)–(3.4). From the discussion preceding the statement of Theorem 3.1, it follows that v is the unique solution to (3.1) and (3.4) of class $C^1([0, 1]) \cap C^2([0, 1] \setminus \{\pi^*\})$, for some $\pi^* \in [\kappa, 1)$. Moreover, in view of (3.21), it follows that v is the only possible solution to (3.1)–(3.4) of this class. So, all we have to show, is that

$$(\mathcal{L}v)(\pi) + \pi - \kappa > 0, \quad \text{for } \pi \in (\pi^*, 1] \quad \text{and} \quad v(\pi) < 0, \quad \text{for } \pi \in [0, \pi^*).$$

The first of these inequalities follows from the observation

$$(\mathcal{L}v)(\pi) + \pi - \kappa = \pi - \kappa > 0,$$

for $\pi \in (\pi^*, 1]$, since $\pi^* > \kappa$. The second inequality is equivalent to

$$(3.27) \quad \frac{g(\pi^*)}{\psi(\pi^*)} < \frac{g(\pi)}{\psi(\pi)}, \quad \text{for } \pi \in [0, \pi^*).$$

However, this inequality follows from the calculation

$$\left(\frac{g(\pi)}{\psi(\pi)}\right)' = -\frac{F(\pi)}{\psi^2(\pi)} < 0, \quad \text{for } \pi \in (0, \pi^*),$$

since by (3.25) and the construction of π^* , it follows that $F(\pi) > 0$ for $\pi \in (0, \pi^*)$.

It remains to prove the optimality of the stopping time T^* given by (3.23), and that v identifies with V . Let $T \in \mathcal{S}$ be an arbitrary stopping time. By Itô's formula and Proposition 4.2,

(3.28)

$$\mathbb{E}^\pi \left[\int_0^{T \wedge \tau_2} (\Pi_t - \kappa) dt \right] = v(\pi) - \mathbb{E}^\pi [v(\Pi_{T \wedge \tau_2}) + \Delta \Pi_{T \wedge \tau_2}] + \mathbb{E}^\pi \left[\int_0^{T \wedge \tau_2} (\mathcal{L}v)(\Pi_t) dt \right],$$

where we have used that the expectation of $\int_0^{T \wedge \tau_2} \sigma(\Pi_t) v'(\Pi_t) dB_t$ is zero since

$$\mathbb{E}^\pi \left[\int_0^{T \wedge \tau_2} \sigma^2(\Pi_t) (v'(\Pi_t))^2 dt \right] \leq \frac{1}{4} \left(\max_{0 \leq \pi \leq 1} |v'(\pi)| \right)^2 \mathbb{E}^\pi [T \wedge \tau_2] < \infty.$$

Since v satisfies (3.1)–(3.4), we conclude from (3.28) that

$$\mathbb{E}^\pi \left[\int_0^T (\Pi_t - \kappa) dt \right] \geq \mathbb{E}^\pi \left[\int_0^{T \wedge \tau_2} (\Pi_t - \kappa) dt \right] \geq v(\pi).$$

Further, from (3.28) and the construction of v and T^* , it follows that

$$\mathbb{E}^\pi \left[\int_0^{T^*} (\Pi_t - \kappa) dt \right] = v(\pi),$$

which implies that T^* is optimal and that v identifies with the value function V . \square

The optimal stopping rule for the disorder problem (2.3) now follows from Theorem 3.1 and the expression for Π given by Proposition 4.1. The optimal time to “raise the alarm” is given by

(3.29)

$$\tau_2 \wedge \inf \left\{ t \geq 0 \mid \frac{p}{1-p} + \lambda_1 \int_0^t e^{-\theta X_u + \frac{1}{2} \theta^2 u + (\lambda_2 - \lambda_1) u} du \geq \frac{\pi^*}{1 - \pi^*} e^{-\theta X_t + \frac{1}{2} \theta^2 t + (\lambda_2 - \lambda_1) t} \right\}.$$

Note that the expression for the stopping time given by (3.29) is equivalent to the expression given by (1.1). The “threshold” parameter $\pi^*(\kappa, 1)$, is determined by the equation $F(\pi^*) = 0$, where the parameter κ is specified through the problem formulation and Lemma 2.3.

4. APPENDIX; DERIVATION OF THE DYNAMICS UNDER PARTIAL INFORMATION

In this section we will characterise the behaviour of the observable process X and the observable event τ_2 with respect to the filtration \mathbf{F} . The setup and analysis is similar to that of Shiryaev [23], [24], and will be based on the change-of-measure method proposed by Davis [5].

Let $(\Omega, \mathcal{F}, P_0)$ be a probability space satisfying the usual assumptions, supporting a standard Brownian motion X , as well as two independent random variables τ and η , both of which are independent of X , with known distributions

$$P_0(\tau_1 > t) = (1 - p)e^{-\lambda_1 t}, \quad P_0(\eta > t) = e^{-\lambda_2 t} \quad \text{for } 0 \leq t < \infty,$$

where $p \in (0, 1)$ and $\lambda_1, \lambda_2 \in (0, \infty)$. Define

$$\tau_2 := \tau_1 + \eta.$$

Further, let $\mathbf{F} := \{\mathcal{F}_t\}_{0 \leq t < \infty}$, where $\mathcal{F}_t := \sigma(X_s + \mathbf{1}_{[\tau_2, \infty)}(s); 0 \leq s \leq t)$. The filtration \mathbf{F} is then the smallest filtration for which X is adapted and τ_2 is a stopping time. Define the larger filtration $\mathbf{G} := \{\mathcal{G}_t\}_{0 \leq t < \infty}$, where $\mathcal{G}_t := \sigma(X_s, \tau_1, \eta; 0 \leq s \leq t)$, i.e. the filtration generated by X given complete information about τ_1 and τ_2 . According to Girsanov's theorem (see, e.g., Theorem 5.1 in Karatzas and Shreve [11]), there exists a probability measure P on (Ω, \mathcal{F}) which is absolutely continuous with respect to P_0 , under which the process

$$W_t := X_t - \theta \int_0^t \mathbf{1}_{[\tau_1, \tau_2)}(s) ds, \quad 0 \leq t < \infty$$

is a standard \mathbf{G} -Brownian motion. The likelihood ratio of this measure with respect to P_0 is given by

$$\begin{aligned} \frac{dP}{dP_0} \Big|_{\mathcal{G}_t} &= \exp\left(\theta \int_0^t \mathbf{1}_{[\tau_1, \tau_2)}(s) dX_s - \frac{1}{2}\theta^2 \int_0^t \mathbf{1}_{[\tau_1, \tau_2)}(s) ds\right) \\ (4.1) \quad &= \mathbf{1}_{[0, \tau_1)}(t) + \frac{L_t}{L_{\tau_1}} \mathbf{1}_{[\tau_1, \tau_2)}(t) + \frac{L_{\tau_2}}{L_{\tau_1}} \mathbf{1}_{[\tau_2, \infty)}(t) =: Z_t, \end{aligned}$$

where

$$L_t := \exp\left(\theta X_t - \frac{1}{2}\theta^2 t\right), \quad 0 \leq t < \infty.$$

The random variables τ_1 and η are \mathcal{G}_0 measurable, from which it follows that under P , τ_1 and η are independent of the \mathbf{G} -Brownian motion W ,

$$P(\tau_1 > t) = \mathbb{E}_0[Z_0 \mathbf{1}_{[0, \tau_1)}(t)] = P_0(\tau_1 > t) = (1 - p)e^{-\lambda_1 t},$$

and

$$P(\eta > t) = \mathbb{E}_0[Z_0 \mathbf{1}_{[0, \eta)}(t)] = P_0(\eta > t) = e^{-\lambda_2 t}.$$

From the previous discussion, we conclude that on the probability space (Ω, \mathcal{F}, P) , the process W , and the random variables τ_1 and τ_2 , have exactly the properties described in Section 2.

The following result provides an explicit expression for Π given by (2.4), and states that Π is a one-dimensional, time-homogeneous diffusion with respect to the filtration \mathbf{F} .

Proposition 4.1. *Let Π be the process given by (2.4). Then Π is a càglàd process, and admits the expression*

$$(4.2) \quad \Pi_t = \frac{pL_t e^{-\lambda_2 t} + (1-p) \int_0^t \frac{L_u}{L_t} \lambda_1 e^{-(\lambda_1 - \lambda_2)u - \lambda_2 t} du}{(1-p) e^{-\lambda_1 t} + pL_t e^{-\lambda_2 t} + (1-p) \int_0^t \frac{L_u}{L_t} \lambda_1 e^{(\lambda_2 - \lambda_1)u - \lambda_2 t} du} \mathbf{1}_{[0, \tau_2]}(t) + \mathbf{1}_{(\tau_2, \infty)}(t),$$

for all \mathbf{F} -predictable stopping times $0 \leq t < \infty$. Moreover, Π satisfies the one-dimensional, time-homogeneous stochastic differential equation

$$(4.3) \quad d\Pi_t = (1 - \Pi_t)(\lambda_1 - \lambda_2 \Pi_t) dt + \theta \Pi_t (1 - \Pi_t) dB_t, \quad \Pi_0 = p,$$

for $0 \leq t \leq \tau_2$, where the standard \mathbf{F} -Brownian motion B , related to X by

$$B_t = X_t + \int_0^t \theta \Pi_s \mathbf{1}_{[0, \tau_2]}(s) ds, \quad 0 \leq t < \infty,$$

is the innovation process.

Proof. For every finite deterministic time t , choose a sequence of \mathbf{F} -predictable stopping times $\{t_n\}_{n=1}^\infty$ announcing t . By (4.1) and Theorem 2.28 in Jacod and Shiryaev [9], it follows that

$$(4.4) \quad \Pi_t = \lim_{n \rightarrow \infty} \frac{\mathbb{E}_0[Z_{t_n} \mathbf{1}_{[\tau_1, \infty)}(t) | \mathcal{F}_{t_n}]}{\mathbb{E}_0[Z_{t_n} | \mathcal{F}_{t_n}]}, \quad \mathbf{P} - a.s..$$

In view of the continuity of L , the identities $\mathbf{1}_{[0, \tau_2]}(t-) = \mathbf{1}_{[0, \tau_2]}(t)$ and $\mathbf{1}_{[\tau_2, \infty)}(t-) = \mathbf{1}_{(\tau_2, \infty)}(t)$, standard calculations show that the right-hand side of (4.4) equals (4.2). Since this holds for every finite and deterministic time t , the expression for Π follows.

As for the proof of the second part, observe that

$$H_t := \frac{\Pi_t \mathbf{1}_{[0, \tau_2]}(t)}{1 - \Pi_t \mathbf{1}_{[0, \tau_2]}(t)} = U_t + V_t, \quad 0 \leq t < \infty,$$

where U and V are given by

$$U_t := \frac{p}{1-p} L_t e^{(\lambda_1 - \lambda_2)t} \mathbf{1}_{[0, \tau_2]}(t) \quad \text{and} \quad V_t := \int_0^t \frac{L_u}{L_t} e^{(\lambda_2 - \lambda_1)u + (\lambda_1 - \lambda_2)t} du \mathbf{1}_{[0, \tau_2]}(t),$$

respectively. By applying Itô's formula, we verify that U and V satisfy

$$\begin{aligned} dU_t &= (\lambda_1 - \lambda_2)U_t dt + \theta U_t dX_t, & U_0 &= \frac{p}{1-p}, \\ dV_t &= [\lambda_1 + (\lambda_1 - \lambda_2)V_t] dt + \theta V_t dX_t, & V_0 &= 0, \end{aligned}$$

for $0 \leq t \leq \tau_2$. These equations imply that H satisfies

$$(4.5) \quad dH_t = [\lambda_1 + (\lambda_1 - \lambda_2)H_t] dt + \theta H_t dX_t, \quad H_0 = \frac{p}{1-p},$$

for $0 \leq t \leq \tau_2$. Moreover, according to (4.2), $\Pi_t = 1$ for $\tau_2 < t < \infty$. Since Π is related to H by $\Pi_t = \frac{H_t}{1+H_t}$, it follows from Itô's formula that Π satisfies (4.3), for $0 \leq t \leq \tau_2$.

Finally, it follows from Lévy's characterisation of Brownian motion (see, e.g., Theorem 38 in Protter [20]) that B is a standard (\mathbf{F}, P) -Brownian motion. \square

We proceed to characterise the observable time τ_2 with respect to the information flow \mathbf{F} . The next result states that the compensator of the observable jump process N is λ_2 times Π , which completely determines the dynamics of N with respect to \mathbf{F} .

Proposition 4.2. *Let N and Π be given by (2.1) and (2.4), respectively. The compensator of N is the \mathbf{F} -predictable process*

$$(4.6) \quad A_t := \int_0^t \lambda_2 \Pi_s ds, \quad 0 \leq t < \infty,$$

which is continuous and increasing, and satisfies $|A_t| \leq 1 + \lambda_2 t$, for all $0 \leq t < \infty$.

Proof. Let t be a finite deterministic time and $\epsilon > 0$ a deterministic positive number. Define a process Ψ with parameter ϵ , by

$$(4.7) \quad \Psi_t(\epsilon) := P(\tau_2 \leq t + \epsilon \mid \mathcal{F}_{t-}) = \mathbb{E}[\mathbf{1}_{[\tau_2, \infty)}(t + \epsilon) \mid \mathcal{F}_{t-}].$$

We claim that for every finite \mathbf{F} -predictable time t ,

$$\Lambda_t := \lim_{\epsilon \downarrow 0} \frac{\Psi_t(\epsilon) - \Psi_t(0+)}{\epsilon} = \lambda_2 \Pi_t \mathbf{1}_{[0, \tau_2]}(t), \quad P\text{-a.s.}$$

Let t be a deterministic finite time. Then rather lengthy, but straightforward calculations show that $\Psi_t(\epsilon) = \Pi_t(\epsilon)/K_t$, where

$$\begin{aligned} \Pi_t(\epsilon) := & \left(\int_t^{t+\epsilon} \lambda_1 e^{-\lambda_1 u} [1 - e^{-\lambda_2(t+\epsilon-u)}] du + p L_t (1 - e^{-\lambda_2 \epsilon}) e^{-\lambda_2 t} \right. \\ & \left. + (1-p) \int_0^t \frac{L_t}{L_u} \lambda_1 e^{-\lambda_1 u} [1 - e^{-\lambda_2(t-u)-\lambda_2 \epsilon}] du \right) \mathbf{1}_{[0, \tau_2]}(t) + \mathbf{1}_{(\tau_2, \infty)}(t), \end{aligned}$$

and

$$K_t := \left((1-p) e^{-\lambda_1 t} + p L_t e^{-\lambda_2 t} + (1-p) \int_0^t \frac{L_t}{L_u} \lambda_1 e^{(\lambda_2 - \lambda_1)u - \lambda_2 t} du \right) \mathbf{1}_{[0, \tau_2]}(t) + \mathbf{1}_{(\tau_2, \infty)}(t).$$

According to standard calculus,

$$K'_t(0+) = \lim_{\epsilon \downarrow 0} \frac{K_t(\epsilon) - K_t(0+)}{\epsilon} = \lambda_2 \left(p L_t e^{-\lambda_2 t} + (1-p) \int_0^t \frac{L_t}{L_u} \lambda_1 e^{(\lambda_2 - \lambda_1)u - \lambda_2 t} du \right) \mathbf{1}_{[0, \tau_2]}(t).$$

The claim (4.7) then follows from the observation $\Lambda_t = K'_t(0+)/L_t = \lambda_2 \Pi_t$, for $0 \leq t \leq \tau_2$.

What remains to prove is that A , given by (4.6), is the compensator of N . First observe that the continuity and the claimed inequality for A follows from the fact that $0 \leq \Pi_t \leq 1$, for every $0 \leq t < \infty$. Moreover, A is \mathbf{F} -predictable and of finite variation, so it suffices to prove that $N - A$ is an \mathbf{F} -martingale. Define

$$A_{[s, t]}^{(n)} := \sum_{k=0}^{2^n - 1} \mathbb{E} \left[N_{s+(k+1)(t-s)2^{-n}} - N_{s+k(t-s)2^{-n}} \mid \mathcal{F}_{(s+k(t-s)2^{-n})-} \right], \quad \text{for } 0 \leq s < t < \infty,$$

and observe that $A_{[s,t]}^{(n)}$ admits the representation

$$A_{[s,t]}^{(n)} = \sum_{k=0}^{2^n-1} \frac{\Psi_{s+k(t-s)2^{-n}}(2^{-n}) - \Psi_{s+k(t-s)2^{-n}}(0+)}{2^{-n}} \cdot 2^{-n}.$$

From (4.7), it follows that

$$\lim_{n \rightarrow \infty} A_{[s,t]}^{(n)} = A_t - A_s, \quad P\text{-a.s.}, \quad \text{for every } 0 \leq s < t < \infty.$$

By the inequality $|N_t - A_{[0,t]}^{(n)}| < (2 + \lambda_2)t$, for all $n = 1, 2, \dots$, and the dominated convergence theorem, we calculate that

$$\begin{aligned} & \mathbb{E}[N_t - N_s - (A_t - A_s) | \mathcal{F}_s] \\ &= \mathbb{E}\left[N_t - N_s - \lim_{n \rightarrow \infty} A_{[s,t]}^{(n)} \mid \mathcal{F}_s\right] \\ &= \lim_{n \rightarrow \infty} \sum_{i=0}^{2^n-1} \mathbb{E}\left[N_{s+(i+1)(t-s)2^{-n}} - N_{s+i(t-s)2^{-n}} \right. \\ & \quad \left. - \mathbb{E}[N_{s+(i+1)(t-s)2^{-n}} - N_{s+i(t-s)2^{-n}} \mid \mathcal{F}_{(s+i(t-s)2^{-n})-}] \mid \mathcal{F}_s\right] \\ &= 0, \end{aligned}$$

which proves that $N - A$ is an \mathbf{F} -martingale. \square

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